Phase-locking between Kuramoto oscillators: robustness to time-varying natural frequencies

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Abstract— **In this paper we analyze the robustness of phase-locking in the Kuramoto system with arbitrary bidirectional interconnection topology. We show that the effects of time-varying natural frequencies encompass the heterogeneity in the ensemble of oscillators, the presence of exogenous disturbances, and the influence of unmodeled dynamics. The analysis, based on a Lyapunov function for the incremental dynamics of the system, provides a general methodology to build explicit bounds on the region of attraction, on the size of admissible inputs, and on the input-to-state gains. As an illustrative application of this method, we show that, in the particular case of the all-to-all coupling, the synchronized state exponentially input-to-state stable provided that all the initial phase differences lie in the same half circle. The approach provides an explicit bound on the convergence rate, thus extending recent results on the exponential synchronization of the finite Kuramoto model. Furthermore, the proposed Lyapunov function for the incremental dynamics allows for a new characterization of the robust asymptotically stable phase-locked states of the unperturbed dynamics in terms of its isolated local minima.**

I. INTRODUCTION

Synchronization has recently found many applications in the modeling and control of physical [1], [2], [3], chemical [4], medical [5], biological [6], and engineering problems [7]. Roughly speaking, an ensemble of interacting agents is said to synchronize when their outputs tend to a common value [8, Chapter 5]. Examples of such a behavior can be found in interconnected neurons [6], [9], [10], chemical oscillators [4], coupled mechanical systems [11] and consensus algorithms [12], [13], [14]. Phase-locking, or frequency locking, is a particular type of synchronization that describes the ability of interconnected oscillators to tune themselves to the same frequency. One of the most widely used mathematical model to analyze this behavior is the Kuramoto model, which was first introduced in [4] to describe globally coupled chemical oscillators, as a generalization of the one originally proposed by Winfree [16]. Later on, many other works generalized these pioneer seminal works [12], [17], [18], [19], [20], [21], [22], [23], [7], [14], [8], [24]. In this paper we consider the Kuramoto system with time-varying natural frequencies and a general interconnection topology. Letting the signal ϖ_i denote the *time-varying natural frequency* of the oscillator $i \in \{1, \ldots, N\}$, and $k = [k_{ij}]_{i,j=1,\ldots,N} \in \mathbb{R}_{\geq 0}^{N \times N}$ represent the *coupling matrix*, each agent i is described by its phase

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θ_i ruled by the following dynamics:

$$\dot{\theta}_i(t) = \varpi_i(t) + \sum_{j=1}^{N} k_{ij} \sin(\theta_j(t) - \theta_i(t)), \ \ \forall t \geq 0. \quad (1)$$

The analysis of robustness with respect to time-varying natural frequencies encompasses different types of perturbations, including agents heterogeneity, influence of exogenous inputs and imprecise modeling. Heterogeneities among oscillators, such as different constant natural frequencies, are known to prevent phase-locking if the coupling strength is too small [20], [19], [25], [26], [24]. Exogenous disturbances, which may include deterministic or stochastic signals [27], can also affect, and even impede, phase-locking. Time-varying natural frequencies may also model the influence of the feedback signals studied in the literature for their desynchronizing features [26], [28], [29], [30], [31], [15], as well as time-varying interconnection topologies an non-sinusoidal coupling. This issue is particularly relevant for the study of interconnected neuronal cells for which little is known on the interconnection topology and synaptic weights between neurons [32].

The robustness of phase-locking in the Kuramoto model has already been partially addressed in the literature both in the case of infinite and finite number of oscillators. On the one hand, the infinite dimensional Kuramoto model allows for an easier analytical treatment of the robustness analysis (see for example [18] for a complete survey). This approach has been used to analyze the effect of delayed [31] and multisite [28] mean-field feedback approach to desynchronization. In the case of stochastic inputs it allows to find the minimum coupling to guarantee phase-locking in the presence of noise [27]. This approach is, however, feasible only in the case of all-to-all interconnection. On the other hand, the finite dimensional case has been the object of both analytical and numerical studies. In particular, [25] proposes a complete numerical analysis of robustness to time-varying natural frequencies, time-varying interconnection topologies and non-sinusoidal coupling. It suggests that phase-locking exhibits some robustness to all these types of perturbations. Analytical studies on the robustness of phase-locking in the finite Kuramoto model have been addressed only for constant natural frequencies [20], [17]. The existence and explicit expression of the fixed points describing stable and unstable phase-locked states is studied in [19]. The Lyapunov approach proposed in [21] for an all-to-all coupling suggests that an analytical study of phase-locking robustness can be deepened. To the best of our knowledge the problem of the robustness of phase-locking with respect to time-varying

natural frequencies has still not been analytically addressed in the finite Kuramoto model with arbitrary bidirectional interconnection topologies.

This paper establishes that phase-locking is locally input-to-state stable (ISS) with respect to small inputs. The proof is based on the existence of an ISS-Lyapunov function for the incremental dynamics of the system. This analysis provides a general methodology to build explicit estimates on the size of the region of convergence, the ISS gain, and the tolerated input bound. It applies to general symmetric interconnection topologies and to any asymptotically stable phase-locked state. As an illustrative application of the main result, we extend some results in [17] to the time-varying case, by proving the exponential ISS of synchronization when all the initial phase differences lie in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and by giving explicit bounds on the convergence rate. The size of the region of convergence, the sufficient bound on the coupling strength and the convergence rate are compared to those obtained in [17]. Furthermore, the Lyapunov function for the incremental dynamics allows for a new characterization of the phase-locked states of the unperturbed system. In particular, when restricted to a suitable invariant manifold, it allows to completely characterize the robust phase-locked states in terms of its isolated local minima.

Notation. For a set $A \subset \mathbb{R}$ and $a \in \mathbb{R}$, $A_{\geq a}$ denotes the set $\{x \in A : x \geq a\}$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm, that is $|x| := \sqrt{\sum_{i=1}^n x_i^2}$, while $|x|_\infty$ denotes its infinity norm, that is $|x|_\infty := \max_{i=1,\dots,n} |x_i|$. We adopt the notation $|x|_2 := |x|$, when we want to explicitly distinguish $|x|$ from $|x|_\infty$. For a set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_\mathcal{A} := \inf_{y \in \mathcal{A}} |y - x|$ denotes the point-to-set distance from x to $\mathcal{A}$. $\mathcal{B}(x, R)$ refers to the closed ball of radius R centered at x in the Euclidean norm, i.e. $\mathcal{B}(x, R) := \{z \in \mathbb{R}^n : |x - z| \leq R\}$; for a subset $\mathcal{A} \subset \mathbb{R}^n$, $\mathcal{B}(\mathcal{A}, R) := \{z \in \mathbb{R}^n : |z|_\mathcal{A} \leq R\}$. $\|u\|$ is the L^1 norm of the signal $u(\cdot)$, that is, if $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ denotes a measurable signal, locally essentially bounded, $\|u\| := \operatorname{esssup}_{t \geq 0} |u(t)|$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for any fixed $t \geq 0$ and $\beta(s, \cdot)$ is continuous decreasing and tends to zero at infinity for any fixed $s \geq 0$. T^n denotes the n-Torus. If $x \in \mathbb{R}^n$, ∇_x is the *gradient vector* with respect to x, i.e. $\nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. Given $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$, $(x \bmod a) := [x_i \bmod a]_{i=1,\dots,n}$, where $\bmod$ denotes the modulo operator. The vector with all unitary components in $\mathbb{R}^n$ is denoted by $\mathbf{1}_n$.

II. ROBUSTNESS OF PHASE-LOCKED SOLUTIONS

A. Robustness analysis

Phase-locking can be formally defined based on the *incremental dynamics* $\dot{\theta}_i - \dot{\theta}_j$ associated to (1). As in [15], a phase-locked solution corresponds to a fixed point of the incremental dynamics.

Definition 1 (Phase-locking / Exact synchronization) A solution θ^* to system (1) is said to be *phase-locked* iff

$$\dot{\theta}_j^*(t) - \dot{\theta}_i^*(t) = 0, \quad \forall\, i, j = 1, \dots, N, \ \forall t \geq 0.$$

It is said to be *exactly synchronized* if it is phase-locked with zero phase differences, that is

$$\theta_i^*(t) - \theta_j^*(t) = 0, \quad \forall\, i, j = 1, \dots, N, \ \forall t \geq 0.$$

In view of this definition, the robustness analysis of phase-locking boils down to the analysis of the fixed points of the dynamics ruling the phase differences $\theta_i - \theta_j$. A similar approach has been exploited in [17] in the case of all-to-all coupling and constant inputs. In contrast to [20], studying the incremental dynamics of the system avoids the use of the grounded Kuramoto model, in which the mean frequency of the ensemble is "grounded" to zero and synchronization corresponds to a fixed point. While the latter is a well defined mathematical object for constant perturbations, its extension to time-varying inputs, which is the subject of the present study, is not clear. Hence, we start by defining the *common drift* $\overline{\omega}$ of the system (1) as

$$\overline{\omega}(t) := \frac{1}{N} \sum_{j=1}^N \varpi_j(t), \quad \forall t \geq 0, \tag{2}$$

and the *grounded input* $\tilde{\omega}$ as $\tilde{\omega} := [\tilde{\omega}_i]_{i=1,\dots,N}$, where

$$\tilde{\omega}_i(t) := \varpi_i(t) - \overline{\omega}(t), \quad \forall i = 1, \dots, N, \ \forall t \geq 0. \tag{3}$$

Noticing that $\varpi_i - \varpi_j = \tilde{\omega}_i - \tilde{\omega}_j$, the evolution equation of the *incremental dynamics* ruled by (1) reads

$$\dot{\theta}_i(t) - \dot{\theta}_j(t) = \tilde{\omega}_i(t) - \tilde{\omega}_j(t) + \tag{4}$$
$$\sum_{l=1}^N k_{il} \sin(\theta_l(t) - \theta_i(t)) - \sum_{l=1}^N k_{jl} \sin(\theta_l(t) - \theta_j(t))$$

for all $i, j = 1, \dots, N$, $i \neq j$, and all $t \geq 0$. In the sequel we use $\tilde{\theta}$ to denote the *incremental variable*:

$$\tilde{\theta} := [\theta_i - \theta_j]_{i,j=1,\dots,N, i \neq j} \in \mathrm{T}^{(N-1)^2}. \tag{5}$$

As expected, the incremental dynamics (4) is independent of $\overline{\omega}$, meaning that it is invariant to common drifts among the oscillators[1]. As stressed in the Introduction, the system (1), and thus its incremental dynamics (4), encompasses both the heterogeneity between agents, the presence of exogenous disturbances and the uncertainties in the interconnection topology. To see this clearly, let ω_i denote the natural frequency of the agent i, let p_i represent its additive external perturbations, and let Δ_{ij} denote the uncertainty on each coupling gain k_{ij}. Then the effects of all these disturbances can be analyzed in a unified manner by letting

$$\varpi_i(t) = \omega_i + p_i(t) + \sum_{j=1}^N \Delta_{ij}(t) \sin(\theta_j(t) - \theta_i(t)). \tag{6}$$

When no inputs are applied, i.e. $\tilde{\omega} = 0$, we expect the solutions of (4) to converge to some asymptotically stable

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[1]This fact can also be interpreted as the invariance of the system (1) with respect to common phase-shift of the ensemble (i.e. $\theta_i \to \theta_i + c$, $\forall i = 1, \dots, N$). See for example [12].

fixed point or, equivalently, the solution of (1) to converge to some asymptotically stable phase-locked solution at least for some coupling matrices k. To make this precise, we start by defining the notion of 0-*asymptotically stable* (0-AS) *phase-locked solutions*, which are described by asymptotically stable fixed points of the incremental dynamics (4) when no inputs are applied.

Definition 2 (0-AS phase-locked solutions) Given any coupling matrix $k \in \mathbb{R}_{\geq 0}^{N \times N}$, let $\mathcal{O}_k$ denote the set of all asymptotically stable fixed points of the unperturbed (*i.e.* $\tilde{\omega} \equiv 0$) incremental dynamics (4). A phase-locked solution θ^* of (1) is said to be 0-*asymptotically stable* if and only if the incremental state $\tilde{\theta}^* := \left[\theta_i^* - \theta_j^*\right]_{i,j=1,\ldots,N, i \neq j}$ belongs to $\mathcal{O}_k$.

A complete characterization of 0-AS phase-locked solutions of (1) for general interconnection topologies can be found in [33] and [14, Chapter 3]. In Section II-D, we characterize the set $\mathcal{O}_k$ in terms of the isolated local minima of a suitable Lyapunov function.

The reason for considering only *asymptotically* stable fixed points of the incremental dynamics stands in the fact that only those are expected to provide some robustness properties (as asymptotic stability implies local robustness with respect to small inputs [34], [35]). On the contrary, (non 0-AS) stable fixed points may correspond to non-robust phase-locked state, as illustrated by the following example.

Example 1 (Non robust phase-locking) Consider the case where $N > 2$ and let $k_{12} = k_{21} > 0$, and $k_{ij} = 0$ for all $(i,j) \in \mathbb{N}_{\leq N} \times \mathbb{N}_{\leq N} \setminus \{(1,2),(2,1)\}$. When $\tilde{\omega} = 0$, the dynamics (4) reads

$$\dot{\theta}_i - \dot{\theta}_j = 0$$

for all $(i,j) \in \mathbb{N}_{\leq N} \times \mathbb{N}_{\leq N} \setminus \{(1,2),(2,1)\}$, and

$$\dot{\theta}_1 - \dot{\theta}_2 = -2k_{12}\sin(\theta_1 - \theta_2).$$

In this case, all the solutions of the form $\theta_1(t) - \theta_2(t) = 0$, for all $t \geq 0$, and $\theta_i(t) - \theta_j(t) = \theta_i(0) - \theta_j(0)$, for all $t \geq 0$ and all $(i,j) \in \mathbb{N}_{\leq N} \times \mathbb{N}_{\leq N} \setminus \{(1,2),(2,1)\}$ are phase-locked. Then can be shown to be stable, but not asymptotically. By adding any (arbitrarily small) constant inputs $\tilde{\omega}_l \neq 0$ to one of the agent $l \in \mathbb{N}_{\leq N} \setminus \{1,2\}$, the system becomes completely desynchronized, since $\dot{\theta}_l - \dot{\theta}_i \equiv \tilde{\omega}_l$ for all $i = 1, \ldots, N, i \neq l$. In particular, the set $\mathcal{O}_k$ is empty for this particular case.

We next recall the definition of local Input-to-State Stability with respect to small inputs [36]. This concept is also referred to as *Total Stability* [35].

Definition 3 (LISS w.r.t. small inputs) For a system of the form $\dot{x} = f(x,u)$, a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *locally input-to-state stable (LISS) with respect to small inputs* iff there exist some constants $\delta, \delta_u > 0$, a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function ρ, such that, for all $|x_0|_{\mathcal{A}} \leq \delta_x$ and all u satisfying $\|u\| \leq \delta_u$, its solution satisfies

$$|x(t)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t) + \rho(\|u\|), \quad \forall t \geq 0.$$

If this estimate holds with $\beta(r,s) = Cre^{-\frac{s}{\tau}}$, where C, τ are positive constants, then $\mathcal{A}$ is said to be *locally exponentially Input-to-State Stable with respect to small inputs*.

Remark 1 (Local Euclidean metric on the n-Torus) Definition 3 is given on $\mathbb{R}^n$, which is little adapted to the context of this article. Its extension to the n-Torus is natural since T^n is locally isometric to $\mathbb{R}^n$ through the identity map $\mathcal{I}$ (*i.e.* $|\theta|_{\mathrm{T}^n} := |\mathcal{I}(\theta)| = |\theta|$). In particular this means that the n-Torus can be provided with the local Euclidean metric and its induced norm. Hence, Definition 3 applies locally in the n-Torus.

The next theorem, whose proof is given in Section III-A, states the LISS of $\mathcal{O}_k$ with respect to small inputs $\tilde{\omega}$.

Theorem 1 (LISS of phase-locking w.r.t. small inputs) Let $k \in \mathbb{R}_{\geq 0}^{N \times N}$ be any symmetric interconnection matrix. Suppose that the set $\mathcal{O}_k$ of Definition 3 is non-empty. Then the system (4) is locally input-to-state stable with respect to small $\tilde{\omega}$. In other words, there exist $\delta_{\tilde{\theta}}, \delta_\omega > 0$, $\beta \in \mathcal{KL}$ and $\rho \in \mathcal{K}_\infty$, such that, for all $\tilde{\omega}$ satisfying $\|\tilde{\omega}\| \leq \delta_\omega$ and all $\tilde{\theta}_0 \in \mathcal{B}(\mathcal{O}_k, \delta_{\tilde{\theta}})$, its solution satisfies

$$|\tilde{\theta}(t)|_{\mathcal{O}_k} \leq \beta(|\tilde{\theta}_0|_{\mathcal{O}_k}, t) + \rho(\|\tilde{\omega}\|), \quad \forall t \geq 0. \qquad (7)$$

Theorem 1 guarantees that, if a given configuration is asymptotically stable for the unperturbed system, then solutions starting sufficiently near from that configuration remain near it at all time, in presence of sufficiently small perturbations $\tilde{\omega}$. Moreover, the steady-state distance of the incremental state $\tilde{\theta}$ from $\mathcal{O}_k$ is somehow "proportional" to the amplitude of $\tilde{\omega}$ with nonlinear gain ρ. This means that the phase-locked states described by $\mathcal{O}_k$ are robust to time-varying natural frequencies, provided they are not too heterogeneous. We stress that, while local ISS with respect to small inputs is a natural consequence of asymptotic stability [34], the size of the constants δ_x and δ_u in Definition 3, defining the robustness domain in terms of initial conditions and inputs amplitude, are potentially infinitesimal. As we show explicitly in the next section in the special case of all-to-all coupling, the Lyapunov analysis used in the proof of Theorem 1 (cf. Section III-A) provides a general *methodology* to build these estimates explicitly. We stress in particular that, while the region of attraction depends on the geometric properties of the fixed points of the unperturbed system, the size of admissible inputs can be made arbitrarily large by taking a sufficiently large coupling strength. This is detailed in the sequel (cf. (26), (28), (32) and (33)).

B. Robustness of the synchronized state in the case of all-to-all coupling

In this section we focus the Lyapunov analysis used in the proof of Theorem 1 to the case of all-to-all coupling. In this case, it is known [33] that the only asymptotically stable phase-locked solution is the *exact synchronization*

$$\tilde{\theta}(t) = 0, \quad \forall t \geq 0, \qquad (8)$$

corresponding to a zero phase difference between each pair of oscillators (cf. Definition 1). The following proposition states the local *exponential* input-to-state stability of the synchronized state with respect to small inputs, and provides explicit bounds on the region of convergence, the size of admissible inputs, the ISS gain, and the convergence rate. Its proof can be found in Section III-B.

Proposition 1 (Exponential LISS of synchronization)
Consider the system (1) with the all-to-all interconnection topology, i.e. $k_{ij} = K > 0$ for all $i, j = 1, \ldots, N$. Then, for all $0 \leq \epsilon \leq \frac{\pi}{2}$, and all $\tilde{\omega}$ satisfying

$$\|\tilde{\omega}\| \leq \delta_\omega^\epsilon := \frac{K\sqrt{N}}{\pi^2}\left(\frac{\pi}{2} - \epsilon\right), \qquad (9)$$

the following facts hold:

1) *the set $\mathcal{D}_\epsilon := \left\{\tilde{\theta} \in T^{(N-1)^2} : |\tilde{\theta}|_\infty \leq \frac{\pi}{2} - \epsilon\right\}$ is forward invariant for the system (4);*
2) *for all $\tilde{\theta}_0 \in \mathcal{D}_0$, the set $\mathcal{D}_\epsilon$ is attractive, and the solution of (4) satisfies*

$$|\tilde{\theta}(t)| \leq \frac{\pi}{2}|\tilde{\theta}_0|e^{-\frac{K}{\pi^2}t} + \frac{\pi^2}{K}\|\tilde{\omega}\|, \quad \forall t \geq 0.$$

Proposition 1 establishes the exponential ISS of the synchronized state in the all-to-all Kuramoto model with respect to time-varying inputs whose amplitudes are smaller than $\frac{K\sqrt{N}}{\pi^2}$. It holds for any initial condition lying in $\mathcal{D}_0$, that is when all the initial phase differences lie in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Moreover, if the inputs amplitude is bounded by δ_ω^ϵ, for some $0 \leq \epsilon \leq \frac{\pi}{2}$, then the set $\mathcal{D}_\epsilon$ is forward invariant and all the solutions starting in $\mathcal{D}_0$ actually converge to $\mathcal{D}_\epsilon$.

Recently, necessary and sufficient conditions for the exponential synchronization of the Kuramoto system with all-to-all coupling and constant different natural frequencies were given in [17]. We stress that the estimated region of attraction provided by Proposition 1 is strictly larger than the one obtained in [17, Theorem 4.1], which does not allow ϵ to be picked as zero. For initial conditions lying in $\mathcal{D}_\epsilon$, with a strictly positive ϵ, it is interesting to compare the convergence rate obtained in Proposition 1, $\frac{K}{\pi^2}$, with the one obtained in [17, Theorem 3.1], $NK\sin(\epsilon)$. While the convergence rate of Proposition 1 is slower than the one obtained in [17, Theorem 3.1] for large ϵ, it provides a better estimate for small values of ϵ. Furthermore, for any fixed amplitude $\|\tilde{\omega}\|$, the bound (9) allows to find the sufficient coupling strength K_ϵ which ensures the attractivity of $\mathcal{D}_\epsilon$:

$$K_\epsilon = \frac{\pi^2}{\left(\frac{\pi}{2} - \epsilon\right)\sqrt{N}}\|\tilde{\omega}\|.$$

Noticing that $\sqrt{N}\max_{i,j=1,\ldots,N}\|\varpi_i - \varpi_j\| \geq \|\tilde{\omega}\|$, we get that

$$K_\epsilon \leq \frac{\pi^2}{\left(\frac{\pi}{2} - \epsilon\right)}\max_{i,j=1,\ldots,N}\|\varpi_i - \varpi_j\|.$$

Since $\left(\frac{\pi}{2} - \epsilon\right) \geq \frac{2}{\pi}\cos(\epsilon)$, for all $0 \leq \epsilon \leq \frac{\pi}{2}$, it results that

$$K_\epsilon \leq \frac{\pi^3}{2\cos(\epsilon)}\max_{i,j=1,\ldots,N}\|\varpi_i - \varpi_j\| < \pi^3 K_{inv},$$

where K_{inv} is the sufficient coupling strength provided in [17, Proof of Theorem 4.1]. This observation shows that, while the estimate K_ϵ may be more restrictive than the one proposed in [17], both are of of the same order, in the sense that $\frac{K_\epsilon}{K_{inv}} < \pi^3$.

In conclusion, Proposition 1 partially extends the main results of [17] to time-varying inputs. On the one hand, it allows to consider a larger set of initial conditions, and bounds the convergence rate by a strictly positive value, independently of the region of attraction. On the other hand it may require larger coupling strength, and, for small regions of attraction, the bound on the convergence rate obtained in Proposition 1 is not as good as the one of [17, Theorem 3.1].

C. Robustness of neural synchrony to mean-field feedback Deep Brain Stimulation

The model (1) encompasses our model of interconnected neurons under mean-field feedback DBS, introduced in [15] and that is referred to as the *Kuramoto system under real mean-field feedback*,

$$\dot{\theta}_i(t) = \omega_i + \sum_{j=1}^{N} k_{ij}\sin(\theta_j - \theta_i)+$$

$$\sum_{j=1}^{N}\gamma_{ij}\sin(\theta_j(t) - \theta_i(t)) - \sum_{j=1}^{N}\gamma_{ij}\sin(\theta_j(t) + \theta_i(t)) \quad (10)$$

for all $i = 1, \ldots, N$ and all $t \geq 0$, where ω_i is the constant natural frequency of the i-th neuron, $k = [k_{ij}]_{i,j=11,\ldots,N} \in \mathbb{R}^{N \times N}$ describes the interconnection between the neurons in the subthalamic nucleus, and $\gamma = [\gamma_{ij}]_{i,j=11,\ldots,N} \in \mathbb{R}^{N \times N}$ is the feedback gain. Indeed, the effect of the real mean-field feedback can be modeled as an exogenous input, that is with

$$\varpi_i(t) =$$

$$\omega_i + \sum_{j=1}^{N}\gamma_{ij}\sin(\theta_j(t) - \theta_i(t)) - \sum_{j=1}^{N}\gamma_{ij}\sin(\theta_j(t) + \theta_i(t)),$$

$$(11)$$

for all $i = 1, \ldots, N$ and all $t \geq 0$. In particular the results of this paper can be used to compute necessary conditions on the mean field feedback gain to guarantee an effective desynchronization. To that end, consider the system (10), and define

$$\overline{\gamma} := \max_{i,j=1,\ldots,N}|\gamma_{ij}|, \qquad (12)$$

i.e. $\overline{\gamma}$ denotes the intensity of the mean-field feedback DBS, and

$$\omega^\perp := \left[\omega_i - \frac{1}{N}\sum_{j=1}^{N}\omega_j\right]_{i=1,\ldots,N}, \qquad (13)$$

i.e. $\omega^\perp$ represents the heterogeneity of the ensemble of neurons. We define the *grounded mean-field input* $\tilde{I}_{MF}$ of the incremental dynamics associated to (10) as

$$\tilde{I}_{MF}(t) := I_{MF}(t) - \overline{I}_{MF}(t), \quad \forall t \geq 0, \qquad (14)$$

where
$$I_{MF}(t) :=$$

$$\left[\sum_{j=1}^{N}\gamma_{ij}\Big(\sin(\theta_i(t)-\theta_j(t))-\sin(\theta_i(t)+\theta_j(t))\Big)\right]_{i=1,\ldots,N},$$

for all $t \geq 0$, represents the input of the mean-field feedback (cf. (11)) and

$$\overline{I}_{MF}(t) := \frac{1}{N}\sum_{i,j=1}^{N}\gamma_{ij}\Big(\sin(\theta_i(t)-\theta_j(t))-\sin(\theta_i(t)+\theta_j(t))\Big),$$

for all $t \geq 0$, represents the common drift among the ensemble of neurons due to the mean-field feedback. The grounded mean-field input $\tilde{I}_{MF}$ is the quantity of interest for the aim of desynchronization. Indeed, as already pointed out in (4), it is this input, along with the intrinsic heterogeneity of the ensemble (*i.e.* $\omega^{\perp}$), which is responsible for the destabilization of the incremental dynamics of the Kuramoto system under real mean-field feedback. The following result underlines the robustness of phase-locking to this particular perturbation. In other words, it provides a negative answer to the question whether mean-field feedback stimulation with arbitrarily small amplitude can effectively desynchronize the STN neurons.

Corollary 1   *Let $k \in \mathbb{R}_{\geq 0}^{N \times N}$ be any symmetric interconnection matrix and $\{\omega_i\}_{i=1,\ldots,N}$ be any (constant) natural frequencies. Let $\gamma \in \mathbb{R}^{N \times N}$ be any feedback gain. Let $\overline{\gamma}$ and $\omega^{\perp}$ be defined as in (12)-(13). Let the set $\mathcal{O}_k$ be defined as in Definition 2 and suppose that it is non-empty. Then there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}_{\infty}$ function σ, a positive constant δ_{ω}, and a neighborhood $\mathcal{P}$ of $\mathcal{O}_k$, such that, for all natural frequencies and all mean-field feedback satisfying*

$$|\omega^{\perp}| + 2\,\overline{\gamma}\,N\sqrt{N} \leq \delta_{\omega},$$

the solution of the incremental dynamics of (10) satisfies, for all $\tilde{\theta}_0 \in \mathcal{P}$,

$$|\tilde{\theta}(t)|_{\mathcal{O}_k} \leq \beta(|\tilde{\theta}_0|_{\mathcal{O}_k}, t) + \sigma(|\omega^{\perp}| + \|\tilde{I}_{MF}\|),$$

where $\tilde{I}_{MF}$ is defined in (14).

Corollary 1 states that the phase-locked states associated to any symmetric interconnection topology are robust to sufficiently small real mean-field feedbacks. The intensity of the tolerable feedback gain $\overline{\gamma}$ depends on the distribution of natural frequencies, reflecting the fact that a heterogeneous ensemble can be more easily brought to an incoherent state.

Energy consumption is a critical issue in the DBS framework [37]. Corollary 1, along with the explicit input bound δ_{ω}, which can be found in the proof of Theorem 1 (cf. (33)), provide a *necessary* condition on the intensity of the DBS through mean-field feedback to achieve effective desynchronisation for a general interconnection between neurons and recording-stimulation setup. Even an approximate knowledge of the distribution of natural frequencies of the neurons in the STN, of their interconnection topology and of the electrical characteristics of the recording-stimulation setup can be used to compute this value, based on the Lyapunov analysis detailed in Section II-D.

Future works will focus on the computation of *sufficient conditions* on this intensity in order to achieve full desynchronization (see [28] for the $N \to \infty$ case with all-to-all coupling and separate stimulation-registration setup).

D. A Lyapunov function for the incremental dynamics

In this section, we introduce the Lyapunov function for the incremental dynamics (4) used in the proof of Theorem 1, that will be referred to as the *incremental Lyapunov function* in the sequel. We start by showing that the incremental dynamics (4) possesses an invariant manifold, that we characterize through some linear relations. This observation allows us to restrict the analysis of the critical points of the Lyapunov function to this manifold. Beyond its technical interest, this analysis shows that phase-locked solutions correspond to these critical points. In particular, it provides an analytic way of computing the set $\mathcal{O}_k$ of Definition 2, completely characterizing the set of robust asymptotically stable phase-locked solutions. Furthermore, we give some partial extensions on existing results on the robustness of phase-locking in the finite Kuramoto model.

The incremental Lyapunov function: We start by introducing the *normalized interconnection matrix* associated to k

$$E = [E_{ij}]_{i,j=1,\ldots,N} := \frac{1}{K}\,[k_{ij}]_{i,j=1,\ldots,N}, \qquad (15)$$

where the scalar K is defined as

$$K = \max_{i,j=1,\ldots,N} k_{ij}. \qquad (16)$$

Inspired by [14, Chapter 3], we consider the *incremental Lyapunov function* $V_I : \mathrm{T}^{(N-1)^2} \to \mathbb{R}_{\geq 0}$ defined by

$$V_I(\tilde{\theta}) = 2\sum_{i=1}^{N}\sum_{j=1}^{N}E_{ij}\sin^2\left(\frac{\theta_i - \theta_j}{2}\right), \qquad (17)$$

where the incremental variable $\tilde{\theta}$ is defined in (5). We stress that V_I is independent of the coupling strength K.

The invariant manifold: Before analyzing the behavior of the function V_I along the solutions of (4), we stress the existence and identify an invariant manifold for the dynamics of interest. The presence of an invariant manifold results from the fact that the components of the incremental variable $\tilde{\theta}$ are not linearly independent. Indeed, we can express $(N-1)(N-2)$ of them in terms of the other $N-1$ independent components. More precisely, by choosing $\varphi_i := \theta_i - \theta_N$, $i = 1, \ldots, N-1$ as the independent variables, it is possible to write, for all $i = 1, \ldots, N$,

$$\theta_i - \theta_N = \varphi_i, \qquad (18a)$$
$$\theta_i - \theta_j = \varphi_i - \varphi_j, \quad \forall j = 1,\ldots,N-1. \quad (18b)$$

These relations can be expressed in a compact form as

$$\tilde{\theta} = \tilde{B}(\varphi) := B\varphi \mod 2\pi, \quad \varphi \in \mathcal{M}, \qquad (19)$$

where $\varphi := [\varphi_i]_{i=1,\ldots,N-1}$, $B \in \mathbb{R}^{(N-1)^2 \times (N-1)}$, $\mathrm{rank}B = N - 1$, $\tilde{B}$ is continuous and continuously differentiable, and $\mathcal{M} \subset \mathrm{T}^{(N-1)^2}$ is the submanifold defined by the

embedding (19). The continuous differentiability of $\tilde{B}$: $\mathcal{M} \to T^{(N-1)^2}$ comes from the fact that $\varphi_i \in T^1$, for all $i = 1, \ldots, N$, and the components of $\tilde{B}(\varphi)$ are linear functions of the form (18). Formally, this means that the system is evolving in the invariant submanifold $\mathcal{M} \subset T^{(N-1)^2}$ of dimension $N-1$. In particular $\mathcal{M}$ is diffeomorphic to T^{N-1}.

Restriction to the invariant manifold: In order to conduct a Lyapunov analysis based on V_I it is important to identify its critical points. Since the system is evolving on the invariant manifold $\mathcal{M}$, only the critical points of the Lyapuonov function V_I restricted to this manifold are of interest. Hence we restrict our attention to the critical points of the restriction of V_I to $\mathcal{M}$, which is defined by the function $V_I|_{\mathcal{M}}$: $T^{N-1} \to \mathbb{R}$ as

$$V_I|_{\mathcal{M}}(\varphi) := V_I(B\varphi), \quad \forall \varphi \in \mathcal{M}. \tag{20}$$

The analysis of the critical points of $V_I|_{\mathcal{M}}$ is not trivial. To simplify this problem, we exploit the fact that the variable φ can be expressed in terms of θ by means of a linear transformation $A \in \mathbb{R}^{(N-1) \times N}$, with rank$A = N - 1$, in such a way that

$$\varphi = \tilde{A}(\theta) = A\theta \mod 2\pi. \tag{21}$$

Based on this, we define the function $V : T^N \to \mathbb{R}$ as

$$V(\theta) = V_I|_{\mathcal{M}}(A\theta). \tag{22}$$

In contrast with $V_I|_{\mathcal{M}}$, the function V owns the advantage that its critical points are already widely studied in the synchronization literature, see for instance [7, Section III] and [14, Chapter 3]. The following lemma allows to reduce the analysis of the critical points of V_I on $\mathcal{M}$ to that of the critical points of V on T^N. Its proof is given in Section IV-E.

Lemma 1 (Computation of the critical points on the invariant manifold) *Let $\mathcal{M}$, $V_I|_{\mathcal{M}}$, A and V be defined by (19)-(22). Then $\theta^* \in T^N$ is a critical point of V (i.e. $\nabla_\theta V(\theta^*) = 0$) if and only if $\varphi^* = A\theta^* \in \mathcal{M}$ is a critical point of $V_I|_{\mathcal{M}}$ (i.e. $\nabla_\varphi V_I|_{\mathcal{M}}(\varphi^*) = 0$). Moreover if θ^* is a local maximum (resp. minimum) of V then φ^* is a local maximum (resp. minimum) of $V_I|_{\mathcal{M}}$. Finally the origin of $\mathcal{M}$ is a local minimum of $V_I|_{\mathcal{M}}$.*

Lyapunov characterization of robust phase-locking: The above development allows to characterize phase-locked states through the incremental Lyapunov function V_I. The following lemma states that the fixed points of the unperturbed incremental dynamics are the critical points of $V_I|_{\mathcal{M}}$, modulo the linear relations (18). That is, recalling that the fixed points of the incremental dynamics describe phase-locked solutions (cf. Definition 1), the critical points of $V_I|_{\mathcal{M}}$ completely characterize phase-locked solutions.

Lemma 2 (Incremental Lyapunov characterization of phase-locking) *Let $k \in \mathbb{R}_{\geq 0}^{N \times N}$ be any symmetric interconnection matrix. Let B and $V_I|_{\mathcal{M}}$ be defined as in (19) and (20). Then $\varphi^* \in \mathcal{M}$ is a critical point of $V_I|_{\mathcal{M}}$ (i.e.*

$\nabla_\varphi V_I|_{\mathcal{M}}(\varphi^*) = 0$) *if and only if $B\varphi^*$ is a fixed point of the unperturbed (i.e. $\tilde{\omega} = 0$) incremental dynamics (4).*

Consequence for the system without inputs: At the light of Lemma 2, we can state the following corollary, which recovers, and partially extends, the result of [14, Proposition 3.3.2] in terms of the incremental dynamics of the system. It states that, for a symmetric interconnection topology, any disturbance with zero grounded input (3) preserves the almost global asymptotic stability of phase-locking for (1).

Corollary 2 (Almost global asymptotic phase-locking) *Let $\varpi : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ be any signal satisfying $\tilde{\omega}(t) = 0$, for all $t \geq 0$, where $\tilde{\omega}$ is defined in (3). If the interconnection matrix $k \in \mathbb{R}_{\geq 0}^{N \times N}$ is symmetric, then almost all trajectories of (1) converge to a stable phase-locked solution.*

We stress that Corollary 2 is an *almost global* result. It follows from the fact that almost all trajectories converge to the set of local minima of $V_I|_{\mathcal{M}}$. From Lemma 2, this set corresponds to stable fixed points of the incremental dynamics, that is to stable phase-locked solutions. The precise proof is omitted here.

Consequences for the robustness of exact synchronization: We now restrict our attention to the exact synchronization, as introduced in Definition 1. This configuration corresponds to the origin of the incremental dynamics and, in particular, of the invariant manifold $\mathcal{M}$. As stressed by Lemma 1, the origin of $\mathcal{M}$ is always a local minimum of $V_I|_{\mathcal{M}}$. The following corollary exploits the incremental Lyapunov analysis to extend known results on the robustness of the exactly synchronized state to external inputs. Its proof directly follows from Theorem 1 and Lemma 1.

Corollary 3 (LISS of the synchronized state) *If the coupling matrix $k \in \mathbb{R}_{\geq 0}^{N \times N}$ is symmetric, and the origin of $\mathcal{M}$ is an isolated critical point of $V_I|_{\mathcal{M}}$, then the origin of $\mathcal{M}$ (exact synchronization) is LISS with respect to small inputs.*

Consequences for the disturbance rejection: In [20, Theorem 2] sufficient conditions are given for the existence of an asymptotically stable synchronized state for the Kuramoto system with general symmetric interconnection and constant different natural frequencies in terms of the algebraic properties of the coupling graph. The proof of Corollary 3 gives an extension of that result to time-varying inputs, by providing sufficient conditions on the coupling strength K, introduced in (16), for the local ISS of the synchronized state. Indeed, Equation (33) in the proof can be inverted to give

$$K > \frac{2\delta_\omega}{\sigma\left(\frac{\delta}{2}\right)}, \tag{23}$$

where δ_ω denotes the inputs amplitude, δ is the minimum distance between two critical sets of the unperturbed incremental dynamics, and σ is a $\mathcal{K}_\infty$ function. We stress that both δ and σ depend only on the normalized interconnection matrix E introduced in (15) and are therefore independent of

the coupling strength K. It follows that any arbitrarily large input $\tilde{\omega}$ can be tolerated if K is taken sufficiently large.

In the following corollary we formally state this result by considering the case when the amplitude of the inputs is bounded and the coupling strength K is seen as a tunable gain. In this situation, the system is *practically stable* in the sense of [38]. That is, given any precision $d > 0$, we can find a sufficiently large coupling strength K that makes the solutions of (1) phase-locked, modulo this prescribed tolerance d. The proof comes directly from [38] in view of Claim 2, by recalling that the Lyapunov function V_I does not depend on the tuning parameter K.

Corollary 4 (Practical phase-locking) *Consider any symmetric interconnection matrix $k \in \mathbb{R}_{\geq 0}^{N \times N}$. Let $\mathcal{O}_k$ be defined as in Definition 2 and suppose that it is non-empty. Then there exists $\delta_{\tilde{\theta}}$ such that, for any $\delta_\omega > 0$ and any $d > 0$ there exist $K > 0$ and $\beta \in \mathcal{KL}$, such that, for all $\tilde{\theta}_0 \in \mathcal{B}(\mathcal{O}_k, \delta_{\tilde{\theta}})$ and all $\|\tilde{\omega}\| \leq \delta_\omega$, the solution of (4) satisfies*

$$|\tilde{\theta}(t)|_{\mathcal{O}_k} \leq \beta(|\tilde{\theta}_0|_{\mathcal{O}_k}, t) + d, \quad \forall t \geq 0.$$

Even if the notation may be confusing, we stress that the set $\mathcal{O}_k$ does not depend on the coupling strength K, but only on the normalized interconnection matrix E defined in (15). We also stress that the constant $\delta_{\tilde{\theta}}$ estimating the domain of attraction in the above result is independent of the tuning gain K. Corollary 4 states that it is possible to arbitrarily tune the steady-state value of the phase differences by choosing a sufficiently large coupling strength K. The proposed Lyapunov function permits to exploit Lyapunov-based practical stability analysis also for the asymptotically stable fixed points, other than synchronization (*i.e.* $\tilde{\theta}^* \neq 0$), of the unperturbed Kuramoto system, which may appear for interconnection topologies that differ from the complete graph [14, Section 3.4].

In the case of all-to-all coupling this result partially extends [17, Section IV] to time-varying (bounded) natural-frequencies. Indeed in this case the same arguments of our proof apply locally (*i.e.* for $\tilde{\theta}$ inside the half circle) to the quadratic Lyapunov functions presented in [17] which can indeed be seen as a small-angles approximation of the one proposed here (see also Proposition 1 above). In the case of all-to-all coupling and constant different natural frequencies, necessary and sufficient conditions for the existence of an asymptotically stable phase-locked solution and its explicit expression are given in [19]. Again, Corollaries 3 and 4 partially generalize this result by giving sufficient conditions for the LISS of the synchronized state and by giving an estimate on the size of the phase differences.

III. PROOFS OF THE MAIN RESULTS

A. Proof of Theorem 1

In order to develop our robustness analysis we consider the Lyapunov function

$$V_I(\tilde{\theta}) = 2 \sum_{i=1}^{N} \sum_{j=1}^{N} E_{ij} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right),$$

already introduce in (17) where the incremental variable $\tilde{\theta}$ is defined in (5), and the normalized interconnection matrix E is defined in (15). The derivative of V_I along the trajectories of the incremental dynamics (4) yields

$$\dot{V}_I(\tilde{\theta}) = (\nabla_{\tilde{\theta}} V_I)^T \dot{\tilde{\theta}}.$$

The following claim, whose proof is given in Section IV, provides an alternative expression for $\dot{V}_I$.

Claim 1 *If k is symmetric, then $\dot{V}_I = -2(K\chi^T\chi + \chi^T\tilde{\omega})$, where $\chi(\tilde{\theta}) := \nabla_\theta V(\tilde{\theta}) = \left[\sum_{j=1}^{N} E_{ij} \sin(\theta_j - \theta_i) \right]_{i=1,...,N}$.*

From Claim 1, we see that if the inputs are small, there are regions of the phase space where the derivative of V_I is negative even in the presence of perturbations. More precisely, it holds that:

$$|\chi| \geq \frac{2|\tilde{\omega}|}{K} \quad \Rightarrow \quad \dot{V}_I \leq -K\chi^T\chi.$$

However, LISS does not follow yet as these regions are given in terms of χ instead of the phase differences $\tilde{\theta}$. In order to estimate these region in terms of $\tilde{\theta}$, we define $\mathcal{F}$ as the set of critical points of $V_I|_{\mathcal{M}}$ (*i.e.* $\mathcal{F} := \{\varphi^* \in \mathcal{M} : \nabla_\varphi V_I|_{\mathcal{M}}(\varphi^*) = 0\}$), where $\mathcal{M}$ and $V_I|_{\mathcal{M}}$ are defined in (19) and (17), respectively. Then, from Lemma 1 and recalling that $\chi = \nabla_\theta V$, it holds that $|\chi| = 0$ if and only if $\tilde{\theta} \in \mathcal{F}$. Since $|\chi|$ is a positive definite function of $\tilde{\theta}$ defined on a compact set, [39, Lemma 4.3], guarantees the existence of a $\mathcal{K}_\infty$ function σ such that, for all $\tilde{\theta} \in \mathrm{T}^{(N-1)^2}$,

$$|\chi| \geq \sigma(|\tilde{\theta}|_\mathcal{F}). \tag{24}$$

Let $\mathcal{U} := \mathcal{F} \setminus \mathcal{O}_k$. In view of Lemma 2, $\mathcal{U}$ denotes the set of all the critical points of $V_I|_{\mathcal{M}}$ which are not asymptotically stable fixed points of the incremental dynamics. Since $\nabla V_I|_{\mathcal{M}}$ is a Lipschitz function defined on a compact space, it can be different from zero only on a finite collection of open sets. That is $\mathcal{U}$ and $\mathcal{O}_k$ can be expressed as the disjoint union of a finite family of closed sets:

$$\mathcal{U} = \bigcup_{i \in I_\mathcal{U}} \nu_i, \quad \mathcal{O}_k = \bigcup_{i \in I_{\mathcal{O}_k}} \{\phi_i\}, \tag{25}$$

where $I_\mathcal{U}, I_{\mathcal{O}_k} \subset \mathbb{N}$ are finite sets, $\{\nu_i, \ i \in I_\mathcal{U}\}$ is a family of closed subsets of $\mathcal{M}$, and $\{\{\phi_i\}, \ i \in I_{\mathcal{O}_k}\}$ is a family of singletons of $\mathcal{M}$. We stress that $a \neq b$ implies $a \cap b = \emptyset$ for any $a, b \in \{\nu_i, \ i \in I_\mathcal{U}\} \bigcup \{\{\phi_i\}, \ i \in I_{\mathcal{O}_k}\} =: \mathcal{F}_S$. Define

$$\delta := \min_{a,b \in \mathcal{F}_S, a \neq b} \inf_{\tilde{\theta} \in a} |\tilde{\theta}|_b, \tag{26}$$

which represents the minimum distance between two critical sets, and, at the light of Lemma 2, between two fixed points of the unpertubed incremental dynamics (1). Note that, since $\mathcal{F}_S$ is finite, $\delta > 0$. Define

$$\delta'_\omega = \frac{K}{2} \sigma\left(\frac{\delta}{2}\right), \tag{27}$$

and let

$$\delta_{\tilde{\theta}} := \frac{\delta}{2}, \tag{28}$$

which gives an estimate of the size of the region of attraction, modulo the shape of the level sets of the Lyapunov function V_I. Then the following claim holds true. Its proof is given in Section IV

Claim 2 *For all $i \in I_{\mathcal{O}_k}$, all $\tilde{\theta} \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$, and all $|\tilde{\omega}| \leq \delta'_{\omega}$, it holds that*

$$|\tilde{\theta} - \phi_i| \geq \sigma^{-1}\left(\frac{2|\tilde{\omega}|}{K}\right) \quad \Rightarrow \quad \dot{V}_I \leq -K\sigma^2(|\tilde{\theta} - \phi_i|).$$

For all $i \in I_{\mathcal{O}_k}$, the function $V_I(\tilde{\theta}) - V_I(\phi_i)$ is zero for $\tilde{\theta} = \phi_i$, and strictly positive for all $\theta \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}}) \setminus \phi_i$. Hence it is positive definite on $\mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$. Noticing that $\mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$ is compact, [39, Lemma 4.3] guarantees the existence of $\mathcal{K}_{\infty}$ functions $\underline{\alpha}_i, \overline{\alpha}_i$ such that, for all $\tilde{\theta} \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$,

$$\underline{\alpha}_i(|\tilde{\theta} - \phi_i|) \leq V_I(\tilde{\theta}) - V_I(\phi_i) \leq \overline{\alpha}_i(|\tilde{\theta} - \phi_i|). \quad (29)$$

Define the following two $\mathcal{K}_{\infty}$ functions

$$\underline{\alpha}(s) := \min_{i \in I_{\mathcal{O}_k}} \underline{\alpha}_i, \quad \overline{\alpha}(s) := \max_{i \in I_{\mathcal{O}_k}} \overline{\alpha}_i, \quad \forall s \geq 0. \quad (30)$$

It then holds that, for all $i \in I_{\mathcal{O}_k}$, and all $\tilde{\theta} \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$

$$\underline{\alpha}(|\tilde{\theta} - \phi_i|) \leq V_I(\tilde{\theta}) - V_I(\phi_i) \leq \overline{\alpha}(|\tilde{\theta} - \phi_i|). \quad (31)$$

In view of Claim 2 and (31), an estimates of the ISS gain is then given by

$$\rho(s) := \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \sigma^{-1}\left(\frac{2}{K}s\right), \quad \forall s \geq 0 \quad (32)$$

where σ is defined in (24). In the same way, the tolerated input bound is given by

$$\delta_{\omega} := \rho^{-1}(\delta_{\tilde{\theta}}) \leq \delta'_{\omega}. \quad (33)$$

From [40, Section 10.4] and Claim 2, it follows that, for all $\|\tilde{\omega}\| \leq \delta_{\omega}$, the set $\mathcal{B}(\mathcal{O}_k, \delta_{\tilde{\theta}})$ is forward invariant for the system (4). Furthermore, invoking [41] and [40, Section 10.4], Claim 2 thus implies LISS with respect to small inputs satisfying $\|\tilde{\omega}\| \leq \delta_{\omega}$, meaning that there exists a class $\mathcal{KL}$ function β such that, for all $\|\tilde{\omega}\| \leq \delta_{\omega}$, and all $\tilde{\theta}_0 \in \mathcal{B}(\mathcal{O}_k, \delta_{\tilde{\theta}})$, the trajectory of (4) satisfies $|\tilde{\theta}(t)| \leq \beta(|\tilde{\theta}_0|, t) + \rho(\|\tilde{\omega}\|)$, for all $t \geq 0$. $\square$

B. Proof of Proposition 1

Input-to-State Gain: We start by computing the ISS gain ρ, defined in (32), in the particular case of all-to-all coupling and show that it can be taken as a linear function. The first step is to compute the function σ, defined in (24), with respect to the origin of the incremental dynamics. That is we have to find a class $\mathcal{K}_{\infty}$ function σ, such that $|\chi(\tilde{\theta})| \geq \sigma(|\tilde{\theta}|)$ for all $\tilde{\theta}$ in some neighborhood of the origin of the incremental dynamics, where χ is defined in Claim 1. The following claim, whose proof is given in Section IV, gives an explicit expression of this function on the set $\mathcal{D}_0 = \{\tilde{\theta} \in \mathbf{T}^{(N-1)^2} : |\tilde{\theta}|_{\infty} \leq \frac{\pi}{2}\}$, as defined in the statement of Proposition 1.

Claim 3 *In the case of all-to-all coupling , the function χ defined in Claim 1 satisfies, for any $\tilde{\theta} \in \mathcal{D}_0$, $|\chi(\tilde{\theta})| \geq \frac{|\tilde{\theta}|}{\pi}$, that is $\sigma(r) = \frac{r}{\pi}$.*

At the light of Claim 3, the ISS gain ρ can be easily computed through (32). Indeed, in the all-to-all case, the entries of the matrix E, introduced in (15), are all 1, and the Lyapunov function V_I, provided in (17), thus becomes

$$V_I(\tilde{\theta}) = 2 \sum_{i,j=1}^{N} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right).$$

Using the fact that $z \geq \sin z \geq \frac{2}{\pi}z$, for all $0 \leq z \leq \frac{\pi}{2}$, it follows that, for all $\tilde{\theta} \in \mathcal{D}_0$,

$$\frac{2}{\pi^2}|\tilde{\theta}|^2 \leq V_I(\tilde{\theta}) \leq \frac{1}{2}|\tilde{\theta}|^2. \quad (34)$$

Recalling the definition of the upper $\overline{\alpha}$ and lower $\underline{\alpha}$ estimates of the Lyapunov function with respect the set of asymptotically stable fixed point (30), and that, in the all-to-all case, this set reduces to the origin, we conclude that

$$\underline{\alpha}(r) = \frac{2}{\pi^2}r^2, \quad \overline{\alpha}(r) = \frac{1}{2}r^2, \quad \forall r \geq 0.$$

In view of Claim 3 and (32), it follows that the ISS gain ρ in the statement of Theorem 1, can be chose as

$$\rho(r) = \frac{\pi^2}{K}r. \quad (35)$$

Input bound and invariant set: For Claim 3, the ISS gain computed in the last section is valid as soon as $\tilde{\theta}$ belongs to $\mathcal{D}_0$. In the following we compute an input bound which guarantees that trajectories starting in $\mathcal{D}_0$ remain inside $\mathcal{D}_0$. For the sake of generality, we actually show the forward invariance of $\mathcal{D}_{\epsilon}$ for any $\epsilon \in \left[0, \frac{\pi}{2}\right]$. To that end, we start by the following technical claim, whose proof is given in Section IV.

Claim 4 *Given any $0 \leq \delta \leq \pi$, the following holds true:*

$$|\tilde{\theta}| \leq \sqrt{N}\delta \quad \Rightarrow \quad \max_{i,j=1,\dots,N} |\theta_i - \theta_j| \leq \delta.$$

At the light of Claim 4, and in view of (7) and (35), we can compute the input bound $\delta_{\omega}^{\epsilon}$ which lets $\mathcal{D}_{\epsilon}$ be invariant for the systems (4) by imposing $\rho(\delta_{\omega}^{\epsilon}) = \sqrt{N}\left(\frac{\pi}{2} - \epsilon\right)$, where $\rho(s) = \frac{\pi^2}{K}s$ is the ISS gain in the statement of Theorem 1. This gives

$$\delta_{\omega}^{\epsilon} = \frac{K\sqrt{N}}{\pi^2}\left(\frac{\pi}{2} - \epsilon\right). \quad (36)$$

Exponential convergence and attractivity of $\mathcal{D}_{\epsilon}$: From Claims 2 and 3, and (34), it holds that, for all $|\tilde{\omega}| \leq \delta_{\omega}^{\epsilon}$, and all $\tilde{\theta} \in \mathcal{D}_0$,

$$|\tilde{\theta}| \geq \frac{2\pi}{K}|\tilde{\omega}| \quad \Rightarrow \quad \dot{V}_I \leq -\frac{K}{\pi^2}|\tilde{\theta}|^2 \leq -\frac{2K}{\pi^2}V_I.$$

Invoking the comparison Lemma [39, Lemma 3.4], it follows that, for all $t \geq 0$ $\min_{0 \leq s \leq t} |\tilde{\theta}(s)| \geq \frac{2\pi}{K}\|\tilde{\omega}\|$,

$$\min_{0 \leq s \leq t} |\tilde{\theta}(s)| \geq \frac{2\pi}{K}\|\tilde{\omega}\| \quad \Rightarrow \quad V_I(\tilde{\theta}(t)) \leq V_I(\tilde{\theta}(0))e^{-\frac{2K}{\pi^2}t}.$$

From (34), this also implies that, for all $t \geq 0$

$$\min_{0 \leq s \leq t} |\tilde{\theta}(s)| \geq \frac{2\pi}{K}\|\tilde{\omega}\| \Rightarrow |\tilde{\theta}(t)| \leq \frac{\pi}{2}|\tilde{\theta}(0)|e^{-\frac{K}{\pi^2}t}.$$

Recalling the explicit expression of ISS gain (35), this implies that the system is exponentially input-to-state stable (see for instance [40, Section 10.4] and [39, Lemma 4.4 and Theorem 4.10]), and in particular that for all $\|\tilde{\omega}\| \leq \delta_\omega^\epsilon$ and all $\tilde{\theta}_0$ in $\mathcal{D}_0$,

$$|\tilde{\theta}(t)| \leq \frac{\pi}{2}|\tilde{\theta}_0|e^{-\frac{K}{\pi^2}t} + \frac{\pi^2}{K}\|\tilde{\omega}\|, \ \forall t \geq 0.$$

Noticing finally that, if $\|\tilde{\omega}\| \leq \delta_\omega^\epsilon$, (36) guarantees that

$$|\tilde{\theta}(t)| \leq \frac{\pi}{2}|\tilde{\theta}_0|e^{-\frac{K}{\pi^2}t} + \sqrt{N}\left(\frac{\pi}{2} - \epsilon\right), \ \forall t \geq 0,$$

Claim 4 implies the attractivity of $\mathcal{D}_\epsilon$ for all $\|\tilde{\omega}\| \leq \delta_\omega^\epsilon$. $\square$

IV. TECHNICAL PROOFS

A. Proof of Claim 1

Consider the derivative of the incremental Lyapunov function V_I, defined in (17), along the trajectories of the incremental dynamics (4):

$$\dot{V}_I(\tilde{\theta}) := (\nabla_{\tilde{\theta}} V_I)^T \dot{\tilde{\theta}} = \sum_{i,j=1}^{N} E_{ij}\sin(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i)$$

$$= -2\sum_{i,j=1}^{N} E_{ij}\sin(\theta_j - \theta_i)\dot{\theta}_i,$$

where the last equality comes from the fact that, if E is a symmetric matrix, then $\sum_{i,j=1}^{N} E_{ij}\sin(\theta_j - \theta_i)\dot{\theta}_j = -\sum_{i,j=1}^{N} E_{ij}\sin(\theta_j - \theta_i)\dot{\theta}_i$. For the same reason it holds that $\overline{\omega}\sum_{i,j=1}^{N} E_{ij}\sin(\theta_j - \theta_i) = 0$. Since, from (1), $\dot{\theta}_i = \overline{\omega} + \tilde{\omega}_i + K\sum_{l=1}^{N} E_{il}\sin(\theta_l - \theta_i)$, we get that

$$\dot{V}_I = -2\sum_{i=1}^{N}\left(\sum_{j=1}^{N} E_{ij}\sin(\theta_j - \theta_i)\right)\left(K\sum_{l=1}^{N} E_{il}\sin(\theta_l - \theta_i) + \tilde{\omega}_i\right),$$

which proves the claim.

B. Proof of Claim 2

From Claim 1 it holds that $\dot{V}_I = -2K|\chi|^2 - 2\chi^T\tilde{\omega} \leq -2K|\chi|^2 + 2|\chi||\tilde{\omega}|$. That is

$$|\chi| \geq \frac{2|\tilde{\omega}|}{K}, \quad \Rightarrow \quad \dot{V}_I \leq -K|\chi|^2.$$

In view of (27)-(28), $|\tilde{\omega}| \leq \delta_\omega'$ implies that $\sigma^{-1}\left(\frac{2|\tilde{\omega}|}{K}\right) \leq \delta_{\tilde{\theta}}$. Recalling that, for all $\tilde{\theta} \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$, $|\tilde{\theta}|_{\mathcal{F}} = |\tilde{\theta} - \phi_i|$, it results that

$$|\tilde{\theta} - \phi_i| \geq \sigma^{-1}\left(\frac{2|\tilde{\omega}|}{K}\right) \quad \Rightarrow \quad |\chi| \geq \frac{2|\tilde{\omega}|}{K},$$

Since (24) ensures that $-|\chi|^2 \leq -\sigma^2(|\tilde{\theta} - \phi_i|)$, we obtain

$$|\tilde{\theta} - \phi_i| \geq \sigma^{-1}\left(\frac{2|\tilde{\omega}|}{K}\right) \quad \Rightarrow \quad \dot{V}_I \leq -K\sigma^2(|\tilde{\theta} - \phi_i|).$$

C. Proof of Claim 3

In the case of all-to-all coupling, the vector χ defined in Claim 1 reads

$$\chi(\tilde{\theta}) = \left[\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\right]_{i=1,\dots,N}.$$

Therefore, the norm inequality $|\tilde{\theta}|_2 \geq |\tilde{\theta}|_\infty$ implies

$$|\chi(\tilde{\theta})|_2 \geq \max_{i=1,\dots,N}\left|\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\right|.$$

Now, since $\tilde{\theta} \in \mathcal{D}_0$, we have $|\theta_i - \theta_j| \leq \frac{\pi}{2}$, which implies that the phases of all oscillators belong to the same quarter of circle. We can thus renumber the indexes of the oscillator phases in such a way that $\theta_i \leq \theta_j$ whenever $i < j$.

First step — For a given $\tilde{\theta}$, in order to find a tight lower bound on $|\chi(\tilde{\theta})|_2$, we are going to show that

$$\max_{i=1,\dots,N}\left|\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\right|$$

$$= \max\left\{\sum_{j=1}^{N}\sin|\theta_j - \theta_1|, \sum_{j=1}^{N}\sin|\theta_j - \theta_N|\right\}.$$

On the one hand, for all $j = 1,\dots,N$, we have $0 \leq \theta_j - \theta_1 \leq \frac{\pi}{2}$ and $0 \leq \theta_N - \theta_j \leq \frac{\pi}{2}$. It follows that

$$\left|\sum_{j=1}^{N}\sin(\theta_j - \theta_1)\right| = \sum_{j=1}^{N}\sin|\theta_j - \theta_1|, \quad (37)$$

and

$$\left|\sum_{j=1}^{N}\sin(\theta_j - \theta_N)\right| = \sum_{j=1}^{N}\sin|\theta_j - \theta_N|. \quad (38)$$

On the other hand, for any $i \notin \{1, N\}$, we have that, for any $j < i$, $0 \leq \theta_i - \theta_j \leq \frac{\pi}{2}$; while, for any $j > i$, $0 \leq \theta_j - \theta_i \leq \frac{\pi}{2}$. That is

$$\left|\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\right| = \left|\sum_{j=i+1}^{N}\sin|\theta_j - \theta_i| - \sum_{j=1}^{i-1}\sin|\theta_j - \theta_i|\right|.$$

Now, for all $\tilde{\theta} \in \mathcal{D}_0$, if $i > j$, it results that $|\theta_j - \theta_i| \leq |\theta_j - \theta_N|$, while, if $i < j$, it results that $|\theta_j - \theta_i| \leq |\theta_j - \theta_1|$. Hence, for all $\tilde{\theta} \in \mathcal{D}_0$,

$$\sum_{j=1}^{i-1}\sin|\theta_j - \theta_i| \leq \sum_{j=1}^{N}\sin|\theta_j - \theta_N|, \quad \forall i \notin \{1, N\},$$

and

$$\sum_{j=i+1}^{N}\sin|\theta_j - \theta_i| \leq \sum_{j=1}^{N}\sin|\theta_j - \theta_1|, \quad \forall i \notin \{1, N\}.$$

Recalling that, for all $a, b > 0$, $|a - b| \leq \max\{a, b\}$, it then follows that, for any $i \notin \{1, N\}$,

$$\left|\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\right|$$

$$\leq \max\left\{\sum_{j=1}^{i-1}\sin|\theta_j - \theta_i|, \ \sum_{j=i+1}^{N}\sin|\theta_j - \theta_i|\right\}$$

$$\leq \max\left\{\sum_{j=1}^{N}\sin|\theta_j - \theta_1|, \sum_{j=1}^{N}\sin|\theta_j - \theta_N|\right\}. \quad (39)$$

Therefore, combining (37), (38), and (39), we obtain

$$\max_{i=1,\ldots,N}\left|\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\right|$$

$$= \max\left\{\sum_{j=1}^{N}\sin|\theta_j - \theta_1|, \sum_{j=1}^{N}\sin|\theta_j - \theta_N|\right\}, \quad (40)$$

which ends the first step of the proof.

Second step — Using the fact that $\sin z \geq \frac{2}{\pi}z$, for all $z \in \left[0, \frac{\pi}{2}\right]$, Equation (40) yields

$$|\chi(\tilde{\theta})|_2 \geq \max\left\{\sum_{j=1}^{N}\sin|\theta_j - \theta_1|, \sum_{j=1}^{N}\sin|\theta_j - \theta_N|\right\}$$

$$\geq \frac{2}{\pi}\max\left\{\sum_{j=1}^{N}|\theta_j - \theta_1|, \sum_{j=1}^{N}|\theta_j - \theta_N|\right\}$$

$$\geq \frac{2}{\pi}\max\left\{\sum_{j=1}^{N}(\theta_j - \theta_1), \sum_{j=1}^{N}(\theta_N - \theta_j)\right\}.$$

Or, equivalently, by defining $\delta := \theta_N - \theta_1$, with $0 \leq \delta \leq \frac{\pi}{2}$,

$$|\chi(\tilde{\theta})|_2 \geq \frac{2}{\pi}\max\left\{\sum_{j=1}^{N}(\theta_j - \theta_1), \sum_{j=1}^{N}[\delta - (\theta_j - \theta_1)]\right\}. \quad (41)$$

For notation purposes, define $I_{\tilde{\theta}} := [0, \delta]^{N-2}$, $x_i := \theta_{i+1} - \theta_1$, for all $i = 1, \ldots, N-2$, and $x := [x_i]_{i=1,\ldots,N-2} \in I_{\tilde{\theta}}$. Then

$$\max\left\{\sum_{j=1}^{N}(\theta_j - \theta_1), \sum_{j=1}^{N}[\delta - (\theta_j - \theta_1)]\right\}$$

$$= \max\left\{\delta + \sum_{i=1}^{N-2}x_i, \ \delta + \sum_{i=1}^{N-2}(\delta - x_i)\right\}$$

$$= \delta + \max\left\{\sum_{i=1}^{N-2}x_i, \ \sum_{i=1}^{N-2}(\delta - x_i)\right\}. \quad (42)$$

In order to obtain the desired bound, we thus have to minimize the function $f : I_{\tilde{\theta}} \to \mathbb{R}_{\geq 0}$, defined by $f(x) := \max\left\{\sum_{i=1}^{N-2}x_i, \ \sum_{i=1}^{N-2}(\delta - x_i)\right\}$. Define the functions

$$a(x) := \sum_{i=1}^{N-2}x_i \quad \text{and} \quad b(x) := \sum_{i=1}^{N-2}(\delta - x_i).$$

And the sets

$$\begin{aligned}
A &:= \{x \in I_{\tilde{\theta}}: \ a(x) > b(x)\} \\
B &:= \{x \in I_{\tilde{\theta}}: \ b(x) > a(x)\} \\
C &:= \{x \in I_{\tilde{\theta}}: \ a(x) = b(x)\}.
\end{aligned}$$

It then results that $I_{\tilde{\theta}} = A \cup B \cup C$, with $A \cap B = B \cap C = C \cap A = \emptyset$. Observe, moreover, that we obviously have $f|_A = a|_A$ and $f|_B = b|_B$. By the fact that, for all $x \in C$, $\sum_{i=1}^{N-2}x_i = \sum_{i=1}^{N-2}(\delta - x_i)$, we have

$$f(x) = a(x) = b(x) = \frac{N-2}{2}\delta, \quad \forall x \in C. \quad (43)$$

Moreover, since $b(x) = (N-2)\delta - a(x)$, it results that

$$a(x) > \frac{N-2}{2}\delta, \quad \forall x \in A,$$

and

$$b(x) > \frac{N-2}{2}\delta, \quad \forall x \in B.$$

That is, since $f|_A = a|_A$, $f|_B = b|_B$, and $A \cup B = I_{\tilde{\theta}} \setminus C$,

$$f(x) > \frac{N-2}{2}\delta, \quad \forall x \in I_{\tilde{\theta}} \setminus C.$$

From (43), this also implies that, for all $x \in I_{\tilde{\theta}}$,

$$f(x) \geq \frac{N-2}{2}\delta. \quad (44)$$

From (41), (42), and (44), we obtain

$$|\chi(\tilde{\theta})|_2 \geq \frac{2}{\pi}\left(\delta + \frac{N-2}{2}\delta\right) = \frac{N}{\pi}\delta.$$

Finally, recalling that $\delta = |\tilde{\theta}|_\infty$, by the norm inequality $|\tilde{\theta}|_\infty \geq \frac{|\tilde{\theta}|_2}{N-1}$, we can conclude that, for all $\tilde{\theta} \in \mathcal{D}_0$,

$$|\chi(\tilde{\theta})|_2 \geq \frac{N}{\pi(N-1)}|\tilde{\theta}|_2 \geq \frac{|\tilde{\theta}|_2}{\pi},$$

which proves the claim.

D. Proof of Claim 4

Since we want to prove that $|\tilde{\theta}|_2 \leq \sqrt{N}\delta \ \Rightarrow \ |\tilde{\theta}|_\infty \leq \delta$, we are going to prove that $|\tilde{\theta}|_\infty \geq \delta \ \Rightarrow \ |\tilde{\theta}|_2 \geq \sqrt{N}\delta$. But, by the monotonicity of the norm, it is enough to show that

$$|\tilde{\theta}|_\infty = \delta \quad \Rightarrow \quad |\tilde{\theta}|_2 \geq \sqrt{N}\delta.$$

In order to do that we minimize the Euclidean norm $|\tilde{\theta}|_2$ (or, equivalently, $|\tilde{\theta}|_2^2$), with the constraint that $|\tilde{\theta}|_\infty = \delta$. For the sake of simplicity, renumber the oscillator phases indexes in such a way that $\theta_i \leq \theta_j$ whenever $i < j$, as in the proof of Claim 3. The problem can then be translated into minimizing $|\tilde{\theta}|_2^2$, with the constraint that $\theta_N - \theta_1 = \delta$. Since the square of the Euclidean norm and the constrained function are smooth, we can apply the method of Lagrange multipliers (see Appendix). That is, we can find critical points of $|\tilde{\theta}|_2^2$, under the constraint $\theta_N - \theta_1 = \delta$, by solving the set of equations

$$\frac{\partial}{\partial\theta_i}F(\theta, \lambda) = 0, \ i = 1, \ldots, N, \quad (45)$$

$$\frac{\partial}{\partial\lambda}F(\theta, \lambda) = 0, \quad (46)$$

where $F(\theta, \lambda) := \sum_{i,j=1}^{N}(\theta_i - \theta_j)^2 - \lambda(\theta_N - \theta_1 - \delta)$. Differentiating with respect to θ_l, for $l \notin \{1, N\}$, gives

$$\sum_{j=1}^{N}(\theta_l - \theta_j) = 0. \quad (47)$$

Differentiating with respect to λ gives the constraint $\theta_N - \theta_1 = \delta$. Differentiating with respect to θ_1 gives $4\sum_{j=1}^{N}(\theta_1 - \theta_j) + \lambda = 0$, differentiating with respect to θ_N gives $4\sum_{j=1}^{N}(\theta_N - \theta_j) - \lambda = 0$, and, by solving with respect to λ, we get

$$\sum_{j=1}^{N}(\theta_1 - \theta_j) + \sum_{j=1}^{N}(\theta_N - \theta_j) = 0. \tag{48}$$

Equations (47), (48), with the constraint $\theta_N - \theta_1 = \delta$, admit a unique solution, modulo a common phase shift among the ensemble (*i.e* $\theta_i \to \theta_i + \alpha$ for all i):

$$\theta_i^* - \theta_j^* = 0, \ \forall (i,j) \notin \{(1,N),(N,1)\}, \tag{49a}$$

$$\theta_i^* - \theta_1^* = \frac{\delta}{2}, \ \theta_i^* - \theta_N^* = -\frac{\delta}{2}, \ \forall i \notin \{1,N\}. \tag{49b}$$

By computing the Hessian of F with respect to the vector $x = (\tilde{\theta}^T, \lambda)^T$, it is easy to show that its symmetric part is positive semidefinite for all x. Hence the solution (49) corresponds to a minimum. To show the uniqueness of this critical point, modulo a common phase shift, note that the set of equations (45) can be rewritten as the linear system

$$\begin{pmatrix} N-1 & -1 & \ldots & -1 & -1 \\ -1 & N-1 & \ldots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \ldots & N-1 & -1 \\ -1 & -1 & \ldots & -1 & N-1 \\ -1 & 0 & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{N-2} \\ \theta_{N-1} \\ \theta_N \end{pmatrix} =:$$

$$=: G\theta = \begin{pmatrix} \frac{\lambda}{4} \\ 0 \\ \vdots \\ 0 \\ -\frac{\lambda}{4} \\ \delta \end{pmatrix}. \tag{50}$$

The matrix $G \in \mathbb{R}^{(N+1)\times N}$ has rank $N-1$, since the minor given by the first N rows is the Laplacian matrix associated to a complete graph, which has rank $N-1$. In particular, it holds that $G\mathbf{1}_N = 0$. Hence the solution to (50) is of the form $\theta^* = \theta_{\perp}^* + \alpha\mathbf{1}_N$, $\alpha \in \mathbb{R}$, where $\theta_{\perp}^*$ belongs to the ortogonal space to $\mathbf{1}_N$, and is uniquely determined by (50), which confirms that the solution (49) is unique, modulo a common phase shift. We can then conclude that, if $|\tilde{\theta}|_\infty = \delta$, then

$$\begin{aligned} |\tilde{\theta}|_2^2 &\geq \sum_{i,j=1}^{N}(\theta_i^* - \theta_j^*)^2 \\ &\geq 2\delta^2 + 2\sum_{j=2}^{N-2}\frac{\delta^2}{4} + 2\sum_{j=2}^{N-2}\frac{\delta^2}{4} \\ &\geq N\delta^2, \end{aligned}$$

which proves the claim.

E. Proof of Lemma 1

By the definition (22) of $V(\theta)$, it holds that $\nabla_\theta V(\theta) = \nabla_\theta V_I|_{\mathcal{M}}(A\theta) = A^T \nabla_{A\theta} V_I|_{\mathcal{M}}(A\theta)$. Hence

$$\nabla_{A\theta} V_I|_{\mathcal{M}}(A\theta) = 0 \quad \Rightarrow \quad \nabla_\theta V(\theta) = 0,$$

by the linearity of A^T. Recalling that, since $\text{rank}A = N-1$, $\ker A^T = 0$, it follows that

$$\nabla_\theta V(\theta) = 0 \quad \Rightarrow \quad \nabla_{A\theta} V_I|_{\mathcal{M}}(A\theta) = 0,$$

which proofs the first part of the lemma.

To proove the second part of the lemma, we note that if θ^* is a local minimum of V then there exists a neighborhood U of θ^* such that $V(\theta) \geq V(\theta^*)$ for all $\theta \in U$. That is, $V_I|_{\mathcal{M}}(A\theta) \geq V_I|_{\mathcal{M}}(A\theta^*)$ for all $\theta \in U$. That is $V_I|_{\mathcal{M}}(\varphi) \geq V_I|_{\mathcal{M}}(\varphi^*)$ for all $\varphi \in W = AU$, where $\varphi^* = A\theta^*$. A similar proof holds for maxima.

The third part of the lemma follows from the fact the function $V_I|_{\mathcal{M}}$ is positive definite and $V_I|_{\mathcal{M}}(0) = 0$. $\square$

F. Proof of Lemma 2

From Lemma 1, it results that $\phi^* \in \mathcal{M}$ is a critical point of $V_I|_{\mathcal{M}}$ if and only if $\theta^* \in \mathrm{T}^N$ is a critical point of V, where $\phi^* = A\theta^*$, and A is defined in (21). Moreover, when $\tilde{\omega} = 0$, it results that the incremental dynamics (4) can be re-written as

$$\dot{\theta}_i - \dot{\theta}_j = \chi_j(\theta) - \chi_i(\theta), \quad \forall i,j = 1,\ldots,N,$$

where $\chi(\theta) = [\chi_i(\theta)]_{i=1,\ldots,N} := \nabla_\theta V(\theta) = \sum_{j=1}^{N} E_{ij}\sin(\theta_j - \theta_i)$, and the normalized interconnection matrix E is defined in (15). Hence, $\chi(\theta^*) = \nabla_\theta V(\theta^*) = 0$ if and only if $\phi^* = A\theta^*$ is a critical point of $V_I|_{\mathcal{M}}$; and $\chi_j(\theta^*) - \chi_i(\theta^*) = 0$, for all $i,j = 1,\ldots,N$, if and only if $B\phi^* = BA\theta^*$ is a fixed point of the unperturbed incremental dynamics, where B is defined in (19). To prove the lemma it thus suffices to show that

$$\chi(\theta^*) = 0 \quad \Leftrightarrow \quad \chi_j(\theta^*) - \chi_i(\theta^*) = 0, \quad \forall i,j = 1,\ldots,N.$$

One implication is straightforward: if $\chi(\theta^*) = 0$, then in particular all of its components are zero, that is $\chi_j(\theta^*) - \chi_i(\theta^*) = 0$, for all $i,j = 1,\ldots,N$. On the other hand, if $\chi_j(\theta^*) - \chi_i(\theta^*) = 0$ for all $i,j = 1,\ldots,N$, then there exists a constant $\overline{\chi}$, such that $\chi_i(\theta^*) = \overline{\chi}$ for all $i = 1,\ldots,N$. Hence, it results that

$$N\overline{\chi} = \sum_{i=1}^{N}\chi_i(\theta^*) = \sum_{i=1}^{N}\frac{\partial V}{\partial \theta_i}(\theta^*) = \sum_{i=1}^{N}\sum_{j=1}^{N}E_{ij}\sin(\theta_j^* - \theta_i^*).$$

Since the interconnection matrix k is symmetric, so is the normalized interconnection matrix E (cf. (15)), and it results that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}E_{ij}\sin(\theta_j^* - \theta_i^*) = 0.$$

Consequently, $\chi_i(\theta^*) = \overline{\chi} = 0$ for all $i = 1,\ldots,N$, which proves the converse implication. $\square$

APPENDIX

An extremum of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, under the constraints $g_i(x) = b_i$, $i = 1, \ldots, m$, where $g_i : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, and $b_i \in \mathbb{R}$ belongs to the image of g_i, for all $i = 1, \ldots, m$, can be found by constructing the Lagrangian function F through the Lagrangian multipliers λ_i, $i = 1, \ldots, m$,

$$F(x, \lambda_1, \ldots, \lambda_m) = f(x) - \sum_{i=1}^{m} \lambda_i (g_i(x) - b_i)$$

and by solving the set of equations

$$\frac{\partial}{\partial x_i} F(x, \lambda_1, \ldots, \lambda_m) = 0,$$

$$\frac{\partial}{\partial \lambda_j} F(x, \lambda_1, \ldots, \lambda_m) = 0,$$

for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$. The optimal value x^*, is found together with the vector of Lagrangian multipliers $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$. See for example [42].

REFERENCES

[1] K. Wiesenfeld, P. Colet, and S. Strogatz, "Synchronization transitions in a disordered Josephson series array," *Phys. Rev. Lett.*, vol. 76, 1996.

[2] ——, "Frequency locking in Josephson arrays: Connection with the Kuramoto model," *Physics Review E*, vol. 57, 1998.

[3] W. Wang, Z. Kiss, and J. Hudson, "Clustering of arrays of chaotic chemical oscillators by feedback and forcing," *Phys. Rev. Lett.*, vol. 86, 2001.

[4] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbolence*. Berlin: Springer, 1984.

[5] C. Peskin, *Mathematical Aspects of Heart Phydiology*. New-York: Courant Institute of Mathematical Sciences, 1975.

[6] A. Winfree, *The Geometry of Biological Times*. New-York: Springer, 1980.

[7] R. Sepulchre, D. A. Paley, and N. E. Leonard, "Stabilization of planar collective motion: All-to-all communication," *IEEE Trans. on Automat. Contr.*, vol. 52, no. 5, pp. 811–824, May 2007.

[8] A. L. Fradkov, *Cybernietical Physics. From Control of Chaos to Quantum Control*, J. S. Kelso, Ed. Berlin Heidelberg: Springer-Verlag, 2007.

[9] E. Izhikevich, "Which model to use for cortical spiking neurons?" *IEEE Trans. on Neural Networks*, vol. 15, pp. 1063–1070, 2004.

[10] ——, *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting*. Cambridge, Mass: MIT Press, 2007.

[11] A. Pavlov, W. N. Victorovich, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems. A Convergent Dynamics Approach*. Birkhäuse, 2006.

[12] L. Scardovi, A. Sarlette, and R. Sepulchre, "Synchronization and balancing on the N-torus," *Syst. & Contr. Letters*, vol. 56, no. 5, pp. 335–341, 2007.

[13] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. on Automat. Contr.*, vol. 49, pp. 1520–1533, 2004.

[14] A. Sarlette, "Geometry and symmetries in coordination control," Ph.D. dissertation, University of Liège, Belgium, Liège, Belgium, 6 January 2009.

[15] A. Franci, A. Chaillet, and W. Pasillas-Lépine, "Existence of phase-locking in coupled Kuramoto oscillators under real mean-field feedback with applications to Deep Brain Stimulation," *Submitted to the Proc. 49th. IEEE Conf. Decision Contr.*, 2010.

[16] A. Winfree, "Biological rhythms and the behaviors of population of coupled oscillators," *Journal of Theoretical Biology*, vol. 16, p. 15, 1967.

[17] N. Chopra and M. W. Spong, "On exponential synchronization of Kuramoto oscillators," *IEEE Trans. on Automat. Contr.*, vol. 54, no. 2, pp. 353–357, 2009.

[18] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, "The Kuramoto model: A simple paradigm for synchronization phenomena," *Reviews of modern physics*, vol. 77, pp. 137–185, 2005.

[19] D. Aeyels and J. Rogge, "Existence of partial entrainment and stability of phase locking behavior of coupled oscillator," *Progress of Theoretical Physics*, vol. 112, no. 6, pp. 921–942, 2004.

[20] A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the Kuramoto model of coupled nonlinear oscillators," *Proc. American Control Conference*, vol. 5, pp. 4296–4301, 2004.

[21] J. van Hemmen and W. Wreszinski, "Lyapunov function for the Kuramoto model on nonlinearly coupled oscillators," *Journal of Statistical Physics*, vol. 72, pp. 145–166, 1993.

[22] E. Brown, P. Holmes, and J. Moehlis, "Globally coupled oscillator networks," in *Perspectives and Problems in Nonlinear Science: A Celebratory Volume in Honor of Larry Sirovich*, 2003.

[23] C. G. Assisi, V. K. Jirsa, and J. A. S. Kelso, "Synchrony and clustering in heterogeneous networks with global coupling and parameter dispersion," *Phys. Rev. Lett.*, vol. 94, no. 1, 2005.

[24] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge, United Kingdom: Cambridge Nonlinear Science Series, 2001.

[25] C. U. D. Cumin, "Generalising the Kuramoto model for the study of neuronal synchronisation in the brain," *Physica D*, vol. 226, pp. 181–196, 2007.

[26] Y. Maistrenko, O. Popovych, and P. Tass, "Desynchronization and chaos in the Kuramoto model," *Lect. Notes Phys.*, vol. 671, pp. 285–306, 2005.

[27] B. C. Daniels, "Synchronization of globally coupled nonlinear oscillators: the rich behavior of the Kuramoto model," *Available at* http://go.owu.edu/p̄hysics/StudentResearch/2005/BryanDaniels/kuramoto_paper.pdf, 2005.

[28] K. Pyragas, O. V. Popovich, and P. A. Tass, "Controlling synchrony in oscillatory networks with a separate stimulation-registration setup," *EPL*, vol. 80, no. 4, 2008.

[29] N. Tukhlina, M. Rosenblum, A. Pikovsky, and J. Kurths, "Feedback suppression of neural synchrony by vanishing stimulation," *Physical Review E*, vol. 75, no. 1, p. 011918(8), 2007.

[30] P. Tass, "A model of desynchronizing deep brain stimulation with a demand-controlled coordinated reset of neural subpopulations," *Biol. Cybern.*, vol. 89, pp. 81–88, 2003.

[31] C. Hauptmann, O. Popovych, and P. Tass, "Delayed feedback control of synchronization in locally coupled neuronal networks," *Neurocomputing*, vol. 65, pp. 759–767, 2005.

[32] M. Bennet and R. Zukin, "Electrical coupling and neuronal synchronization in the mammalian brain," *Neuron*, vol. 41, no. 4, pp. 495–511, 2004.

[33] R. Sepulchre, D. Paley, and N. E. Leonard, "Stabilization of planar collective motion with limited communication," *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 706–719, 2008.

[34] A. Loria and E. Panteley, "Cascade nonlinear time-varying systems: analysis and design," in *Advanced topics in control systems theory*, ser. Lecture Notes in Control and Information Sciences, F. Lamnabhi-Lagarrigue, A. Loria, and E. Panteley, Eds. Springer Verlag, 2005.

[35] I. J. Malkin, "Theory of stability of motion," U.S. Atomic energy commission, Tech. Rep., 1958.

[36] E. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. on Automat. Contr.*, vol. 20, 1999.

[37] M. Rodriguez-Oroz, J. Obeso, A. Lang, J.-L. Houeto, P. Pollak, S. Rehncrona, J. Kulisevsky, A. Albanese, J. Volkmann, M. Hariz, N. Quinn, J. Speelman, J. Guridi, I. Zamarbide, A. Gironell, J. Molet, B. Pascual-Sedano, B. Pidoux, Y. Agid, J. Xie, A. Benabid, A. Lozano, J. Saint-Cyr, L. Romito, M. Contarino, M. Scerrati, V. Fraix, and N. V. Blercom, "Bilateral deep brain stimulation in Parkinson's disease: a multicentre study with 4 years follow-up," *Brain*, vol. 128, pp. 2240–2249, 2005.

[38] A. Chaillet and A. Loría, "Uniform global practical asymptotic stability for non-autonomous cascaded systems," *European Journal of Control*, vol. 12, no. 6, pp. 595–605, 2006.

[39] H. Khalil, *Nonlinear systems*. New York: Prentice Hall, 3rd ed., 2001.

[40] A. Isidori, *Nonlinear control system II*. Springer Verlag, 1999.

[41] E. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Syst. & Contr. Letters*, vol. 24, pp. 351–359, 1995.

[42] G. Bliss, *Lectures on the calculus of variations*. Chicago Univ. Press, 1947.