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Non-Atomic Games for Multi-User Systems

Nicolas Bonneau, Mérouane Debbah, Eitan Altman, and Are Hjørungnes

Abstract—In this contribution, the performance of a multi-user system is analyzed in the context of frequency selective fading channels. Using game theoretic tools, a useful framework is provided in order to determine the optimal power allocation when users know only their own channel (while perfect channel state information is assumed at the base station). This scenario illustrates the case of decentralized schemes, where limited information on the network is available at the terminal. Various receivers are considered, namely the matched filter, the MMSE filter and the optimum filter. The goal of this paper is to extend previous work, and to derive simple expressions for the non-cooperative Nash equilibrium as the number of mobiles becomes large and the spreading length increases. To that end two asymptotic methodologies are combined. The first is asymptotic random matrix theory which allows us to obtain explicit expressions of the impact of all other mobiles on any given tagged mobile. The second is the theory of non-atomic games which computes good approximations of the Nash equilibrium as the number of mobiles grows.

Index Terms—

I. INTRODUCTION

RESOURCE allocation is of major interest in the context of multi-user systems. In uplink multi-user systems, it is important for users to transmit with enough power to achieve their requested quality of service, but also to minimize the amount of interference caused to other users. Thus, an efficient power allocation mechanism prevents an excessive consumption of the limited resources of the users.

The most straightforward way to design a power allocation (PA) mechanism is as a centralized procedure, with the base station receiving training sequences from the users and signaling back the optimal power allocation for each user. Power control schemes in cellular systems were first introduced for TDMA/FDMA [1], [2]; more recently, an optimal scheme was derived for Code Division Multiple Access (CDMA) [3]. In order to achieve the optimal capacity, the users may also be sorted according to some rule of precedence [4]. However, this involves a non negligible overhead and numerous non informational transmissions. In addition, the complexity of

centralized schemes increases drastically with the number of users. As discussed in [5], centralized algorithms generally do not have a practical use for real systems, but provide useful bounds on the performance that can be attained by distributed algorithms.

A way to avoid the constraints of a centralized procedure is to implement a decentralized one where each user calculates its estimation of the optimal transmission power according to its local knowledge of the system. This is, for example, the case in ad-hoc networks applications. Most of the time, a distributed algorithm means an iterative version of a centralized one. Mobiles update their power allocation according to some rule based on the limited information they retrieve from the system. Supposing that an optimal power allocation exists, a distributed iterative algorithm is derived from a differential equation in [6] and its convergence is proven analytically. A distributed version of the algorithm of [2] is presented in [7]. Building on these results, a general framework for power control in cellular systems is given in [8]. A review of different methods of centralized and distributed power control in CDMA systems is given in [5].

In this context, a natural framework is game theory, which studies competition (as well as cooperation) between independent actors. Tools of game theory have already been frequently used as a central framework for modeling competition and cooperation in networking, see for example [9] and references therein. Building on the framework of [8], a game theoretic approach was introduced in [10], [11]. Numerous works on power allocation games have followed since, a selection of which we present in Sec. II. In particular, this contribution can be considered as an extension of previous work such as [12], as detailed in Sec. II.

Game theory can be used to treat the case of any number of players. However, as the size of the system increases, the number of parameters increases drastically and it is difficult to gain insight on the expressions obtained.

In order to obtain expressions depending only on few parameters, we consider the system in an asymptotic setting, letting both the number of users and the spreading factor tend to infinity with a fixed ratio. We use tools of random matrix theory [13] to analyze the system in this limit. Random matrix theory is a field of mathematical physics that has been recently applied to wireless communications to analyze various measures of interest such as capacity or Signal to Interference plus Noise Ratio (SINR). Interestingly, it enables to single out the main parameters of interest that determine the performance in numerous models of communication systems with more or less involved models of attenuation [14], [15], [16], [17]. In addition, these asymptotic results provide good approximations for the practical finite size case, as shown by

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simulations.

In the asymptotic regime, the non-cooperative game becomes a non-atomic one, in which the impact (through interference) of any single mobile on the performance of other mobiles is negligible. In the networking game context, the related solution concept is often called Wardrop equilibrium [18]; it is often much easier to compute than the original Nash equilibrium [9], and yet, the former equilibrium is a good approximation for the latter, see details in [19]. In this work, we derive the non-atomic equilibrium, which generally corresponds to a non-uniform PA.

The non-atomic Nash equilibrium is studied in this paper for several linear receivers, namely the matched filter and the MMSE filter, as well as non-linear filters, such as the successive interference cancellation (SIC) [20] version of those filters. However, in order to perform SIC, the users need to know their decoding order, in order to adjust their rates. In this paper, we introduce ways of obtaining an ordering of the users in a distributed manner. The ordering can be determined simply in a distributed manner under weak hypotheses. This gives rise to a different kind of PA, that depends explicitly on the order in which the users are decoded.

Moreover, we quantify the gain of the non-uniform PA with respect to uniform PA, according to the number of paths. The originality of the paper lies in the fact that we show that as the number of paths increases, the optimal PA becomes more and more uniform due to the ergodic behavior of all the CDMA channels. This is reminiscent of an effect (“channel hardening”) already revealed in MIMO [21]. The highest gain (in terms of utility) is obtained in the case of flat fading (which also favors dis-uniform power allocation between the users).

The layout of this paper is the following. First, a detailed account of related works is made in Sec. II. A short presentation of the model is given in Sec. III. Asymptotic SINR and capacity expressions are given in Sec. IV. The particular game played between users is introduced in Sec. V, along with the existence of a Nash equilibrium. Theoretical results for the power allocation are derived in Sec. VI for unordered users and Sec. VII when there is an ordering of the users. Analytical results are matched with simulations in Sec. VIII. Conclusions are provided in Sec. IX.

II. RELATED WORK

This section is dedicated to present some of the works that use game theory for power control. We remind that a Nash equilibrium is a stable solution, where no player has an incentive to deviate unilaterally, while a Pareto equilibrium is a cooperative dominating solution, where there is no way to improve the performance of a player without harming another one. Generally, both concepts do not coincide. Following the general presentation of power allocation games in [10], [11], an abundance of works can be found on the subject.

In particular, the utility generally considered in those articles is justified in [22] where the author describes a widely applicable model “from first principles”. Conditions under which the utility will allow to obtain non-trivial Nash equilibria (i.e., users actually transmit at the equilibrium) are derived. The utility consisting of throughput-to-power ratio (detailed

in Sec. V) is shown to satisfy these conditions. In addition, it possesses a propriety of reliability in the sense that the transmission occurs at non-negligible rates at the equilibrium. This kind of utility function had been introduced in previous works, with an economic leaning [23], [24], [25].

Unfortunately, Nash equilibria often lead to inefficient allocations, in the sense that higher rates (Pareto equilibria) could be obtained for all mobiles if they cooperated. To alleviate this problem, in addition to the non-cooperative game setting, [24] introduces a pricing strategy to force users to transmit at a socially optimal rate. They obtain communication at Pareto equilibrium.

Another pricing mechanism is investigated in [25]. It leads to the design of update algorithms depending only on a few system parameters available to the users. Their convergence is also proven and shown by simulations. The paper is limited to the matched filter, and fading values are known and constant. Selective fading and other filters are not considered.

In this contribution, we consider a different kind of utility, that does *not* involve pricing. In [12], defining the utility as advised in [22] as the ratio of the throughput to the transmission power, the authors obtain results of existence and unicity of a Nash equilibrium for a CDMA system. They extend this work to the case of multiple carriers in [26]. In particular, it is shown that users will select and only transmit over their best carrier. As far as the attenuation is concerned, the consideration is restricted to flat fading in [12] and in [26] (each carrier being flat fading in the latter). However, wireless transmissions generally suffer from the effect of multiple paths, thus becoming frequency-selective. The goal of this paper is to determine the influence of the number of paths (or the selectivity of the channel) on the performance of PA.

This work is an extension of [12] in the case of frequency-selective fading, in the framework of multi-user systems. We do not consider multiple carriers, as in [26], and the results are very different to those obtained in that work. The extension is not trivial and involves advanced results on random matrices with non-equal variances due to Girko [27] whereas classical results rely on the work of Silverstein [28]. A part of this work was previously published as a conference paper [29]. Moreover, in addition to the linear filters studied in [12], we study the enhancements provided by the optimum and successive interference cancellation filters.

III. MODEL

We consider a single uplink CDMA multi-user system cell, i.e., inter-cell interference free case. The spreading length is denoted N . The number of users in the cell is K . The load is $\alpha = K/N$. We make the hypothesis that the users employ Gaussian i.i.d. codes with zero mean and variance $1/N$ [30]. This hypothesis enables us to state simply our results, however almost all of the results are valid for any distribution of the codes as long as it has mean zero and variance $1/N$ [17]. Similarly to [30], the received signal \mathbf{y} of size $N \times 1$ at the base station is given by

$$\mathbf{y} = \sum_{k=1}^K \mathbf{H}_k \mathbf{w}_k \sqrt{P_k} s_k + \mathbf{n} = (\mathbf{H} \sqrt{\mathbf{P}} \odot \mathbf{W}) \mathbf{s} + \mathbf{n}. \quad (1)$$

In (1), \mathbf{H} is the frequency selective fading matrix, of size $N \times K$. $\sqrt{\mathbf{P}}$ is the root square of the diagonal power control matrix, of size $K \times K$. The diagonal elements of $\sqrt{\mathbf{P}}$ are the square roots of the transmitted powers of the users. \mathbf{W} is an $N \times K$ random spreading matrix. \mathbf{s}_k is a $K \times 1$ vector containing the signals of the users, with covariance matrix \mathbf{I}_K . \mathbf{n} is an $N \times 1$ Additive White Gaussian Noise (AWGN) vector with covariance matrix $\sigma^2 \mathbf{I}_N$.

We assume that the reader is familiar with random matrix theory concepts, which can be found in [13]. In the following, we will assume that the frequency selective fading matrix \mathbf{H} behaves ergodically. The two-dimensional channel profile of $\mathbf{H}\sqrt{\mathbf{P}}$ is denoted $\rho(f, x) = P(x) |h(f, x)|^2$, $f \in [0, 1]$, $x \in [0, \alpha]$. f is the frequency index and x is the user index. This enables us to use Th. 2.1 in [17] in order to obtain expressions for the SINR. It is also assumed that the power of all users is upper bounded by P_{\max} , i.e., the square of all diagonal elements of $\sqrt{\mathbf{P}}$ is upper bounded by P_{\max} , and the square norm of the fading, on all paths, for all users, is upper bounded by h_{\max} .

IV. ASYMPTOTIC SINR EXPRESSIONS

Let \mathbf{h}_k be the k -th column of \mathbf{H} , and $\mathbf{H}_{(-k)}$ be \mathbf{H} with \mathbf{h}_k removed. Similarly, let \mathbf{w}_k be the k -th column of \mathbf{W} , and $\mathbf{W}_{(-k)}$ be \mathbf{W} with \mathbf{w}_k removed. Let $\sqrt{\mathbf{P}}_{(-k)}$ be $\sqrt{\mathbf{P}}$ with the k -th column and line removed. Let $\mathbf{G}_{(-k)} = \mathbf{H}_{(-k)} \sqrt{\mathbf{P}}_{(-k)} \odot \mathbf{W}_{(-k)}$.

A. Matched Filter

The Matched filter is the simplest linear filter. Supposing perfect SIC at the receiver, the matched filter for the k -th user is given by the $N \times 1$ vector $\mathbf{g}_k = \sqrt{P_k} (\mathbf{h}_k \odot \mathbf{w}_k)$. This leads to the following expression for the SINR of user k

$$\text{SINR}_k = \frac{|\mathbf{g}_k^H \mathbf{g}_k|^2}{\sigma^2 \mathbf{g}_k^H \mathbf{g}_k + \mathbf{g}_k^H \left(\mathbf{G}_{(-k)} \mathbf{G}_{(-k)}^H \right) \mathbf{g}_k}.$$

Proposition 1: [17] As $N, K \rightarrow \infty$ with $K/N \rightarrow \alpha$, the SINR of user k at the output of the matched filter is given by

$$\text{SINR}_k = \beta^{\text{MF}} \left(\frac{k}{N} \right)$$

where $\beta^{\text{MF}} : [0, \alpha] \rightarrow \mathbb{R}$ is given by

$$\beta^{\text{MF}}(x) = P(x) \frac{(H(x))^2}{\sigma^2 H(x) + \int_0^\alpha \int_0^1 P(y) |h(f, y)|^2 |h(f, x)|^2 df dy} \quad (2)$$

and $H(x) = \int_0^1 |h(f, x)|^2 df$.

Denoting $\text{SINR}_k = \beta_k^{\text{MF}}$, Prop. 1 enables us to extract an approximation of the value of the SINR of user k in the finite size case

$$\beta_k^{\text{MF}} = \frac{P_k \left(\frac{1}{N} \sum_{n=1}^N |h_{nk}|^2 \right)^2}{\frac{\sigma^2}{N} \sum_{n=1}^N |h_{nk}|^2 + \frac{1}{N^2} \sum_{j \neq k} \sum_{n=1}^N P_j |h_{nj}|^2 |h_{nk}|^2}. \quad (3)$$

We observe that $P_k \frac{\partial \beta_k^{\text{MF}}}{\partial P_k} = \beta_k^{\text{MF}}$.

B. MMSE Filter

The MMSE filter is the filter that maximizes the SINR over all linear filters. Supposing perfect SIC at the receiver, the MMSE filter for the k -th user is given by $\mathbf{g}_k^{\text{MMSE}} = \mathbf{R}^{-1} \mathbf{g}_k$, where $\mathbf{R} = \left(\mathbf{H}\sqrt{\mathbf{P}} \odot \mathbf{W} \right) \left(\mathbf{H}\sqrt{\mathbf{P}} \odot \mathbf{W} \right)^H + \sigma^2 \mathbf{I}_N$. This leads to the following expression for the SINR of user k [15]

$$\text{SINR}_k = \mathbf{g}_k^H \left(\mathbf{G}_{(-k)} \mathbf{G}_{(-k)}^H + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{g}_k. \quad (4)$$

Proposition 2: [17] As $N, K \rightarrow \infty$ with $K/N \rightarrow \alpha$, the SINR of user k at the output of the MMSE receiver is given by:

$$\text{SINR}_k = \beta^{\text{MMSE}} \left(\frac{k}{N} \right)$$

where $\beta^{\text{MMSE}} : [0, \alpha] \rightarrow \mathbb{R}$ is a function defined by the implicit equation

$$\beta^{\text{MMSE}}(x) = P(x) \int_0^1 \frac{|h(f, x)|^2 df}{\sigma^2 + \int_0^\alpha \frac{P(y) |h(f, y)|^2 dy}{1 + \beta^{\text{MMSE}}(y)}}. \quad (5)$$

Denoting $\text{SINR}_k = \beta_k^{\text{MMSE}}$, Prop. 2 enables us to extract an approximation of the value of the SINR of user k in the finite size case

$$\beta_k^{\text{MMSE}} = P_k \frac{1}{N} \sum_{n=1}^N |h_{nk}|^2 \frac{1}{\sigma^2 + \frac{1}{N} \sum_{j \neq k} \frac{P_j |h_{nj}|^2}{1 + \beta_j^{\text{MMSE}}}}. \quad (6)$$

From (4), we observe that $P_k \frac{\partial \beta_k^{\text{MMSE}}}{\partial P_k} = \beta_k^{\text{MMSE}}$.

From Prop. 2, the capacity of user k is $C_k^{\text{MMSE}} = \frac{1}{N} \log_2 (1 + \beta_k^{\text{MMSE}})$ and the global capacity of the system is

$$C^{\text{MMSE}} = \int_0^\alpha \log_2 (1 + \beta^{\text{MMSE}}(x)) dx. \quad (7)$$

C. Optimal Filter

The term optimal filter designates a filter capable of decoding the received signal at the bound given by Shannon's capacity. Hence, it is difficult to define an SINR associated to it. However, results of random matrix theory can still be applied. Let $\mathbf{Y} = \mathbf{H}\sqrt{\mathbf{P}} \odot \mathbf{W}$. The definition of Shannon's capacity per dimension for our system is

$$C_{(N)}^{\text{OPT}} = \frac{1}{N} \log_2 \det \left(\mathbf{I}_N + \frac{1}{\sigma^2} \mathbf{Y} \mathbf{Y}^H \right). \quad (8)$$

As $N, K \rightarrow \infty$ with $K/N \rightarrow \alpha$,

$$C_{(N)}^{\text{OPT}} \rightarrow \int \log_2 \left(1 + \frac{1}{\sigma^2} t \right) \nu(dt) \quad (9)$$

where ν is the empirical eigenvalue distribution of $\mathbf{Y} \mathbf{Y}^H$. If we differentiate the asymptotic value C^{OPT} of (9) with respect to σ^2 , we obtain (after some derivations)

$$\frac{\partial C^{\text{OPT}}}{\partial \sigma^2} = \log_2(e) \left(m^\nu(-\sigma^2) - \frac{1}{\sigma^2} \right) \quad (10)$$

where $m^\nu(\cdot)$ is the Stieltjes transform of the empirical eigenvalue distribution of $\mathbf{Y}\mathbf{Y}^H$. From Th. 2.1 in [17], $m^\nu(\cdot)$ is given by $m^\nu(z) = \int_0^1 u(f, z) df$ where $u(f, z)$ is given by

$$u(f, z) = \frac{1}{\int_0^\alpha \frac{\rho^{\mathbf{H}\sqrt{\mathbf{P}}}(f, x) dx}{1 + \int_0^1 \frac{\rho^{\mathbf{H}\sqrt{\mathbf{P}}}(f', x) u(f', z) df'}{z}}}. \quad (11)$$

with $\rho^{\mathbf{H}\sqrt{\mathbf{P}}}(f, x) = \rho(f, x) = P(x) |h(f, x)|^2$. Given that if $\sigma^2 = +\infty$, $C^{\text{OPT}} = 0$, it is immediate to obtain C^{OPT} from (10) as

$$C^{\text{OPT}} = \log_2(e) \int_{\sigma^2}^{+\infty} m^\nu(-z) - \frac{1}{z} dz. \quad (12)$$

Proposition 3: C^{OPT} and C^{MMSE} are related through the following equality

$$C^{\text{OPT}} = C^{\text{MMSE}} - \log_2(e) \int_0^\alpha \frac{\beta^{\text{MMSE}}(x)}{1 + \beta^{\text{MMSE}}(x)} dx + \int_0^1 \log_2 \left(1 + \frac{1}{\sigma^2} \int_0^\alpha \frac{\rho(f, x)}{1 + \beta^{\text{MMSE}}(x)} dx \right) df. \quad (13)$$

Proof: See Appendix A. ■

The additional term in the right-hand side of (13) corresponds to the non-linear processing gain. It quantifies the gain in terms of capacity that can be achieved between pure linear MMSE and non-linear filtering.

Assuming perfect cancellation of decoded users, successive interference cancellation with MMSE filter achieves the optimal capacity [31]. Hence the following proposition.

Proposition 4: [17] As $N, K \rightarrow \infty$ with $K/N \rightarrow \alpha$, the optimal capacity is given by:

$$C^{\text{OPT}} = \int_0^\alpha \log_2(1 + \beta^{\text{SIC}}(x)) dx$$

where $\beta^{\text{SIC}} : [0, \alpha] \rightarrow \mathbb{R}$ is a function defined by the implicit equation

$$\beta^{\text{SIC}}(x) = P(x) \int_0^1 \frac{|h(f, x)|^2 df}{\sigma^2 + \int_0^x \frac{P(y) |h(f, y)|^2 dy}{1 + \beta^{\text{SIC}}(y)}}. \quad (14)$$

Prop. 4 enables us to extract an expression that is analog to the SINR for the optimal filter. Similarly to the case of β^{MMSE} in Sec. IV-B, this expression obeys the property $P_k \frac{\partial \beta_k^{\text{SIC}}}{\partial P_k} = \beta_k^{\text{SIC}}$. From now on, we denote $\text{SINR}_k = \beta_k$, whichever filter is actually used.

V. GAMES AND EQUILIBRIA

A. Power Allocation Game

A game with a unique strategy set for all users is defined by a triple $\{S, \mathbb{P}, (u_k)_{k \in S}\}$ where S is the set of *players*, \mathbb{P} is the set of *strategies*, and $(u_k)_{k \in S}$ is the set of *utility functions*, $u_k : \mathbb{P}^{|S|} \rightarrow \mathbb{R}$.

In our setting, the players are simply the users, indexed by the set $S^{(K)} = \{1, \dots, K\}$. The strategy for a mobile is its power allocation P_k , which we will assume belongs to a compact interval $\mathbb{P} = [0, P_{\max}] \subseteq \mathbb{R}$. The utility measures the gain of a user as a result of the strategy this user plays. In [22], the author derives what he calls Throughput to Power

Ratio (TPR) under minimal requirements. The utility of user k is expressed as [22]

$$u_k = \frac{\gamma_k}{P_k}. \quad (15)$$

We denote $\gamma_k = \gamma(\beta_k)$, where $\gamma(\cdot)$ is the same function for all users. In (15), γ is a relevant performance measure function, which is at least C^2 and should satisfy conditions detailed in [22] in order to obtain an “interesting” equilibrium, i.e., for which the equilibrium power allocation is not 0 for all users.

As a performance measure, we consider the goodput $\gamma(\beta_k)$, which is proportional to $(1 - e^{-\beta_k})^M$ where M is the number of bits transmitted in a CDMA packet. Remark that the usual definition of goodput would rather be considered proportional to $q(\beta_k) = (1 - \text{BER}_k)^M$, where BER is the bit error rate. However, this quantity is not zero when the transmitted power is zero. Using this function in the utility would lead to the unsatisfying conclusion that mobiles should not transmit at all, since the (improbable) event of a correct guess gives them infinite utility [10]. Therefore, an adapted version of the goodput is adopted, where a factor 2 is added before the BER. The performance measure considered is hence proportional to $q_2(\beta_k) = (1 - 2\text{BER}_k)^M$, leading to the expression above. This function has the desirable property $q_2(0) = 0$ and its shape follows closely the shape of the original goodput $q(\cdot)$. This is a relevant performance measure, as each mobile wants to use its (limited) battery power to transmit the maximum possible amount of information.

This utility is expressed in *bits per joule*. In the non-cooperative game setting, each user wants to selfishly maximize its utility. A Nash equilibrium is obtained when no user can benefit by unilaterally deviating from its strategy.

To obtain the maximum utility achievable by user k , we differentiate u_k with respect to the power P_k and equate to 0. We obtain

$$P_k \frac{\partial \beta_k}{\partial P_k} \gamma'(\beta_k) - \gamma(\beta_k) = 0. \quad (16)$$

For all filters under consideration, $P_k \frac{\partial \beta_k}{\partial P_k} = \beta_k$, thus, (16) reduces to an equation on β_k

$$\beta_k \gamma'(\beta_k) - \gamma(\beta_k) = 0. \quad (17)$$

Eq. (17) is particularly interesting in the case when there exists a unique solution β^* .

The existence of a solution to (17) is guaranteed as long as the function $\gamma(\cdot)$ is a quasiconcave function of the SINR, i.e., there exists a point below which the function is non-decreasing, and above which the function is non-increasing [24], [22]. In addition, we assume that the function $\gamma(\cdot)$ takes value $\gamma(0) = 0$, so that users cannot achieve an infinite utility by not transmitting. This occurs for several functions $\gamma(\cdot)$ of interest, in particular the goodput [12], which we will use for simulations. Unfortunately, the capacity cannot be used as a function $\gamma(\cdot)$, since it leads to the trivial result $\beta^* = 0$ for this utility function. The uniqueness of the solution β^* to (17) is due to fact that the SINR of each user is a strictly increasing function of its transmit power. Given the target SINR β^* , we obtain the strategy of users in the next section.

VI. POWER ALLOCATION IN THE NASH EQUILIBRIUM

A. Flat Fading

In this subsection, we show that the results of [12] for matched and MMSE filters are a special case of our setting when $L = 1$ (flat fading case). In addition, we derive the power allocation for the optimum filter. When there is only one path, for each user k , denoted by its index $\frac{k}{N} = x \in [0, \alpha]$, $h(f, x)$ does not depend on f . Given the target SINR β^* , we have explicit expressions of the power with which user k transmits for the various receivers.

In Appendix B, we show that the influence of the strategy of a player on the payoffs of other players is (asymptotically) “small”. It justifies the fact that we can obtain an equilibrium in the asymptotic setting, without the need for players to possess all the information on the system. Their local information is sufficient. In the asymptotic limit, we obtain results similar to Wardrop equilibrium: the strategy used by each user does not influence the strategy of other users.

1) *Matched filter*: From Prop. 1,

$$P_k = \frac{\beta^* \left(\sigma^2 + \frac{1}{N} \sum_{j=1, j \neq k}^K P_j |h_j|^2 \right)}{|h_k|^2}. \quad (18)$$

Summing (18) over $k = 1, \dots, K$, we obtain a closed form expression for the minimum power with which user k transmits when using the matched filter

$$P_k = \frac{1}{|h_k|^2} \frac{\sigma^2 \beta^*}{1 - \alpha \beta^*} \text{ for } \alpha < \frac{1}{\beta^*}. \quad (19)$$

2) *MMSE filter*: From Prop. 2,

$$P_k = \frac{\beta^* \left(\sigma^2 + \frac{1}{1+\beta^*} \frac{1}{N} \sum_{j=1, j \neq k}^K P_j |h_j|^2 \right)}{|h_k|^2}. \quad (20)$$

Summing (20) over $k = 1, \dots, K$, we obtain a closed form expression for the minimum power with which user k transmits when using the MMSE filter

$$P_k = \frac{1}{|h_k|^2} \frac{\sigma^2 \beta^*}{1 - \alpha \frac{\beta^*}{1+\beta^*}} \text{ for } \alpha < 1 + \frac{1}{\beta^*}. \quad (21)$$

Both (19) and (21) are the same results as in [12].

3) *Optimum filter*: Each user maximizes its utility for a SINR equal to β^* . However, in the case of the optimum filter, the SINR is not defined directly. It is nevertheless possible to extract an equivalent quantity from the expression of the capacity, since the value of the capacity of user k at the equilibrium is given by $C^* = \frac{1}{N} \log_2(1 + \beta^*)$.

Proposition 5: The power allocation is given by

$$P_k = \frac{1}{|h_k|^2} \frac{\sigma^2 \beta^+}{1 - \alpha \frac{\beta^+}{1+\beta^+}} \text{ for } \alpha < 1 + \frac{1}{\beta^+} \quad (22)$$

where β^+ is the solution to

$$\begin{aligned} & \alpha \log_2(1 + \beta^+) - \alpha \log_2(e) \frac{\beta^+}{1 + \beta^+} \\ & + \log_2 \left(1 + \frac{1}{1 + \beta^+} \frac{\alpha \beta^+}{1 - \alpha \frac{\beta^+}{1 + \beta^+}} \right) \\ & = \alpha \log_2(1 + \beta^*). \end{aligned} \quad (23)$$

Proof: See Appendix C. ■

B. Frequency Selective Fading

In the context of frequency selective fading, for each user k , denoted by its index $\frac{k}{N} = x \in [0, \alpha]$, there are $L > 1$ paths with respective attenuations $h_\ell(x)$, $\ell = 1, \dots, L$, which are i.i.d. random variables with some known distribution. We suppose that $h_\ell(x)$ has mean zero, and the distributions of the real part and imaginary part of $h_\ell(x)$ are even functions, as for example the Gaussian distribution, which we consider in the simulations. $h(f, x)$ depends on f through $h(f, x) = \sum_{\ell=1}^L h_\ell(x) e^{-2\pi i f(\ell-1)}$, where $i^2 = -1$. Given the target SINR β^* , the Nash equilibrium power allocation is determined by implicit equations for the various receivers.

1) *Matched filter*: Denoting $h_{nk} = h\left(\frac{n-1}{N}, \frac{k}{N}\right)$,

$$P_k = \beta^* \frac{\frac{\sigma^2}{N} \sum_{n=1}^N |h_{nk}|^2 + \frac{1}{N} \sum_{n=1}^N |h_{nk}|^2 \frac{1}{N} \sum_{j \neq k}^K P_j |h_{nj}|^2}{\left(\frac{1}{N} \sum_{n=1}^N |h_{nk}|^2 \right)^2}. \quad (24)$$

In this expression, the power allocation of user k seems to depend on the power allocation and fading realization of all the other users. However, when the number of users tends to infinity, the strategy of any single user does not have any influence on the payoff of user k , as shown in Appendix B. Hence, the appropriate framework is non-atomic games. The expression $\frac{1}{N} \sum_{j=1}^K P_j |h_{nj}|^2$ is asymptotically a constant (not depending on n), denoted Ω .

$$\Omega = \frac{\alpha \beta^* \sigma^2 \frac{1}{K} \sum_{j=1}^K \frac{|h_{nj}|^2}{E_j}}{1 - \alpha \beta^* \frac{1}{K} \sum_{j=1}^K \frac{|h_{nj}|^2}{E_j}}, \quad (25)$$

where $E_j = \frac{1}{N} \sum_{m=1}^N |h_{mj}|^2$. As $K \rightarrow \infty$, we apply the Central Limit Theorem to the sum of random variables $\frac{1}{K} \sum_{j=1}^K \frac{|h_{nj}|^2}{E_j}$. It tends to its expectation, which is 1 (see Appendix D).

It follows that asymptotically $\Omega = \frac{\alpha \beta^* \sigma^2}{1 - \alpha \beta^*}$ (and simulations in Sec. VIII prove that this approximation is valid for moderate finite values of N). From (24), we obtain a formula similar to (19)

$$P_k = \frac{1}{E_k} \frac{\sigma^2 \beta^*}{1 - \alpha \beta^*} \text{ for } \alpha < \frac{1}{\beta^*}. \quad (26)$$

2) *MMSE filter*: The power allocation is

$$P_k = \frac{\beta^*}{\frac{1}{N} \sum_{n=1}^N \frac{|h_{nk}|^2}{\sigma^2 + \frac{1}{1+\beta^*} \frac{1}{N} \sum_{j=1, j \neq k}^K P_j |h_{nj}|^2}}. \quad (27)$$

As previously, when the number of users tends to infinity, $\frac{1}{N} \sum_{j=1}^K P_j |h_{nj}|^2$ is asymptotically a constant. We obtain a formula similar to (21)

$$P_k = \frac{1}{E_k} \frac{\sigma^2 \beta^+}{1 - \alpha \frac{\beta^+}{1+\beta^+}} \text{ for } \alpha < 1 + \frac{1}{\beta^+}. \quad (28)$$

3) *Optimum filter*: Each user maximizes its utility for a SINR equal to β^* . However, in the case of the optimum filter, the SINR is not defined directly. It is nevertheless possible to extract an equivalent quantity from the expression of the capacity, since the value of the capacity of user k at the equilibrium is given by $C^* = \frac{1}{N} \log_2(1 + \beta^*)$.

Proposition 6: Asymptotically, as $N, K \rightarrow \infty$, the power allocation is given by

$$P_k = \frac{1}{E_k} \frac{\sigma^2 \beta^+}{1 - \alpha \frac{\beta^+}{1 + \beta^+}} \text{ for } \alpha < 1 + \frac{1}{\beta^+}, \quad (29)$$

where β^+ is the solution to

$$\begin{aligned} \alpha \log_2(1 + \beta^+) - \alpha \log_2(e) \frac{\beta^+}{1 + \beta^+} \\ + \log_2 \left(1 + \frac{1}{1 + \beta^+} \frac{\alpha \beta^+}{1 - \alpha \frac{\beta^+}{1 + \beta^+}} \right) \\ = \alpha \log_2(1 + \beta^*). \end{aligned} \quad (30)$$

Proof: The proof is similar to the proof of Prop. 5. ■

We observe that for all filters considered, the optimal PA is a constant times the inverse of the *total* energy of the channel E_j . Via Parseval's Theorem, $E_j = \sum_{\ell=1}^L |h_\ell(\frac{j}{N})|^2$. It is a sum of i.i.d. random variables. As the number of paths increases, the optimal PA tends to a uniform PA. This is an effect similar to "channel hardening" [21]: as the number of paths increases, the variance of the distribution of the channel energy decreases and the Nash equilibrium PA becomes more and more uniform for all users.

VII. SUCCESSIVE INTERFERENCE CANCELLATION

The optimal filter gives a bound on the performance that can be achieved through (non-linear) filtering at the base station. In order to improve the performance of the system, we introduce Successive Interference Cancellation (SIC) [20] at the base station. Under the assumption of perfect decoding, SIC improves immensely the performance of linear filters (Matched Filter or MMSE Filter). The MMSE SIC filter actually achieves the optimum filter bound, under the assumption of perfect decoding. The principle of SIC receivers is quite simple: users are ordered and are decoded successively. At each step, supposing that the user has been encoded at the appropriate decoding rate, the signal is decoded and its contribution to the interference is then perfectly subtracted. This removes some of the inter-user interference and, therefore, increases the SINR of the following decoded users.

The challenge is that the users must transmit at the appropriate rate to avoid the catastrophic occurrence of imperfect decoding. Usually, the ordering of users is done in a centralized way, at the base station, which advertises it to the users. However, for the protocol to remain distributed, users should be able to decide, based on their local information, at which rate to transmit.

At equilibrium, the rate is determined by the SINR β^* , and it is the transmission power of the user that is determined according to its rank of decoding. The equilibrium PA can be determined in a simple manner when the number of multipaths is finite ($L < \infty$) and the number of users is very high ($K \rightarrow \infty$). In Sec. VII-A, we make use of the fact that the whole law of E_j is realized in this case, so that users automatically know their rank of decoding. Another manner to give a (random) ordering of decoding is to introduce an additional degree of liberty in the system. In Sec. VII-B, we develop a coordination mechanism that enables users to learn their rank of decoding

in a simple way. In the following, we assume that each user has a unique i.d. number j ranging between 1 to K .

A. Ordering when $K \rightarrow \infty$

If the number of users $K \rightarrow \infty$, with L fixed, the whole law of the total channel energy will be realized. Assume the base station advertises to the users that they will be decoded by decreasing total channel energy. Each user knows, according to the realization of its fading, its rank in the decoding order given by K times 1 minus the cumulative distribution function $D(\cdot)$ of the total channel energy E_j :

$$\text{rank}_j = K(1 - D(E_j)).$$

In case that the base station advertises to the users that they will be decoded by increasing total channel energy, user j will have rank $\text{rank}_j = KD(E_j)$.

B. Coordination Mechanisms

We wish to introduce a simple mechanism that enables players to coordinate and to know in which order they will be decoded. We are inspired by the notion of correlated equilibrium introduced by R. Aumann in [32] and further studied in [33], [34], [35]. Correlated equilibrium represents a generalization of Nash equilibrium. The important feature of [32] is the presence of an *arbitrator*. An arbitrator needs not have any intelligence or knowledge of the game, it needs only to send random (private or public) signals to the players that are independent of all other data in the game. In the context of non-cooperative games, each player has the possibility not to consider the signal(s) it receives. Coordination between players turns out to be useful also in the case of cooperative optimization. The signals enable joint randomization between the strategies of the players, possibly resulting in equilibria with higher payoffs. The concept of correlated equilibrium was recently introduced in a networking context in [36], where the authors consider a simple ALOHA setting.

The simplest and most intuitive coordination mechanism is given by a common signal which users as well as the base station overhear before each transmission. There are $K!$ possible permutations of K users. Hence, the arbitrator broadcasts a signal to the users belonging to the set $\{0, \dots, K! - 1\}$. Each of these numbers corresponds to a permutation π of $\{1, \dots, K\}$ that gives the (random) ordering of decoding as $\text{rank}_j = \pi(j)$. The users can then adjust their transmit power according to this ordering. In terms of size of the message, this is equivalent to the case when the base station decides the decoding order and broadcasts it to the users, or sends K individual messages of $\ln(K)$ bits containing the rank, since $\ln(K!) = K \ln(K) + o(K \ln(K))$. However, there is no need of either any knowledge of the system or computations at the base station with this coordination mechanism.

C. SIC Power Allocations

In both cases, once the users know their order, they can calculate their transmit power according to the filter that is used. The equilibrium still occurs when all users reach the

SINR β^* . A single user will not benefit by deviating, since it would decrease its utility. From now on, index k denotes the rank of decoding.

In the case of the matched filter with SIC, the SINR of the user decoded at rank k is

$$\beta_k^{\text{MF}} = \frac{P_k \left(\frac{1}{N} \sum_{n=1}^N |h_{nk}|^2 \right)^2}{\frac{\sigma^2}{N} \sum_{n=1}^N |h_{nk}|^2 + \frac{1}{N^2} \sum_{j>k} \sum_{n=1}^N P_j |h_{nj}|^2 |h_{nk}|^2}. \quad (31)$$

From (31), we get the equilibrium PA of user k as

$$P_k = \beta^* \cdot \frac{\frac{\sigma^2}{N} \sum_{n=1}^N |h_{nk}|^2 + \frac{1}{N^2} \sum_{j>k} \sum_{n=1}^N P_j |h_{nj}|^2 |h_{nk}|^2}{\left(\frac{1}{N} \sum_{n=1}^N |h_{nk}|^2 \right)^2}. \quad (32)$$

In the case of the MMSE filter with SIC, the SINR of the user decoded at rank k is

$$\beta_k^{\text{MMSE}} = P_k \frac{1}{N} \sum_{n=1}^N |h_{nk}|^2 \frac{1}{\sigma^2 + \frac{1}{N} \sum_{j>k} \frac{P_j |h_{nj}|^2}{1 + \beta_j^{\text{MMSE}}}}. \quad (33)$$

From (33), we get the equilibrium PA of user k as

$$P_k = \frac{\beta^*}{\frac{1}{N} \sum_{n=1}^N \frac{|h_{nk}|^2}{\sigma^2 + \frac{1}{1 + \beta^*} \sum_{j>k}^K P_j |h_{nj}|^2}}. \quad (34)$$

For flat fading, a simple recursion gives the equilibrium PA (see Appendix E).

$$P_k^{\text{MF}} = \frac{\sigma^2 \beta^*}{|h_k|^2} \left(1 + \frac{1}{N} \beta^* \right)^{K-k}, \quad (35)$$

$$P_k^{\text{MMSE}} = \frac{\sigma^2 \beta^*}{|h_k|^2} \left(1 + \frac{1}{N} \frac{\beta^*}{1 + \beta^*} \right)^{K-k}. \quad (36)$$

As far as frequency-selective fading is concerned, this gives us the form of the asymptotic expressions. Asymptotically, the power allocation of one user will not depend on the PA of the other users, as shown in Appendix B. With a similar reasoning as in Sec. VI, the expressions mimic (35) and (36) with the total channel energy E_k replacing $|h_k|^2$, i.e.,

$$P_k^{\text{MF}} = \frac{\sigma^2 \beta^*}{E_k} \left(1 + \frac{1}{N} \beta^* \right)^{K-k}, \quad (37)$$

$$P_k^{\text{MMSE}} = \frac{\sigma^2 \beta^*}{E_k} \left(1 + \frac{1}{N} \frac{\beta^*}{1 + \beta^*} \right)^{K-k}. \quad (38)$$

These expressions are also validated by simulations.

Since MMSE SIC with perfect decoding is equivalent to the optimum filter, we thus obtain a second possible equilibrium PA for the optimum filter. In Sec. VIII, we investigate which is the PA which minimizes total amount of power needed to transmit at equilibrium SINR. In the case of automatic ordering of the users, one question is whether it is best to order the users by increasing or decreasing total fading energy. The answer is the following: it is always best to decode the users by decreasing total channel energy $E_1 < \dots < E_k$ (see Appendix F).

An interesting feature of equilibrium PA (37) and (38) is that there is no limitation on the number of users than can

be accommodated by the system, contrary to the previous case of (26), (28), and (29). The limitation is only imposed by the increasing power needed for each new user decoded last, which grows without bound as an exponential.

VIII. NUMERICAL RESULTS

In the following, we consider that P_{\max} is chosen sufficiently high so that users can actually transmit at the equilibrium PA values. For the simulations, we consider the usual case of Rayleigh fading. Although Rayleigh distribution is not bounded from above, simulations show that the results still hold.

We consider a CDMA system with $K = 32$ users and a spreading factor $N = 256$. The noise variance is $\sigma^2 = 10^{-10}$. For a number of bits in a CDMA packet $M = 100$, the goodput is $\gamma(\beta) = (1 - e^{-\beta})^{100}$ (see [12]), and $\beta^* = 6.48$. The capacity achieved at the Nash Equilibrium is $C = \alpha \log_2(1 + \beta^*) = 0.39$ bits/s. Unfortunately, the capacity itself cannot be used as a relevant performance measure in the definition of the utility, because, in this case, the maximal utility is obtained when not sending.

We have performed simulations over 10000 realizations. Fig. 1 shows the good fit of theoretic values calculated directly from (26), (28), and (29) with those simulations. The values of the utility do not depend on the number of multipaths. We see that optimum filter requires the minimal power, and matched filter the maximal power to achieve the required goodput.

In Fig. 2, we have plotted the average utility versus the number of multipaths L . Multipaths are supposed to be i.i.d. Rayleigh distributed with variance $1/L$, in order for the channels to have the same energy. Two cases are considered: the utility obtained in the Nash equilibrium, according to the PA given by (24) and (27), and the utility in the case where all nodes transmit at the same power. For comparison purposes, the sum of the uniform powers is equal to the sum of the powers used in the Nash equilibrium. In addition, simulations (not reproduced here) show that this value gives the higher average utility for a uniform PA. The utility does not vary with L in the Nash equilibrium: the Central Limit Theorem applies to the utility, which is a constant times the random variable E_k in the Nash equilibrium. The utility with uniform powers is always inferior to the utility in the Nash equilibrium. However, as L increases, the gap decreases, as the variance of E_k decreases, and the equilibrium PA becomes uniform.

In Fig. 3, we have plotted the average of the inverse power of the users in the Nash equilibrium for each of the investigated schemes. We plot the average inverse power because of the direct relation to the utility for the users. The higher this average, the higher the utility for the user. The SIC filters are always more efficient than their linear counterparts. However, for a load $\alpha < 0.12$ and optimum filter,¹ it is better to use the first variation of PA (29) than use MMSE SIC (38). This relation is reversed when $\alpha > 0.12$. In addition to the theoretical curves, Monte-Carlo simulations were performed both with random ordering (circles) and

¹The value of α is obtained as solution of the equation $\alpha \beta^* \frac{\beta^*}{1 + \beta^*} (1 - \alpha \frac{\beta^*}{1 + \beta^*}) = \beta^+ (1 - \exp(-\alpha \frac{\beta^*}{1 + \beta^*}))$.

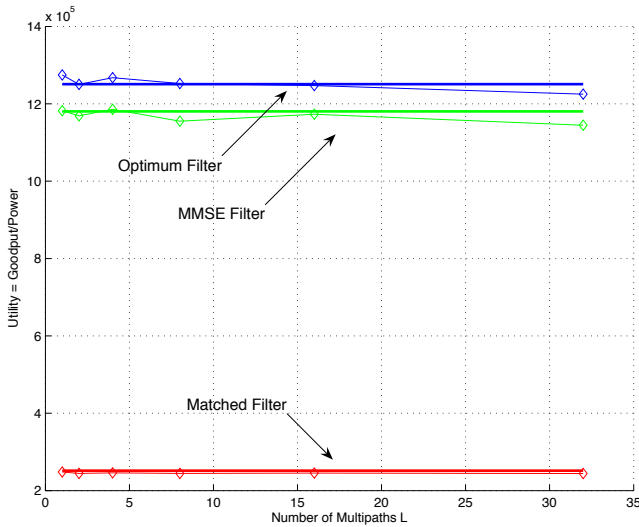


Fig. 1. Comparison of theoretic values and simulations for utilities in the Nash equilibrium.

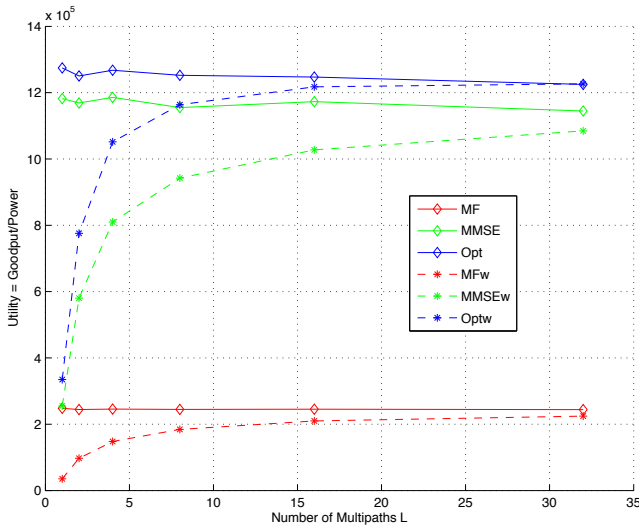


Fig. 2. Simulation of utilities in the Nash equilibrium and constant power allocations versus L . Curves with a 'w' suffix correspond to the uniform PA case.

ordering by decreasing total channel energy (crosses), for $L = 8$ multipaths. Simulations show that the optimal ordering improves the power efficiency of the successive interference cancellation filters.

In Fig. 4, we investigate the amelioration provided by optimal ordering as a function of the number of multipaths. The simulations are done for $K = 128$ users, in order to be in the “interesting” zone $\alpha > 0.12$. As expected, as the number of paths increases, the total channel energy is more and more the same for each channel and the gain provided by ordering the users decreases. However, when the number of users is very large and they benefit from automatic ordering, we see that the utility with the MMSE SIC equilibrium PA is the maximal utility that can be obtained in the non-cooperative setting.

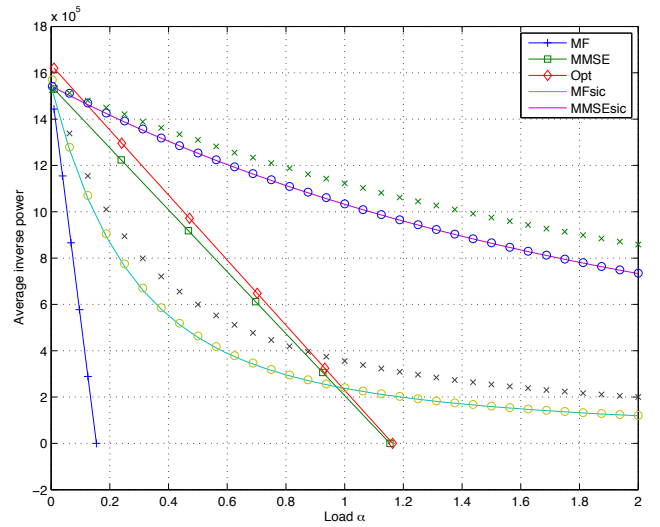


Fig. 3. Average inverse power used by the different filters. Monte-Carlo simulations were performed both with random ordering (\circ) and ordering by decreasing total channel energy (\times).

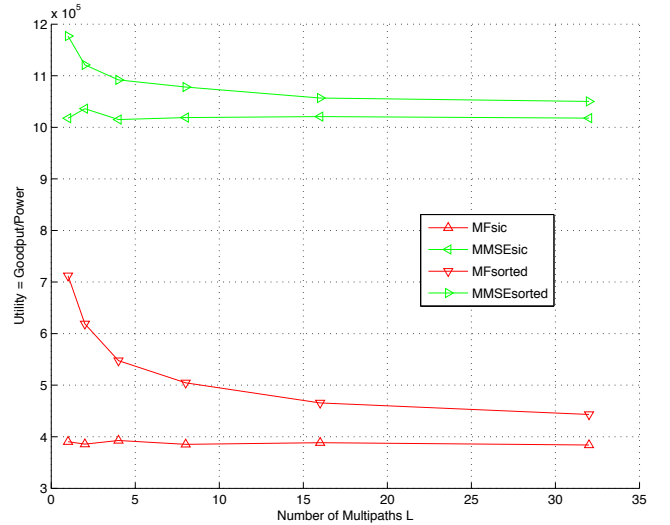


Fig. 4. Simulation of utilities in the Nash equilibrium with SIC filter with and without optimal ordering, versus L .

IX. CONCLUSIONS

Using tools of random matrices, we have derived the equilibrium power allocation in a game-theoretic framework applied to asymptotic CDMA with cyclic prefix, under frequency-selective fading. Three receivers are considered: matched filter, MMSE, and optimum filter (given by Shannon’s capacity). In addition, distributed ordering mechanisms are introduced and the successive interference cancellation variants of the linear filters are studied. For each user, this power allocation depends only on the total energy of the channel of the user under consideration. For a frequency-flat channel, the power allocation among users is non-uniform, whereas when the number of multipaths increases, the power allocation tends more and more to a uniform one.

APPENDIX A
PROOF OF PROP. 3

Notice that when $\sigma^2 \rightarrow \infty$, $C^{\text{OPT}} = 0$, $C^{\text{MMSE}} = 0$, and $\beta^{\text{MMSE}}(x) = \beta(x) = 0$. Thus, we only have to prove that the derivatives of either side of (13) are equal.

Using $\rho(f, x) = P(x) |h(f, x)|^2$, (5) can be rewritten

$$\beta(x) = \int_0^1 \frac{\rho(f, x) df}{\sigma^2 + \int_0^\alpha \frac{\rho(f, y) dy}{1 + \beta(y)}}. \quad (39)$$

From (11), $\int_0^1 \rho(f, x) u(f, -\sigma^2) df$ satisfies the same implicit equation (39) as $\beta(x)$ and, thus,

$$u(f, -\sigma^2) = \frac{1}{\int_0^\alpha \frac{\rho(f, y) dy}{1 + \beta(y)} + \sigma^2}. \quad (40)$$

Using (39) and (40), we can rewrite

$$\begin{aligned} & \int_0^1 u(f, -\sigma^2) df - \frac{1}{\sigma^2} \\ &= \int_0^1 \frac{1}{\int_0^\alpha \frac{\rho(f, y) dy}{1 + \beta(y)} + \sigma^2} df - \int_0^1 \frac{1}{\sigma^2} df \\ &= \int_0^1 \frac{-\int_0^\alpha \frac{\rho(f, x) dx}{1 + \beta(x)}}{\sigma^2 \left(\int_0^\alpha \frac{\rho(f, y) dy}{1 + \beta(y)} + \sigma^2 \right)} df \\ &= \int_0^\alpha \frac{-1}{\sigma^2 (1 + \beta(x))} \int_0^1 \frac{\rho(f, x) df}{\int_0^\alpha \frac{\rho(f, y) dy}{1 + \beta(y)} + \sigma^2} dx \\ &= - \int_0^\alpha \frac{\beta(x)}{\sigma^2 (1 + \beta(x))} dx. \end{aligned}$$

Thus, from (10)

$$\frac{\partial C^{\text{OPT}}}{\partial \sigma^2} = -\log_2(e) \int_0^\alpha \frac{\beta(x)}{\sigma^2 (1 + \beta(x))} dx. \quad (41)$$

Differentiating (7) with respect to σ^2 , we obtain

$$\frac{\partial C^{\text{MMSE}}}{\partial \sigma^2} = \log_2(e) \int_0^\alpha \frac{1}{1 + \beta(x)} \frac{\partial \beta}{\partial \sigma^2}(x) dx. \quad (42)$$

Let $\pi(x) = \frac{1}{\sigma^2(1 + \beta(x))}$. From (41) and (42), we obtain

$$\begin{aligned} & \frac{\partial C^{\text{OPT}}}{\partial \sigma^2} - \frac{\partial C^{\text{MMSE}}}{\partial \sigma^2} \\ &= -\log_2(e) \int_0^\alpha \left(\beta(x) + \sigma^2 \frac{\partial \beta}{\partial \sigma^2}(x) \right) \pi(x) dx. \quad (43) \end{aligned}$$

From (5), we have

$$\begin{aligned} & \int_0^\alpha \sigma^2 \beta(x) \frac{\partial \pi}{\partial \sigma^2}(x) dx \\ &= \int_0^\alpha \int_0^1 \frac{\sigma^2 \rho(f, x) df}{\sigma^2 + \int_0^\alpha \sigma^2 \rho(f, y) \pi(y) dy} \frac{\partial \pi}{\partial \sigma^2}(x) dx \\ &= \int_0^1 \frac{\int_0^\alpha \rho(f, x) \frac{\partial \pi}{\partial \sigma^2}(x) dx}{1 + \int_0^\alpha \rho(f, y) \pi(y) dy} df \\ &= \frac{1}{\log_2(e)} \frac{\partial}{\partial \sigma^2} \int_0^1 \log_2 \left(1 + \int_0^\alpha \rho(f, y) \pi(y) dy \right) df. \end{aligned}$$

Observing that

$$\begin{aligned} & \int_0^\alpha \left(\beta(x) + \sigma^2 \frac{\partial \beta}{\partial \sigma^2}(x) \right) \pi(x) + \sigma^2 \beta(x) \frac{\partial \pi}{\partial \sigma^2}(x) dx \\ &= \frac{\partial}{\partial \sigma^2} \int_0^\alpha \sigma^2 \beta(x) \pi(x) dx, \end{aligned}$$

we obtain (13) from Prop. 3.

APPENDIX B
INFLUENCE OF OTHER PLAYERS' STRATEGIES

We want to prove that asymptotically, in the game $\{S^{(K)}, \mathbb{P}, (u_k)_{k \in S^{(K)}}\}$, the strategy of a single player does not have any influence on the payoff of the other players. In other words, for all $k \neq i \in S^{(K)}$, for all $\mathbf{p} = (P_1, \dots, P_K) \in \mathbb{P}^K$, for all $P'_i \in \mathbb{P}$,

$$|u_k(\mathbf{p}) - u_k(P'_i, \mathbf{p}_{(-i)})| \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Remember that $u_k = \frac{\gamma(\beta_k)}{P_k}$, and γ is at least C^2 . Let $(\beta_1, \dots, \beta_K)$ be the SINRs associated with the power allocation \mathbf{p} and $(\beta'_1, \dots, \beta'_K)$ the SINRs associated with the power allocation $(P'_i, \mathbf{p}_{(-i)})$. Then a simple Taylor expansion of γ in β'_k gives

$$\gamma(\beta'_k) = \gamma(\beta_k) + (\beta'_k - \beta_k) \frac{\partial \gamma}{\partial \beta}(\beta_k) + o(\beta'_k - \beta_k). \quad (44)$$

According to (44), it is sufficient to show that

$$\left| \frac{\beta'_k - \beta_k}{P_k} \right| \rightarrow 0, \text{ as } N \rightarrow \infty. \quad (45)$$

a) Matched Filter: For the matched filter, the inequality is obtained directly from (3). The denominator of (3) is always greater than $\frac{\sigma^2}{N} \sum_{n=1}^N |h_{nk}|^2$. Hence,

$$\begin{aligned} \left| \frac{\beta'_k - \beta_k}{P_k} \right| &\leq \left| \frac{P_k \frac{1}{N} (P'_i - P_i) \frac{1}{N} \sum_{n=1}^N |h_{ni}|^2 |h_{nk}|^2}{P_k \sigma^4} \right| \\ &\leq \frac{P_{\max} h_{\max}^2}{\sigma^4 N}. \end{aligned}$$

b) MMSE Filter: For the MMSE filter, the inequality is obtained from (4), Lemma 1 from [30] and Lemma 2.1 from [37], which we both reproduce below for convenience.

Lemma 1: [30] Let \mathbf{C} be a $N \times N$ complex matrix with uniformly bounded spectral radius for all N : $\sup_N (|\mathbf{C}|) < \infty$. Let $\mathbf{w} = \frac{1}{\sqrt{N}} [w_1, \dots, w_N]^T$ where $\{w_i\}_{i=1 \dots N}$ are i.i.d. complex random variables with zero mean, unit variance and finite eighth moment. Then:

$$\mathbb{E} \left[\left| \mathbf{w}^H \mathbf{C} \mathbf{w} - \frac{1}{N} \text{tr} \mathbf{C} \right|^4 \right] \leq \frac{C}{N^2}$$

where C is a constant that does not depend on N or \mathbf{C} .

Lemma 2: [37] Let $\sigma^2 > 0$, \mathbf{A} and \mathbf{B} $N \times N$ with \mathbf{B} Hermitian nonnegative definite, and $\mathbf{q} \in \mathbb{C}^N$. Then

$$\text{tr} \left(((\mathbf{B} + \sigma^2 \mathbf{I})^{-1} - (\mathbf{B} + \mathbf{q} \mathbf{q}^H + \sigma^2 \mathbf{I})^{-1}) \mathbf{A} \right) \leq \frac{\|\mathbf{A}\|}{\sigma^2}.$$

In Lemma 2, $\|\mathbf{A}\|$ is the spectral norm of \mathbf{A} , i.e., the square root of the largest singular value of \mathbf{A} .

From (4), we can write

$$\beta_k = P_k \mathbf{w}_k^H \mathbf{H}_k^H \left(\mathbf{G}_{(-k)} \mathbf{G}_{(-k)}^H + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{H}_k \mathbf{w}_k,$$

$$\beta'_k = P_k \mathbf{w}_k^H \mathbf{H}_k^H \left(\mathbf{G}_{(-k)}' \mathbf{G}_{(-k)}'^H + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{H}_k \mathbf{w}_k$$

where $\mathbf{G}_{(-k)}' \mathbf{G}_{(-k)}'^H = \mathbf{G}_{(-k)} \mathbf{G}_{(-k)}^H + (P'_i - P_i)(\mathbf{h}_i \odot \mathbf{w}_i)(\mathbf{h}_i \odot \mathbf{w}_i)^H$.

A corollary of Lemma 1 is that for either matrix $\mathbf{C} = \mathbf{H}_k^H \left(\mathbf{G}_{(-k)} \mathbf{G}_{(-k)}^H + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{H}_k$ or matrix $\mathbf{C} = \mathbf{H}_k^H \left(\mathbf{G}_{(-k)}' \mathbf{G}_{(-k)}'^H + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{H}_k$, we obtain [30]

$$\left| \mathbf{w}_k^H \mathbf{C} \mathbf{w}_k - \frac{1}{N} \text{tr} \mathbf{C} \right| \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Matrix $\mathbf{B} = \mathbf{G}_{(-k)} \mathbf{G}_{(-k)}^H$ is Hermitian nonnegative definite, as for all $\mathbf{w} \in \mathbb{C}^N$, $\mathbf{w}^H \mathbf{G}_{(-k)} \mathbf{G}_{(-k)}^H \mathbf{w} = \|\mathbf{G}_{(-k)} \mathbf{w}\|^2 \geq 0$. Diagonal matrix $\mathbf{A} = \mathbf{H}_k \mathbf{H}_k^H$ has spectral norm $\|\mathbf{H}_k \mathbf{H}_k^H\| \leq h_{\max}^2$. Using Lemmas 1 and 2, as $N \rightarrow \infty$, we obtain

$$\left| \frac{\beta'_k - \beta_k}{P_k} \right| \rightarrow 0, \text{ as } N \rightarrow \infty.$$

c) Optimum and SIC Filters: The analog of the SINR derived for the optimum filter stems from the MMSE filter with SIC. The SINR for SIC filters have similar expressions with less interfering users appearing in the denominator. Hence, the result is immediate.

APPENDIX C PROOF OF PROP. 5

Given C^* , we can use (13) to obtain a Nash equilibrium power allocation in the following way. We rewrite (13) assuming that the target SINR for the MMSE filter is β^+ .

$$\alpha \log_2(1 + \beta^+) - \alpha \log_2(e) \frac{\beta^+}{1 + \beta^+}$$

$$+ \log_2 \left(1 + \frac{1}{\sigma^2(1 + \beta^+)} \int_0^\alpha P(y) |h(y)|^2 dy \right)$$

$$= \alpha \log_2(1 + \beta^*). \quad (46)$$

In the left-hand side of (46), $P(y)$ is given by a MMSE power allocation similar to the one given by (21). Hence, the term $\int_0^\alpha P(y) |h(y)|^2 dy$ in (46) does not depend on the actual realizations of the channels. Replacing β^* by β^+ in (20), we obtain that $\int_0^\alpha P(y) |h(y)|^2 dy = \frac{\alpha \sigma^2 \beta^+}{1 - \alpha \frac{\beta^+}{1 + \beta^+}}$, which gives us (23). Replacing β^* by β^+ in (21), we obtain the power allocation (22).

APPENDIX D EXPECTATION OF THE RANDOM VARIABLE

Under the hypotheses on $h_{n,j}$ of Sec. VI-B, we show that the expectation of the random variable $\frac{1}{K} \sum_{j=1}^K \frac{|h_{n,j}|^2}{E_j}$ is equal to 1. By expanding the expression of $h_{n,j}$, this is equivalent to showing that the expectation of $\frac{h_\ell \left(\frac{j}{N}\right) h_{\ell'} \left(\frac{j'}{N}\right)}{E_j}$ is equal to

0. Denoting by $p(\cdot)$ the distribution of $h_\ell = h_\ell \left(\frac{j}{N}\right)$, this expectation is equal to the L -dimensional integral of

$$\frac{h_\ell h_{\ell'}}{|h_\ell|^2 + |h_{\ell'}|^2 + \sum_{k \neq \ell, \ell'} |h_k|^2} p(h_\ell) p(h_{\ell'}) \prod_{k \neq \ell, \ell'} p(h_k)$$

which is an odd function of h_ℓ . Its integral is therefore 0, which proves the desired result.

APPENDIX E PROOF OF (35) AND (36)

Denote $m_k = P_{K-k} |h_{K-k}|$. From (32), with flat fading, the sequence $\{m_k\}_{k \in S(K)}$ satisfies $m_0 = \beta^* \sigma^2$ and $m_{k+1} = \beta^* \sigma^2 + \frac{\beta^*}{N} \sum_{j=0}^k m_j$. Using the fact that $\sum_{i=j}^k \binom{i}{j} = \binom{k+1}{j+1}$, it is immediate to prove by recurrence that

$$m_k = \beta^* \sigma^2 \sum_{j=0}^k \binom{k}{j} \frac{1}{N^j} \beta^{*j} = \beta^* \sigma^2 \left(1 + \frac{1}{N} \beta^* \right)^k.$$

Hence formula (35). The demonstration is exactly similar for (36) from the recursion $m_0 = \beta^* \sigma^2$ and $m_{k+1} = \beta^* \sigma^2 + \frac{\beta^*}{(1 + \beta^*)N} \sum_{j=0}^k m_j$.

APPENDIX F OPTIMAL ORDERING OF USERS

We determine the ordering that makes use of the least total power for equilibrium PA (35) (the case is similar for (36), (37), and (38)). Let the ordering of the users be such as $|h_1|^2 < \dots < |h_K|^2$. Let π be any permutation of $\{1, \dots, K\}$. Let $a_{ij} = \left(1 + \frac{1}{N} \beta^* \right)^{K-i} - \left(1 + \frac{1}{N} \beta^* \right)^{K-j}$.

Then showing that the optimal ordering is such as $|h_1|^2 < \dots < |h_K|^2$ is equivalent to showing that for any π

$$\sum_{k=1}^K \frac{1}{|h_k|^2} a_{k\pi(k)} > 0. \quad (47)$$

Consider first a cyclic permutation. By the definition of a_{ij} , the sum of the $a_{k\pi(k)}$ is equal to zero: $\sum_{k=1}^K a_{k\pi(k)} = 0$. The first coefficient $a_{1\pi(1)}$ is positive. It is affected coefficient $\frac{1}{|h_1|^2}$, which is the greatest coefficient in the sum in (47). Hence, the sum in (47) is positive.

Permutation π can be decomposed as a product of disjoint permutation cycles. Each cycle determines a subset of indexes k , these subsets form a partition of $\{1, \dots, K\}$. With a similar reasoning as precedently, replacing index 1 with the smallest index in the cycle, the sum over the indexes k pertaining to a cycle of $\frac{1}{|h_k|^2} a_{k\pi(k)}$ is positive. Hence the global sum of (47) is also positive.

It can be proven in a similar way that the same ordering maximizes the sum of inverse powers of the users.

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