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ROBUSTNESS OF ISS SYSTEMS TO INPUTS WITH LIMITED MOVING AVERAGE, WITH APPLICATION TO SPACECRAFT FORMATIONS

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Abstract: We provide a theoretical framework that fits realistic challenges related to spacecraft formation with disturbances. We show that the input-to-state stability of such systems guarantees some robustness with respect to a class of signals with bounded average-energy, which encompasses the typical disturbances acting on spacecraft formations. Solutions are shown to converge to the desired formation, up to an offset which is somewhat proportional to the considered moving average of disturbances. The approach provides a tighter evaluation of the disturbances' influence, which allows for the use of more parsimonious control gains.

1 INTRODUCTION

Spacecraft formation control is a relatively new and active field of research. Formations, characterized by the ability to maintain relative positions without real-time ground commands, are motivated by the aim of placing measuring equipment further apart than what is possible on a single spacecraft. This is desirable as the resolution of measurements often are proportional to the baseline length, meaning that either a large monolithic spacecraft or a formation of smaller, but accurately controlled spacecraft, may be used. Monolithic spacecraft architecture that satisfy the demand of resolution are often both impractical and costly to develop and to launch. On the other hand, smaller spacecrafts may be standardized and have lower development cost. In addition they may be of a lower collective weight and/or of smaller collective size such that cheaper launch vehicles can be used. There is also the possibility for them to piggyback with other commercial spacecraft. These advantages, come at the cost of an increased complexity. From a control design perspective, a crucial challenge is to maintain a predefined relative trajectory, even in presence of disturbances. Most of these disturbances are hard to model in a precise manner. Only statistical or averaged characteristics of the perturbing signals (*e.g.* amplitude, energy, average energy, etc.) are typically available. These perturbing signals may have diverse origins:

- *Intervehicle interference*. In close formation or spacecraft rendezvous, thruster firings and exhaust gases may influence other spacecraft.
- *Solar wind and radiation.* Particles and radiation expelled from the sun influence the spacecraft and are highly dependent on the solar activity (Wertz, 1978), which is difficult to predict (Hanslmeier et al., 1999).
- *Small debris.* While large debris would typically mean the end of the mission, some space trash, including paint flakes, dust, coolant and even small needles[1], is small enough to "only" deteriorate the performance, see (NASA, 1999).
- *Micrometeroids.* The damages caused by micrometeroids may be limited due to their tiny size, but constant high velocity impacts also degrade the performance of the spacecraft through momentum transfer (Schäfer, 2006).
- *Gravitational disturbances.* Even gravitational

[1] Project West Ford was a test carried out in the early 1960s, where 480 million needles were placed in orbit, with the aim to create an artificial ionosphere above the Earth to allow global radio communication, (Overhage and Radford, 1964).

models including higher order zonal harmonics, can only achieve a limited level of accuracy due to the shape and inhomogeneity of the Earth. In addition comes the gravitational perturbation due to other gravitating bodies such as the Sun and the Moon.

- *Actuator mismatch.* There will commonly be a mismatch between the actuation computed by the control algorithm, and the actual actuation that the thrusters can provide. This mismatch is particularly present if the control algorithm is based on continuous dynamics, without taking into account pulse based thrusters.

Nonlinear control theory provides instruments to guarantee a prescribed precision in spite of these disturbances. Input-to-state stability (ISS) is a concept introduced in (Sontag, 1989), which has been thoroughly treated in the literature: see for instance the survey (Sontag, 2008) and references therein. Roughly speaking, this robustness property ensures asymptotic stability, up to a term that is "proportional" to the *amplitude* of the disturbing signal. Similarly, its integral extension, iISS (Sontag, 1998), links the convergence of the state to a measure of the *energy* that is fed by the disturbance into the system. However, in the original works on ISS and iISS, both these notions require that these indicators (amplitude or energy) be finite to guarantee some robustness. In particular, while this concept has proved useful in many control application, ISS may yield very conservative estimates when the disturbing signals come with high amplitude even if their *moving average* is reasonable.

These limitations have already been pointed out and partially addressed in the literature. In (Angeli and Nešić, 2001), the notions of "Power ISS" and "Power iISS" are introduced to estimate more tightly the influence of the power or moving average of the exogenous input on the *power* of the state. Under the assumption of local stability for the zero-input system, these properties are shown to be actually equivalent to ISS and iISS respectively. Nonetheless, for a generic class of input signals, no hard bound on the state norm can be derived for this work.

Other works have focused on quantitative aspects of ISS, such as (Praly and Wang, 1996), (Grüne, 2002) and (Grüne, 2004). All these three papers solve the problem by introducing a "memory fading" effect in the input term of the ISS formulation. In (Praly and Wang, 1996) the perturbation is first fed into a linear scalar system whose output then enters the right hand side of the ISS estimate. The resulting property is referred to as exp-ISS and is shown to be equivalent to ISS. In (Grüne, 2002) and (Grüne, 2004) the concept of input-to-state dynamical stability (ISDS) is introduced and exploited. In the ISDS state estimate, the value of the perturbation at each time instant is used as the initial value of a one-dimensional system, thus generalizing the original idea of Praly and Wang. The quantitative knowledge of how past values of the input signal influence the system allows, in particular, to guarantee an explicit decay rate of the state for vanishing perturbations.

In this paper, our objective is to guarantee hard bound on the state norm for ISS systems in presence of signals with possibly unbounded amplitude and/or energy. We enlarge the class of signals to which ISS systems are robust, by simply conducting a tighter analysis on these systems. In the spirit of (Angeli and Nešić, 2001), and in contrast to most previous works on ISS and iISS, the considered class of disturbances is defined based on their moving average. We show that any ISS system is robust to such a class of perturbations. When an explicitly Lyapunov function is known, we explicitly estimate the maximum disturbances' moving average that can be tolerated for a given precision. These results are presented in Section 2. We then apply this new analysis result to the control of spacecraft formations. To this end, we exploit the Lyapunov function available for such systems to identify the class of signals to which the formation is robust. This class includes all kind of perturbing effects described above. This study is detailed, and illustrated by simulations, in Section 3.

Notation and terminology

A continuous function $\alpha : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is strictly increasing and $\alpha(0) = 0$. If, in addition, $\alpha(s) \to \infty$ as $s \to \infty$, then α is of class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$). A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in \mathbb{R}_{\geq 0}$, and $\beta(s, \cdot)$ is decreasing and tends to zero as s tends to infinity. The solutions of the differential equation $\dot{x} = f(x, u)$ with initial condition $x_0 \in \mathbb{R}^n$ is denoted by $x(\cdot; x_0, u)$. We use $|\cdot|$ for the Euclidean norm of vectors and the induced norm of matrices. The closed ball in $\mathbb{R}^n$ of radius δ centered at the origin is denoted by $\mathcal{B}_\delta$, i.e. $\mathcal{B}_\delta := \{x \in \mathbb{R}^n : |x| \leq \delta\}$. $|\cdot|_\delta$ denotes the distance to the ball $\mathcal{B}_\delta$, that is $|x|_\delta := \inf_{z \in \mathcal{B}_\delta} |x - z|$. $\mathcal{U}$ denotes the set of all measurable locally essentially bounded signals $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$. For a signal $u \in \mathcal{U}$, $\|u\|_\infty := \text{ess sup}_{t \geq 0} |u(t)|$. The maximum and minimum eigenvalue of a symmetric matrix A is denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. I_n and 0_n denote the identity and null matrices of $\mathbb{R}^{n \times n}$ respectively.

2 ISS SYSTEMS AND SIGNALS WITH LOW MOVING AVERAGE

2.1 Preliminaries

We start by recalling some classical definitions related to the stability and robustness of nonlinear systems of the form

$$\dot{x} = f(x,u), \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is locally Lipschitz and satisfies $f(0,0) = 0$.

Definition 1 *Let δ be a positive constant and u be a given signal in $\mathcal{U}$. The ball $\mathcal{B}_\delta$ is said to be globally asymptotically stable (GAS) for (1) if there exists a class $\mathcal{KL}$ function β such that the solution of (1) from any initial state $x_0 \in \mathbb{R}^n$ satisfies*

$$|x(t;x_0,u)| \leq \delta + \beta(|x_0|,t), \quad \forall t \geq 0. \tag{2}$$

Definition 2 *The ball $\mathcal{B}_\delta$ is said to be globally exponentially stable (GES) for (1) if the conditions of Definition 1 hold with $\beta(r,s) = k_1 r e^{-k_2 s}$ for some positive constants k_1 and k_2.*

We next recall the definition of ISS, originally introduced in (Sontag, 1989).

Definition 3 *The system $\dot{x} = f(x,u)$ is said to be input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$, the solution of (1) satisfies*

$$|x(t;x_0,u)| \leq \beta(|x_0|,t) + \gamma(\|u\|_\infty), \quad \forall t \geq 0. \tag{3}$$

ISS thus imposes an asymptotic decay of the norm of the state up to a function of the *amplitude* $\|u\|_\infty$ of the input signal.

We also recall the following well-known Lyapunov characterization of ISS, originally established in (Praly and Wang, 1996) and thus extending the original characterization proposed by Sontag in (Sontag and Wang, 1995).

Proposition 1 *The system (1) is ISS if and only if there exist $\underline{\alpha}, \overline{\alpha}, \gamma \in \mathcal{K}_\infty$ and $\kappa > 0$ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^p$,*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{4}$$

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\kappa V(x) + \gamma(|u|). \tag{5}$$

γ is then called a supply rate *for (1).*

Remark 1 *Since ISS implies iISS (cf. (Sontag, 1998)), it can be shown that the solutions of any ISS system with supply rate γ satisfies, for all $x_0 \in \mathbb{R}^n$,*

$$|x(t;x_0,u)| \leq \beta(|x_0|,t) + \eta\left(\int_0^t \gamma(|u(\tau)|)d\tau\right), \quad \forall t \geq 0, \tag{6}$$

where $\beta \in \mathcal{KL}$ and $\eta \in \mathcal{K}_\infty$. The above integral can be seen as a measure, through the function γ, of the energy *of the input signal u.*

The above remark establishes a link between a measure of the energy fed into the system and the norm of the state: for ISS (and iISS) systems, if this input energy is small, then the state will eventually be small. However, Inequalities (3) and (6) do not provide any information on the behavior of the system when the amplitude (for (3)) and/or the energy (for (6)) of the input signal is not finite.

From an applicative viewpoint, the precision guaranteed by (3) and (6) involve the *maximum value* and the *total energy* of the input. These estimates may be conservative and thus lead to the design of greedy control laws, with negative consequences on the energy consumption and actuators solicitation. This issue is particularly relevant for spacecraft formations in view of the inherent fuel limitation and limited power of the thrusters.

(Angeli and Nešić, 2001) has started to tackle this problem by introducing ISS and iISS-like properties for input signals with limited *power*, thus not necessarily bounded in amplitude nor in energy. For systems that are stable when no input is applied, the authors show that ISS (resp. iISS) is equivalent to "power ISS" (resp. "power iISS") and "moving average ISS" (resp. "moving average iISS"). In general terms, these properties evaluate the influence of the *amplitude* (resp. the *energy*) of the input signal on the *power* or *moving average* of the state. However, as stressed by the authors themselves, these estimates do not guarantee in general any *hard bound* on the state norm. Here, we consider a slightly more restrictive class of input signals under which such a hard bound can be guaranteed. Namely, we consider input signals with bounded moving average.

Definition 4 *Given some constants $E, T > 0$ and some function $\gamma \in \mathcal{K}$, the set $\mathcal{W}_\gamma(E,T)$ denotes the set of all signals $u \in \mathcal{U}$ satisfying*

$$\int_t^{t+T} \gamma(|u(s)|)ds \leq E, \quad \forall t \in \mathbb{R}_{\geq 0}.$$

The main concern here is the measure E of the maximum energy that can be fed into the system over a moving time window of given length T. These quantities are the only information on the disturbances that will be taken into account in the control design. More parsimonious control laws than those based on their amplitude or energy can therefore be expected. We stress that signals of this class are not necessarily globally essentially bounded, nor are they required to have a finite energy, as illustrated by the

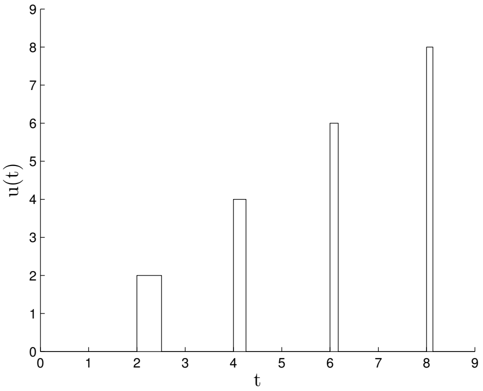

Figure 1: An example of unbounded signal with bounded moving average.

following examples. Robustness to this class of signals thus constitutes an extension of the typical properties of ISS systems.

Example 1

1. *Unbounded signals: given any* $T > 0$ *and any* $\gamma \in \mathcal{K}$*, the following signal belongs to* $\mathcal{W}_\gamma(1,T)$ *and satisfies* $\limsup_{t\to\infty} |u(t)| = +\infty$*:*

$$u(t) := \begin{cases} 2k & \text{if} \quad t \in [2kT; 2kT + \frac{1}{2k}],\, k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

The signal for $T = 1$ *is illustrated in Figure 1.*

2. *Essentially bounded signals: given any* $T > 0$ *and any* $\gamma \in \mathcal{K}$*, if* $\|u\|_\infty$ *is finite then it holds that* $u \in \mathcal{W}_\gamma(T\gamma(\|u\|_\infty), T)$*. We stress that this includes signals with infinite energy (think for instance of constant non-zero signals).*

2.2 Robustness of ISS systems to signals in the class $\mathcal{W}$

The following result establishes that the impact of an exogenous signal on the qualitative behavior of an ISS systems is negligible if the moving average of this signal is sufficiently low.

Theorem 1 *Assume that the system* $\dot{x} = f(x,u)$ *is ISS. Then there exists a function* $\gamma \in \mathcal{K}_\infty$ *and, given any precision* $\delta > 0$ *and any time window* $T > 0$*, there exists a positive average energy* $E(T,\delta)$ *such that the ball* $\mathcal{B}_\delta$ *is GAS for any* $u \in \mathcal{W}_\gamma(E,T)$*.*

The above result, proved in Section 4, adds another brick in the wall of nice properties induced by ISS, cf. (Sontag, 2008) and references therein. It ensures that, provided that a steady-state error δ can be tolerated, every ISS system is robust to a class of disturbances with sufficiently small moving average.

If an ISS Lyapunov function is known for the system, then an explicit bound on the tolerable average excitation can be provided based on the prooflines of Theorem 1. More precisely, we state the following result.

Corollary 1 *Assume there exists a continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, class* $\mathcal{K}_\infty$ *functions* γ*,* $\underline{\alpha}$ *and* $\overline{\alpha}$ *and a positive constant* κ *such that (4) and (5) hold for all* $x \in \mathbb{R}^n$ *and all* $u \in \mathbb{R}^p$*. Given any precision* $\delta > 0$ *and any time window* $T > 0$*, let* E *denote any average energy satisfying*

$$E(T,\delta) \leq \frac{\underline{\alpha}(\delta)}{2} \frac{e^{\kappa T} - 1}{2e^{\kappa T} - 1}\,. \tag{7}$$

Then the ball $\mathcal{B}_\delta$ *is GAS for* $\dot{x} = f(x,u)$ *for any* $u \in \mathcal{W}_\gamma(E,T)$*.*

The above statement shows that, by knowing a Lyapunov function associated to the ISS of a system, and in particular its dissipation rate γ, one is able to explicitly identify the class $\mathcal{W}_\gamma(E,T)$ to which it is robust up to the prescribed precision δ.

In a similar way, we can state sufficient condition for global exponential stability of some neighborhood of the origin. This result follows also trivially from the proof of Theorem 1.

Corollary 2 *If the conditions of Corollary 1 are satisfied with* $\underline{\alpha}(s) = \underline{c}s^p$ *and* $\overline{\alpha}(s) = \overline{c}s^p$*, with* $\underline{c}, \overline{c}, p$ *positive constants, then, given any* $T, \delta > 0$*, the ball* $\mathcal{B}_\delta$ *is GES for (1) with any signal* $u \in \mathcal{W}_\gamma(E,T)$ *provided that*

$$E(T,\delta) \leq \frac{\underline{c}\delta^p}{2} \frac{e^{\kappa T} - 1}{2e^{\kappa T} - 1}\,.$$

3 ILLUSTRATION: SPACECRAFT FORMATION CONTROL

We now exploit the results developed in Section 2 to demonstrate the robustness of a spacecraft formation control in a leader-follower configuration, when only position is measured. The focus on output feedback in this illustration is motivated by the fact that velocity measurements in space may not be easily achieved, *e.g.* because the spacecraft cannot be equipped with the necessary sensors for such measurements due to space constraints or budget limits. The models described in this section have strong resemblance with the model of a robot manipulator. Our control design is therefore be based on control algorithms already validated for robot manipulators, in particular (Berghuis and Nijmeijer, 1993) and (Paden and Panja, 1988). We stress that he proposed study is made for two spacecraft only, but can easily be extended to formations involving more spacecrafts.

3.1 Spacecraft models

The spacecraft models presented in this section are similar to the ones derived in (Ploen et al., 2004). All coordinates, both for the leader and the follower spacecraft, are expressed in an orbital frame, which origin relative to the center of Earth is given by $\vec{r}_o$, and satisfies Newton's gravitational law

$$\ddot{\vec{r}}_o = -\frac{\mu}{|\vec{r}_o|^3}\vec{r}_o\,,$$

μ being the gravitational constant of Earth. The unit vectors are such that $\vec{o}_1 := \vec{r}_o/|\vec{r}_o|$ points in the anti-nadir direction, $\vec{o}_3 := (\vec{r}_o \times \dot{\vec{r}}_o)/|\vec{r}_o \times \dot{\vec{r}}_o|$ points in the direction of the orbit normal, and finally $\vec{o}_2 := \vec{o}_3 \times \vec{o}_1$ completes the right-handed orthogonal frame. We let ν_o denote the true-anomaly of this reference frame and assume the following:

Assumption 1 *The true anomaly rate $\dot{\nu}_o$ and true anomaly rate-of-change $\ddot{\nu}_o$ of the reference frame satisfy $\|\dot{\nu}_o\|_\infty \le \beta_{\dot{\nu}_o}$ and $\|\ddot{\nu}_o\|_\infty \le \beta_{\ddot{\nu}_o}$, for some positive constants $\beta_{\dot{\nu}_o}$ and $\beta_{\ddot{\nu}_o}$.*

Note that this assumption is naturally satisfied when the reference frame is following a Keplerian orbit, but it also holds for any sufficiently smooth reference trajectory. We define the following quantities:

$$C(\dot{\nu}_o) := 2\dot{\nu}_o\bar{C}, \quad \bar{C} := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D(\dot{\nu}_o, \ddot{\nu}_o) := \dot{\nu}_o^2\bar{D} + \ddot{\nu}_o\bar{C}, \quad \bar{D} := \mathrm{diag}(-1,-1,0),$$

and

$$n(r_o, p) := \mu\left(\frac{r_o + p}{|r_o + p|^3} - \frac{r_o}{|r_o|^3}\right).$$

In the above reference frame, the dynamics ruling the evolution of the coordinate $p \in \mathbb{R}^3$ of the leader spacecraft is then given by

$$\ddot{p} + C(\dot{\nu}_o)\,\dot{p} + D(\dot{\nu}_o, \ddot{\nu}_o)\,p + n(r_o, p) = F_l \tag{8}$$

with $F_l := (u_l + d_l)/m_l$, while the evolution of the relative position ρ of the follower spacecraft with respect to the leader is given by

$$\ddot{\rho} + C(\dot{\nu}_o)\,\dot{\rho} + D(\dot{\nu}_o, \ddot{\nu}_o)\,\rho + n(r_o + p, \rho) = F_f - F_l\,, \tag{9}$$

where $F_f := (u_f + d_f)/m_f$, and where subscripts l and f stand for the leader and follower spacecraft respectively, m_l and m_f are the spacecrafts' masses, u_l and u_f are the control inputs, and d_l and d_f denote all exogenous perturbations acting on the spacecrafts (*e.g.*, as detailed in the Introduction; intervehicle interference, small impacts, solar wind, etc.).

3.2 Control of the leader spacecraft

We now propose a controller whose goal is to make the leader spacecraft follow a given trajectory $p_d : \mathbb{R}_{\ge 0} \to \mathbb{R}^3$ relative to the reference frame. In other words, its aim is to decrease the tracking error defined as $e_l := p - p_d$. To derive this controller, we rely on the position p of the leader only. No measurement on its velocity is required. The latter will be estimated through the derivative of the some position estimate $\hat{p}$ in order to avoid brute force derivation of the measurement p. We therefore define $\tilde{p} := p - \hat{p}$ as the estimation error. Similarly to (Berghuis, 1993), the controller is given by:

$$\begin{aligned} u_l = & m_l\Big[\ddot{p}_d + C(\dot{\nu}_o)\,\dot{p}_d + D(\dot{\nu}_o, \ddot{\nu}_o)\,p + n(r_o, p) \\ & - k_l(\dot{p}_0 - \dot{p}_r)\Big] \end{aligned} \tag{10}$$

$$\dot{p}_r = \dot{p}_d - \ell_l e_l \tag{11}$$

$$\dot{p}_0 = \dot{\hat{p}} - \ell_l\tilde{p}, \tag{12}$$

where k_l and ℓ_l denote positive gains. The velocity estimator is given by

$$\dot{\hat{p}} = a_l + (l_l + \ell_l)\,\tilde{p} \tag{13}$$

$$\dot{a}_l = \ddot{p}_d + l_l\ell_l\tilde{p}\,, \tag{14}$$

where l_l denotes another positive gain. Define $X_l := \left(e_l^\top, \dot{e}_l^\top, \tilde{p}^\top, \dot{\tilde{p}}^\top\right)^\top \in \mathbb{R}^{12}$ and $d := (d_l^\top, d_f^\top)^\top \in \mathbb{R}^6$. Then the leader dynamics takes the form of a perturbed linear time-varying system:

$$\dot{X}_l = A_l(\dot{\nu}_o(t))X_l + B_l d\,, \tag{15}$$

where $A_l \in \mathbb{R}^{12\times 12}$ and $B_l \in \mathbb{R}^{12\times 6}$ refer to the following matrices

$$A_l(\dot{\nu}_o) := \begin{bmatrix} 0_3 & I_3 & 0_3 & 0_3 \\ a_{21} & a_{22}(\dot{\nu}_o) & a_{23} & a_{24} \\ 0_3 & 0_3 & 0_3 & I_3 \\ a_{41} & a_{42}(\dot{\nu}_o) & a_{43} & a_{44} \end{bmatrix}, \tag{16}$$

$$B_l := \frac{1}{m_l}\begin{bmatrix} 0_3 & 0_3 \\ I_3 & 0_3 \\ 0_3 & 0_3 \\ I_3 & 0_3 \end{bmatrix},$$

where out of notational compactness, the following matrices are defined: $a_{21} := a_{41} := -k_l\ell_l I_3$, $a_{22} := a_{42} := -C(\dot{\nu}_o) - k_l I_3$, $a_{23} := k_l\ell_l I_3$, $a_{24} := k_l I_3$, $a_{43} := (k_l - l_l)\ell_l I_3$ and $a_{44} := (k_l - l_l - \ell_l)I_3$.

3.3 Control of the follower spacecraft

We next propose a controller to make the follower spacecraft track a desired trajectory $\rho_d : \mathbb{R}_{\ge 0} \to \mathbb{R}^3$ relative to the leader. In the same way as for the leader spacecraft, let $\hat{\dot{\rho}} \in \mathbb{R}^3$ denote the estimated velocity of the follower with respect to the leader, let $e_f := \rho - \rho_d$ denote the tracking error and let $\tilde{\rho} := \rho - \hat{\rho}$ be the estimation error. We use the following control law:

$$\begin{aligned} u_f =& m_f\Big[\ddot{p}_d + \ddot{\rho}_d + C(\dot{\nu}_o)(\dot{p}_d + \dot{\rho}_d) \\ & + D(\dot{\nu}_o, \ddot{\nu}_o)(p+\rho) + n(r_o+p,\rho) + n(r_o,p) \\ & - k_l(\dot{p}_0 - \dot{p}_r) - k_f(\dot{\rho}_0 - \dot{\rho}_r)\Big] \qquad (17) \\ \dot{\rho}_r =& \dot{\rho}_d - \ell_f e_f \qquad (18) \\ \dot{\rho}_0 =& \hat{\dot{\rho}} - \ell_f \tilde{\rho}, \qquad (19) \end{aligned}$$

with the observer being given by

$$\begin{aligned} \hat{\dot{\rho}} &= a_f + (l_f + \ell_f)\tilde{\rho} \qquad (20) \\ \dot{a}_f &= \ddot{\rho}_d + l_f \ell_f \tilde{\rho} \qquad (21) \end{aligned}$$

where k_f, l_f and ℓ_f denote positive tuning gains. We stress that, in order to implement (17), (11)-(14) must also be implemented in follower spacecraft control algorithm. Define $X_f := (e_f^\top, \dot{e}_f^\top, \tilde{\rho}^\top, \tilde{\dot{\rho}}^\top)^\top \in \mathbb{R}^{12}$. Combining (9) and (17)-(21) and inserting the leader spacecraft controller u_l (10), we can summarize the follower spacecraft's dynamics by

$$\dot{X}_f = A_f(\dot{\nu}_o(t))X_f + B_f d\,, \qquad (22)$$

where $A_f(\dot{\nu}_o)$ can be obtained from $A_l(\dot{\nu}_o)$ (cf. (16)) by simply substituting the subscripts l by f in the expression of the submatrices a_{ij}, and

$$B_f := \frac{1}{m_l m_f}\begin{bmatrix} 0_3 & 0_3 \\ -m_f I_3 & m_l I_3 \\ 0_3 & 0_3 \\ -m_f I_3 & m_l I_3 \end{bmatrix}.$$

3.4 Robustness analysis of the overall formation

We are now ready to state the following result, which establishes the robustness of the controlled formation to a wide class of disturbances.

Proposition 2 *Let Assumption 1 hold. Let the controller of the leader spacecraft be given by (10)-(14) and the controller of the follower spacecraft be given by (17)-(21) with, for each* $i \in \{l, f\}$, $l_i \geq 2k_i$, $k_i > 2k_i^\star$ *and (for simplicity)* $\ell_i \geq 1$, *where*

$$k_i^\star := \ell_i + \beta_{\dot{\nu}_o}\sqrt{2\ell_i^2+1} + \left(1 + \frac{m_f^2}{m_l^2}\right)\frac{2\left(l_i^2+1\right)}{m_i^2}. \qquad (23)$$

Given any precision $\delta > 0$ *and any time window* $T > 0$, *consider any average energy satisfying*

$$E \leq \frac{1}{4}\min_{i\in\{l,f\}}\left\{\ell_i^2 - \frac{1}{2}\sqrt{4\ell_i^4+1} + \frac{1}{2}\right\}\delta^2\frac{e^{\kappa T}-1}{2e^{\kappa T}-1}, \qquad (24)$$

where

$$\kappa := \frac{\min_{i\in\{l,f\}} k_i^\star / \max_{i\in\{l,f\}}\left\{\frac{k_i}{\ell_i}\right\}}{\max_{i\in\{l,f\}}\left\{\ell_i^2 + \frac{1}{2}\sqrt{4\ell_i^4+1} + \frac{1}{2}\right\}}. \qquad (25)$$

Then, for any $d \in \mathcal{W}_\gamma(E,T)$ *where* $\gamma(s) := s^2$, *the ball* $\mathcal{B}_\delta$ *is GES for the overall formation summarized by (15) and (22).*

Proof. Let the overall dynamics be condensed into $\dot{X} = AX + Bd$ with $X := (X_1^\top, X_2^\top)^\top$, $A := \mathrm{diag}(A_l, A_f)$ and $B := (B_l^\top, B_f^\top)^\top$. The proof is done by applying Corollary 2. Consider the Lyapunov function candidate

$$V(X) := \frac{1}{2}\sum_{i\in\{l,f\}} V_i(X_i)$$

where $V_i(X_i) := X_i^\top W_i^\top R_i W_i X_i$, $R_i := \mathrm{diag}((2k_i/\ell_i - 1)I_3, I_3, 2k_i/\ell_i I_3, I_3)$ and

$$W_i := \begin{bmatrix} \ell_i I_3 & 0_3 & 0_3 & 0_3 \\ \ell_i I_3 & I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & \ell_i I_3 & 0_3 \\ 0_3 & 0_3 & \ell_i I_3 & I_3 \end{bmatrix}.$$

It can be shown that the time derivative of the Lyapunov function candidate can be written as

$$\begin{aligned} \dot{V} &= \sum_{i\in\{l,f\}} X_i^\top W_i^\top R_i W_i A_i X_i + X_i^\top W_i^\top R_i W B_i d \\ &= -\left(\sum_{i\in\{l,f\}} X_i^\top (Q_i + S_i) X_i - X_i^\top W_i^\top R_i W_i B_i d\right) \end{aligned}$$

where $Q_i := \mathrm{diag}(k_i\ell_i^2 I_3, (k_i-\ell_i)I_3, k_i\ell_i^2 I_3, k_i I_3)$,

$$S_i := \frac{1}{2}\begin{bmatrix} 0_3 & C(\dot{\nu}_o)\ell_i & 0_3 & 0_3 \\ C^\top(\dot{\nu}_o)\ell_i & 0_3 & C^\top(\dot{\nu}_o)\ell_i & C^\top(\dot{\nu}_o) \\ 0_3 & C(\dot{\nu}_o)\ell_i & \ell^2 s_i & \ell s_i \\ 0_3 & C(\dot{\nu}_o) & \ell s_i & s_i \end{bmatrix},$$

where $s_i := 2(l_i - 2k_i)I_3$. Since $l_i \geq 2k_i$, $-X_i^\top S_i X_i \leq \|\dot{\nu}_o\|_\infty(2\ell_i^2+1)^{1/2}|X_i|^2$. Furthermore, $\lambda_{\min}(Q_i) = \min\{k_i - \ell_i, k_i\ell_i^2\} = k_i - \ell_i$ for $\ell_i \geq 1$, $|W_l^\top R_l W_l B_l| = (2(l_l^2+1))^{1/2}/m_l$, $|W_f^\top R_f W_f B_f| = (2(m_f^2+m_l^2)(l_f^2+1))^{1/2}/(m_l m_f)$, and invoking Assumption 1, we get that the derivative of the Lyapunov function can be upper bounded as:

$$\begin{aligned} \dot{V} \;\leq\; & -\sum_{i\in\{l,f\}}\left(k_i - \ell_i - \beta_{\dot{\nu}_o}\sqrt{2\ell_i^2+1}|X_i|^2\right) \\ & + \frac{\sqrt{2\left(l_l^2+1\right)}}{m_l}|X_l|\,|d| \\ & + \frac{\sqrt{2(m_f^2+m_l^2)(l_f^2+1)}}{m_l m_f}|X_f|\,|d|\,. \end{aligned}$$

By Young's inequality it follows that

$$\dot{V} \leq -\sum_{i\in\{l,f\}} \left[k_i - \ell_i - \beta_{\dot{\nu}_o}\sqrt{2\ell_i^2+1} - \left(1+\frac{m_f^2}{m_l^2}\right)\frac{2\left(l_i^2+1\right)}{m_i^2}\right]|X_i|^2 + |d|^2.$$

If we chose $k_l > 2k_l^\star$ and $k_f > 2k_f^\star$ as given in the statement of Proposition 2, R_l, R_f, Q_l, Q_f are all positive definite matrices. Furthermore, it can be shown that $\underline{c}|X|^2 \leq V(X) \leq \overline{c}|X|^2$, where

$$\underline{c} := \frac{1}{2}\min_{i\in\{l,f\}}\left\{\ell_i^2 - \frac{1}{2}\sqrt{4\ell_i^4+1} + \frac{1}{2}\right\} \tag{26}$$

$$\overline{c} := \max_{i\in\{l,f\}}\left\{\frac{k_i}{\ell_i}\right\}\max_{i\in\{l,f\}}\left\{\ell_i^2 + \frac{1}{2}\sqrt{4\ell_i^4+1} + \frac{1}{2}\right\}. \tag{27}$$

Using these inequalities, we get that

$$\begin{aligned} \dot{V} &\leq -\min_{i\in\{l,f\}}\{k_i^\star\}\left(|X_l|^2 + \left|X_f\right|^2\right) + |d|^2 \\ &\leq -\kappa V(x) + |d|^2 \end{aligned}$$

with the constant κ defined in (25). Hence, the conditions of Corollary 2 are satisfied, with $\underline{c}$ and $\overline{c}$ defined in (26)-(27) and $\gamma(s) = s^2$, and the conclusion follows.

3.5 Simulations

Let the reference orbit be an eccentric orbit with radius of perigee $r_p = 10^7 m$ and radius of apogee $r_a = 3\times 10^7 m$, which can be generated by numerical integration of

$$\ddot{r}_o = -\frac{\mu}{|r_o|^3} r_o, \tag{28}$$

with $r_o(0) = (r_p, 0, 0)$ and $\dot{r}_o(0) = (0, v_p, 0)$, and where

$$v_p = \sqrt{2\mu\left(\frac{1}{r_p} - \frac{1}{(r_p+r_a)}\right)}.$$

The true anomaly ν_o of the reference frame can be obtained by numerical integration of the equation

$$\ddot{\nu}_o(t) = \frac{-2\mu e_o\left(1+e_o\cos\nu_o(t)\right)^3 \sin\nu_o(t)}{\left(\frac{1}{2}(r_p+r_a)(1-e_o^2)\right)^3}.$$

From this expression, and the eccentricity, which can be calculated from r_a and r_p to be $e_o = 0.5$, we see that the constant $\beta_{\ddot{\nu}_o}$ in Assumption 1 can be chosen as $\beta_{\ddot{\nu}_o} = 4\times 10^{-7}$. From the analytical equivalent for $\dot{\nu}_o$,

$$\dot{\nu}_o(t) = \frac{\sqrt{\mu}\left(1+e_o\cos\nu_o(t)\right)^2}{\left(\frac{1}{2}(r_p+r_a)(1-e_o^2)\right)^{3/2}},$$

we see that the constant $\beta_{\dot{\nu}_o}$ in Assumption 1 can be chosen as $\beta_{\dot{\nu}_o} = 8\times 10^{-4}$. Since the reference frame is initially at perigee, $\nu_o(0) = 0$ and $\dot{\nu}_o(0) = v_p/r_p$. For simplicity, we choose the desired trajectory of the leader spacecraft to coincide with the reference orbit, *i.e.* $p_d(\cdot) \equiv (0,0,0)^\top$. The initial values of the leader spacecraft are $p_l(0) = (2,-2,3)^\top$ and $\dot{p}_l(0) = (0.4,-0.8,-0.2)^\top$. The initial values of the observer are chosen as $\hat{p}(0) = (0,0,0)^\top$ and $a_l(0) = (0,0,0)^\top$.

The reference trajectory of the follower spacecraft are chosen as the solutions of a special case of the Clohessy-Wiltshire equations, cf. (Clohessy and Wiltshire, 1960). We use

$$\rho_d(t) = \begin{bmatrix} 10\cos\nu_o(t) \\ -20\sin\nu_o(t) \\ 0 \end{bmatrix}. \tag{29}$$

This choice imposes that the two spacecrafts evolve in the same orbital plane, and that the follower spacecraft will make a full rotation about the leader spacecraft per orbit around the Earth. The initial values of the follower spacecraft are $\rho(0) = (9,-1,2)^\top$ and $\dot{\rho}(0) = (-0.3, 0.2, 0.6)^\top$. The initial parameters of the observer are chosen to be $\hat{\rho}(0) = \rho_d(0) = (10,0,0)^\top$ and $a_f(0) = (0,0,0)^\top$. We use $m_f = m_l = 25$ kg both in the model and the control structure.

The choice of control gains are based on the analysis in Section 3. First we pick $\ell_i = 1$, $i\in\{l,f\}$. Then, by using that $\beta_{\dot{\nu}_o} = 8\times 10^{-4}$, we find that $k_i^\star = 1.0014 + 0.0064(l_i^2+1)$ from (23). Since k_i should satisfy $k_i > 2k_i^\star$ and $l_i \geq 2k_i$, we chose $k_i = 2.3$ and $l_i = 4.6$, $i\in\{l,f\}$. With these choices, we find from (25) that $\kappa \approx 0.1899$. Over a 10 second interval (*i.e.* T=10), the average excitation must satisfy $E(T,\delta) \leq 0.0439\,\delta^2$, according to (24). We consider two types of disturbances acting on the spacecraft: "impacts" and continuous disturbances. The "impacts" have random amplitude, but with maximum of 1.5 N in each direction of the Cartesian frame. For simplicity, we assume that at most one impact can occur over each 10 second interval, and we assume that the duration of each impact is 0.1s. The continuous part is taken as sinusoids, also acting in each direction of the Cartesian frame, and are chosen to be $(0.1\sin 0.01t, 0.25\sin 0.03t, 0.3\sin 0.04t)^\top$ for both spacecraft. The motivation for choosing the same kind of continuous disturbance for both spacecraft, is that this disturbance is typically due to gravitational perturbation, which at least for close formations, have the same effect on both spacecraft. Notice from (9) that the relative dynamics are influenced by disturbances acting on the leader and follower spacecraft, so the effect of the continuous part of the disturbance on the relative dynamics is zero. It can easily

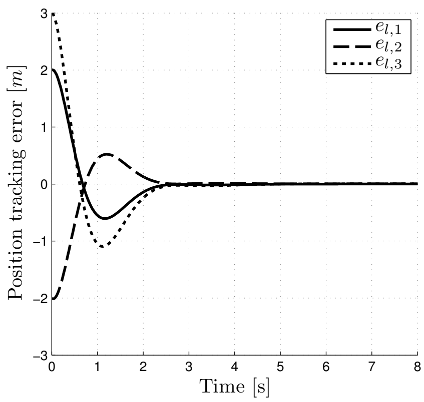

Figure 2: Position tracking error of the leader spacecraft

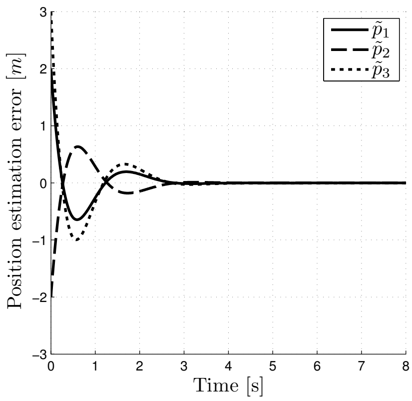

Figure 3: Position estimation error of the leader spacecraft

be shown that the disturbances satisfy the following:

$$\int_t^{t+10} |d(\tau)|^2 d\tau \leq 1.42\,, \quad \forall t \geq 0\,.$$

Figure 2, 3 and 4 show the position tracking error, position estimation error and control history of

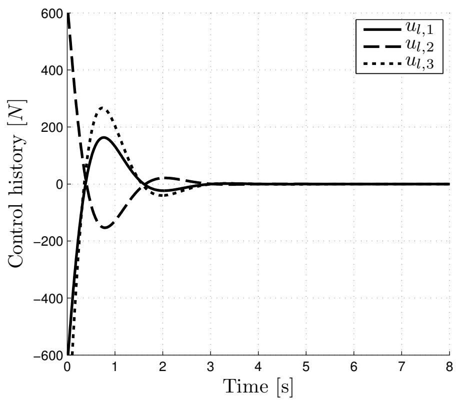

Figure 4: Control actuation of the leader spacecraft

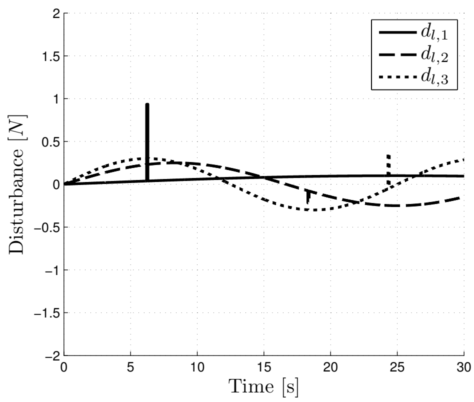

Figure 5: Disturbances acting on the leader spacecraft

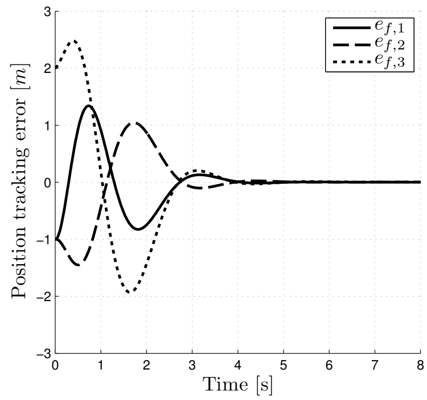

Figure 6: Position tracking error of the follower spacecraft

the leader spacecraft, whereas Figure 6, 7 and 8 are the equivalent figures for the follower spacecraft. Figure 5 and 9 show the effect of d_l and $d_l - d_f$ acting on the formation. Notice in Figure 9 that the effect of the continuous part of the disturbance is canceled out (since we consider relative dynamics and both spacecraft are influenced by the same continuous disturbance), whereas the effect of the impacts has increased compared to the effect of the impacts on the leader spacecraft. The control gains have been chosen based on the Lyapunov analysis. This yields in general very conservative constraints on the choice of control gains, and also conservative estimates of the disturbances the control system is able to handle. As shown in Figure 4, and in particular Figure 8, this leads to large transients in the actuation. We stress that the control gains proposed by this approach is still much smaller that those obtained through a classical ISS approach (*i.e.* relying on the disturbance magnitude).

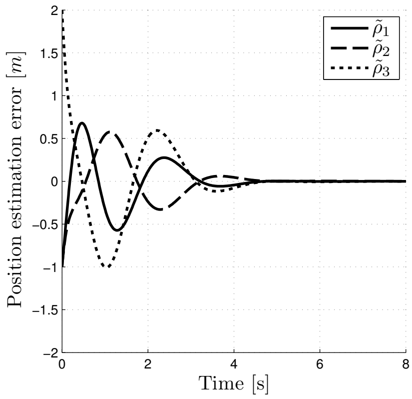

Figure 7: Position estimation error of the follower spacecraft

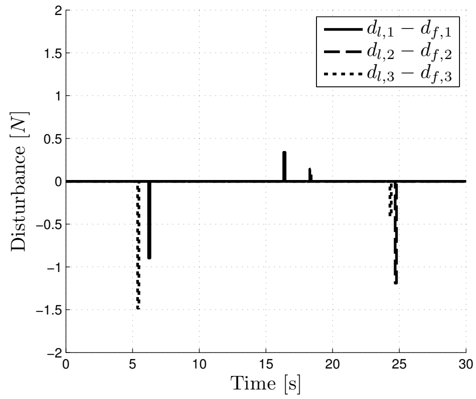

Figure 9: Disturbances acting on the follower spacecraft

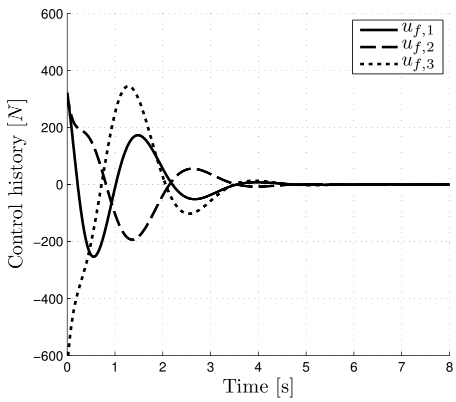

Figure 8: Control actuation of the leader spacecraft

4 Proof of Theorem 1

In view of (Praly and Wang, 1996, Lemma 11) and (Angeli et al., 2000, Remark 2.4), there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ and γ, and a positive constant κ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{30}$$

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\kappa V(x) + \gamma(|u|)\,. \tag{31}$$

Let $w(t) := V(x(t;x_0,u))$. Then it holds in view of (31) that

$$\begin{array}{rcl} \dot{w}(t) & = & \dot{V}(x(t;x_0,u)) \\ & \leq & -\kappa V(x(t;x_0,u)) + \gamma(|u(t)|) \\ & \leq & -\kappa w(t) + \gamma(|u(t)|)\,. \end{array}$$

In particular, it holds that, for all $t \geq 0$,

$$w(t) \leq w(0)e^{-\kappa t} + \int_0^t \gamma(|u(s)|)ds\,. \tag{32}$$

Assuming that u belongs to the class $\mathcal{W}_\gamma(E,T)$, for some arbitrary constants $E, T > 0$, it follows that

$$w(T) \leq w(0)e^{-\kappa T} + \int_0^T \gamma(|u(s)|)ds \leq w(0)e^{-\kappa T} + E\,.$$

Considering this inequality recursively, it follows that, for each $\ell \in \mathbb{N}_{\geq 1}$,

$$\begin{array}{rcl} w(\ell T) & \leq & w(0)e^{-\ell\kappa T} + E\displaystyle\sum_{j=0}^{k-1} e^{-j\kappa T} \\ & \leq & w(0)e^{-\ell\kappa T} + E\displaystyle\sum_{j\geq 0} e^{-j\kappa T} \\ & \leq & w(0)e^{-\ell\kappa T} + E\dfrac{e^{\kappa T}}{e^{\kappa T}-1}\,. \end{array} \tag{33}$$

Given any $t \geq 0$, pick ℓ as $\lfloor t/T \rfloor$ and define $t' := t - \ell T$. Note that $t' \in [0,T]$. It follows from (32) that

$$w(t) \leq w(\ell T)e^{-\kappa t'} + \int_{\ell T}^t \gamma(|u(s)|)ds \leq w(\ell T)e^{-\kappa t'} + E\,,$$

which, in view of (33), implies that

$$\begin{array}{rcl} w(t) & \leq & \left(w(0)e^{-\ell\kappa T} + E\dfrac{e^{\kappa T}}{e^{\kappa T}-1} \right) e^{-t'} + E \\ & \leq & w(0)e^{-k(\ell T + t')} + E\left(1 + \dfrac{e^{\kappa T}}{e^{\kappa T}-1}\right) \\ & \leq & w(0)e^{-\kappa t} + \dfrac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,. \end{array}$$

Recalling that $w(t) = V(x(t;x_0,u))$, it follows that

$$V(x(t;x_0,u)) \leq V(x_0)e^{-\kappa t} + \frac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,,$$

which implies, in view of (30), that

$$\underline{\alpha}(|x(t;x_0,u)|) \leq \overline{\alpha}(|x_0|)e^{-\kappa t} + \frac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,,$$

Recalling that $\underline{\alpha}^{-1}(a+b) \le \underline{\alpha}^{-1}(2a) + \underline{\alpha}^{-1}(2b)$ as $\underline{\alpha} \in \mathcal{K}_\infty$, we finally obtain that, given any $x_0 \in \mathbb{R}^n$, any $u \in \mathcal{W}_\gamma(E,T)$ and any $t \ge 0$,

$$|x(t;x_0,u)| \le \underline{\alpha}^{-1}\left(2\overline{\alpha}(|x_0|)e^{-\kappa t}\right) + \underline{\alpha}^{-1}\left(2E\frac{2e^{\kappa T}-1}{e^{\kappa T}-1}\right). \tag{34}$$

Given any $T, \delta \ge 0$, the following choice of E:

$$E(T,\delta) \le \frac{\underline{\alpha}(\delta)}{2}\frac{e^{\kappa T}-1}{2e^{\kappa T}-1}. \tag{35}$$

ensures that

$$\underline{\alpha}^{-1}\left(2E\frac{2e^{\kappa T}-1}{e^{\kappa T}-1}\right) \le \delta$$

and the conclusion follows in view of (34) with the $\mathcal{KL}$ function

$$\beta(s,t) := \underline{\alpha}^{-1}\left(2\overline{\alpha}(s)e^{-\kappa t}\right), \quad \forall s,t \ge 0.$$

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