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***Abstract*—We consider the uplink mobile association game with a large number of mobile terminals. Traditional techniques consider the discrete modelization but these models lead to high combinatorial complexities.**

Thanks to optimal transport theory we are able to characterize the cell formation as well as the equilibrium properties of the network where intelligent mobile terminals decide by themselves to which base station upload their information.

We determine closed form expressions for the cell formation and we illustrate numerically these cell formations in the one- and two-dimensional cases.

I. Introduction

Future wireless networks will be composed by intelligent mobile terminals capable of accessing multiple radio access technologies and capable of deciding by themselves the wireless access technology to use and the access point to which to connect. Within this context, we study the uplink mobile association game where we determine the cells corresponding to each base station, or the locations at which intelligent mobile terminals prefer to connect to a given base station rather than to others. We consider that these capabilities should be taken into account in the design and strategic planning of wireless networks.

Starting from the seminal paper of Hotelling [9] a large area of research on location games has been developed. His paper [9] introduced the notion of spatial competition in a duopoly situation. Plastria [13] presented an overview of the research on locating one or more new facilities in an environment where competing facilities already exist. Gabszewicz and Thisse [8] provided another general overview on location games. Altman et al. [1] studied the duopoly situation in the uplink case in a line segment and realized that with the particular cost structure that arises in the cellular context more complex cells are obtained at equilibrium. The authors of [14] and [15] examined the downlink mobile association problem under different policies. Our work focuses on the uplink case and in a more general situation without any assumption on the symmetry of the location of the users. We solve this problem for the one- and two-dimensional case. Traditional techniques have been focus on the discrete modeling of these networks (where it is assumed that the location of the users is known and deterministic). The main problem with these techniques is the highly combinatorial complexity leading to the curse of dimensionality problems [16]. The discrete modeling approach is commonly adopted for the detailed planning and analysis of the network. The continuum modeling approach is used for the initial phase of planning and modeling in broad-scale regional studies. In this setting, the focus is on the general trend and pattern of the distribution of the location of the users (which can be given in an hour-by-hour or day-by-day situation or other temporal pattern). The continuum modeling approach has many advantages over the discrete approach in macroscopic studies of dense networks. First, it reduces the problem size for dense networks. The problem size in the continuum model depends only on the method that is adopted to approximate the modeled region, but not on the actual network itself. This reduction in problem size saves computational time and memory. Second, less data is required to model the set-up in a continuum model. As continuum modeling can be characterized by a small number of spatial variables, it can be set-up with a much smaller amount of data than the discrete modeling approach, which requires data for all of the included links. This makes the continuum model convenient for macroscopic studies in the initial phase of design since the collection of data in this phase is time consuming and labor intensive, and the resources to undertake it are generally not available, which means there is usually insufficient data on the system to set up a detailed model. Finally, the continuum modeling approach gives a better understanding of the global characteristics of a network.

Within this context, we propose a new framework for the mobile association problem using optimal transport theory (See [17] and references therein), a theory initiated by Monge [12] and Kantorovich [10] that has prove to be useful on many economical contexts [3], [5], [4]. It is a stylized model but it contains important dependences, such as the number of mobiles associated to each base station and on the distance between the mobile terminal and the associated base station. The continuation of this work should include many other important concepts in order to be applied in a real network, such as call dropping and call blocking probabilities.

The remaining of this paper is organized as follows. Section II presents the formulation of the uplink mobile association problem. In Section III we explain some basic concepts and properties in optimal transport theory. In Section IV we

TABLE I
NOTATION

N	Total number of MTs in the network
K	Total number of BSs
f	Deployment distribution of MTs
(x_i, y_i)	Position of the i-th BS
C_i	Cell determined by the i-th BS
N_i	Number of MTs associated to the i-th BS
M_i	Number of carriers offered by the i-th BS
κ_i	Penalization function of non-service
h_i	Channel gain function over the i-th cell
ξ_i	Path loss exponent over the i-th cell

give some numerical examples with uniform and non-uniform distribution of the users and both one- and two-dimensional settings. Section V concludes this work.

II. The Model

To the reader convenience, we summarize the notation used on our work in Table I.

Consider a grid area network D on the two-dimensional plane which will be the geographical reference of the network. The mobile terminals (MTs) are distributed according to an integrable distribution $f(x,y)$ scaled in the grid area network D so that $\iint_{\mathrm{D}} f(x,y)\,dx\,dy = 1$. Then, the number of MTs in some part of the network area $A \subseteq D$, denoted by $N(A)$ will be

$$N(A) = N\left(\iint_A f(x,y)\,dx\,dy\right)$$

where N is the total number of MTs in the network. The integral on the right hand side between brackets takes into account the proportion of MTs distributed on the area A. We assume that in this grid area network there are K base stations (BSs): $\mathrm{BS}_1, \mathrm{BS}_2, \ldots, \mathrm{BS}_K$ located at positions $(x_1,y_1),(x_2,y_2)\ldots,(x_K,y_K)$.

We consider that when a MT, located at position (x,y), transmits to a BS_i, it uses power $P_i(x,y)$. Each MT is going to select only one BS according to some policy to be defined. We denote C_i the cell associated to BS_i that we want to determine. We denote by N_i the quantity of mobiles that are assigned to base station BS_i. If the quantity of mobiles is greater than some constant, denoted M (for example, the maximum number of possible carriers in Wi-Max is around 2048, so in this case we can consider $\mathrm{M} = 2048$) then we consider a penalization cost function

given by $$\kappa_i(N_i) = \begin{cases} 0 & \text{if } N_i \le M, \\ \bar{\kappa}_i(N_i - M) & \text{if } N_i > M. \end{cases}$$

We assume that $\bar{\kappa}_i$ is a continuous and non-decreasing function. We analyze the case $N_i \le M$ but thanks to the problem (P1) described and solved in the following, we can directly generalize this result by considering $s_i := \kappa_i$. As each cell C_i of the network contain a large number of MT continuously distributed with a distribution of $f(x,y)$ then the quantity of mobiles assigned to base station BS_i will be $N_i = N\iint_{C_i} f(x,y)\,dx\,dy$. Notice that $\sum_{i=1}^K N_i = N$ so each MT is associated to one and only one BS in the network. The power received at the BS_i from a MT located at position (x,y) is given by $P_i(x,y)h_i(x,y)$ where $h_i(x,y)$ is the channel gain. We shall further assume that it corresponds to the path loss given by $h_i(x,y) = (\sqrt{R^2 + d_i(x,y)^2})^{-\xi}$ where ξ is the path loss exponent, R is the height of the base station, and $d_i(x,y)$ is the Euclidean distance between a MT at position (x,y) and BS_i located at (x_i,y_i), *i.e.*, $d_i(x,y) = \sqrt{(x_i - x)^2 + (y_i - y)^2}$.

In the one-dimensional case, Altman *et al.* [1] derived that the SINR density of a MT located at x transmitting to base station BS_i located at y is given by

$$\mathrm{SINR}_i(x) = \frac{(\sqrt{R^2 + (y-x)^2})^{-\xi}}{\int_D (\sqrt{R^2 + (y-z)^2})^{-\xi}\,dz + \sigma^2}.$$

where ξ is the path loss exponent and σ is a noise parameter. In this case, the authors of [1] considered the specific case where there is a uniform distribution of the MTs and a constant power. We generalize their setting by considering a general deployment distribution of mobile terminals $f(x)$ and a general power distribution which may depend on the position x, given by $P_i(x)$. Then the problem reads

$$\mathrm{SINR}_i(x) = \frac{P_i(x)(\sqrt{R^2 + (y-x)^2})^{-\xi}}{\int_D P_i(z)(\sqrt{R^2 + (y-z)^2})^{-\xi} f(z)\,dz + \sigma^2}.$$

Following their derivation, this can be generalized for the two-dimensional case

$$\mathrm{SINR}_i(x,y) = \frac{P_i(x,y)(\sqrt{R^2 + d_i(x,y)^2})^{-\xi}}{P_{\mathrm{total}} + \sigma^2},$$

where $d_i(x,y)$ is the distance between BS_i and a MT located at (x,y), and

$$P_{\mathrm{total}} := \iint_D P_i(x,y)(\sqrt{R^2 + d_i(x,y)^2})^{-\xi} f(x,y)\,dx\,dy.$$

We want to guarantee an average SINR of $\Theta(x,y)$ to a MT located at position (x,y). This condition is written as

$$\frac{P_i(x,y)(\sqrt{R^2 + d_i(x,y)^2})^{-\xi}}{P_{\mathrm{total}} + \sigma^2} \ge \Theta(x,y).$$

Then as the constraint will be reached it follows that

$$P_i(x,y) = \Theta(x,y)(P_{\mathrm{total}} + \sigma^2)(\sqrt{R^2 + d_i(x,y)^2})^{+\xi}.$$

And then our problem reads

$$\operatorname*{Min}_{C_i} \sum_{i=1}^K \iint_{C_i} P_i(x,y) f(x,y)\,dx\,dy$$

We denote this problem as (UL) and replacing the power is written as

$$\operatorname*{Min}_{C_i} \sum_{i=1}^K \iint_{C_i} \Theta(x,y)(P_{\mathrm{total}} + \sigma^2)(\sqrt{R^2 + d_i(x,y)^2})^{+\xi} f(x,y)\,dx\,dy$$

which is similar to an optimal transport problem.

In order to solve the problem of the uplink case (UL) we will make use of Optimal Transport Theory, a theory that has prove to be useful on many economical context [3], [5], [4], as well as in the road traffic community [6], and the telecommunication community [15].

III. Basics in Optimal Transport Theory

The theory of mass transportation, also called optimal transport theory, goes back to the original works by Monge in 1781 [12], and later in 1942 by Kantorovich [10].

The work by Brenier [2] has renewed the interest for the subject and since then many works have been done in this topic (see [17] and references therein).

The original problem of Monge can be interpreted as the question:

> "How do you best move given piles of sand to fill up given holes of the same total volume?".

In our setting, this problem is of main importance. Suppose that mobile terminals are sending information to base stations in a grid area network and positions of mobile terminals and base stations are given.

> What is the "best move" of information from the MTs to the BSs?

Both questions share similarities as we will see.

The general mathematical framework to deal with this problem is a little technical but we encourage to jump the details and to focus on the main ideas.

The framework is the following:

We first consider a grid area network D in the one-dimensional case. As an example, the function $f(t)$ will represent the proportion of information sent by mobile terminals

$$d\mu(t) := f(t)\, dt.$$

The function $g(s)$ will represent the proportion of information received by a base station at location s

$$d\nu(s) := g(s)\, ds.$$

The function T (called transport map) is the function that transfers information from location s to location t. It assigns mobile terminals to base stations and transport information from mobile terminals to base stations. Then the conditions that each mobile terminal satisfies its uplink demand is written

$$\int_A g(y)\, dy = \int_{\{x\,:\,T(x)\in A\}} f(x)\, dx$$

for all continuous function F, where X is the support[1] of function f and we denote this condition (following the optimal transport theory notation) as

$$T\#\mu = \nu.$$

which is an equation of conservation of the information. Notice that, in communication systems there exists packet loss so in general this constraint may not be satisfied, but considering an estimation of the packet loss by sending standard packets test, this constraint can be modified in the reception measure μ. If we can not obtain a good estimation of this reception measure, we can consider it in its current form as a conservative policy.

In the original problem, Monge considered that the cost of moving a commodity from position x to a position y depends on the distance $c(|x-y|)$. Then the cost of moving a commodity from position x through T to its new position $T(x)$ will be $c(|x-T(x)|)$. For the global optimization problem, we consider the additive total cost over the network, which in the continuum setting will be given by

$$\mathrm{Min} \int_{\mathrm{D}} c(|x-T(x)|)\, f(x)\, dx \quad \text{such that} \quad T\#\mu = \nu.$$

where μ and ν are probability measures and $T : \mathrm{D} \to \mathrm{D}$ is an integrable function. This problem is known as Monge's problem in optimal transport theory.

The main difficulty in solving Monge's problem is the highly non-linear structure of the objective function. For examples on the limitations on Monge's modelization, see [15]. We pointed out the limitations of Monge's problem that motivated Kantorovich to consider another modeling of this problem in [10].

Kantorovich noticed that the problem of transportation from one location to another can be seen as "graphs of functions" (called transport plans) in the product space (See Fig. 1).

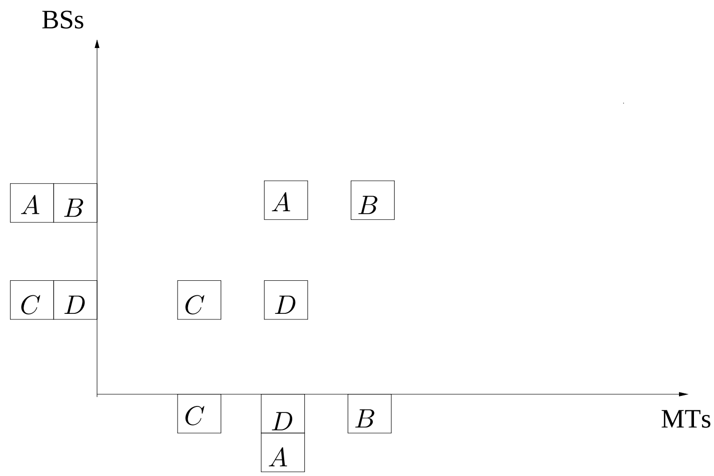

Fig. 1. Kantorovich considered "graphs" where the projection in the first axis coincide with the mobile terminal position ($\mathrm{MT}_1 = 3.5$, $\mathrm{MT}_2 = 5$ and $\mathrm{MT}_3 = 6.5$) and the second axis coincides with the base station position ($\mathrm{BS}_1 = 4$ and $\mathrm{BS}_2 = 6$).

The idea is to minimize over the space of graphs g. $g = (\mathrm{Id} \times T)$ in the product space. Then with the condition that each mobile terminal satisfies its uplink demand and that the information is received at the base stations, Kantorovich's problem reads

$$\underset{g\in\Pi(\mu,\nu)}{\mathrm{Min}} \iint_{\mathrm{D}\times\mathrm{D}} c(x,y)\, dg(x,y)$$

where

$$\Pi(\mu,\nu) = \{g \,:\, \pi_1\#g = \mu \text{ and } \pi_2\#g = \nu\}$$

[1]The support of a function f is the closure of the set of points where the function is not zero, *i.e.*, support $(f) = \overline{\{t \,:\, f(t) \neq 0\}}$

is denoted the ensemble of transport plans g, $\pi_1(x, y)$ stands for the projection on the first axis x, and $\pi_2(x, y)$ stands for the projection on the second axis y.

The relationship between Monge and Kantorovich problems is that every transport map T of Monge's problem determines a transport plan $g_T = (\mathrm{Id} \times T)\#\mu$ in Kantorovich's problem with the same cost (where Id denotes the identity). However, Kantorovich's problem considers more functions than the ones coming from Monge's problem (which can always be viewed as the product of the identity and the map T), so we can choose from a bigger set $\Pi(\mu, \nu)$.

Then every solution of Kantorovich's problem is a lower bound to Monge's problem, *i.e.*,

$$\underset{g \in \Pi(\mu,\nu)}{\mathrm{Min}} \iint_{\mathrm{D}\times\mathrm{D}} c(x, y)\, dg(x, y) \leq$$
$$\leq \underset{T\#\mu=\nu}{\mathrm{Min}} \int_{\mathrm{D}} c(|x - T(x)|)\, f(x)\, dx.$$

Theorem Consider the cost function $c(|x - y|) = |x - y|^p$. Let μ and ν be probability measures in D and fix $p \geq 1$. We assume that μ can be written[2] as $d\mu = f(x)\, dx$. Then the optimal value of Monge's problem coincides with the optimal value of Kantorovich's problem, *i.e.*, $M_p(\mu, \nu) = W_p(\mu, \nu)$ and there exists an optimal transport map from μ to ν, which is also unique almost everywhere if $p > 1$.

This result is very difficult to obtain and it has been proved only recently (see [2] for the case $p = 2$, and the references at [17] for the other cases).

The case that we are interested in can be characterized because the image of the transport plan is a discrete finite set.

Thanks to optimal transport theory we are able to characterize the partitions on very general settings. For doing so, consider locations $(x_1, y_1)\dots,(x_K, y_K)$, the Euclidean distance $d_i(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2}$, and F a continuous function.

Theorem Consider the problem $(\mathrm{P1})$

$$\underset{C_i}{\mathrm{Min}} \sum_{i=1}^{K} \iint_{C_i} \left[F(d_i(x, y)) + s_i \left(\iint_{C_i} f(\omega, z)\, d\omega\, dz \right) \right] f(x, y)\, dx\, dy,$$

where C_i is the cell partition of D. Suppose that s_i are continuously differentiable, non-decreasing, and convex functions. The problem $(\mathrm{P1})$ admits a solution that verifies

$$(\mathrm{S1}) \begin{cases} C_i = & \{x : F(d_i(x, y)) + s_i(N_i) + N_i \cdot s_i'(N_i) \\ & \leq F(d_j(x, y)) + s_j(N_j) + N_j \cdot s_j'(N_j)\} \\ N_i = & \iint_{C_i} f(\omega, z)\, d\omega\, dz. \end{cases}$$

Theorem Consider the problem $(\mathrm{P2})$

$$\underset{C_i}{\mathrm{Min}} \sum_{i=1}^{K} \iint_{C_i} \left[F(d_i(x, y)) \cdot m_i \left(\iint_{C_i} f(\omega, z)\, d\omega\, dz \right) \right] f(x, y)\, dx\, dy,$$

[2]The exact condition is that μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$ where d is the dimension of the space.

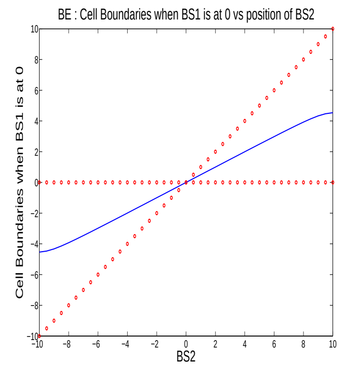

Fig. 2. 2 BSs

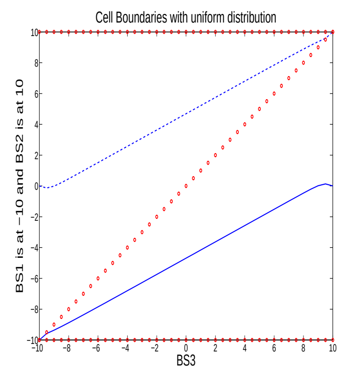

Fig. 3. 3 BSs

where C_i is the cell partition of D. Suppose that m_i are derivable. The problem $(\mathrm{P2})$ admits a solution that verifies

$$(\mathrm{S2}) \begin{cases} C_i = & \{x : m_i(N_i)F(d_i(x, y))\, f(x, y) + U_i(x, y) \\ & \leq m_j(N_j)F(d_j(x, y))\, f(x, y) + U_j(x, y)\} \\ U_i = & m_i'(N_i) \iint_{C_i} F(d_i(x, y)) f(x, y)\, dx\, dy \\ N_i = & \iint_{C_i} f(\omega, z)\, d\omega\, dz. \end{cases}$$

Notice that in problem $(\mathrm{P1})$ if the functions $s_i \equiv 0$ the solution of the system $(\mathrm{S1})$ becomes the well known Voronoi cells. In problem $(\mathrm{P2})$ if we have that the functions $h_i \equiv 1$ we find again the Voronoi cells. However in all the other cases the Voronoi configuration is not optimal. The proofs of both theorems are given in [14].

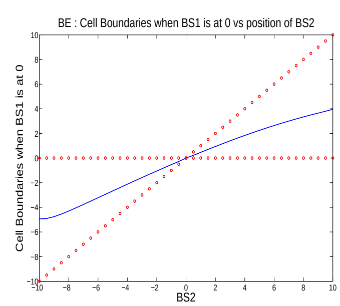

Fig. 4. Cell Boundaries in Non-Uniform distribution

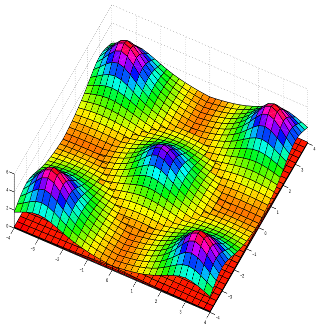

Fig. 5. Uniform distribution of users

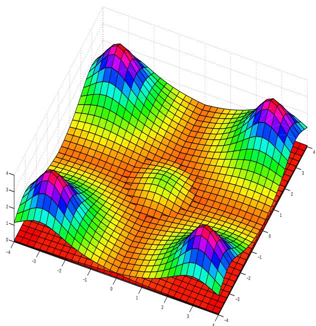

Fig. 6. Non-uniform distribution of users

IV. Validation of our theoretical model

A. One-dimensional case

We first consider the one-dimensional case and we consider a uniform distribution of users in the interval $[-10, 10]$. We set the noise parameter $\sigma = 0.3$. In Fig. 2, we fix one base station BS_2 at position 0 and take as parameter the position of base station BS_1. We consider the path loss exponent of $\xi = 2$. Red lines shows the positions of the BSs. We are able to determine the cell boundary (solid blue curve) from BS_1 and BS_2 at different positions. In Fig. 3 we fix two base stations $\mathrm{BS}_1 = -10$ and $\mathrm{BS}_2 = 10$ and we take as parameter the position of base station BS_3. Red lines shows the positions of the BSs. We determine the cell boundary (solid blue curve) from BS_1 and BS_3 and the cell boundary (dashed blue curve) from BS_2 and BS_3.

B. Two-dimensional case: Uniform and Non-Uniform distribution of users

We consider the two-dimensional case. We consider the square $[-4, 4] \times [-4, 4]$ and the noise parameter $\sigma = 0.3$. We set five base stations at positions $\mathrm{BS}_1 = (-3, -3)$, $\mathrm{BS}_2 = (3, -3)$, $\mathrm{BS}_3 = (-3, 3)$, $\mathrm{BS}_4 = (3, 3)$, and $\mathrm{BS}_5 = (0, 0)$. We determine the cell boundaries for the uniform distribution of MTs (see Fig. 5) and we compare it to the cell boundaries for the non-uniform distribution of MTs given by $f(x, y) = (L^2 - (x^2 + y^2))/K$ where K is a normalization factor. The latter situation can be interpreted as the situation when mobile terminals are more concentrated in the center and less concentrated in suburban areas as in Paris, New York or London. We observe that the cell size of the base station BS_5 at the center is smaller than the others at the suburban areas. This can be explained by the fact that as the density of users is more concentrated in the center the interference is greater in the center than in the suburban areas and then the SINR is smaller in the center. However the quantity of users is greater than in the suburban areas.

V. Conclusions

We have studied a uplink mobile association game. We determined the location at which intelligent mobile terminals prefer to connect to a given base station rather than to others. Thanks to our proposed approach using optimal transport theory for mobile association we are able to completely characterize the mobile association and the cell formation under different policies from the mobile terminals point of view and as well as from the global system point of view.

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