Exploiting Packet Size in Uncertain Nonlinear Networked Control Systems

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Abstract

This paper addresses the problem of stabilizing uncertain nonlinear plants over a shared limited-bandwidth packet-switching network. While conventional control loops are designed to work with circuit-switching networks, where dedicated communication channels provide almost constant bit rate and delay, many networks, such as Ethernet, organize data transmission in packets, carrying larger amount of information at less predictable rates. To avoid the bandwidth waste due to the relatively large overhead inherent to packet transmission, we exploit the packet payload to carry longer control sequences. To this aim we adopt a model-based approach to remotely compute a predictive control signal on a suitable time horizon, which leads to effectively reducing the bandwidth required to guarantee stability. We consider networks for which both the time between consecutive accesses to each node (MATI) and the transmission and processing delays (MAD) for measurements and control packets are bounded. Communications are assumed to be ruled by a rather general protocol model, which encompasses many protocols used in practice. As a distinct improvement over the state of the art, our result is shown to be robust with respect to sector-bounded uncertainties in the plant model. Namely, an explicit bound on the combined effects of MATI and MAD is provided as a function of the basin of attraction and the model accuracy. A case study is presented to appreciate the improvements induced by the packet-based control strategy over existing methods.

1 Introduction

Industrial manufacturing is witnessing an ever more extensive use of communication networks to support automated scheduling, control and diagnostic activities [12], [27]. The possibility offered by networks of replacing traditional point-to-point connections with more complex and dynamic schemes, opens unprecedented opportunities for factory control and management. Alongside allowing a pervasive adoption of decentralization and cooperation, networks convey many advantages in terms of flexibility, scalability and robustness. The adoption of a distributed networked architecture can induce a remarkable reduction of costs and delays for both installation and maintenance. These advantages justify the increasing interest in control over networks (see for instance [4], [1], [2], [6], [24]).

In general terms, a Networked Control System (NCS) is a system in which sensors, actuators and controllers are spatially distributed and exchange information through a shared, digital, finite capacity channel. The use of the network as a communication medium and the distributed nature of the system make traditional control theory not always applicable. Issues such as quantization errors, data dropouts, variable transmission intervals, variable communication delays, and constrained access to the network, can no longer be ignored [8]. The NCS literature has separately addressed many of these problems, and sometimes the combinations thereof. An excellent discussion of the state-of-the-art is reported in [7], and the reader is referred there for a detailed analysis of the literature towards the mentioned communication constraints. An essential aspect of NCS, not considered in detail in [7], is the packet-switching nature of many networks. As opposed to conventional control loops, which are designed to work with circuit-switching networks where dedicated communication channels provide almost constant bit rate and delay, networks such

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as Ethernet organize data transmission in packets, carrying larger amount of information at less predictable rates.

The organization of control information in data packets, which have relatively large transmission overhead, substantially alter the bandwidth/performance trade-off of traditional design. For instance, important data-rate theorems [9], [14], [15] expressing a fundamental relationship between the degree of instability of a given physical system and the minimum bit rate required to stabilize it, do not account for the fact that data come in packets with a minimum size (e.g. 84 bytes in Ethernet). To simplify, transmitting a 16 bits record every millisecond requires as much bandwidth in average as sending a packet of 84 bytes every 48 milliseconds; however, the implications on the effective sampling rate and feedback control performance are apparent. How to recover part of this performance is an objective of this study.

A second aspect inherent to packet-switching networks is transmission overhead. For instance, every Ethernet packet carries 38 bytes of headers and interframe separations, and useless information is necessarily padded into the payload to reach the minimum required packet length. As a consequence, transmitting a few bits per packet has essentially the same bandwidth cost as transmitting hundreds of them. A new, specific trade-off hence arises between packet rate and packet dimension for a given estimation/control task.

While the above aspects have been observed and described in the early literature on NCS (see e.g. the surveys [22], [10], [8]), only recently have appeared results which address them explicitly in controller design. The goal can be succinctly described as to decrease the network utilization (in terms of bandwidth, or packets per unit of time) without compromising control performance. To achieve this, [11] pioneered the idea of exploiting the empty portion of packet payload to carry feedforward control sequences, computed in advance on the basis of a model-based scheme. Following developments along these lines generalized the technique to address nonlinear systems [19], time-varying delays and packet dropouts [18], [17], as well as the constraints imposed by communication protocols on state measurement access [5].

In this paper we also adopt the feedforward approach to send in a packet not only the control value to be applied at a specific instant, but also a prediction of the control law valid on a given time-horizon, so as to better exploit the payload. In the same spirit of other model-based approaches (e.g. [11], [19], [18], [17]), the control sequence is obtained by simulating an (imprecise) model of the closed-loop plant. The internal state of the model is asynchronously updated by means of the measurements of the plant state provided by sensors. Due to their spatial distribution, only portions of the model state can be updated in each instant. Therefore, we consider the constrained access to the network to be ruled by a protocol deciding which sensor can communicate at each instant. The large control-packet, sent by the remote controller, is stored in an embedded memory on the plant side. Based on a local re-synchronization, made possible by a time-stamping of measurements, this strategy also allows to compensate the effect of bounded communication delays in the control loop. We build our model upon the powerful hybrid formalism introduced in [16], and we consider network imperfections affecting both sides of the control loop. We provide explicit bounds on the Maximum Allowable Delay (MAD [7]) and on the Maximum Allowable Transfer Interval (MATI [23], i.e. the maximum duration between two successive communications) ensuring the semiglobal exponential stability of the NCS.

The main contribution of this paper is a control strategy for packet-switching networks ensuring the stability of an uncertain nonlinear NCS affected by varying transmission intervals, varying (and potentially large) delays, and constrained access to the network. Unlike the commonly assumed small-delays hypothesis (see for instance [7]), we can compensate for delays larger than the transmission interval. It should be noticed that the authors of [7] consider the development of a model for NCSs accounting for variable transmission intervals, potentially large and varying delays and constrained network access, as a hard problem.

A line of work close to ours is reported in [18], where the problem of stabilizing a nonlinear NCS with feedforward control sequences is addressed. Such sequences are computed by means of an approximate discrete-time plant model. Authors assume that the approximation algorithm is the only source of uncertainty in the model and that the inaccuracy of such a model can be reduced at will in order to achieve the desired MATI. In this paper, instead, we consider a robustness problem, where the plant uncertainty is a given, and we provide a bound on the MATI in terms of the model inaccuracy (measured through its local Lipschitz constant).

Preliminary results concerning our approach were presented in [5]. The present paper extends that in at least three relevant aspects. We consider here uniformly global exponentially stable (UGES) protocols, rather than the conservative class of *invariably* UGES protocols (which do not include, for instance, the very common Round Robin protocol) assumed in [5]. We significantly extend the class of nonlinear plants by imposing only *local* Lipschitz conditions, instead of global ones in [5]. Finally, we take directly into account, in the computation of the MATI, the accuracy of the model used to build the prediction.

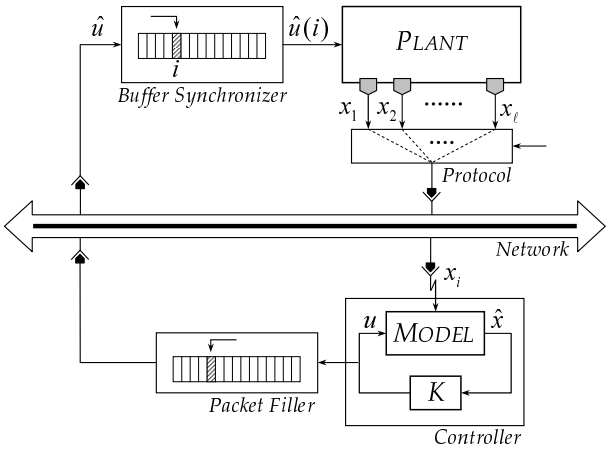

Fig. 1. Networked Control System with packet-switching network and protocol.

2 Problem Statement

Notation: Given a set $A \subset \mathbb{R}$ and $a \in A$, $A_{\geq a}$ denotes the set $\{s \in A \mid s \geq a\}$. Given a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $n \in \mathbb{N}_{\geq 1}$, $|x|$ denotes its Euclidean norm, i.e. $|x| \triangleq \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. Given $R \geq 0$, B_R denotes the closed ball of radius R centered in zero: $B_R \triangleq \{x \in \mathbb{R}^n \mid |x| \leq R\}$. Given a locally essentially bounded signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\|u\|_{\mathcal{L}_\infty} \triangleq \operatorname{ess\,sup}_{t \geq 0} |u(t)|$. We use mod to denote the modulo operator, i.e. given $m, n \in \mathbb{N}$, $m \bmod n = p$ if and only if there exists $r \in \mathbb{N}$ such that $m = rn + p$ with $p < n$. We define the floor function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ as $\lfloor x \rfloor \triangleq \max\{m \in \mathbb{Z} \mid m \leq x\}$.

2.1 Network Model

We consider a NCS constituted of a remote controller receiving measurements from and sending commands to a physical plant through a shared communication channel (see Figure 1). Control sequences are sent over the digital network as packets. An elementary embedded control device receives, decodes, synchronizes these packets and applies control commands to the plant. Measurements are taken by physically distributed sensors and sent towards the controller as packets encoded with sufficient precision to neglect quantization effects. Sensors are assumed to be embedded with the plant and hence synchronized with it. Due to the distributed nature of the sensors, we also assume that the measurement part of the network is partitioned in ℓ nodes and only a *unique* node at a time can send its information (i.e. only partial knowledge of the plant state is available at each time instants).

We consider that measurements are taken and sent at instants $\{\tau_i^m\}_{i \in \mathbb{N}}$, and are received by the remote controller at instants $\{\tau_i^m + T_i^m\}_{i \in \mathbb{N}}$. In other words, $\{T_i^m\}_{i \in \mathbb{N}}$ denote the (possibly time-varying) measurement data delays, which cover both processing and transmission delays on the measurement chain. In the same way, control commands are computed, encoded into packets and sent over the network at time instants $\{\tau_j^c\}_{j \in \mathbb{N}}$. They reach the plant at instants $\{\tau_j^c + T_j^c\}_{j \in \mathbb{N}}$, where $\{T_j^c\}_{j \in \mathbb{N}}$ denote the (possibly time-varying) control data delays accounting both for computation and transmission delays from the remote controller to the plant.

Assumption 1 (Network) *The communication network satisfies the following properties:*

i) (MATI) *There exist two constants $\tau_m, \tau_c \geq 0$ such that $\tau_{i+1}^m - \tau_i^m \leq \tau_m$ and $\tau_{j+1}^c - \tau_j^c \leq \tau_c$, $\forall i, j \in \mathbb{N}$;*

ii) (MAD) *There exist two constants $T_m, T_c \geq 0$ such that $T_i^m \leq T_m$ and $T_j^c \leq T_c$, $\forall i, j \in \mathbb{N}$;*

iii) (No Zeno phenomenon) *There exist constants $\varepsilon_m > 0$ and $\varepsilon_c > 0$ such that $\varepsilon_m \leq \tau_{i+1}^m - \tau_i^m$, $\forall i \in \mathbb{N}$ and $\varepsilon_c \leq \tau_{j+1}^c - \tau_j^c$, $\forall j \in \mathbb{N}$.*

Item i) in the previous assumptions imposes that the MATI between two consecutive accesses to the network is bounded both for measurements and control. Item ii) imposes that the MAD, both on measurements and control side, is bounded. Item iii) imposes that the minimum time interval between two consecutive accesses to the network by the nodes is lower bounded away from zero, and similarly for the control side. The objective of this paper is to provide explicit bounds on the MATIs (τ_m and τ_c) and on the MADs (T_m and T_c) to guarantee exponential stability of the closed-loop NCS based on a specific control procedure.

2.2 Protocol Model

The access to the network is ruled by a protocol choosing, at each instant τ_i^m, which node communicates its data. Decisions can be taken either according to the time index i (static protocol) or based on the value of the error e between the state estimate $\hat{x}$ and the available state measurements x from sensors (dynamic protocol). More precisely, in the spirit of [16], we model the network protocol as a time-varying discrete-time system involving the error $\mathbb{R}^n \ni e \triangleq \hat{x} - x$, $n \in \mathbb{N}_{\geq 1}$, that this type of communication generates:

$$e(i+1) = h(i, e(i)), \quad \forall i \in \mathbb{N}, \tag{1}$$

where $h : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$. If the network were able to send the measurement of the whole state at each time instant τ_i^m, then the function h would be identically zero; this is an assumption commonly posed in the literature on NCSs (see for instance [3], [26], [11], [20], [25], [13], [19], [18], [17]) where network effects are mostly modeled as sampling and delays. This assumption may no longer be justified when sensors are physically distributed.

Purely static protocols involve a function h which takes as an argument the time index i only. An example of such protocols is the Round Robin (RR) protocol, which executes a cyclic inspection of each node. On the opposite, some network protocols purely rely on the current value of the error, in which case h is independent of i: this is the case of the Try-Once-Discard (TOD) protocol [23]. The objective of most communication protocols is to decrease some function of the transmission error e at each transmitted packet. A particularly relevant class of such protocols is the one that ensures an exponential decay of this error. We recall here a slightly modified version of the definition in [16] to focus on the class of protocols we deal with in this work.

Assumption 2 (UGES Protocol) *The protocol modeled by the discrete-time system (1) is* uniformly globally exponentially stable (UGES) *and admits an associated Lyapunov function with bounded gradient. That is, there exist a function $W_0 : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ locally Lipschitz in the second argument, and constants $\underline{a}, \overline{a}, c > 0$ and $\rho_0 \in [0, 1)$ such that, for all $e \in \mathbb{R}^n$ and all $i \in \mathbb{N}$,*

$$\underline{a} \left| e \right| \leq W_0(i, e) \leq \overline{a} \left| e \right| \tag{2}$$

$$W_0(i + 1, h(i, e)) \leq \rho_0 W_0(i, e) \ , \tag{3}$$

and for almost all $e \in \mathbb{R}^n$ and all $i \in \mathbb{N}$

$$\left| \frac{\partial W_0}{\partial e}(i, e) \right| \leq c. \tag{4}$$

It is worth stressing that the UGES protocols considered here are not necessarily *invariably* UGES, as assumed in [5]. The latter property is rather restrictive, as it excludes, for instance, the commonly adopted Round Robin protocol.

Remark 1 *We do not explicitly consider packet dropouts here. However, its inclusion is possible without modifying the overall framework if some additional assumptions are made. Dropouts in a plant-to-controller channel governed by an invariably UGES protocol (see [5]) are easily dealt with by considering a scaled MATI, as proposed in Remark II.4 in [7]. It should be noticed however that the MATI scaling approach does not apply if an UGES, but not invariably UGES, protocol is considered. Bounded packet dropouts in the controller-to-plant channel can be tolerated in our framework if consecutive feedforward control packets overlap sufficiently (this will be more clear in the sequel).*

2.3 The plant and its model

We assume that a nominal feedback controller is given, which would be able, in the absence of the effects induced by the network, to globally exponentially stabilize the real plant. More precisely, we assume the following.

Assumption 3 (Nominal GES) *There exists a continuously differentiable function $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ such that the closed-loop system*

$$\dot{x} = f(x, u) \tag{5}$$
$$u = \kappa(x) \tag{6}$$

is globally exponentially stable (GES)*, so that there exists a differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and constants $\underline{\alpha}, \overline{\alpha}, \alpha, d > 0$ such that the following conditions hold for all $x \in \mathbb{R}^n$*

$$\underline{\alpha} \left| x \right|^2 \leq V(x) \leq \overline{\alpha} \left| x \right|^2$$

$$\frac{\partial V}{\partial x}(x) f(x, \kappa(x)) \leq -\alpha \left| x \right|^2$$

$$\left| \frac{\partial V}{\partial x}(x) \right| \leq d \left| x \right| .$$

In order to compute the control signal, the remote controller makes use of a state estimate based on an approximate model $\hat{f}$ of the plant f. Both the plant and its model are considered to be zero at the origin ($f(0, \kappa(0)) = \hat{f}(0, \kappa(0)) = 0$). The strategy developed in this paper relies on the assumption that the plant, its model and the nominal controller are all locally Lipschitz.

Assumption 4 (Local Lipschitz) *Given some constants $R_x, R_u > 0$, there exist some positive constants λ_f and λ_κ* [1] *such that for all $x_1, x_2 \in B_{R_x}$ and all $u_1, u_2 \in B_{R_u}$, the following inequalities hold*

$$\left| f(x_1, u_1) - f(x_2, u_2) \right| \leq \lambda_f \left(\left| x_1 - x_2 \right| + \left| u_1 - u_2 \right| \right) \tag{7}$$
$$\left| \kappa(x_1) - \kappa(x_2) \right| \leq \lambda_\kappa \left| x_1 - x_2 \right| . \tag{8}$$

It is worth noting that the previous assumption represents a further important relaxation with respect to [5], where all involved vector fields were assumed to be *globally* Lipschitz.

Finally, we assume that the plant model inaccuracy is sector-bounded.

Assumption 5 (Sector-Bounded Model Inaccuracy) *Given $R_x, R_u > 0$, there exists a nonnegative constant $\lambda_{f\hat{f}}$ such that for all $x \in B_{R_x}$ and all $u \in B_{R_u}$,*

$$\left| \hat{f}(x, u) - f(x, u) \right| \leq \lambda_{f\hat{f}} \left(\left| x \right| + \left| u \right| \right) . \tag{9}$$

[1] We stress that λ_κ can be chosen independently of R_u.

The constant $\lambda_{f\hat{f}}$ thus measures the model accuracy: the closer the model $\hat{f}$ is to the real system f, the smaller is $\lambda_{f\hat{f}}$ (in the ideal case of perfect modeling, it would be zero). Note that Assumption 5 allows to cope with both parametric uncertainties and unmodeled dynamics.

3 A model-based strategy

3.1 Modeling the overall setup

We develop here a model-based strategy exploiting the relatively large payload of a packet. At each reception of a new measurement, the remote controller updates an estimate of the current state of the plant and computes a prediction of the control signal over a fixed time horizon T_0^p by numerically running the model $\hat{f}$. This signal is then coded and sent in a single packet at the next network access. When received by the plant, it is decoded and re-synchronized by the embedded computer, based on the time-stamping of the original measurement. We assume here that the plant and its sensors have a common clock; however, we also stress that in our strategy there is no need for clock synchronization between the plant and the remote controller.

In order to guarantee that a relevant control signal is always available, the fixed time horizon on which each state prediction is achieved is chosen as

$$T_0^p \geq T_c + T_m + \tau_m + \tau_c. \tag{10}$$

This prediction horizon guarantees, in view of Assumption 1, that a control sequence corresponding to the present time is always loaded in the memory of the embedded controller.

For sake of mathematical rigor, we introduce first a model accounting for infinitely many state variables and infinitely many duplicates of the model $\hat{f}$. In Section 3.2, we show how to properly reduce them to a finite number. Therefore, for any measurement taken at $\tau_i^m,\ i \in \mathbb{N}$, we consider a new estimate state variable $\hat{x}_i$, valid over the time interval $[\tau_i^m, \tau_i^m + T_0^p]$, whose evolution is given by

$$\begin{aligned} \dot{\hat{x}}_i(t) &= \hat{f}(\hat{x}_i(t), \kappa(\hat{x}_i(t))), \qquad \forall t \in [\tau_i^m, \tau_i^m + T_0^p] \\ \hat{x}_i(\tau_i^{m+}) &= x(\tau_i^m) + h(i, \hat{x}_{i-1}(\tau_i^m) - x(\tau_i^m)). \end{aligned} \tag{11}$$

Each variable is updated at time τ_i^{m+} according to the protocol h. Usually, when dealing with a unique variable, the update of an estimate is performed by means of the error between the measurement and the variable itself. In our case, instead, a new estimate variable $\hat{x}_i$ is created at each τ_i^m, with the previous variable $\hat{x}_{i-1}$ containing the latest value of the estimate. Hence, the error we compute at time τ_i^m is between the measurement made on $x(\tau_i^m)$ and the previous estimate variable $\hat{x}_{i-1}(\tau_i^m)$. In this way all measurements are used to continuously update the internal model.

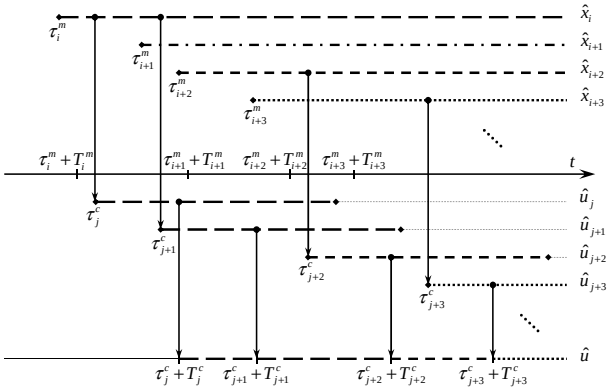

Fig. 2. Excerpt of an infinite sequence of estimate variables and control feedforward signals along with the control signal applied to the plant.

The infinite sequence of evolutions for the simulated dynamics (11) is schematically depicted at the top of Figure 2, above the time line. Each simulated evolution is represented by a straight line starting at times $\tau_i^m,\ i \in \mathbb{N}$ (explicitly reported at their left). Different line styles represent different evolutions for the estimate variables. The time line reports the instants $\tau_i^m + T_i^m,\ i \in \mathbb{N}$ at which the measurements $x(\tau_i^m)$ reach the controller. It is important to remark that the dynamics (11) actually evolves in a virtual (simulated) time. The measurement $x(\tau_i^m)$ reaches the controller only at $\tau_i^m + T_i^m$ and then triggers the simulation of the dynamics (11) for a virtual time interval $[\tau_i^m, \tau_i^m + T_0^p]$. The actual time spent for this simulation and for the computation of the predicted control signal is, in fact, part of the delay T_j^c. What we have done in (11), is to consider the estimate dynamics 'stretched' on the real time as if it ran concurrently with the plant. This notation trick allows us to cast the overall system in a compact model similar to the one in [16].

At each instant τ_j^c a new control signal $u_j(t)$ is computed. It is based on the estimate variable $\hat{x}_{\gamma(j)}$, where $\gamma(j)$ denotes the index of the latest measurement received before τ_j^c. More precisely, the function $\gamma : \mathbb{N} \to \mathbb{N}$ is defined as

$$\gamma(j) \triangleq \max\left\{ i \in \mathbb{N} \mid \tau_i^m + T_i^m < \tau_j^c \right\}, \quad \forall j \in \mathbb{N}.$$

It can be easily verified that, in view of Assumption 1, the time horizon for the control signal has to satisfy

$$T_0^c \geq T_c + \tau_c \tag{12}$$

in order to guarantee that a valid control signal is always available to the embedded controller. Note that the required time horizon T_0^c for the control signal is smaller than the time horizon T_0^p used for prediction, as it does not need to account for measurement MAD and MATI.

We thus define an infinite number of feedforward control signals as

$$\hat{u}_j(t) = \kappa(\hat{x}_{\gamma(j)}(t)), \qquad \forall t \in [\tau_j^c, \tau_j^c + T_0^c], \forall j \in \mathbb{N}.$$

At each reception of a new control packet (i.e. at instants $\tau_j^c + T_j^c$), the buffer of the embedded controller is updated. Consequently, the control signal applied to the plant is given by

$$\hat{u}(t) = \hat{u}_j(t), \qquad \forall t \in [\tau_j^c + T_j^c, \tau_{j+1}^c + T_{j+1}^c). \quad (13)$$

Both the feedforward signals $\hat{u}_j$ and the control $\hat{u}$ are depicted at the bottom of Figure 2. Line styles are consistent with those of the estimate evolutions used to build the control signals. Vertical arrows show which estimate variable $\hat{x}_{\gamma(j)}$ is chosen for the computation of the feedforward signal $\hat{u}_j$ at time instant τ_j^c, and which control signal $\hat{u}_j$ is used at $\tau_j^c + T_j^c$ to update the embedded controller. In the particular example of Figure 2, it can be noticed that $\hat{u}_j$ and $\hat{u}_{j+1}$ are computed with respect to the same estimate $\hat{x}_i$ since $\gamma(j) = \gamma(j+1) = i$. On the other hand, $\hat{x}_{i+1}$ is not directly used by any control since $\gamma(j+2) = i+2$.

3.2 A reduced NCS model

The model considered so far makes use of infinitely many state estimate variables $\hat{x}_i$ and control signals $\hat{u}_j$. They can be reduced to a finite number by noticing that they are all defined over compact time intervals and that "old" variables are no longer used after a while. State estimates variables are stored in a finite memory and new values are cyclically written on dismissed variables. We must prevent that a variable is accidentally reset while still in use for the computation of a control signal. In particular, $\hat{x}_{\gamma(j)}$ cannot be reset during the interval $[\tau_{\gamma(j)}^m, \tau_{j+1}^c + T_{j+1}^c]$. Hence, the dimension of such a memory, in terms of number of variables, is given by the maximum number of measurements that can be received during the life horizon T_0^p of an estimate variable. Recalling that T_0^p accounts also for the interval during which no measurements are received, whose length is bounded by τ_m, the dimension N of the memory is given by

$$N \triangleq \left\lfloor \frac{T_0^p - \tau_m}{\varepsilon_m} \right\rfloor + 1. \quad (14)$$

Therefore, we use only N state variables x_{c_r}, $r \in \{1, \ldots, N\}$, to store the state estimates. They are cyclically updated according to the following relation

$$x_{c_r}(t) \triangleq \hat{x}_i(t) \quad \text{iff } \eta(t, i, r) = 1,$$

where $\eta : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \{1, \ldots, N\} \to \{0, 1\}$ is the function defined as

$$\eta(t, i, r) \triangleq \begin{cases} 1 \text{ if } t \in (\tau_i^m, \tau_{i+1}^m] \text{ and } \mu(i) = r \\ 0 \text{ otherwise,} \end{cases} \quad (15)$$

which identifies the index of the relevant state estimate, and $\mu : \mathbb{N} \to \{1, \ldots, N\}$ is defined as

$$\mu(i) \triangleq ((i-1) \bmod N) + 1 \quad (16)$$

to make a cyclic update of the state estimates in the memory. By means of the vectors $\bar{x}, x_c, e \in \mathbb{R}^{Nn}$ defined as $\bar{x} \triangleq [x^T, \ldots, x^T]^T$, $x_c \triangleq [x_{c_1}^T, \ldots, x_{c_N}^T]^T$ and $e = [e_1^T, \ldots, e_N^T]^T \triangleq x_c - \bar{x}$, the closed-loop dynamics of the NCS can be compactly written as

$$\dot{x} = F(t, \bar{x}, e) \quad (17a)$$
$$\dot{e} = G(t, \bar{x}, e) \quad (17b)$$
$$e(\tau_i^{m+}) = H(i, e(\tau_i^m)), \quad (17c)$$

where

$$F(t, \bar{x}, e) = f(x, u(t, e + \bar{x})) \quad (18a)$$

$$G(t, \bar{x}, e) = \begin{bmatrix} \hat{f}(e_1 + x, \kappa(e_1 + x)) - f(x, u(t, e + \bar{x})) \\ \vdots \\ \hat{f}(e_N + x, \kappa(e_N + x)) - f(x, u(t, e + \bar{x})) \end{bmatrix} \quad (18b)$$

$$H(i, e) = \begin{bmatrix} e_1 + (h(i, e_N) - e_1)\, \eta(t, i, 1) \\ e_2 + (h(i, e_1) - e_2)\, \eta(t, i, 2) \\ \vdots \\ e_N + (h(i, e_{N-1}) - e_N)\, \eta(t, i, N) \end{bmatrix}. \quad (18c)$$

The control signal u in (18a) and (18b) is given by [2]

$$u(t, x_c) \triangleq \sum_{k=1}^{N} \kappa(x_{c_k}) \nu(t, j, k), \qquad \forall j \in \mathbb{N}, \quad (19)$$

where the function $\nu : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \{1, \ldots, N\} \to \{0, 1\}$ is defined as

$$\nu(t, j, k) \triangleq \begin{cases} 1 & \text{if } t \in (\tau_j^c + T_j^c, \tau_{j+1}^c + T_{j+1}^c] \\ & \text{and } \mu(\gamma(j)) = k \\ 0 & \text{otherwise.} \end{cases}$$

[2] Since $\nu(t, j, k) \neq 0$ only when $\mu(\gamma(j)) = k$, the control input in (19) is independent of j contrarily to what the notation suggests.

This compact notation carries the advantage to involve a finite number of state variables and to fit the framework of [16]. Note that the control signal in (13) now reads $\hat{u}(t) = u(t, x_c(t))$.

4 Main results

We start by proving that the obtained protocol (17c) and (18c) inherits the UGES property from the original one (1). All proofs are deferred to Section 6.

Proposition 1 *Under Assumption 2, the protocol modeled by the discrete-time system (17c) and (18c) is UGES and admits an associated Lyapunov function* $W : \mathbb{N} \times \mathbb{R}^{Nn} \to \mathbb{R}_{\geq 0}$ *given by*

$$W(k, e) \triangleq \sum_{r=1}^{N} W_0(k, e_r)\eta(t, k, r)\,,$$

where η *is defined in (15), and satisfying for all* $k \in \mathbb{N}$ *and all* $e \in \mathbb{R}^{Nn}$*:*

$$a_L\,|e| \leq W(k, e) \leq a_H\,|e| \tag{20}$$

$$W(k+1, H(k, e)) \leq \rho_0 W(k, e) \tag{21}$$

$$\left|\frac{\partial W}{\partial e}(k, e)\right| \leq c\,, \tag{22}$$

with $a_L \triangleq \underline{a}$ *for* $N = 1$ *and* $a_L \triangleq \frac{a}{N}\min\left\{1, \left(\frac{a}{\overline{a}}\right)^2 \frac{1}{\rho_0}\right\}$ *for* $N > 1$*, and* $a_H \triangleq \overline{a}$*.*

Let us now present a local result on the exponential stability of the NCS (17). It provides an explicit bound (cf. (23) below) on the measurement MATI τ_m in terms of the characteristic parameters of the network-free closed-loop system, the protocol, the regularity assumptions on the dynamics and the model precision.

Theorem 1 *Assume that Assumptions 1-3 hold. Given some* $R > 0$*, fix* $R_x = R$ *and* $R_u = \lambda_\kappa R$ *and suppose that Assumptions 4-5 hold with these constants. Let* $\underline{a}$*,* $\overline{a}$*,* ρ_0*,* c*,* $\underline{\alpha}$*,* $\overline{\alpha}$*,* α*,* d*,* $\lambda_{f\hat{f}}$*,* λ_f*,* λ_κ*,* a_L*,* a_H *be generated by these assumptions and by Proposition 1. Assume that the following conditions on* τ_m*,* τ_c*,* T_m*,* T_c*,* ε_m *hold*

$$\tau_m \in (0, \tau_m^\star),\ \ \tau_m^\star \triangleq \frac{1}{L}\ln\left(\frac{H\gamma_2 + a_L L}{H\gamma_2 + a_L\rho_0 L}\right) \tag{23}$$

$$N = \left\lfloor \frac{T_c + T_m + \tau_c}{\varepsilon_m} \right\rfloor + 1 \tag{24}$$

$$\tau_m \geq \varepsilon_m \tag{25}$$

where

$$L \triangleq \frac{c}{a_L}\Big(\sqrt{N}\lambda_{f\hat{f}}(1 + \lambda_\kappa) + \sqrt{N}\lambda_f + \left(\sqrt{N-1} + N - 1\right)\lambda_f\lambda_\kappa\Big) \tag{26}$$

$$H \triangleq cN\lambda_{f\hat{f}}\,(1 + \lambda_\kappa) \tag{27}$$

$$\gamma_2 \triangleq \frac{d}{\alpha}\sqrt{\frac{\overline{\alpha}}{\underline{\alpha}}}\lambda_f\lambda_\kappa. \tag{28}$$

Then, the origin of the NCS (17) is exponentially stable with radius of attraction

$$\tilde{R} = \frac{R}{K} \tag{29}$$

where

$$K \triangleq \frac{\sqrt{2}}{1 - \gamma_1\gamma_2}\max\left\{k_2\,(1 + \gamma_1)\,,\ k_1\,(1 + \gamma_2)\right\} \tag{30}$$

$$\gamma_1 \triangleq \frac{\exp\left(L\tau_m\right) - 1}{a_L L\left(1 - \rho_0\exp\left(L\tau_m\right)\right)}H \tag{31}$$

$$k_1 \triangleq \frac{a_H}{\rho_0 a_L} \tag{32}$$

$$k_2 \triangleq \sqrt{\frac{\overline{\alpha}}{\underline{\alpha}}}. \tag{33}$$

It is important to remark that the bound (23) on the measurement MATI is also related to the dimension of the memory N, whose definition (24), obtained by (14) for $T_0^p = T_c + T_m + \tau_m + \tau_c$, embeds the other relevant communication parameters: MADs and control MATI. The pair (23)-(24) thus imposes a trade-off between the two MATIs and the MADs. The packet-based strategy aims at enlarging the control MATI τ_c, but a larger τ_c could require a larger memory N and hence could produce a lower measurement MATI τ_m. Moreover, conditions (24)-(25) bind the four relevant parameters (i.e. T_c, T_m, τ_c and τ_m) together and with the constant ε_m, bounding the minimum time between two consecutive accesses to the network. In particular, they require that the communication MATI τ_m is not smaller than ε_m. Furthermore, depending on the parameter R for which Assumptions 4 and 5 hold, an explicit estimate $\tilde{R}$ of the radius of attraction can be computed, cf. (29). Note that, since Theorem 1 guarantees only local properties, Assumption 3 could be relaxed to *local* exponential stability of the nominal plant, over a sufficiently large domain.

The following proposition establishes that the MATI and memory requirements of the previous theorem can always be satisfied.

Proposition 2 *Given any* $R > 0$*, the parameters* τ_m*,* τ_c*,* T_m*,* T_c*,* ε_m *can always be picked small enough to satisfy conditions (23), (24) and (25).*

In general, the radius of attraction $\tilde{R}$ of the resulting NCS guaranteed by Theorem 1 cannot be arbitrarily specified due to the possible dependency of the constants L and H (and consequently K) in the parameter R ruling the domain on which Assumptions 4 and 5 hold. To see this more clearly, consider, for instance, the case of K proportional to R. Relation (29) shows that, in this case, the radius of the initial condition $\tilde{R}$ would be a constant irrespective of the amplitude of R. One could even imagine that, in some situations, $\tilde{R}$ actually shrinks when R is enlarged. Hence, in order to ensure that the set of initial conditions can be arbitrarily enlarged, we must add some constraints on the growth rate of the constant K or, equivalently, on some of the Lipschitz constants. After reporting a definition of semiglobal exponential stability which is adapted to our NCS framework, we present our main result in this regard in Theorem 2.

Definition 1 *The NCS (17) is said to be* semiglobally exponentially stable *if, for any $\tilde{R} > 0$, there exist positive constants $\tau_m^\star(\tilde{R})$, $\tau_c^\star(\tilde{R})$, $T_m^\star(\tilde{R}), T_c^\star(\tilde{R})$ and $\varepsilon_m^\star(\tilde{R})$, as introduced in Assumption 1, such that its origin is exponentially stable on $B_{\tilde{R}}$.*

Theorem 2 *Suppose that Assumptions 1-4 hold for all $R_x, R_u > 0$ and that there exists $\sigma \in (0,1)$ such that*

$$\lim_{s\to\infty} \frac{\lambda_f(s)\lambda_\kappa(s)}{s^\sigma} < \infty. \tag{34}$$

Then, the NCS (17) is semiglobally exponentially stable.

The above result guarantees that, provided sufficient regularity of the dynamics involved (i.e. Lipschitz constants sublinear in the size of the domain over which they are computed), any prescribed compact domain of attraction can be reached if MADs and MATIs are small enough.

5 Case Study

The exploitation of the packet payload and the model-based predictive strategy presented in this paper can improve the MATI bounds obtained by sending a single control value in each packet. Let us illustrate such improvement by comparing our bounds with those computed in [21] for a Ch-47 Tandem-Rotor Helicopter. The linearized model describing the helicopter can be written as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{bmatrix}$, $A = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix}$.

The exponentially stabilizing static output feedback is given by $u = Ky$ with $K = \begin{bmatrix} -12.7177 & -45.0824 \\ 63.5123 & 25.9144 \end{bmatrix}$. We assume the state x is transmitted by means of two links ($\ell = 2$) ruled by the Round Robin protocol. The bound on the MATI provided in [16] is $\tau_{[16]}^\star = 1.20\ 10^{-5}$ s and the improvement in [21] provides $\tau_{[21]}^\star = 2.81\ 10^{-4}$ s. In the ideal case of a linear system with equidistant transmissions and Round Robin protocol, the *exact* MATI can be computed following the argument in [16, Section VII-A]. For the present case study we get $\tau_{\text{single}}^\star \simeq 1.13\ 10^{-3}$ s.

In order to appreciate the improvement induced by the packet payload exploitation, let us compute the MATI bound according to expression (23). For the comparison to be fair, we assume zero delays ($T_m = T_c = 0$). We also fix $\varepsilon_m \simeq \tau_m^\star$ and τ_c slightly less than ε_m (just to keep $N = 1$ in (24)). The relevant protocol parameters, according to Assumption 2, are $\underline{a} = 1$, $\overline{a} = \sqrt{\ell}$, $\rho_0 = \sqrt{(\ell-1)/\ell}$, $c = \sqrt{\ell}$ (see [16]). As concerns the nominal stability parameters in Assumption 3, we have $\underline{\alpha} = 9\ 10^{-4}$, $\overline{\alpha} = 399.22$, $\alpha = 1$, $d = 2\overline{\alpha}$ (see [21]). The Lipschitz constants required by Assumption 4 are $\lambda_f = \max\{|A|, |B|\} = 43.8904$ and [3] $\lambda_\kappa = |KC| \simeq 2980$. We assume also to have a perfect model: $\hat{f} = f$ and $\lambda_{f\hat{f}} = 0$ (see Assumption 5). With $N = 1$, the constants in (26) and (27) become $L = \frac{c}{a_L}\left(\lambda_{f\hat{f}}(1+\lambda_\kappa) + \lambda_f\right) = \frac{c}{\underline{a}}\lambda_f$ and $H = c\lambda_{f\hat{f}}(1+\lambda_\kappa) = 0$. Thus, the MATI bound (23) is given by

$$\tau_m^\star = \frac{\underline{a}}{c\lambda_f} \ln\left(\frac{1}{\rho_0}\right) \simeq 5.58\ 10^{-3} \text{ s},$$

i.e. about 20 times larger than $\tau_{[21]}^\star$ - or, in other terms, our method would require sending ca. 20 times less packets to stabilize the system. In reality, the exploitation of the packet payload often induces a more substantial improvement than that shown by the bounds above. If we

[3] In computing λ_κ we accounted for the fact that we send the state vector instead of the output vector.

compare the exact MATI [4] achievable with our predictive control strategy with a classical single control value technique, we have $\tau^\star_{\text{mult}} \simeq 1.3105$ s, hence a theoretical improvement of $\frac{\tau^\star_{\text{mult}}}{\tau^\star_{\text{single}}} \simeq 1160$.

6 Proofs

6.1 Proof of Proposition 1

For $N = 1$, $W(k,e) = W_0(k,e)$ and the thesis follows from Assumption 2. For $N > 1$, let us consider any $s \in \{1,\dots,N-1\}$ satisfying $\eta(t,k,s) = 1$ (with $\eta(\cdot)$ defined in (15)) for some $t \geq 0$ and some $k \in \mathbb{N}_{\geq 1}$. Then $W(k,e) = W_0(k,e_s)$ and for $k+1$ we have $\eta(t,k+1,s+1) = 1$ and $W(k+1,H(k,e)) = W_0(k+1,h(k,e_s)) \leq \rho_0 W_0(k,e_s) = \rho_0 W(k,e_s)$. The inequality $W(k,e) = W_0(k,e_s) \leq \overline{a}|e_s| \leq \overline{a}|e|$ is easily verified (thus $a_H = \overline{a}$), while the other inequality requires to consider the evolution of the system.

Recall that at step k the $(s+1)$-th variable is updated, hence the s-th variable is left unchanged for $N-1$ periods, the $(s-1)$-th variable for $N-2$ periods, etc. Summarizing, the following relations hold:

$$
\begin{aligned}
e_{s-1}(k) &= h(k-1, e_{s-2}(k-1)) \\
e_{s-2}(k) &= e_{s-2}(k-1) = h(k-2, e_{s-3}(k-2)) \\
&\vdots \\
e_1(k) &= e_1(k-1) = \cdots = e_1(k-s+2) \\
&= h(k-s+1, e_N(k-s+1)) \\
e_N(k) &= e_N(k-1) = \cdots = e_N(k-s+1) \\
&= h(k-s, e_{N-1}(k-s)) \\
&\vdots \\
e_{s+1}(k) &= e_{s+1}(k-1) = \cdots = e_{s+1}(k-N+2) \\
&= h(k-N+1, e_s(k-N+1)) \\
e_s(k) &= e_s(k-1) = \cdots = e_s(k-N+1) \\
&= h(k-N, e_{s-1}(k-N)).
\end{aligned}
$$

This permanency allows us to write $N-1$ of the previous relations referring to the variables computed at instant $k-1$, and hence to suppress the dependency from k: $e_{s-1} = h(k-1, e_{s-2}) \cdots e_{s+1} = h(k-N+1, e_s)$. By the previous relations we have $W_0(k,e_{s-1}) = W_0(k, h(k-1,e_{s-2})) \leq \rho_0 W_0(k-1,e_{s-2})$, and consequently $W_0(k-1,e_{s-2}) \geq \frac{W_0(k,e_{s-1})}{\rho_0}$. Analogously we can write

$$
W_0(k-N+1,e_s) \geq \frac{W_0(k-N+2,e_{s+1})}{\rho_0} \tag{35}
$$

[4] Even in this case we used the argument in [16, Section VII-A].

and iterating $W_0(k-N+1,e_s) \geq \frac{W_0(k-N+2,e_{s+1})}{\rho_0} \geq \frac{W_0(k-N+3,e_{s+2})}{\rho_0^2} \cdots \geq \frac{W_0(k,e_{s-1})}{\rho_0^{N-1}}$. Recalling that for all k we can write $\underline{a}|e| \leq W_0(k,e) \leq \overline{a}|e|$, then for all e, q and k we have

$$
W_0(q,e) \geq \frac{\underline{a}}{\overline{a}} W_0(k,e). \tag{36}
$$

Whereby, using (35) and (36) we have $W_0(k,e_s) \geq \frac{\underline{a}}{\overline{a}} W_0(k-N+1,e_s) \geq \frac{\underline{a}}{\overline{a}\rho_0} W_0(k-N+2,e_{s+1}) \geq \left(\frac{\underline{a}}{\overline{a}}\right)^2 \frac{1}{\rho_0} W_0(k,e_{s+1})$, and iterating $W_0(k,e_s) \geq \left(\frac{\underline{a}}{\overline{a}}\right)^2 \frac{1}{\rho_0} W_0(k,e_{s+1}) \cdots \geq \left(\frac{\underline{a}}{\overline{a}}\right)^2 \frac{1}{\rho_0^{N-2}} W_0(k,e_{s-2}) \geq \frac{\underline{a}}{\overline{a}} \frac{1}{\rho_0^{N-1}} W_0(k,e_{s-1})$. Using the previous relations we can write the sought inequality and compute a_L:

$$
\begin{aligned}
W(k,e) &= W_0(k,e_s) \\
&= \frac{1}{N} \sum_{r=1}^{N} W_0(k,e_s) \\
&\geq \frac{1}{N} \left(W_0(k,e_s) + \left(\frac{\underline{a}}{\overline{a}}\right)^2 \frac{1}{\rho_0} W_0(k,e_{s+1}) + \cdots \right. \\
&\quad \left. + \frac{\underline{a}}{\overline{a}} \frac{1}{\rho_0^{N-1}} W_0(k,e_{s-1}) \right) \\
&\geq \frac{\underline{a}}{N} \left(|e_s| + \left(\frac{\underline{a}}{\overline{a}}\right)^2 \frac{1}{\rho_0} |e_{s+1}| + \cdots + \frac{\underline{a}}{\overline{a}} \frac{1}{\rho_0^{N-1}} |e_{s-1}| \right) \\
&\geq \frac{\underline{a}}{N} \min\left\{ 1, \left(\frac{\underline{a}}{\overline{a}}\right)^2 \frac{1}{\rho_0}, \cdots, \frac{\underline{a}}{\overline{a}} \frac{1}{\rho_0^{N-1}} \right\} |e| \\
&\geq \frac{\underline{a}}{N} \min\left\{ 1, \left(\frac{\underline{a}}{\overline{a}}\right)^2 \frac{1}{\rho_0} \right\} |e|,
\end{aligned}
$$

where we used the fact that $\frac{\underline{a}}{\overline{a}}, \rho_0 < 1$.

Finally we write

$$
\begin{aligned}
\left| \frac{\partial W}{\partial e}(k,e) \right| &= \left| \frac{\partial W_0}{\partial e}(k,e_s) \right| \\
&= \left| \left[0, \cdots, 0, \frac{\partial W_0}{\partial e_s}(k,e_s), 0, \cdots, 0 \right]^T \right| \leq c.
\end{aligned}
$$

6.2 Proof of Theorem 1

The proof consists of the following 4 steps:

(1) Show that the system (17b) is locally input-to-state exponentially stable with linear gain from x to e, provided that solutions remain inside the domain $B_{\tilde{R}}$;
(2) Show that the system (17a) is locally input-to-state exponentially stable with linear gain from e to x, provided that solutions remain inside the domain $B_{\tilde{R}}$;

(3) Show by means of small gain arguments that the overall system (17) is locally exponentially stable inside $B_{\tilde{R}}$;
(4) Show that solutions remain indeed inside $B_{\tilde{R}}$.

Step 1 Let us consider any $r \in \{1,\dots,N\}$ such that $\nu(t,j,r) = 1$ for some $t \geq 0$ and some $j \in \mathbb{N}$. In view of (19) we can simply write $u(t, e+\bar{x}) = \kappa(e_r + x)$. Since $|e_s| \leq |e|$ for all $s \in \{1,\dots,N\}$, in light of Assumptions 4 and 5, we have for all $\left|(x^T(t), e^T(t))\right| \leq R$

$$\frac{\partial W}{\partial e} G(t,\bar{x},e,u(t)) \leq \left|\frac{\partial W}{\partial e}\right| |G(t,\bar{x},e,u(t))|$$

$$\begin{aligned}
&\leq c\sum_{s=1}^{N}\left|\hat{f}(e_s+x,\kappa(e_s+x)) - f(x,\kappa(e_r+x))\right| \\
&\leq c\sum_{s=1}^{N}\left|\hat{f}(e_s+x,\kappa(e_s+x)) - f(e_s+x,\kappa(e_s+x))\right| \\
&+ c\sum_{s=1}^{N}|f(e_s+x,\kappa(e_s+x)) - f(x,\kappa(e_r+x))| \\
&\leq c\sum_{s=1}^{N}\lambda_{f\hat{f}}\left(|e_s| + |x| + \lambda_\kappa\left(|e_s|+|x|\right)\right) \\
&+ c\sum_{s=1}^{N}\lambda_f\left(|e_s| + \lambda_\kappa|e_s - e_r|\right) \\
&\leq cN\lambda_{f\hat{f}}(1+\lambda_\kappa)|x| \\
&+ c\left(\sqrt{N}\lambda_{f\hat{f}}(1+\lambda_\kappa) + \sqrt{N}\lambda_f\right. \\
&\left.+ \left(\sqrt{N-1}+N-1\right)\lambda_f\lambda_\kappa\right)|e| \\
&\leq |\tilde{y}| + LW(i,e),
\end{aligned}$$

where L is given by (26) and $\tilde{y} \triangleq Hx$ with H given by (27). In the light of footnote 8 in [16] we have that, for all $i \in \mathbb{N}$ and almost all t, $\frac{d}{dt}W(i,e(t)) \leq LW(i,e(t)) + |\tilde{y}(t)|$. From Proposition 6 and the proof of Proposition 7 in [16], we conclude the input-output stability of system (17b) from $\tilde{y}$ to W with exp-$\mathcal{KL}$ function and linear gain, provided that $\left|(x^T(t), e^T(t))\right| \leq R$ at all time. More precisely, for any $t \in [t_{s_i}, t_{s_{i+1}}]$ and $i \geq k \geq 0$ arbitrarily chosen, we have that $t - t_{s_k} \leq (i-k+1)\tau_m$ and, as long as $\left|(x^T(t), e^T(t))\right| \leq R$,

$$\begin{aligned}
W(i,e(t)) &\leq \exp(L\tau_m)\lambda_{(k+1)\to i}W(k,e(t_{s_k})) \\
&+ \frac{\exp(L\tau_m)-1}{L\left(1-\rho_0\exp(L\tau_m)\right)}\left\|\tilde{y}[t_{s_k},t]\right\|_{\mathcal{L}_\infty},
\end{aligned}$$

with

$$\begin{aligned}
\lambda_{(k+1)\to i} &\triangleq \left(\rho_0\exp(L\tau_m)\right)^{i-k} \\
&\leq \frac{\exp\left(\ln\left(\rho_0\exp(L\tau_m)\right)\frac{t-t_{s_k}}{\tau_m}\right)}{\rho_0\exp(L\tau_m)}.
\end{aligned}$$

Note that (23) ensures in particular that $\rho_0\exp(L\tau_m) < 1$. The previous expression holds in particular for $k = 0$. Recalling the definition of $\tilde{y}$ and the inequalities (20), we can write, as long as $\left|(x^T(t), e^T(t))\right| \leq R$,

$$|e(t)| \leq k_1 \mathrm{e}^{-\lambda_1(t-t_{s_0})}|e_0| + \gamma_1\left\|x[t_{s_0},t]\right\|_{\mathcal{L}_\infty}, \quad (37)$$

with γ_1 and k_1 given by (31) and (32) respectively, and $\lambda_1 \triangleq -\frac{\ln(\rho_0\exp(L\tau_m))}{\tau_m} > 0$.

Step 2 Let us consider the Lyapunov function V of Assumption 3. In view of Assumption 4, the total derivative of V along the solutions of (17a) yields for all $\left|(x^T(t), e^T(t))\right| \leq R$

$$\frac{\partial V}{\partial x}F(t,\bar{x},e,u(t)) = \frac{\partial V}{\partial x}f(x,\kappa(e_r+x))$$

$$\begin{aligned}
&= \frac{\partial V}{\partial x}f(x,\kappa(x)) + \frac{\partial V}{\partial x}\left[f(x,\kappa(e_r+x)) - f(x,\kappa(x))\right] \\
&\leq -\alpha|x|^2 + d\lambda_f\lambda_\kappa|x||e_r| \\
&\leq -\alpha|x|^2 + d\lambda_f\lambda_\kappa|x||e_r| + \left(\sqrt{\frac{\alpha}{2}}|x| - \frac{d\lambda_f\lambda_\kappa}{\sqrt{2\alpha}}|e_r|\right)^2 \\
&= -\frac{\alpha}{2}|x|^2 + \frac{d^2\lambda_f^2\lambda_\kappa^2}{2\alpha}|e_r|^2 \\
&\leq -\frac{\alpha}{2\overline{\alpha}}V(x) + \frac{d^2\lambda_f^2\lambda_\kappa^2}{2\alpha}|e|^2. \quad (38)
\end{aligned}$$

Whereby, applying the comparison lemma, we get, as long as $\left|(x^T(t), e^T(t))\right| \leq R$,

$$\begin{aligned}
V(x(t)) &\leq \mathrm{e}^{-\frac{\alpha}{2\overline{\alpha}}(t-t_{s_0})}V(x(t_{s_0})) \\
&+ \frac{d^2\lambda_f^2\lambda_\kappa^2}{2\alpha}\int_{t_{s_0}}^{t}\mathrm{e}^{-\frac{\alpha}{2\overline{\alpha}}(t-s)}|e(s)|^2\,ds \\
&\leq \mathrm{e}^{-\frac{\alpha}{2\overline{\alpha}}(t-t_{s_0})}V(x_0) \\
&+ \frac{d^2\lambda_f^2\lambda_\kappa^2}{\alpha^2}\left(1-\mathrm{e}^{-\frac{\alpha}{2\overline{\alpha}}(t-t_{s_0})}\right)\left\|e[t_{s_0},t]\right\|_{\mathcal{L}_\infty}.
\end{aligned}$$

Hence, recalling Assumption 3, we can write, as long as $\left|(x^T(t), e^T(t))\right| \leq R$,

$$|x(t)| \leq k_2\mathrm{e}^{-\lambda_2(t-t_{s_0})}|x_0| + \gamma_2\left\|e[t_{s_0},t]\right\|_{\mathcal{L}_\infty}, \quad (39)$$

with γ_2 and k_2 given by (28) and (33) respectively, and $\lambda_2 \triangleq \frac{\alpha}{4\overline{\alpha}} > 0$.

Step 3 By means of a local version of Corollary 1 in [16], we can conclude, for the fixed $R > 0$, that the NCS system (17) is locally exponentially stable if the small gain condition $\gamma_1\gamma_2 < 1$ is verified and $\left|(x^T(t), e^T(t))\right| \leq R$ for all t. In view of (28) and (31), it is easy to see that the previous inequality is satisfied for every $\tau_m \in (0, \tau_m^\star)$ compatible with conditions (23), (24) and (25). The value of $\tau_m^\star$ in (23) can be found by solving in τ_m the small gain condition. Moreover, in order for the definition (14) (or equivalently (24)) of the memory N to be consistent and for the meaning of τ_m as MATI to be preserved, it must hold $\tau_m \geq \varepsilon_m$, namely condition (25).

Step 4 Finally, we can compute the set of initial conditions for which trajectories remain inside B_R. Recalling the inequalities (37) and (39) we can write

$$\begin{aligned} \|x[t_{s_0}, t]\|_{\mathcal{L}_\infty} &\leq k_2 |x_0| + \gamma_2 \|e[t_{s_0}, t]\|_{\mathcal{L}_\infty} \\ \|e[t_{s_0}, t]\|_{\mathcal{L}_\infty} &\leq k_1 |e_0| + \gamma_1 \|x[t_{s_0}, t]\|_{\mathcal{L}_\infty} \ , \end{aligned}$$

and

$$\left\| \begin{matrix} x[t_{s_0}, t] \\ e[t_{s_0}, t] \end{matrix} \right\|_{\mathcal{L}_\infty} \leq \|x[t_{s_0}, t]\|_{\mathcal{L}_\infty} + \|e[t_{s_0}, t]\|_{\mathcal{L}_\infty} \leq K \left| \begin{matrix} x_0 \\ e_0 \end{matrix} \right|$$

with K given by (30). Consequently, in order to ensure that the evolution of the system does not exit the ball B_R, it is sufficient to impose that $K\left|(x_0^T, e_0^T)\right| < R$, or equivalently that $(x_0^T, e_0^T) \in B_{\tilde{R}} \subseteq B_R$ with $\tilde{R} = \frac{R}{K}$ (cf. (29)).

6.3 Proof of Proposition 2

Let us name τ_m^1 the value assumed by $\tau_m^\star$ (see (23)) for $N = 1$. Conditions (23), (24) and (25) are satisfied for every τ_m, T_c, T_m, τ_c, ε_m such that $\tau_m \in (0, \tau_m^1)$, $\varepsilon_m \in (0, \tau_m]$ and $0 < T_c + T_m + \tau_c < \varepsilon_m$, which is always feasible for sufficiently small values of these parameters.

6.4 Proof of Theorem 2

This proof strongly relies on that of Theorem 1. According to Definition 1, we must show that for any arbitrarily fixed set of initial conditions we can find suitable values for the parameters τ_m, T_c, T_m, τ_c, ε_m ensuring the exponential stability of the NCS on the chosen set. Let us consider $R > 0$ as a free variable. By the small gain condition $\gamma_1(R)\gamma_2(R) < 1$ (cf. Step 3 of the proof of Theorem 1) and recalling expression (31) of $\gamma_1(R)$, we have that, for any $\delta \in (0, 1)$ independent of R, we can find a constant

$$\tau_m^\star(R, \delta) \triangleq \frac{1}{L(R)} \ln\left(\frac{H(R)\gamma_2(R) + (1-\delta)a_L L(R)}{H(R)\gamma_2(R) + (1-\delta)a_L\rho_0 L(R)}\right),$$

such that for every $\tau_m \in (0, \tau_m^\star(R, \delta)]$ $\gamma_1(R)\gamma_2(R) \leq 1 - \delta$. In a way similar to Proposition 2 we can show that it is always possible, for any fixed R, to find a set of parameters τ_m, T_c, T_m, τ_c, ε_m satisfying the previous condition and the conditions (24) and (25).

All the Lipschitz constants of Assumptions 4 and 5 are non-decreasing functions of R. If $\lim_{R\to\infty} \gamma_2(R) = \bar{\gamma}_2$ constant, then $\tau_m^\star(R, \delta) \geq \tau_{m\bar{\gamma}_2}^\star(R, \delta)$ with

$$\tau_{m\bar{\gamma}_2}^\star(R, \delta) \triangleq \frac{1}{L(R)} \ln\left(\frac{H(R)\bar{\gamma}_2 + (1-\delta)a_L L(R)}{H(R)\bar{\gamma}_2 + (1-\delta)a_L\rho_0 L(R)}\right)$$

and for every $\tau_m \in (0, \tau_{m\bar{\gamma}_2}^\star(R, \delta))$ we have $\gamma_1(R) < \frac{1-\delta}{\bar{\gamma}_2}$. This means that $K(R)$ of (30) is bounded by the constant $\bar{K} \triangleq \frac{\sqrt{2}}{\delta} \max\left\{k_2\left(1 + \frac{1-\delta}{\bar{\gamma}_2}\right),\ k_1\left(1 + \bar{\gamma}_2\right)\right\}$, thus allowing the radius $\tilde{R}$ of the set of initial conditions to be arbitrarily chosen. Indeed, once $\tilde{R}$ is fixed, R can easily be computed as $R = \bar{K}\tilde{R}$ (cf. (29)). Such a value of R is a function of $\tilde{R}$, hence, it can be used to explicitly compute the parameters $\tau_m^\star(\tilde{R})$, $T_c^\star(\tilde{R})$, $T_m^\star(\tilde{R})$, $\tau_c^\star(\tilde{R})$, $\varepsilon_m^\star(\tilde{R})$ required by Definition 1.

If, instead, $\lim_{R\to\infty} \gamma_2(R) = \infty$, we can chose $\tau_m \in (0, \tau_m^\star(R, \delta))$ such that $\lim_{R\to\infty} \gamma_1(R) = 0$ to ensure that $\gamma_1(R)\gamma_2(R) \leq 1 - \delta$. By (28) we see that $\gamma_2(R)$ is a non-decreasing function of R. Hence, there exists an $\bar{R}$ such that for any $R > \bar{R}$, $\max\{k_2\left(1 + \gamma_1(R)\right),\ k_1\left(1 + \gamma_2(R)\right)\} = k_1\left(1 + \gamma_2(R)\right)$. Let us consider the case of $R > \bar{R}$. By the condition (29), for $\tilde{R}$ to be arbitrary enlargeable, it must hold $\lim_{R\to\infty} \frac{R}{K(R)} = \infty$. By the previous relations we have that, for any $R > \bar{R}$ and any $\tau_m \in (0, \tau_m^\star(R, \delta))$, $K(R) \leq \frac{\sqrt{2}}{\delta} k_1\left(1 + \gamma_2(R)\right)$. Using the condition (34) on the growth rate of $\lambda_f(R)\lambda_\kappa(R)$ and the expression (28) of $\gamma_2(R)$, we see that $\lim_{R\to\infty} \frac{\gamma_2(R)}{R^\sigma} < \infty$ for some $\sigma \in (0, 1)$. Hence, $\lim_{R\to\infty} \frac{R}{K(R)} \geq \lim_{R\to\infty} \frac{R}{\frac{\sqrt{2}}{\delta} k_1(1+\gamma_2(R))} = \lim_{R\to\infty} R^{1-\sigma} \frac{R^\sigma}{\frac{\sqrt{2}}{\delta} k_1(1+\gamma_2(R))} = \infty$ as desired. Due to the ultimate nature of the condition (34), it can be violated for small values of R, say for $R < \hat{R}$. Let us then define $R^\star = \max(\bar{R}, \hat{R})$. If for every $R < R^\star$ we fix all the Lipschitz constants to the value assumed for $R = R^\star$, then, for any fixed $\tau_m \in (0, \tau_m^\star(R^\star, \delta))$, $K(R^\star)$ is a constant independent of R. We can fix as before the other parameters T_c, T_m, τ_c. Let us define $\tilde{R}^\star = \frac{R^\star}{K(R^\star)}$. For any radius $\tilde{R}$ of the set of the initial conditions such that $\tilde{R} \in (0, \tilde{R}^\star]$, the values required by Definition 1, are fixed independently of $\tilde{R}$ in terms of $R^\star$ (or equivalently in terms of $\tilde{R}^\star$). For any $\tilde{R} > \tilde{R}^\star$ we can find a solution R to the inequality $R \geq \frac{\sqrt{2}}{\delta} k_1\left(1 + \gamma_2(R)\right)\tilde{R}$, which is finite if $\tilde{R}$ is finite. Once again, such $\tilde{R}$ can be used to compute the parameters $\tau_m^\star(\tilde{R})$, $T_c^\star(\tilde{R})$, $T_m^\star(\tilde{R})$, $\tau_c^\star(\tilde{R})$, $\varepsilon_m^\star(\tilde{R})$ required by Definition 1.

7 Conclusions

The problem of stabilizing nonlinear time-invariant plants over a limited-bandwidth packet-switching network has been considered. Traditional control schemes, designed for circuit-switching networks, send small pieces of data very frequently, thus driving the network towards bandwidth saturation. On the other hand, in a packet-switching network the adoption of feedforward control sequences allows to send larger packets less frequently. To this aim, we presented a model-based approach to remotely compute a predictive control signal on a given time horizon. We considered a robustness problem, where the plant uncertainty is given a priori, and we provided a bound on the combined effects of the MATI and MAD as a function of the basin of attraction and the model precision. The improvement to the MATI induced by our control strategy has been verified by means of a case study. Our future research will focus on the exploitation of the packetization of measurements to further reduce the bandwidth occupation and to better cope with model parameter variations.

References

[1] P. Antsaklis and J. Baillieul. Special issue on networked control systems. *IEEE Trans. on Automat. Contr.*, 49(9), 2004.

[2] P. Antsaklis and J. Baillieul. Special issue on technology of networked control systems. *Proc. of IEEE*, 95(1), 2007.

[3] M. S. Branicky, S. M. Phillips, and W. Zhang. Stability of networked control systems: Explicit analysis of delay. In *Proc. American Control Conf.*, pages 2352–2357, Chicago, USA, 2000.

[4] L. G. Bushnell. Special issue on networks and control. *IEEE Contr. Syst. Mag.*, 21(1), 2001.

[5] A. Chaillet and A. Bicchi. Delay compensation in packet-switching networked controlled systems. In *Proc. IEEE Int. Conf. on Decision and Control*, pages 3620–3625, 2008.

[6] M. D. Di Benedetto, K. H. Johansson, M. Johansson, and F. Santucci. Special issue: Industrial control over wireless networks (ICWN08). *International Journal of Robust and Nonlinear Control*, 20(2), 2010.

[7] W. P. M. H. Heemels, A. R. Teel, N. van de Wouw, and D. Nešić. Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance. *IEEE Trans. on Automat. Contr.*, 55(8):1781–1796, 2010.

[8] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proc. of IEEE*, 95(1):138–162, 2007.

[9] J. P. Hespanha, A. Ortega, and L. Vasudevan. Towards the control of linear systems with minimum bit-rate. In *Proc. of the Int. Symp. on the Mathematical Theory of Networks and Syst.*, Aug. 2002.

[10] F.-L. Lian, J. R. Moyne, and D. M. Tilbury. Performance evaluation of control networks: Ethernet, ControlNet, and DeviceNet. *IEEE Contr. Syst. Mag.*, 21(1):66–83, 2001.

[11] L. A. Montestruque and P. Antsaklis. Stability of model-based networked control systems with time-varying transmission times. *IEEE Trans. on Automat. Contr.*, 49(9):1562–1572, 2004.

[12] J. R. Moyne and D. M. Tilbury. The emergence of industrial control networks for manufacturing control, diagnostics, and safety data. *Proc. of IEEE*, 95(1):29–47, 2007.

[13] P. Naghshtabrizi and J. P. Hespanha. Stability of networked control systems with variable sampling and delay. In *Proc. of the Annual Allerton Conf. on Communication, Control, and Computing*, 2006.

[14] G. N. Nair and R. J. Evans. Exponential stabilisability of finite-dimensional linear systems with limited data rates. *Automatica*, 39(4):585–593, 2003.

[15] G. N. Nair and R. J. Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM J. Contr. Optim.*, 43(2):413–436, 2004.

[16] D. Nešić and A. R. Teel. Input-output stability properties of networked control systems. *IEEE Trans. on Automat. Contr.*, 49(10):1650–1667, 2004.

[17] G. Pin and T. Parisini. Networked predictive control of constrained nonlinear systems: Recursive feasibility and input-to-state stability analysis. In *Proc. American Control Conf.*, pages 2327–2334, St. Louis, MO, USA, June 2009.

[18] I.G. Polushin, P.X. Liu, and C-H. Lung. On the model-based approach to nonlinear networked control systems. *Automatica*, 44(9):2409–2414, 2008.

[19] D. E. Quevedo, E. I. Silva, and G. C. Goodwin. Packetized predictive control over erasure channels. In *Proc. American Control Conf.*, pages 1003–1008, NY City, USA, July 2007.

[20] A. Seuret, E. Fridman, and J. P. Richard. Exponential stabilization of delay neutral systems under sampled-data control. In *Proc. 13th Mediterranean Conference on Control and Automation*, Limassol, Cyprus, June 2005.

[21] M. Tabbara, D. Nešić, and A. R. Teel. Stability of wireless and wireline networked control systems. *IEEE Trans. on Automat. Contr.*, 52(9):1615–1630, 2007.

[22] G. C. Walsh and H. Ye. Scheduling of networked control systems. *IEEE Contr. Syst. Mag.*, 21(1):57–65, 2001.

[23] G. C. Walsh, H. Ye, and L. Bushnell. Stability analysis of networked control systems. In *Proc. American Control Conf.*, 1999.

[24] F.-Y. Wang and D. Liu. *Networked Control Systems: Theory and Applications*. 2008.

[25] D. Yue, Q. L. Han, and J. Lam. Networked-based robust H_∞ control of systems with uncertainty. *Automatica*, 41:999–1007, 2005.

[26] P. V. Zhivoglyadov and R. H. Middleton. Networked control design for linear systems. *Automatica*, 39:743–750, 2003.

[27] R. Zurawski. Special issue on industrial communication systems. *Proc. of IEEE*, 93(6), 2005.