Determination of the Behaviour of Self-Sampled Digital Phase-Locked Loops

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***Abstract*—This paper deals with the stability of so-called "self-sampled" digital phase-locked-loops (PLLs). These systems are meant to be used as the nodes of autonomous clock distribution networks, where clock signals are locally generated in each node and each node is synchronized with its neighbours. Despite the absence of an absolute reference clock, it is possible to use the local irregular clock to trigger the operations of the digital loop filter. In this paper, we show that, in this mode of operation, PLLs can be modeled as autonomous piecewise-linear systems. We investigate what filter coefficients to choose in order to ensure stability and, hence synchronization. Two methods are explored, the first based on transient simulations, the second on linear matrix inequalities. It is shown that the second method yields much more conservative results than the first but that it cannot apply to all design options of self-sampled PLLs.**

I. INTRODUCTION

Networks of synchronized oscillators are an alternative approach to classical tree-like clock distribution methods in large-scale synchronous systems-on-chips (SOCs). Each node of the network may for example consist of a phase-locked loop (PLL) trying to match the phase of its nearest neighbors. If neighbour-to-neighbour phase synchronization is reached, even the most distant parts of the SOC see the same clock signal and, hence, operate synchronously. The main advantages of this approach are that it requires no absolute reference clock and that it imposes much less design constraints than tree-based distribution methods. This concept was introduced in [1] and an implementation was proposed in [2]. However, this architecture had little success with designers of digital circuits, probably because it was based on analog techniques. The HODISS project, funded by the ANR ARFU program, aims at pursuing the seminal work of [1] and [2] into the digital domain, in order to benefit from the noise-immunity and the greater flexibility of digital components.

This work come in the wake of a previous paper in which so-called "self-sampled" PLLs (SS-PLLs) were introduced [3]. An SS-PLL is a digital PLL whose loop-filter operations are triggered by the (rising) edges of the local clock, i.e. the clock output by the PLL itself. This mode of operation is an approach to circumvent the absence of an absolute reference clock in non-synchronized autonomous networks of oscillators. The main characteristic of SS-PLLs, as opposed to classical ones [4], is that their dynamics are governed by two linear equations (as opposed to only one in the classical case), depending on whether the local clock is lagging or leading. It is not possible to use the tools of linear analysis to determine whether an SS-PLL is stable (and, hence, synchronizes) or not. In [3], the stability of an SS-PLL was investigated by means of transient simulations. In this paper, we compare this approach with a more rigorous but more conservative approach, based on piecewise-quadratic Lyapunov functions (PQLFs), which requires solving linear matrix inequalities (LMIs). The governing equations of SS-PLLs are given in section II. The theory of PQLFs is addressed in section III. In section IV, the two approaches are compared and their relative advantages and shortcomings are discussed.

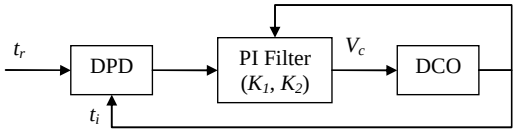

Figure 1. Block-diagram of a digital self-sampled PLL.

II. GENERAL MODEL OF SS-PLLS

An SS-PLL is represented in Fig. 1. It is composed of a digital phase detector (DPD), a proportional-integral (PI) filter and a digitally controlled oscillator (DCO). The PI filter is driven by the rising edges of the output of the DCO. The DPD is linear and outputs the time difference between the moment when arrives a rising edge of the reference clock and those of the local clock. Supposing that the frequency of the DCO is not far from that of the master clock (i.e. that the PLL is in the lock-in domain), the self-sampled PLL can be described by the following model.

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Let $t_r[n]$ (respectively $t_i[n]$) be the time when the n^{th} rising edge of the reference clock (respectively of the DCO output) occurs. These quantities are governed by:

$$t_r[n+1]=t_r[n]+T_r,\quad t_i[n+1]=t_i[n]+T_i+V_c[n] \qquad (1)$$

where T_r (resp. T_i) is the period of the reference (resp. local) clock, V_c is the control voltage of the DCO and all quantities are non-dimensional. The error output by the DPD at time $t_i[n]$ depends on whether the n^{th} rising edge of the reference clock has occurred or not. In other words, if $e_{ri}[n]=t_r[n]-t_i[n]\le 0$, the error is correctly output by the DPD at time $t_i[n]$. Otherwise, the error cannot be calculated and it must be replaced by a prediction, $\hat{e}_{ri}[n]$. The control voltage is then governed by:

$$V_c[n]=V_c[n-1]+K_1\varepsilon[n]+K_2\varepsilon[n-1] \qquad (2\text{-a})$$

or

$$V_c[n]=V_c[n-1]+K_1\varepsilon[n]+K_2 e_{ri}[n-1], \qquad (2\text{-b})$$

where

$$\varepsilon[n]=\begin{cases} e_{ri}[n] & \text{if } e_{ri}[n]\le 0 \\ \hat{e}_{ri}[n] & \text{otherwise} \end{cases} \qquad (3)$$

and a simple choice for the predicted error is $\hat{e}_{ri}[n]=e_{ri}[n-1]$. V_c is governed by (2-a) or (2-b) depending on whether the predicted error is propagated in the filter or not [1]. Substituting V_c in (1) by (2-a), respectively (2-b), one can show that the error is respectively governed by either:

$$e_{ri}[n+1]=2e_{ri}[n]-K_1\varepsilon[n]-e_{ri}[n-1]-K_2\varepsilon[n-1] \qquad (4\text{-a})$$

or

$$e_{ri}[n+1]=2e_{ri}[n]-K_1\varepsilon[n]-(1+K_2)e_{ri}[n-1]. \qquad (4\text{-b})$$

Note that (4-a) (resp. (4-b)) may be recast as four (resp. two) separate linear equations where only $e_{ri}[n+1]$ and its past values appear, each equation corresponding to a possible value of $\varepsilon[n]$ and $\varepsilon[n-1]$. The SS-PLL synchronizes when $e_{ri}[n]$ goes to zero or, equivalently, when the piecewise-linear systems (PLSs) defined by (4-a) and (4-b) are stable. Determining the stability of such systems is a notoriously difficult task: for example [3], for certain values of K_1 and K_2, an SS-PLL governed by (4-b) may be stable even though the equation corresponding to $e_{ri}[n]>0$:

$$e_{ri}[n+1]=2e_{ri}[n]-(1+K_1+K_2)e_{ri}[n-1] \qquad (5)$$

is that of an unstable autonomous system. In [3], some sufficient stability conditions on K_1 and K_2 were established for (4-b) but transient simulations showed that these conditions, based on analytical considerations, seemed very conservative. However, transient simulations cannot be completely trusted, because the behaviour of a PLS may depend on its initial state and all possible initial states cannot be tested. To circumvent this shortcoming, it is possible, in some cases, to use the approach introduced in the next section.

III. PIECEWISE-QUADRATIC STABILITY OF PLSS

A classical approach to the determination of the stability of nonlinear systems is via Lyapunov functions. A Lyapunov function is a positive function of the states of a system whose value decreases along all the possible trajectories in state-space of the system. The existence of a Lyapunov function is a sufficient condition for proving the stability of an autonomous system. Except in the most trivial cases, there exists no generic method to construct or check for the existence of such a function. However, in the particular case of PLSs, the problem of finding a Lyapunov function can be broken down into several sub-problems. The main results [5-6] are summed up hereafter.

A discrete-time PLS can be represented for its analysis by:

$$\mathbf{x}[n+1]=\mathbf{A}_i\mathbf{x}[n],\ \mathbf{x}\in S_i, \qquad (6)$$

where $\mathbf{x}\in R^n$ is the state of the system, $\{S_i\}_{i\in I}\subset R^n$ is a partition of the state-space in a number of closed polyhedral subspaces, I is the set of the indices of the subspaces and $\mathbf{A}_i$ the matrix of the i^{th} local model of the system. Let us also define Ω the set representing all the possible transitions from one region to another, such as:

$$\Omega=\{i,\ j/\mathbf{x}[n]\in S_i,\mathbf{x}[n+1]\in S_j,\ j\ne i\} \qquad (7)$$

In some cases, it is possible to prove the stability of PLSs by finding a common quadratic Lyapunov function (CQLF), i.e. a function $V(\mathbf{x})=\mathbf{x}^T\mathbf{P}\mathbf{x}$, $\mathbf{P}=\mathbf{P}^T>0$, such that

$$\mathbf{A}_i{}^T\mathbf{P}\mathbf{A}_i-\mathbf{P}<0,\ \forall i\in I. \qquad (8)$$

Determining the existence of a CQLF can be done by solving the set of linear matrix inequalities (LMIs) (8), which can be achieved with software such as Matlab. However, many PLSs are stable, even though no CQLF exists. It may then be possible to prove stability by constructing piecewise-quadratic Lyapunov functions [5-6], $V_i(\mathbf{x})=\mathbf{x}^T\mathbf{P}_i\mathbf{x}$, $i\in I$, so that the following relaxed stability conditions:

$$\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i-\mathbf{P}_i+\mathbf{M}_i<0,\ \forall i\in I \qquad (9)$$

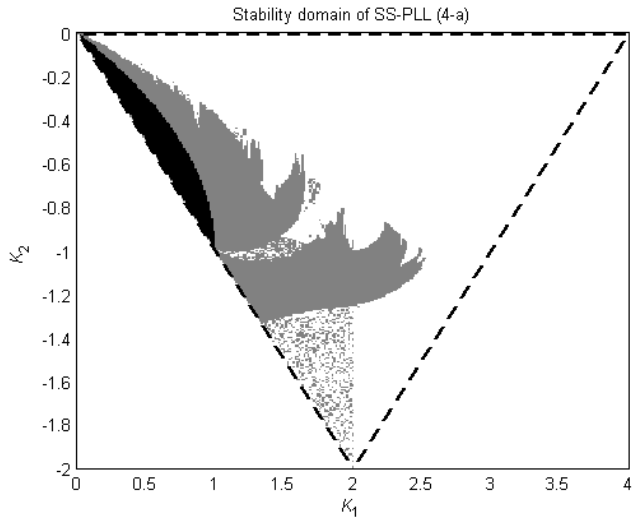

Figure 2. Stability domain of SS-PLL (4-a) given by PQLF (black area) and transient simulation (grey area).

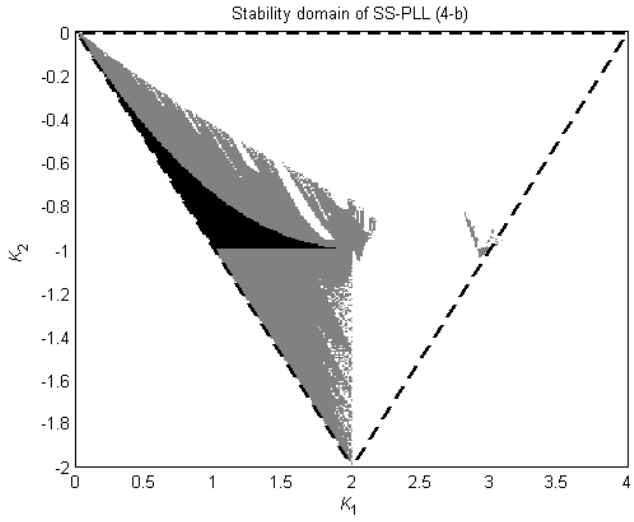

Figure 3. Stability domain of SS-PLL (4-b) given by [3] (black area) and transient simulation (grey area).

are satisfied. In (9), $\mathbf{M}_i$ is a matrix such that $\mathbf{x}^T\mathbf{M}_i\mathbf{x} \geq 0$, $\forall \mathbf{x} \in S_i$, which can be constructed as follows. Since the cells S_i are polyhedral, it is easy to build for each of them a matrix $\mathbf{E}_i$ such that, $\forall \mathbf{x} \in S_i$, $\mathbf{E}_i\mathbf{x}$ has non-negative entries. Then, for any positive matrix $\mathbf{U}_i$ (i.e. any matrix with non-negative entries):

$$\mathbf{x}^T\mathbf{E}_i^T\mathbf{U}_i\mathbf{E}_i\mathbf{x} \geq 0\,,\ \forall \mathbf{x} \in S_i \tag{10}$$

and $\mathbf{M}_i$ can be chosen as $\mathbf{M}_i = \mathbf{E}_i^T\mathbf{U}_i\mathbf{E}$. The main result of Feng's work [6] applied to discrete-time PLSs without an affine term is summarized in the following theorem.

Theorem (Feng)

Consider the discrete-time PLS (6). If there exist some symmetric matrices $\mathbf{P}_i$, $\mathbf{U}_i$, $\mathbf{W}_i$ and $\mathbf{Q}_{ij}$, $i, j \in I$ such that $\mathbf{U}_i$, $\mathbf{W}_i$, and $\mathbf{Q}_{ij}$ are positive and the following LMIs are respected:

$$\begin{aligned} &0 < \mathbf{P}_i - \mathbf{E}_i^T\mathbf{U}_i\mathbf{E}_i\ ,\ i \in I \\ &\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i - \mathbf{P}_i + \mathbf{E}_i^T\mathbf{W}_i\mathbf{E}_i < 0\ ,\ i \in I \\ &\mathbf{A}_i^T\mathbf{P}_j\mathbf{A}_i - \mathbf{P}_i + \mathbf{E}_i^T\mathbf{Q}_{ij}\mathbf{E}_i < 0\ ,\ \{i, j\} \in \Omega \end{aligned} \tag{11}$$

then the origin of the PLS is asymptotically stable. Moreover, the function:

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}_i\mathbf{x}\,,\ \mathbf{x} \in S_i \tag{12}$$

is a Lyapunov function for the system.

IV. SYNCHRONIZATION OF SS-PLLS

Let us define the "stability domain" of an SS-PLL as the set of values of K_1 and K_2 for which the SS-PLL is stable and, thus, synchronizes. In this section, we try to determine the stability domain of SS-PLLs governed by (4-a) or (4-b) associated with the prediction $\hat{e}_{ri}[n] = e_{ri}[n-1]$. Equation (4-a) can be recast into the canonical form (6) by setting:

$$\mathbf{x}[n] = \left(e_{ri}[n] \quad e_{ri}[n-1] \quad e_{ri}[n-2]\right)^T, \tag{13}$$

$$\mathbf{A_1} = \begin{bmatrix} 2-K_1 & -1-K_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ if } e_{ri}[n] \text{ and } e_{ri}[n-1] \leq 0\,, \tag{14}$$

$$\mathbf{A}_2 = \begin{bmatrix} 2 & -1-K_1-K_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ if } e_{ri}[n] > 0 \text{ and } e_{ri}[n-1] \leq 0\,, \tag{15}$$

$$\mathbf{A}_3 = \begin{bmatrix} 2 & -1-K_1 & -K_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ if } e_{ri}[n] \text{ and } e_{ri}[n-1] > 0\,, \tag{16}$$

$$\mathbf{A}_4 = \begin{bmatrix} 2-K_1 & -1 & -K_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ if } e_{ri}[n] \leq 0 \text{ and } e_{ri}[n-1] > 0\,. \tag{17}$$

Equation (4-b) can be recast into canonical form using $\mathbf{A}_1$ if $e_{ri}[n] \le 0$ and $\mathbf{A}_2$ if $e_{ri}[n] > 0$. Now, it must be noted that it is not possible to find a PQLF for a system which stays for more than one time-step in an unstable cell (i.e. a cell S_i whose matrix $\mathbf{A}_i$ has a singular value larger than 1). Since, from [3], $\mathbf{A}_2$ is always unstable, it is impossible to find a PQLF for an SS-PLL governed by (4-b) and $\hat{e}_{ri}[n] = e_{ri}[n-1]$; on the other hand, some sufficient conditions on K_1 and K_2 can be derived analytically [3]. If the SS-PLL is governed by (4-a) and $\hat{e}_{ri}[n] = e_{ri}[n-1]$, the PQLF approach can be used, because, by construction, the system can only stay for one time-step in S_2 (and in S_4). The matrix inequalities can be solved with Matlab.

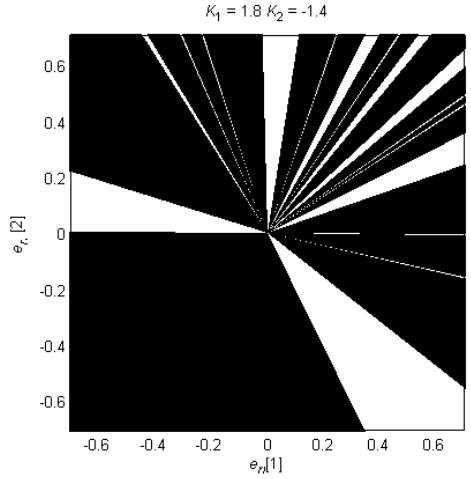

Figure 4. Basins of attraction of an SS-PLL. The white (resp. black) domain corresponds to initial conditions leading to stable (resp. unstable) trajectories.

In Fig. 2 (resp. 3), the stability domain of an SS-PLL governed by (4-a) (resp. (4-b)) and $\hat{e}_{ri}[n] = e_{ri}[n-1]$ is represented. In Fig. 2, the black subset corresponds to the values of K_1 and K_2 for which a PQLF could be found, whereas, in Fig. 3, it corresponds to the values for which the PLL can be analytically shown to synchronize [3]. The grey subset corresponds to the values of K_1 and K_2 for which the transient simulation of (4-a) or (4-b) converges in less than 10^3 iterations. The criterion for convergence is that the final value of the error e_{ri} be smaller than 10^{-5}. The initial values of e_{ri} follow a normal distribution $N(0,1)$. In both figures, the triangle delimited by the dashed lines corresponds to the stability domain of a "classical" PLL, for which $\varepsilon[n] = e_{ri}[n]$. As expected, the black area is always a subset of the grey area. Choosing K_1 and K_2 in the grey subset should always be made with caution. As can be seen from Fig. 2 or 3, the border between the grey subset and the white (unstable) subset is very difficult to define. Furthermore, as was mentioned in section II, some points belonging to the grey subset may in fact correspond to unstable systems. For example, consider a PLL governed by (4-b), with $K_1 = 1.8$ and $K_2 = -1.4$. Testing different initial conditions of the form $e_{ri}[1] = \sin\alpha$ and $e_{ri}[2] = \cos\alpha$, we obtain the basins of attraction shown in Fig. 4. Depending on the initial conditions, this SS-PLL may synchronize or not. However, for small values of $e_{ri}[n]$, the system may easily switch from a stable behaviour to an unstable one (because of round-off errors, for example).

V. CONCLUSION

SS-PLLs, the basic building blocks of clock distribution networks based on digital PLLs, were described in this paper. We showed that SS-PLLs can be modeled as piecewise-linear systems and tested two approaches in order to determine their stability domains. None of these approaches is very conclusive. On the one hand, the LMI-based approach is rigorous but cannot be made to apply to all SS-PLLs. Moreover, it yields very conservative results and is quite costly in computing time. On the other hand, determining stability of SS-PLLs through transient simulations is rather hazardous: as we have shown, some SS-PLLs may exhibit stable or unstable behaviour depending on their initial conditions. While it was relatively simple to test a large number of initial conditions for a single SS-PLL, this approach is not realistic in the context of a large network of SS-PLLs. It might then be necessary to explore robust simulation methods [7], which may give results when PQLFs cannot be found. However, these methods are notoriously costly and may not adapt easily to the context of large networks of PLLs. In practice, another solution would be to design the PLLs so that they adapt the values of the filter coefficients until a consensus is reached. Other predictions $\hat{e}_{ri}[n]$ may also be tested in order to increase the stability domain. A more realistic model - including saturation of the DPD and quantization of the DCO - of the SS-PLL is the subject of ongoing work, as is determination of the lock-in range.

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