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Abstract: The goal of prognosis on a structure, system or component (SSC) is to predict whether the SSC can perform its function up to the end of its life and in case it cannot, estimate the Time to Failure (TTF), *i.e.*, the lifetime remaining between the present and the instance when it can no longer perform its function. Such prediction on the loss of functionality changes dynamically as time goes by and is typically based on measurements of parameters representative of the SSC state. Uncertainties from two different sources affect the prediction: randomness due to variability inherent in the SSC degradation behavior (aleatory uncertainty) and imprecision due to incomplete knowledge and information on the SSC failure mechanisms (epistemic uncertainty). Such uncertainties must be adequately represented and propagated in order for the prognostic results to have operational significance, e.g. in terms of maintenance and renovation decisions. This work addresses the problem of predicting the reliability and TTF of a SSC, as measurements of parameters representative of its state become available in time. The representation and propagation of the uncertainties associated to the prediction are done alternatively by a pure probabilistic method and an hybrid Monte Carlo and possibilistic method. A case study is considered, regarding a component which is randomly degrading in time according to a stochastic fatigue crack growth model of literature; the maximum level of degradation beyond which failure occurs is affected by epistemic uncertainty.

***Keywords**: Prognostics, reliability prediction, Time to failure prediction, uncertainty representation and propagation, probability, possibility, Monte Carlo simulation, fatigue crack growth.*

Nomenclature

NPP	Nuclear Power Plant
RAMS	Reliability, Availability, Maintenance, Safety
SSC	Structure, System or Component
TTF	Time to Failure
TTR	Time to Repair
UOD	Universe of Discourse
x	Generic model variable
X	Universe of discourse of variable x
$\Pr$	Probability function
Π	Possibility function
N	Necessity function
$p(x)$	Probability density function

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$\pi(x)$	Possibility distribution
$P(X)$	Power set of X
$\mathrm{P}(\pi)$	Probability family
A	Subset of X
A_α	α -cut of A
$d\alpha$	Infinitesimal probability
$\lvert A_\alpha \rvert$	Width of A_α
$\underline{a}_\alpha, \overline{a}_\alpha$	Limits of A_α
$\underline{F}(x), \overline{F}(x)$	Bounding cumulative probability distributions
$f(\cdot)$	Output of the prognostic model
Y_j	Uncertain input variable $j=1,...,n$
y_j^i	i -th realization of Y_j , $i=1,...,m$
k	Number of aleatory variables
$p_{Y_j}(y)$	Probability density function of variable Y_j
$\pi_{Y_j}(y)$	Possibilistic distribution of variable Y_j
π_i^f	i -th possibilistic random realization of $f(\cdot)$
$\underline{f}_\alpha^i, \overline{f}_\alpha^i$	α -cut of the i -th random realization π_i^f
$Bel(A)$	Belief measure of A
$Pl(A)$	Plausibility measure of A
u	Generic value of f
h	Crack depth
L	Load cycles
C, η	Constants related to material properties
ΔK	Stress intensity amplitude
β	Empirical constant
$\omega = N(0, \sigma_\omega^2)$	White Gaussian noise
$h(t)$	Degradation value at the time instant t
$H_{\max}$	Degradation upper threshold
T_{miss}	Mission time
p_d	Degradation probability
H_p	Current degradation value
t_p	Current time value

X_S Uncertain Boolean variable

1. Introduction

Prognosis is an important and challenging task in Reliability, Availability, Maintainability and Safety (RAMS). The primary goal of a prognostic system is to indicate whether the structure, system or component (SSC) of interest can perform its function throughout its lifetime with reasonable assurance and, in case it cannot, to estimate its Time To Failure (TTF), i.e. the lifetime remaining before it can no longer perform its function. The prediction is more effective if informed by measurements of parameters representative of the state of the SSC during its life.

The predictive task must adequately account for the uncertainty associated to the future behavior of the SSC under analysis. Sources of uncertainty derive from: (1) randomness due to inherent variability in the SSC degradation behavior (aleatory uncertainty) and (2) imprecision due to incomplete knowledge and information on the parameters used to model the degradation and failure processes (epistemic uncertainty).

Two methods for the representation and propagation of these uncertainties are considered in this work: (1) a pure probabilistic method and (2) an hybrid Monte Carlo and possibilistic method. In the context of the pure probabilistic approach, both epistemic and aleatory uncertainties are represented in terms of probability distributions and a double-randomization Monte Carlo simulation is carried out in which epistemic variables are sampled in the outer loop and aleatory variables are sampled in the inner loop, to propagate the uncertainty through the model onto the prediction of the SSC reliability and TTF [1]. By so doing, the method obtains a different cumulative distribution of the reliability and TTF for each realization of the epistemic variables, thus maintaining separated the two types of uncertainty. In this way, all information on the uncertainty in the TTF is preserved, although its interpretation may not be straightforward in practical terms. Also, a probabilistic representation of the epistemic uncertainty may not be appropriate when sufficient data is not available for statistical analysis, even if one relies on expert judgments within a subjective view of probability [2], [3].

Some of the limitations of the above pure probabilistic approach to uncertainty representation and propagation can be overcome by an hybrid method in which the aleatory uncertainties are properly represented by probability distributions while the epistemic uncertainties are represented by possibility distributions [2], [3]. Monte Carlo sampling of the random variables is repeatedly performed to process the aleatory uncertainty in the behavior of the SSC and possibilistic interval analysis is carried out at each sampling to process the epistemic uncertainty. The method leads to the computation of a possibilistic random distribution representing the SSC reliability or TTF for each considered realization of the aleatory variables. Finally, the obtained possibilistic random distributions can be combined (1) into a set of limiting cumulative distributions characterized by different degrees of confidence [2] or (2) accordingly to the Dempster-Shafer Theory, into Belief and Plausibility measures that can be interpreted as “rational averages” of the above mentioned limiting cumulative distributions [3].

For exemplification, the prognostics is carried out with reference to a case study regarding a component which is randomly degrading in time according to a stochastic fatigue crack growth model of literature [4]. This degradation process is typical of, for example, components in the pressure boundaries of a nuclear power plant (NPP) [5] and may initiate from the foundry flaws which are always present in large cast components subject to fatigue loading caused by allowed power variations or incidental transients; once initiated, the degradation process can propagate under stressful operating conditions, up to limits threatening the components' structural integrity [6]. The prediction of the reliability and remaining useful lifetime of the component is based on the knowledge of the value of a degradation parameter that is assumed to be measured at the present time. If the process is such to lead its value above a given limit, then the SSC fails. The maximum level of degradation beyond which structural integrity is lost is considered to be affected by epistemic uncertainty.

The paper is organized as follows. Section 2 recalls the probabilistic and the possibilistic viewpoints for to the representation of epistemic uncertainty; little detail is given on the well known probabilistic view and its difficulty in representing epistemic uncertainty in cases of limited knowledge and information, whereas relatively more details are provided to illustrate the possibilistic view. In Section 3, the pure probabilistic method and the hybrid Monte Carlo and possibilistic method for uncertainty propagation are resumed. The case study of fatigue crack propagation is presented in Section 4 and the corresponding prognostic results are reported and commented in Section 5. Finally, some conclusions on the advantages and limitations of the two methods of uncertainty treatment are given in Section 6.

2. Representing uncertainty in prognostic tasks

When modeling the degradation and failure behavior of an SSC, uncertainty can be considered essentially of two different types: randomness due to inherent variability in the physical behavior and imprecision due to lack of knowledge and information on it. The former type of uncertainty is often referred to as objective, aleatory, stochastic whereas the latter is often referred to as subjective, epistemic, state-of-knowledge [7], [8].

The distinction between aleatory and epistemic uncertainty plays a relevant role in prognostics. The aleatory uncertainty affects the time evolution of the degradation process of the SSC, whereas epistemic uncertainty arises from incomplete knowledge of fixed but poorly known parameter values which enter the models used for the evaluation of the SSC reliability and remaining useful lifetime.

While the aleatory uncertainty is appropriately represented by probability distributions, current scientific discussions dispute the advantages and disadvantages of two possible representations of the epistemic uncertainty, probabilistic and possibilistic [1], [3].

2.1. Probabilistic representation of the epistemic uncertainty

In the current RAMS practice, epistemic uncertainty is often represented by means of probability distributions as is done for aleatory uncertainty. When sufficient data is not available for statistical analysis, one may adopt a subjective view of probability based on expert judgment. For example, in risk assessments and RAMS analyses it is common practice to represent by means of lognormal distributions the epistemic uncertainty on the parameters of the probability distributions of the TTF and time to repair (TTR) of a component [1].

However, there might be some limitations to the probabilistic representation of epistemic uncertainty under limited knowledge. Let us consider, for example, an extreme case in which the available information on a model parameter x is only that its value is located somewhere between a value $x_{\min}$ and a value $x_{\max}$. In this case, a uniform probability distribution between $x_{\min}$ and $x_{\max}$, $p(x)=1/(x_{\max}-x_{\min})$, $\forall x \in [x_{\min}, x_{\max}]$ is typically assumed to represent the uncertainty on x within a probabilistic scheme [8]. This choice of the uniform distribution appeals to i) *Laplace principle of insufficient* *reason* according to which all that is equally plausible is equally probable and to ii) the *maximum entropy approach* [9]. On the other hand, it may be criticizable since the uniform distribution implies that the degrees of probability of the different values of x depend on the equal range $X = [x_{\min}, x_{\max}]$. Further, the uniform distribution results in the following relationship [8]:

$$\Pr\left(x \in [x_{\min}, x_{mean}]\right) = \Pr\left(x \in [x_{mean}, x_{\max}]\right) \quad (1)$$

with $x_{mean} = (x_{\min} + x_{\max})/2$; but, if no information is available to characterize the uncertainty under study, then no relation between $\Pr\left(x \in [x_{\min}, x_{mean}]\right)$ and $\Pr\left(x \in [x_{mean}, x_{\max}]\right)$ should be supported.

From the above considerations, it seems that lack of knowledge may not be satisfactorily represented by specific functional choices of probability distributions, like the uniform, but it should somehow be expressed in terms of the full set of possible probability distributions on X so that the probability of a value $x \in X$ is allowed to take any value in $[0,1]$.

2.2. Possibilistic representation of the epistemic uncertainty

Due to the potential limitations associated to a probabilistic representation of epistemic uncertainty under limited information, a number of alternative representation frameworks have emerged, e.g. fuzzy set theory [10], evidence theory [11], possibility theory [12], interval analysis [13] and imprecise probability [28]. Possibility theory, in particular, is attractive for the prognostic task here of interest, because of its representation power and relative mathematical simplicity. It is similar to probability theory in that it is based on set functions but differs in that it relies on two cumulative functions, called possibility (Π) and necessity (N) measures, instead of only one, to represent the uncertainty.

The basic notion upon which the theory is founded is the possibility distribution of an uncertain variable (not necessarily random) which assigns to each real number x in a range X (Universe of Discourse, UOD) a degree of possibility $\pi(x) \in [0,1]$ of being the correct value [14].

A possibility distribution is thus an upper, semi-continuous mapping from the real line to the unit interval describing what an analyst knows about the more or less plausible values x of the uncertain variable ranging on X. These values are mutually exclusive, since the uncertain variable can take on only one true value. This also gives the normalization condition:

$$\exists x \in X : \pi(x) = 1 \quad (2)$$

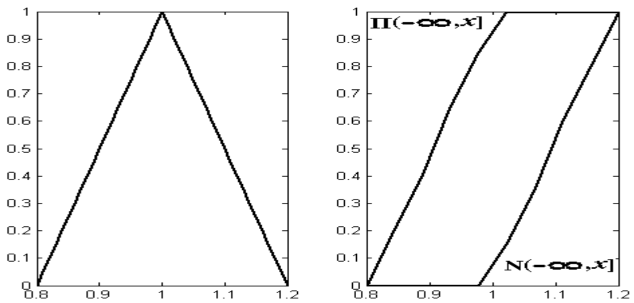
which is a claim that at least one value is viewed as totally possible, a much weaker statement than $p(x) = 1$.

The possibility and necessity measures $\Pi(A)$, $N(A)$ for all subsets A in the *power set* $P(X)$ of X are defined by the associated possibility distribution $\pi(x)$ through the following maximization and minimization relationships, respectively:

$$\Pi(A) = \sup_{x \in A} \pi(x) \quad (3)$$

$$N(A) = 1 - \Pi(\bar{A}) = \inf_{x \notin A}(1 - \pi(x)) \quad (4)$$

The relation between the possibility distribution $\pi(x)$ of a continuous uncertain variable x and the corresponding necessity and possibility measures of the set $A = (-\infty, x]$ is illustrated in Figure 1.



(a) **(b)**

Figure 1: Example of: **(a)** triangular possibilistic distribution and **(b)** corresponding possibility and necessity measures of the set $A = (-\infty, x]$

A necessity value $N(A) = 1$ means that A is certainly true and it implies that $\Pi(A) = 1$; a possibility value $\Pi(A) = 0$ means that A is impossible, certainly false and it implies that $N(A) = 0$. $\Pi(A) = 1$ means that A is possible, that it is not surprising if A occurs, whereas $N(A)$ remains unconstrained; $N(A) = 0$ means that A is unnecessary, that it is not surprising if A does not occur, whereas $\Pi(A)$ remains unconstrained.

A unimodal numerical possibility distribution may also be viewed as a set of nested confidence intervals A_α, which are the $\alpha - cuts$ of $\pi(x)$ [3]. The degree of certainty that $[\underline{a}_\alpha, \bar{a}_\alpha]$ contains the value of the uncertain variable is $N([\underline{a}_\alpha, \bar{a}_\alpha])$, which is equal to $1-\alpha$ if $\pi(x)$ is continuous. Furthermore, the family of nested confidence intervals $[(A_1, \lambda_1), (A_2, \lambda_2), \cdots, (A_m, \lambda_m)]$, where $\lambda_i = N(A_i)$, can be reconstructed from the possibility distribution $\pi(x)$ [14].

Under this view, a pair (A, λ) supplied by an expert is interpreted as stating that the *subjective probability* $p(A)$ is at least equal to λ [15]. In particular, the $\alpha - cut$ of a continuous possibility distribution can be interpreted as the inequality $\Pr(\text{uncertain variable} \in [\underline{a}_\alpha, \bar{a}_\alpha] \geq 1-\alpha)$ or equivalently as $p(\text{uncertain variable} \notin [\underline{a}_\alpha, \bar{a}_\alpha] \leq \alpha)$.

Thus, we can interpret any pair of dual necessity/possibility functions as lower and upper probabilities induced from specific probability families: degrees of necessity are equated to lower probability bounds and degrees of possibility to upper probability bounds.

Formally, let π be a possibility distribution inducing a pair of necessity/possibility functions $[N,\Pi]$. The probability family $\mathrm{P}(\pi)$ is defined as [15]:

$$\mathrm{P}(\pi)=\{p,\forall A \text{ measurable}: N(A)\leq p(A)\}=\{p,\forall A \text{ measurable}: p(A)\leq \Pi(A)\} \tag{5}$$

In this case, the probability family $\mathrm{P}(\pi)$ is entirely determined by the probability intervals it generates:

$$\sup_p p(A)=\Pi(A) \tag{6}$$

$$\inf_p p(A)=N(A) \tag{7}$$

Similarly, suppose pairs (A_i,λ_i) are supplied by an expert as subjective probabilities that $p(A_i)$ is at least equal to λ_i, where A_i is a measurable set. The probability family $\mathrm{P}(\pi)$ is defined as:

$$\mathrm{P}(\pi)=\{p,\forall A_i:\lambda_i\leq p(A_i)\} \tag{8}$$

In this view, given a continuous uncertain variable x, it is possible to interpret $N(-\infty,x]$ and $\Pi(-\infty,x]$ as the lower and upper limiting cumulative distributions $\underline{F}(x)$ and $\overline{F}(x)$, respectively [15].

To exemplify a case in which a possibilistic representation of the epistemic uncertainty may be useful, let us consider an uncertain parameter x for whose definition we know that the parameter can take values only in the range $[0.8,1.2]$ and the most likely value is 1 [16]. This case can be represented by the triangular possibility distribution reported in Figure 1(a) where to the most likely value is assigned a level of possibility equal to one while the certain interval $[0.8,1.2]$ is the “support” of the possibility distribution, implying that values located outside the interval are considered impossible.

Finally, let us return to the same case of “ignorance” discussed from a probabilistic point of view in the previous Section 2.1, and consider an uncertain model parameter x for which the only available information is that its true value is located between $x_{\min}$ and $x_{\max}$. Within the possibilistic framework, the uncertainty on x is represented by a possibility distribution $\pi(x)=1$, $\forall x\in[x_{\min},x_{\max}]$ and $\pi(x)=0$, otherwise. Such distribution simply describes the fact that any value $x\in[x_{\min},x_{\max}]$ is possible and for any set $A\in P(X)$ with $A\neq X$, $N(A)=0$ while $\Pi(A)=1$. Moreover, according to the interpretation of eq. (4), for any set $A\subseteq[x_{\min},x_{\max}]$ the only information available is that $0\leq P(A)\leq 1$.

From this example, it seems that imprecision or partial information can be adequately described in terms of possibility distributions representing families of probability distributions [14]. Actually, the connection between the possibility and probability theories could be very powerful in providing an integrated framework of representation and analysis of uncertainties of both the aleatory and epistemic type [3].

2.3. Transformation from a possibilistic distribution to a probabilistic distribution

For the purposes of comparing probability and possibility representations of uncertainty, it is useful to consider a method for transforming a possibility distribution into a probability distribution. Obviously, it remains that the probabilistic and the possibilistic representations of uncertainty are not equivalent: the possibilistic representation is weaker than the probabilistic one because it explicitly handles imprecision (e.g. incomplete knowledge) and because its measures are based on an ordinal structure rather than an additive one [17]. Remembering this, given that a possibility measure encodes a family of probability measures, the transformation aims at finding the probability measure which preserves as much as possible of the uncertainty in the original possibility distribution.

Considering the possibility distribution $\pi(x)$ on $X=[x_{\min},x_{\max}]$, its transformation into the corresponding probability distribution $p(x)$ according to [18] is:

$$p(x)=\int_0^{\pi(x)}\frac{d\alpha}{|A_\alpha|} \qquad \forall x\in[x_{\min},x_{\max}] \tag{9}$$

where $|A_\alpha|$ is the width of the $\alpha-cut\ A_\alpha$, i.e. if $A_\alpha=\left[\underline{a}_\alpha,\overline{a}_\alpha\right]$ then $|A_\alpha|=\overline{a}_\alpha-\underline{a}_\alpha$. Indeed, selecting α in $[0,1]$ is done with infinitesimal probability $d\alpha$ and yields to the uniform density p_α such that $p_\alpha(x)=1/|A_\alpha|$. For $x\in A_\alpha$ the infinitesimal probability of x in A_α is thus equal to $dx\cdot d\alpha/|A_\alpha|$ [18].

For instance, if the possibility-probability transformation is applied to the triangular possibility distribution of Figure 1(a) with support $[-1,+1]$ and modal value 0, then the probability density function $p(x)=-1/2Log(x)$ is obtained [18], [19].

3. Uncertainty propagation

Uncertainty in the input variables and parameters of a model must be adequately propagated to the output, to provide the relevant representation of its uncertainty.

Let us consider a model whose output is a function $f(\cdot)$ of n uncertain variables Y_j, $j=1,\cdots,n$, of which the first k are affected by aleatory uncertainty and the last $n-k$ are affected by epistemic uncertainty. This Section briefly presents two methods for the propagation of such a mixed uncertain information: (1) a purely probabilistic method and (2) an hybrid Monte Carlo – Possibilistic method.

3.1. The Purely Probabilistic Method of Uncertainty Propagation

The uncertainty affecting both the aleatory variables Y_j, $j=1,...,k$ and the epistemic variables Y_j, $j=k+1,...,n$ is represented by known probabilistic density functions $p_{Y_i}(y_i)$, $i=1,...,n$.

The propagation of the uncertainty in the input variables through the function $f(\cdot)$ is based on a double-randomization Monte Carlo simulation built on two sampling loops [1]: (1) the epistemic variables are sampled in the outer loop and (2) the aleatory variables are sampled in the inner loop.

The operative steps of the procedure are:

1. sample the i_e-*th* realization $\left(y_{k+1}^{i_e},...,y_n^{i_e}\right)$ of the epistemic variables $\left(Y_{k+1},...,Y_n\right)$ from their respective distributions $p_{Y_{k+1}}(y_{k+1})$, …, $p_{Y_n}(y_n)$;
2. sample the i_a-*th* realization $\left(y_1^{i_e,i_a},...,y_k^{i_e,i_a}\right)$ of the aleatory variables from their respective distributions $p_{Y_1}(y_1)$, …, $p_{Y_k}(y_k)$;
3. compute the output of the model corresponding to the i_e-th realization of the epistemic variables and the i_a-th realization of the aleatory variables:
$$u^{i_e,i_a} = f\left(y_1^{i_e,i_a},...,y_k^{i_e,i_a},y_{k+1}^{i_e},...,y_n^{i_e}\right);$$
4. repeat steps 2 and 3 m_a times. Then, the cumulative distribution $F^{i_e}\left(u\middle|y_{k+1}^{i_e},...,y_n^{i_e}\right)$ of the output u of the model, conditioned by the sampled values $\left(y_{k+1}^{i_e},...,y_n^{i_e}\right)$ of the epistemic variables, is estimated from u^{i_e,i_a}, $i_e = 1,...,m_e$, $i_a = 1,...,m_a$.
5. repeat steps 2 to 4 m_e times to generate m_e cumulative distributions $F^{i_e}\left(u\middle|y_{k+1}^{i_e},...,y_n^{i_e}\right)$ of the model output.

Notice that the purely probabilistic method keeps separated the two types of uncertainty: the aleatory uncertainty is captured in the slope of the obtained cumulative distributions of u, the epistemic one is contained in the spectrum of different cumulative distributions of u. On the other hand, the interpretation of the set of cumulative distributions obtained may not be straightforward in practical terms given the difficulty of extracting concise information on the output uncertainty from such a representation. In this respect a possible way to represent the epistemic uncertainty is to fix a given percentile of the cumulative distributions $F^{i_e}\left(u\middle|y_{k+1}^{i_e},...,y_n^{i_e}\right)$, i.e. 95%, and then build the probabilistic distribution of this percentile from the different realizations of the epistemic variables.

3.2. An hybrid Monte Carlo and possibilistic method for uncertainty propagation

The hybrid Monte Carlo and possibilistic method for uncertainty propagation considers that the uncertainty on the k random variables (aleatory uncertainty) is represented by the probability distributions $p_{Y_j}(y)$ and the uncertainty on the $n-k$ possibilistic variables (epistemic uncertainty) is represented by the possibility distributions $\pi_{Y_j}(y)$ measuring the degree of possibility that the linguistic variables Y_j be equal to *y*. For the propagation of such mixed uncertainty information, the Monte Carlo technique [20] can be combined with the possibility theory [21] by means of the following two main steps [3]:

i. repeated Monte Carlo sampling of the random variables to process aleatory uncertainty;

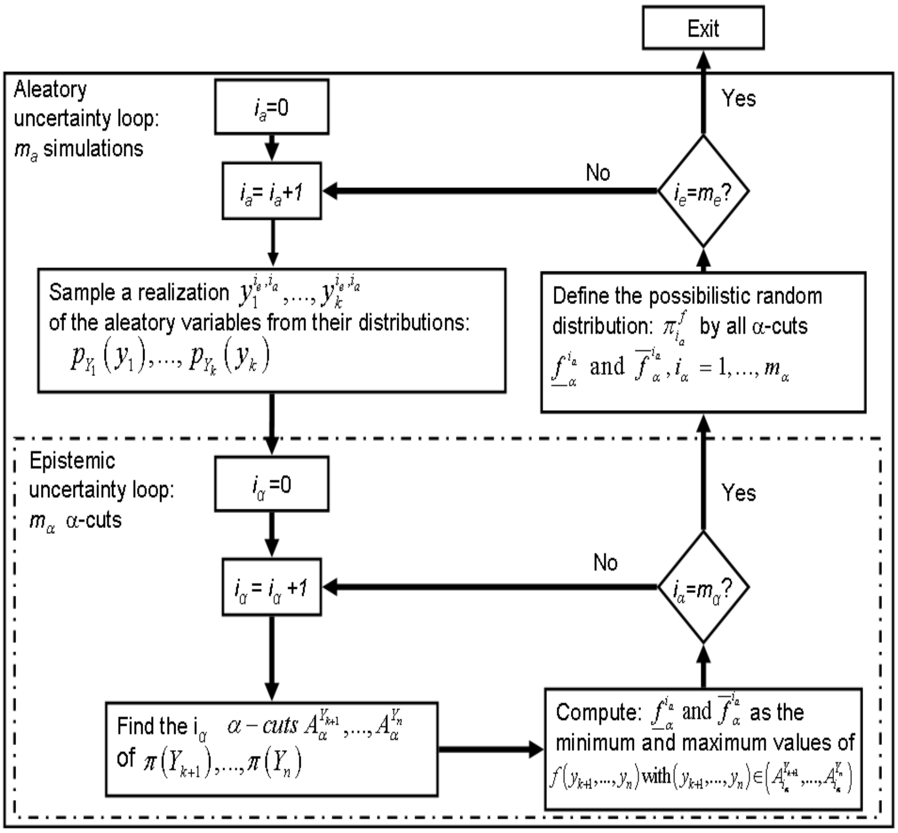


Figure 2: The hybrid method for uncertainty propagation

ii. possibilistic interval analysis to process the epistemic uncertainty.

For the generic $i_a - th \quad k - tuple$ of random values, $i_a = 1,..., m_a$, sampled by Monte Carlo from the probability distributions of the aleatory variables, a possibilistic distribution $\pi_{i_a}^f$ estimate of $f(Y)$ is constructed by fuzzy interval analysis. After m_a repeated samplings of the aleatory variables, the possibilistic set estimates $\pi_{i_a}^f$, $i_a = 1,..., m_a$ are combined to give an estimate of $f(Y)$ as a random possibility distribution in the sense of [22].

The operative steps of the procedure are (Figure 2):

1. sample the $i_a - th$ realization $\left(y_1^{i_a}, \cdots, y_k^{i_a}\right)$ of the probabilistic variable vector $\left(Y_1, \cdots, Y_k\right)$;
2. select a possibility value $\alpha \in [0,1]$ and the corresponding $\alpha - cuts$ of $A_\alpha^{Y_{k+1}},..., A_\alpha^{Y_n}$ the possibility distributions $\left(\pi_{Y_{k+1}}, \cdots, \pi_{Y_n}\right)$ as intervals of possible values of the possibilistic variables $\left(Y_{k+1}, \cdots, Y_n\right)$;

3. compute the smallest and largest values of $f\left(y_1^{i_a},\cdots,y_k^{i_a},Y_{k+1},\cdots,Y_n\right)$, denoted by $\underline{f}_\alpha^{i_a}$ and $\overline{f}_\alpha^{i_a}$ respectively, considering the fixed values $\left(y_1^{i_a},\cdots,y_k^{i_a}\right)$ sampled for the random variables $\left(Y_1,\cdots,Y_k\right)$ and all values of the possibilistic variables $\left(Y_{k+1},\cdots,Y_n\right)$ in the $\alpha-cuts$ $A_\alpha^{Y_{k+1}},...,A_\alpha^{Y_n}$ of their possibility distributions $\left(\pi_{Y_{k+1}},\cdots,\pi_{Y_n}\right)$. Then, take the extreme values $\underline{f}_\alpha^{i_a}$ and $\overline{f}_\alpha^{i_a}$ as the lower and upper limits of the $\alpha-cut$ of $f\left(y_1^{i_a},\cdots,y_k^{i_a},Y_{k+1},\cdots,Y_n\right)$;
4. return to step 2 and repeat for another α-cut; after having repeated steps 2-3 for all the m_α α-cuts of interest, the fuzzy random realization (fuzzy interval) $\pi_{i_a}^f$ of $f\left(Y\right)$ is obtained as the collection of the values $\underline{f}_\alpha^{i_\alpha}$ and $\overline{f}_\alpha^{i_\alpha}$; in other words, $\pi_{i_a}^f$ is defined by all its α-cut intervals $\left[\underline{f}_\alpha^{i_a},\overline{f}_\alpha^{i_a}\right]$;
5. return to step 1 to generate a new realization of the random variables. After having repeated steps 1-4 for m_a times, exit.

At the end of the procedure an ensemble of realizations of random possibility distributions is obtained, i.e. $\left(\pi_1^f,...,\pi_{m_a}^f\right)$. In this work, the information contained in the obtained random possibility distributions $\left(\pi_1^f,...,\pi_{m_a}^f\right)$ are aggregated according to two different methods: (1) the joint aggregation method [3] and (2) the Ferson aggregation method [2].

3.2.1. The joint aggregation method

First of all, for each set $A\in P\left(X\right)$, at each realization of the aleatory variables $i_a=1,...,m_a$ it is possible to obtain the possibility measure $\Pi_{i_a}^f(A)$ and the necessity measure $N_{i_a}^f(A)$ from the corresponding possibility distribution $\pi_{i_a}^f\left(u\right)$ obtained at the end of the previous Section by using eqs. (3) and (4).

Then, the m_a different realizations of possibility and necessity of measures $N_{i_a}^f\left(A\right)$ and $\Pi_{i_a}^f\left(A\right)$ can be combined to obtain the believe $Bel\left(A\right)$ and the plausibility $Pl(A)$ measures by using, respectively [3]:

$$Bel\left(A\right)=\sum_{i_a}p_{i_a}N_{i_a}^f\left(A\right) \tag{10}$$

$$Pl\left(A\right)=\sum_{i_a}p_{i_a}\Pi_{i_a}^f\left(A\right) \tag{11}$$

where p_{i_a} is the probability of sampling the i_a-th realization $\left(y_1^{i_a},\cdots,y_k^{i_a}\right)$ of the random variable vector $\left(Y_1,\cdots,Y_k\right)$.

For each set A, this technique thus computes the probability-weighted average of the possibility measures associated with each output fuzzy interval. Considering a generic value u of $f\left(Y\right)$, a pair of bounding, average cumulative distributions

$\underline{F}(u) = Bel\left(f(Y) \in (-\infty, u]\right)$, $\bar{F}(u) = Pl\left(f(Y) \in (-\infty, u]\right)$ can then be computed [3]. Let the *core* and the *support* of a possibilistic distribution $\pi^f(u)$ be the crisp sets of all points of X such that $\pi^f(u)$ is, respectively, equal to 1 and non zero. $Pl\left(f(Y) \in (-\infty, u]\right) = 1$ if and only if $\Pi_i^f\left(f(Y) \in (-\infty, u]\right) = 1, \forall i = 1, \cdots, m$, i.e. for $u > u^* = \max_i\left\{\inf\left(core\left(\pi_i^f\right)\right)\right\}$. Similarly, $Pl\left(f(Y) \in (-\infty, u]\right) = 0$ if and only if $\Pi_i^f\left(f(Y) \in (-\infty, u]\right) = 0, \forall i = 1, \cdots, m$, i.e. for $u \le u_* = \min_i\left\{\inf\left(support\left(\pi_i^f\right)\right)\right\}$.

3.2.2. The Ferson aggregation method

In [2], a different technique for the aggregation of the random possibility distributions $\pi_1^f, ..., \pi_{m_a}^f$ obtained in Section 3.2.1 is proposed.

The procedure, called Ferson aggregation method, follows the next steps [2], [3]:

(1) Select a possibility value α and consider the corresponding $\alpha - cut$ $A_\alpha^{i_a} = \left[\underline{f}_\alpha^{i_a}, \overline{f}_\alpha^{i_a}\right]$ of the possibility distribution $\pi_{i_a}^f(u)$.

(2) Define a new possibility distribution $\pi_{i_a}^\alpha(u)$:

$$\begin{cases} \pi_{i_a}^\alpha(u) = 1 \text{ if } u \in A_\alpha^{i_a} \\ \pi_{i_a}^\alpha(u) = 0 \text{ otherwise} \end{cases} \tag{12}$$

(3) Compute the possibility and the necessity measures associated to $\pi_{i_a}^\alpha$, $\Pi_{i_a}^\alpha(A)$ and $N_{i_a}^\alpha(A)$ respectively, for any set $A = (-\infty, u]$ according to eqs. (3) and (4). From a practical point of view, this is equivalent to computing the possibility and the necessity measures as:

$$\Pi_{i_a}^\alpha\left((-\infty, u]\right) = \begin{cases} 1, \text{ if } u \ge \underline{f}_\alpha^{i_a} \\ 0, \text{ otherwise} \end{cases} \tag{13}$$

$$N_{i_a}^\alpha\left((-\infty, u]\right) = \begin{cases} 1, \text{ if } u \ge \overline{f}_\alpha^{i_a} \\ 0, \text{ otherwise} \end{cases} \tag{14}$$

(4) Repeat steps 1-3 for all $i_a = 1, ..., m_a$ and finally compute an upper ($\overline{F_\alpha}$) and a lower ($\underline{F}_\alpha$) cumulative distributions for the α -cut chosen at step 1:

$$\overline{F_\alpha}(u) = \sum_{i_a} p_{i_a} \Pi_{i_a}^\alpha\left((-\infty, u]\right) \tag{15}$$

$$\underline{F}_\alpha(u) = \sum_{i_a} p_{i_a} N_{i_a}^\alpha\left((-\infty, u]\right) \tag{16}$$

Notice that since $A_1^{i_a} \subseteq A_\alpha^{i_a} \subseteq A_0^{i_a}$ then:

$$\Pi_{i_a}^1\left((-\infty, u]\right) \le \Pi_{i_a}^\alpha\left((-\infty, u]\right) \le \Pi_{i_a}^0\left((-\infty, u]\right) \tag{17}$$

$$N_{i_a}^1\left((-\infty, u]\right) \le N_{i_a}^\alpha\left((-\infty, u]\right) \le N_{i_a}^0\left((-\infty, u]\right) \tag{18}$$

and thus:

$$\overline{F}_{i_a}^{1}\left(u\right) \le \overline{F}_{i_a}^{\alpha}\left(u\right) \le \overline{F}_{i_a}^{0}\left(u\right) \tag{19}$$

$$\underline{F}_{i_a}^{1}\left(u\right) \le \underline{F}_{i_a}^{\alpha}\left(u\right) \le \underline{F}_{i_a}^{0}\left(u\right) \tag{20}$$

This kind of representation captures the aleatory variability and epistemic imprecision of a random fuzzy interval in a parameterized way through the $\alpha - cuts$ and displays extreme pairs of cumulative distributions $\overline{F_\alpha}$ and $\underline{F}_\alpha$. The gap between $\overline{F_\alpha}$ and $\underline{F}_\alpha$ represents the imprecision due to epistemic variables; the slopes of $\overline{F_\alpha}$ and $\underline{F}_\alpha$ characterize the variability of the results caused by the aleatory uncertainty. A confident user who assumes high precision would work with the cores of the fuzzy intervals ($\alpha - cut = 1$), whereas a cautious one may choose $\alpha - cut = 0$ to work with their supports.

For a comparison between the Ferson and the joint aggregation methods (Section 3.2.1), consider that the latter produces upper and lower cumulative distributions which span the ranges between $\overline{F_0}$ and $\overline{F_1}$ and between $\underline{F}_0$ and $\underline{F}_1$, respectively:

$$\overline{F_1}\left(u\right) \le Pl\left(\left(-\infty, u\right]\right) \le \overline{F_0}\left(u\right) \tag{21}$$

$$\underline{F}_0\left(u\right) \le Bel\left(\left(-\infty, u\right]\right) \le \underline{F}_1\left(u\right) \tag{22}$$

In particular, in [23] it is demonstrated that the plausibility and belief measures obtained by the joint aggregation method are the mean values of the upper and lower boundary cumulative distributions of the Ferson method, respectively:

$$Pl\left(\left(-\infty, u\right]\right) = \int_0^1 \overline{F}_\alpha\left(u\right) d\alpha \tag{23}$$

$$Bel\left(\left(-\infty, u\right]\right) = \int_0^1 \underline{F}_\alpha\left(u\right) d\alpha \tag{24}$$

Finally notice that also the Ferson method treats aleatory and epistemic uncertainties separately as does the purely probabilistic method (Section 3.1). Nevertheless, it uses a probabilistic representation for the aleatory uncertainty and a possibilistic representation for the epistemic uncertainty.

4. Case study

Let us consider the process of crack growth in a component subject to fatigue. The common Paris-Erdogan model is adopted for describing the evolution of the crack depth h as a function of the load cycles L [24]:

$$\frac{dh}{dL} = C(\Delta K)^\eta \tag{25}$$

where C and η are constants related to the material properties [25], [26], which can be estimated from experimental data [27] and ΔK is the stress intensity amplitude, roughly proportional to the square root of h [26]:

$$\Delta K = \beta\sqrt{h} \tag{26}$$

where β is again a constant which may be determined from experimental data.

The intrinsic stochasticity of the process may be inserted in the model by modifying eq. (26) as follows [26]:

$$\frac{dh}{dL} = e^{\omega} C(\beta\sqrt{h})^{\eta} \tag{27}$$

where $\omega \sim N(0, \sigma_{\omega}^2)$ is a white Gaussian noise. For ΔL sufficiently small, the state-space model (30) can be discretized to give:

$$h_k = h_{k-1} + e^{\omega_k} C(\Delta K)^{\eta} \Delta L \tag{28}$$

which represents a non-linear Markov process with independent, non-stationary degradation increments.

This degradation process is typical of, for example, components in the pressure boundaries of a nuclear power plant [5] and may initiate from the foundry flaws which are always present in large cast components subject to fatigue loading caused by allowed power variations or incidental transients; once initiated, the degradation process can propagate under stressful operating conditions, up to limits threatening the components' structural integrity [6].

In this work, it is assumed that the level of degradation affecting the component is indicated by the crack depth h. At each time step, the degradation level can either remain constant or be incremented according to eq. (28). The probability p_d that the component degrades between two successive time instants is assumed to be known.

The component is designed to accomplish its function for a certain time window, called mission time T_{miss} and it is considered failed when the value of the degradation exceeds a certain threshold, $H_{\max}$. With respect to the degradation threshold $H_{\max}$, its value is affected by epistemic uncertainty due to limited information: according to an expert, $H_{\max}$ lies between 9 and 11 with a most likely value of 10, in arbitrary units.

Table 1 reports the values of the parameters defining the degradation process considered in this work.

Table 1. Numerical values of the case study parameters

Parameter	Value
ω_k	$N(0,1.7)$
C	0.005
β	1
η	1.3
p_d	0.25
T_{miss}	500 time units
$H_{\max}$	$\in [9,11]$, most likely value = 10

5. Prognostic Results

In this Section, the methods for uncertainty representation and propagation described in Section 3 are applied to the case study introduced in Section 4 to address the associated prognostic task of predicting the component performance during the mission time. The problem of estimating the component reliability, i.e. the probability that the component survives until the mission time, is firstly addressed in Section 5.1. Then, in Section 5.2, the problem of estimating the TTF, i.e. the remaining lifetime from the present instant until the component fails, is considered. Both estimations are performed assuming that at the present time t_p a measurement of the current degradation level is available: this is given by the crack depth $h(t_p)$. The results that follow refer to the case $h(1)=1$, in arbitrary units.

5.1. Reliability estimation

Let us introduce an uncertain Boolean variable X_S indicating the state of the component at the mission time by 0 (failure) and 1 (success). Such variable depends on the crack depth at the mission time $h(T_{miss})$, which is affected by aleatory uncertainty, and on the failure threshold $H_{\max}$, which is affected by epistemic uncertainty:

$$X_S = f(h(T_{miss}), H_{\max}) = \begin{cases} 1 & \text{if} \quad h(T_{miss}) < H_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

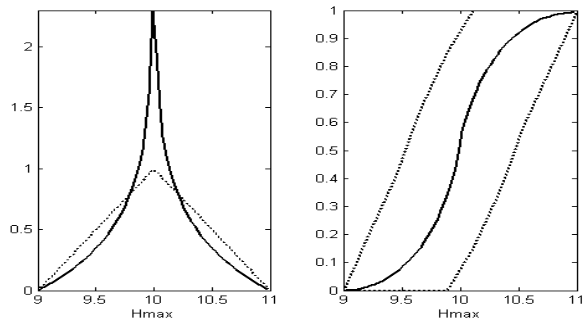
Assuming that no repairs are allowed on the component, the objective is to estimate the component reliability, i.e. the probability that the component has not failed prior to the mission time, i.e. $p(X_S = 1)$.

5.1.1. Application of the hybrid Monte Carlo-possibilistic method

Following the example of Section 2.2, the epistemic uncertainty associated to the available scarce information on the value of the threshold $H_{\max}$ is represented by the triangular possibility distribution $\pi(H_{\max})$ of Figure 3(a) (dotted line); Figure 3(b) reports the corresponding limiting cumulative function.

The hybrid procedure of Section 3.2 for the propagation of the uncertainty in the estimation of the component $p(X_S = 1)$ is here followed step by step:

1. with respect to the aleatory degradation process, $m_a = 10^4$ realizations have been simulated by Monte Carlo and the crack depths at the mission time $h(T_{miss})$ have been collected. Figure 4 reports an example of 4 simulated realizations of the degradation process. Then, for each simulated degradation process, steps 2-4 below have been performed;



(a) **(b)**

Figure 3: (a) Possibilistic distribution of $H_{\max}$ (dotted line) and its probabilistic transformation (continuous line); (b) corresponding necessity and possibility measures of the set $(-\infty, H_{\max}]$ (dotted line) and cumulative distribution function of $H_{\max}$ considered by the purely probabilistic method (continuous line).

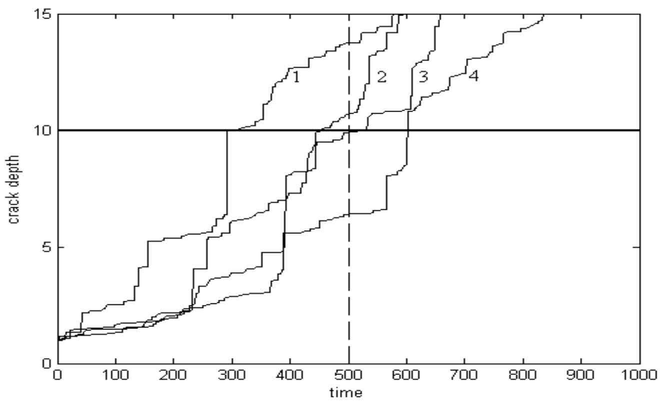


Figure 4: Four different realizations of degradation evolution

2. with respect to the degradation threshold $H_{\max}$, $m_\alpha = 21$ values of α have been considered $(\alpha = 0, 0.05, ..., 1)$; the $\alpha - cuts$ $\left[\underline{H}_{\max}^{\alpha}, \overline{H}_{\max}^{\alpha}\right]$ of the possibilistic distribution of $H_{\max}$ have been found and for each $\alpha - cut$, steps 3-5 below have been performed;
3. for the $i_a - th$ simulation of the degradation evolution and the $\alpha - cut$ of $\pi(H_{\max})$, the smallest $\underline{f}_{\alpha}^{i_a}$ and largest $\overline{f}_{\alpha}^{i_a}$ values of the state $X_S = f(h_{i_a}(T_{miss}), H_{\max})$ of the component at the mission time have been computed considering the fixed value of $h_{i_a}(T_{miss})$ obtained in step 1 and all values of $H_{\max}$ in the considered $\alpha - cut$ intervals $\left[\underline{H}_{\max}^{\alpha}, \overline{H}_{\max}^{\alpha}\right]$. In particular, three cases may occur (Figure 5, first column): (1) $h_{i_a}(T_{miss}) < \underline{H}_{\max}^{\alpha}$ resulting in $\underline{f}_{\alpha}^{i_a} = \overline{f}_{\alpha}^{i_a} = 1$ (Figure 5, first row); (2) $h_{i_a}(T_{miss}) > \overline{H}_{\max}^{\alpha}$ resulting in $\underline{f}_{\alpha}^{i_a} = \overline{f}_{\alpha}^{i_a} = 0$ (Figure 5, fourth row); (3) $\underline{H}_{\max}^{\alpha} < h_{i_a}(T_{miss}) < \overline{H}_{\max}^{\alpha}$ resulting in $\underline{f}_{\alpha}^{i_a}$ =0, $\overline{f}_{\alpha}^{i_a}$ =1 (Figure 5, second and third rows);
4. after repeating step 3 for all the $m_\alpha = 21$ $\alpha - cuts$, the possibilistic random distribution $\pi_{i_a}^{f}$ is constructed as the collection of its 21 $\alpha - cuts$ $\left[\underline{f}_{\alpha}^{i_a}, \overline{f}_{\alpha}^{i_a}\right]$. Figure 5, second column, reports the $\pi_{i_a}^{f}$ obtained for the four simulated degradation evolutions of Figure 4:

- the realization 1 (fourth row in Figure 5) ends with $h(T_{miss}) = 13.74$ and thus corresponds to a failed component independently from the considered $\alpha - cut$ of $H_{\max}$; this leads to a possibility distribution equal to a singleton in 0;
- in the simulation of the second and third realizations (second and third rows of Figure 4), the state at the mission time depends on the considered $\alpha - cut$ of $H_{\max}$; in particular, for small α the state is uncertain as represented by a possibility larger

than 0 of both success and failure, while for large values of α the state is of failure in the second realization simulation and success in the third one, which end with $h(T_{miss}) = 9.88$ and 10.70, respectively;

- the realization 4 ends with a crack depth $h(T_{miss}) = 6.37$ which corresponds to a success state independently of the considered $\alpha - cut$ of $H_{\max}$, i.e. to a possibility distribution which is singleton in 1;

5. steps 2 to 4 have been repeated for $m_a = 10^4$ realizations of the degradation process obtaining $m_a = 10^4$ corresponding possibilistic random distributions $\pi_{i_a}^f$, $i_a = 1,...,10^4$;
6. the obtained 10^4 distributions $\pi_{i_a}^f$, $i_a = 1,...,10^4$ are aggregated by the joint aggregation or Ferson methods (Sections 3.2.1 and 3.2.2, respectively).

According to the joint aggregation method (Section 3.2.1), the believe and plausibility are computed by simple averaging of the necessity N and possibility Π measures (Figure 5, column 4 and 3, respectively) obtained in the different simulations, i.e.:

$$Bel(success) = Bel(X_S = 1) = \frac{\sum_{i_a=1}^{m_a} N_{i_a}^f (success)}{m_a}$$

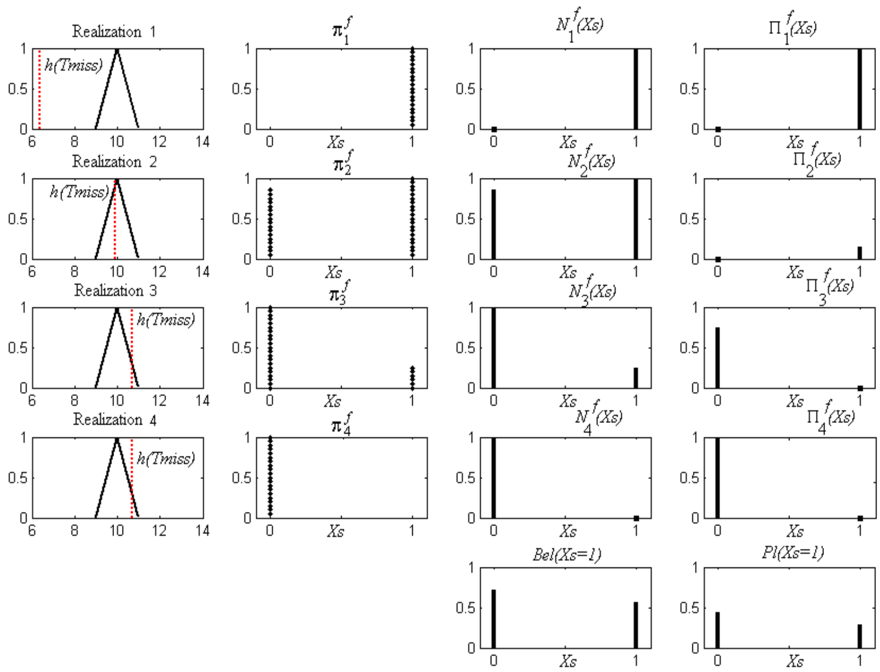


Figure 5: Application of the hybrid method of uncertainty representation (columns 1-2) and joint aggregation method for the uncertainty propagation (column 3-4), with regards to the four realizations of the degradation process shown in Figure 4

$$Pl(success) = Pl(X_S = 1) = \frac{\sum_{i_a=1}^{m_a} \Pi_{i_a}^f (success)}{m_a}$$

where the $N_{i_a}^f(success)$ and the $\Pi_{i_a}^f(success)$ are obtained by eqs. (3) and (4).

From the $m_a = 10^4$ simulations it turns out that $Bel(success) = 0.896$ and $Pl(success) = 0.927$. This result merges the aleatory uncertainty represented by the different realizations of the stochastic degradation process with the epistemic uncertainty represented by the possibilistic random distributions $\pi_{i_a}^f$, $i_a = 1,...,10^4$. In line with the interpretation of the believe and plausibility functions as limiting probability values, one may say that the probability that the non-repairable component survives the degradation process up to the mission time is between 0.896 and 0.927.

Alternatively, according to Ferson aggregation method (Section 3.2.2), firstly a new possibility distribution $\pi_{i_a}^\alpha$ is obtained from $\pi_{i_a}^f$ using eq. (12), $i_a = 1,...,10^4$. With reference for example to the $0.95-cut$ (dotted line in Figure 6, second column) the new possibility distributions $\pi_{i_a}^{0.95}$ for the four random realizations of the degradation process of Figure 4 are represented in Figure 6 third column. Notice that the first and second simulated degradation evolutions (first and second row of Figure 6) are characterized by a $0.95-cut$ of $\pi_{i_a}^f$ given only by $X_S = 1$, thus leading to $\pi_1^{0.95}$ and $\pi_2^{0.95}$ being a singleton in $X_S = 1$; on the contrary, the third and fourth simulated degradation processes are characterized by a $0.95-cut$ of $\pi_{i_a}^f$ formed only by $X_S = 0$, thus leading to $\pi_3^{0.95}$ and $\pi_4^{0.95}$ being a singleton in $X_S = 0$.

The third and fourth columns of Figure 6 show the necessity and possibility measures corresponding to $\pi_{i_a}^f$.

Finally, the lower and upper cumulative discrete distributions $\underline{F}^{0.95}(X_S)$ and $\overline{F}^{0.95}(X_S)$ are obtained as means of the values of $N_{i_a}^{0.95}[0, X_S]$ and $\Pi_{i_a}^{0.95}[0, X_S]$ for the 4 simulated degradation processes (last row of Figure 6). Notice that the limiting interval for the probability of $X_S = 0$ is $[\underline{F}(0), \overline{F}(0)]$ and thus the limiting interval for the probability of $X_S = 1$ is $[1-\underline{F}(0), 1-\overline{F}(0)]$.

Applying the Ferson aggregation method to all $m_a = 10^4$ random possibility distributions $\pi_{i_a}^f(X_S)$, $i_a = 1,...,10^4$ and considering all the 21 $\alpha-cuts$ of $\pi(H_{\max})$ $(\alpha = 0, 0.05,...,1)$, the limiting bounds of the component reliability reported in Figure 7(a) are obtained. The ordinates report the values of α in correspondence of the upper and lower bounds of the component reliability values in abscissa.

In this case, to interpret the results it is necessary to choose an $\alpha-cut$ $[\underline{H}_{\max}^\alpha, \overline{H}_{\max}^\alpha]$ of $\pi(H_{\max})$. In particular, the degree of certainty that $[\underline{H}_{\max}^\alpha, \overline{H}_{\max}^\alpha]$

contains the value of the uncertain value $H_{\max}$ is $N\left(\left[\underline{H}_{\max}^{\alpha}, \overline{H}_{\max}^{\alpha}\right]\right)$, which is equal to $(1-\alpha)$. In this view, the $0-cut$ of $\pi(H_{\max})$ should be considered if one wants to be completely sure (i.e., with degree of certainty equal to 1) to include the correct value of $H_{\max}$ in the propagation of the uncertainty, which gives rise to a component reliability between $[0.873, 0.940]$. Otherwise, if the analyst wants to reduce the epistemic uncertainty on the component reliability, then he/she should reduce $(1-\alpha)$ (the degree of certainty). For example, considering a $0.95-cut$, the reliability lies in the interval $[0.912, 0.916]$.

5.1.2. Application of the purely probabilistic method

For applying the purely probabilistic method, firstly the probability distribution describing the epistemic uncertainty on $H_{\max}$ is obtained by applying the transformation described in Section 2.3 to the triangular possibility distribution considered in the previous Section (Figure 3(a), dotted line). The obtained probability density function is shown in Figure 3(a) (continuous line) and the corresponding cumulative distribution function in Figure 3(b) (continuous line). The propagation of the uncertainty can then be performed according to the double-randomization method of Section 3.1 characterized by a separation between the epistemic uncertainty (outer randomization loop) and the aleatory uncertainty (inner randomization loop). Upon performing $m_a = 10^4$ simulations of the degradation process for each of the $m_e = 10^3$ values of $H_{\max}$ sampled from the probability distribution of Figure 3, the obtained 10^3 component reliabilities are used to build the cumulative function of Figure 7(b).

For the interpretation of the results, it is possible to choose a level of confidence

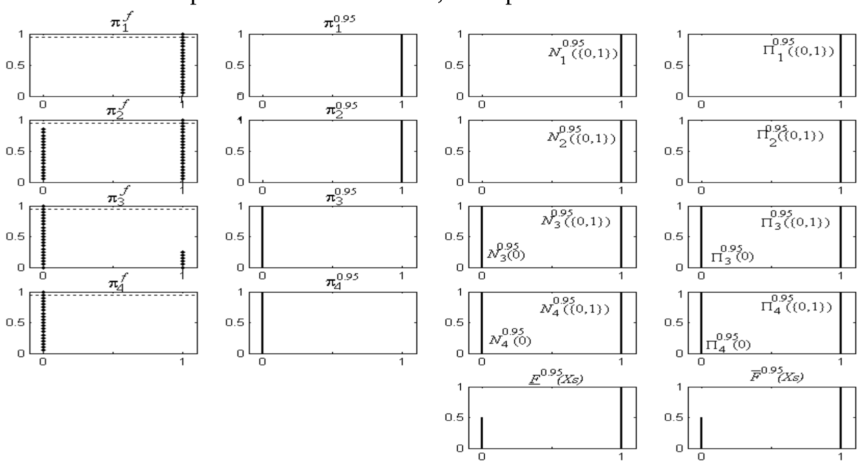


Figure 6: Application of the hybrid method of uncertainty representation (column 1) and Ferson aggregation method for the uncertainty propagation (column 2-4), with regards to the four realizations of the degradation process shown in Figure 4

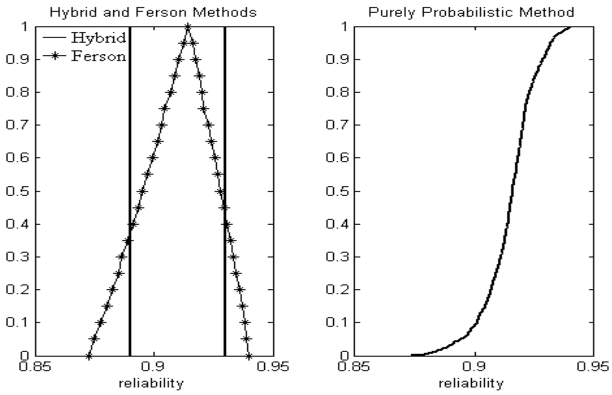


Figure 7: Comparison of the results obtained with: (a) joint aggregation method (continuous line), Ferson aggregation method (asterisk) and (b) purely probabilistic method

$(1-\beta)$ and then provide the confidence interval given by the percentiles $\left(\frac{\beta}{2}\right)$ and $\left(1-\frac{\beta}{2}\right)$. For example for a confidence of $(1-\beta)=0.95$, the resulting confidence interval for the component reliability is $[0.889, 0.935]$.

5.1.3. Comparison of the results

Let us first compare the representation of the available uncertain information given by the probabilistic and possibilistic methods. Figure 7(b) shows that while the possibilistic representation of the uncertainty on $H_{\max}$ allows us to consider all the possible cumulative distributions lying between the believe and the plausibility measures of the set $A=[0,u)$, the probabilistic representation forces us to consider only one specific cumulative distribution, albeit conservatively the one containing as much uncertainty as possible (thanks to the transformation of Section 2.3). In this application characterized by a rather limited available information on the value of $H_{\max}$, the probabilistic density function tends for this reason to force information in the representation.

With respect to the uncertainty propagation and representation techniques, notice that both the purely probabilistic and the Ferson methods keep separate the epistemic uncertainty from the aleatory uncertainty whereas the hybrid aggregation method merges them. From a practical point of view, the interpretation of the results obtained by the hybrid method is straightforward (the component reliability is between [0.896,0.927]), while the purely probabilistic and the Ferson methods require to fix the level of confidence that the analyst wants in the estimation. Notice, however, that the interpretation of the parameter $(1-\alpha)$ of the Ferson method and the parameter $(1-\beta)$ of the purely probabilistic method is different, the former indicating a degree of certainty that the interval of values of $H_{\max}$ that is considered in the uncertainty propagation contains the true $H_{\max}$, the latter being amenable of the classical interpretation of confidence level in statistics. Hence, fixing $(1-\beta)$ does not imply a choice on the interval of values of the epistemic variable that will be propagated further in the analysis, whereas fixing $(1-\alpha)$ does not imply a choice on the probability that the obtained confidence interval of the success probability will contain its true value. Due to the different interpretation of the parameters, fixing a confidence level, for example $(1-\alpha)=(1-\beta)=0.95$, leads to different intervals for the component reliability, i.e. [0.875,0.940] for the Ferson method and [0.889,0.935] for the purely probabilistic method.

5.2. Time To Failure estimation

If the reliability prediction made at time t_p on the basis of the crack depth measurement $h(t_p)$ is not satisfactory because the probability that the component fails to perform its function up to T_{miss} is too low, then one should proceed to the estimation of the component TTF in order to establish a proper maintenance intervention or component replacement. Given that in the case study considered in Section 4 a component is functioning until the value of its degradation $h(t)$ exceeds the value of the failure upper threshold $H_{\max}$, the TTF is a function of the crack depth time evolution $h(t)$, of the value of the failure threshold $H_{\max}$ and of the present time value t_p at which the crack depth measurement $h(t_p)$ is collected:

$$TTF = f\left(h(t), H_{\max}, t_p\right) = t_e - t_p, \text{ where } t_e = \min\left(\left\{t \middle| h(t) \geq H_{\max}\right\}\right) \tag{30}$$

5.2.1. Application of the hybrid Monte Carlo-possibilistic method

Considering as before the triangular possibility distribution $\pi(H_{\max})$ of Figure 3(a) for representing the epistemic uncertainty associated to the information on the value of the threshold $H_{\max}$ and the measurement of crack depth $h(1)=1$, the hybrid procedure of Section 3.2 for the propagation of the uncertainty in the estimation of the $TTF = f(h(t), H_{\max}, t_p)$ proceeds along the same sequence of computational steps 1-6 illustrated in Section 5.1 for the estimation of the component reliability.

Figure 8, first column, reports the $\pi_{i_a}^f$ of the TTF obtained for the four simulated degradation evolutions of Figure 4.

By the joint aggregation method of Section 3.2.1, the believe and plausibility functions are again obtained by simple averaging of the necessity N and possibility Π measures obtained in the different $m_a = 10^4$ simulations (Figure 9(a)).

For a confidence of $\beta = 0.95$, given that $Pl([0,962]) = 0.95$ and $Bel([0,1015]) = 0.95$, it is possible to conclude that the upper limit of the 95% one sided-confidence interval of the TTF is between $[962,1015]$. Notice that the width of this interval is caused by the epistemic uncertainties on the parameters $H_{\max}$, i.e. if $H_{\max}$ were perfectly known, then $Bel([0,t])$ and $Pl([0,t])$ would coincide. Furthermore, the analyst has to choose only a confidence parameter related to the aleatory uncertainty, i.e. the

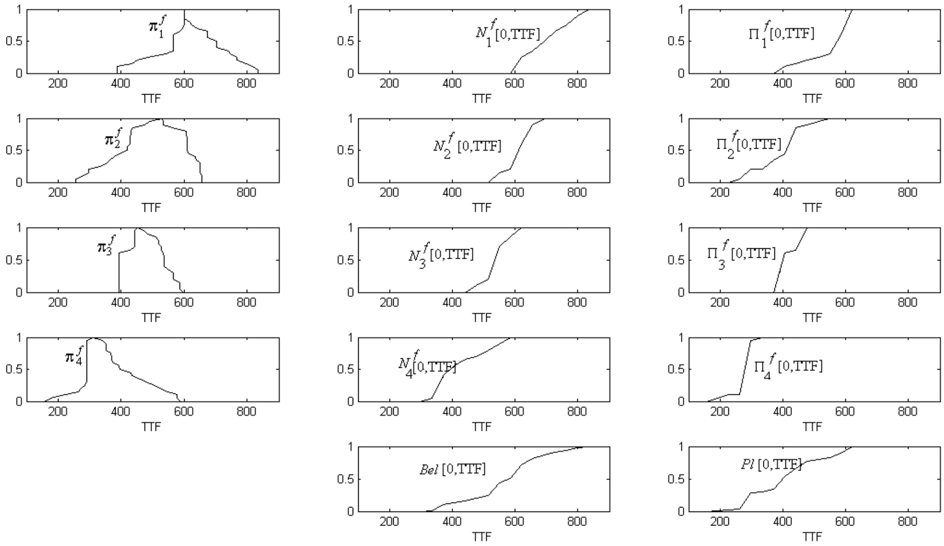
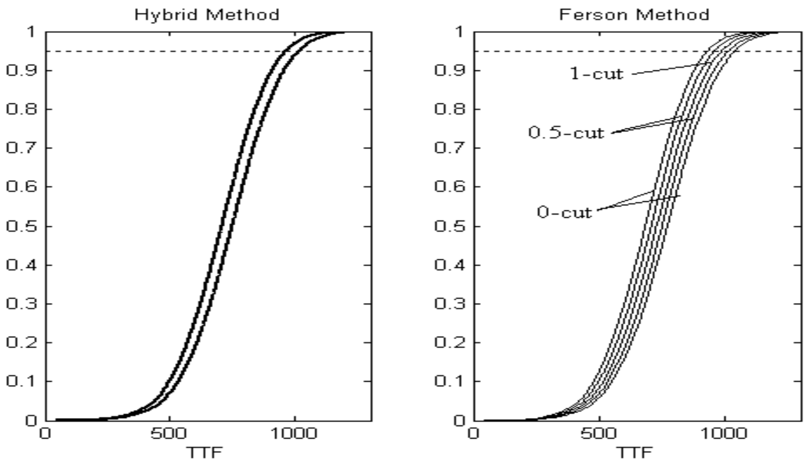


Figure 8: Application of the hybrid method of uncertainty representation (column 1) and joint aggregation method for the uncertainty propagation (columns 2-3), with regards to the four realizations of the degradation process shown in Figure 4



(a) **(b)**

Figure 9: Comparison of the results obtained with: **(a)** joint aggregation method, **(b)** Ferson aggregation method

confidence level β, whereas he/she is not asked to provide a confidence level on the epistemic uncertainty.

Alternatively, with the Ferson aggregation method of Section 3.2.2 and a $0.95-cut$ (dotted line in Figure 10, first column) the possibility distributions $\pi_{i_a}^{0.95}$ represented in Figure 10, second column are obtained for the four random realizations of the degradation process of Figure 4. The third and fourth columns of Figure 10 show the necessity $N_{i_a}^{0.95}$ and possibility $\Pi_{i_a}^{0.95}$ measures corresponding to the $0.95-cut$ of the $\pi\left(H_{\max}\right)$.

Finally, the lower and upper cumulative distributions $\underline{F}^{0.95}\left(t\right)$ and $\overline{F}^{0.95}\left(t\right)$ are obtained as means of the values of $N_{i_a}^{0.95}$ and $\Pi_{i_a}^{0.95}$ for the 4 simulated degradation processes (last row of Figure 10).

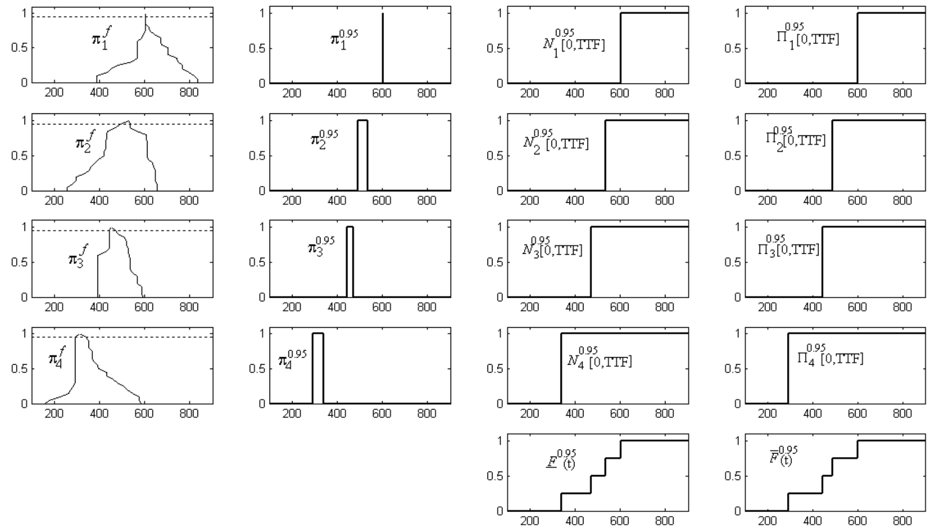


Figure 10: Application of the hybrid method of uncertainty representation (column 1) and Ferson aggregation method for the uncertainty propagation (columns 2-4), with regards to the four realizations of the degradation process shown in Figure 4.

As before, applying the Ferson aggregation method to all $m_a = 10^4$ random possibility distributions $\pi_{i_a}^{f}\left(TTF\right)$, $i_a = 1,...,10^4$, 21 pairs of limiting cumulative distributions of the TTF for the considered 21 $\alpha - cuts$ of $\pi\left(H_{\max}\right)$, $\alpha = 0, 0.05, ..., 1$, are obtained. For example, Figure 9(b) reports the cumulative boundary distributions obtained for the $0-cut$, $0.5-cut$ and $1-cut$. To be completely sure (i.e. with degree of certainty equal to 1) of including the correct value of $H_{\max}$ in the propagation of the uncertainty then the $0-cut$ of $\pi\left(H_{\max}\right)$ should be considered, which gives rise to the two bounding cumulative distribution functions reported in Figure 9(b). Figure 11 reports the limiting bounds of the upper limit of the 95% one-sided confidence level of the TTF function of the $\alpha - cut$. For example, considering a $0.95-cut$, at 95% confidence the upper limit for the TTF lies in the interval $\left[983, 988\right]$ (dotted line in Figure 11). Notice that the analyst in this case should provide two confidence values, α for the epistemic uncertainty affecting $H_{\max}$ and β for the aleatory uncertainties in the degradation process.

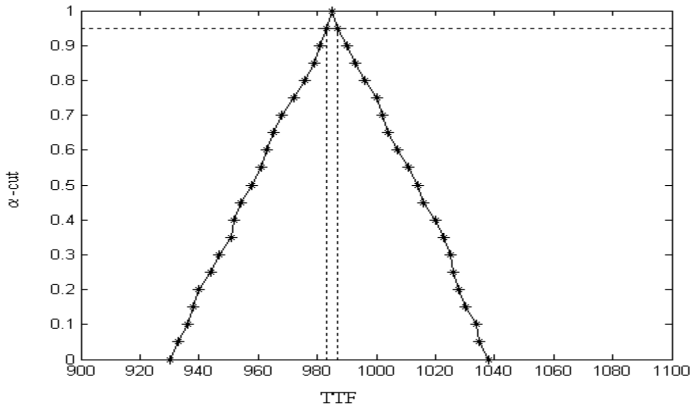


Figure 11: Limiting bounds of the upper limit of the 95% one sided confidence level of the TTF computed with the Ferson aggregation method

5.2.2. Application of the purely probabilistic method

Upon performing $m_a = 10^4$ simulations of the degradation process for each of the $m_e = 10^3$ values of $H_{\max}$ sampled from the probability distribution of Figure 3, 10^3 cumulative distributions are obtained. Figure 12(a) reports, as example, 10 of the resulting cumulative distributions.

If one accepts to merge the epistemic and aleatory uncertainties for a synthetic representation of the results, considering that each of the obtained cumulative distributions is the conditional probability of the TTF given a certain value of $H_{\max}$, it is possible to compute the cumulative distribution of the TTF (Figure 12(b)) by:

$$F(t) = \Pr(TTF \leq t) = \int F(t \mid H_{\max}) g(H_{\max}) dH_{\max} \tag{31}$$

where $g(H_{\max})$ is the probability density function of $H_{\max}$ (Figure 5(a), continuous line). For the interpretation of the results, it is possible to choose a level of confidence β and then provide the upper limit of the one-sided confidence level of the TTF. For example for a confidence level $\beta = 0.95$ (dotted line in Figure 12), the resulting upper limit is 988, which means that with 95% probability the TTF is lower or equal than 988.

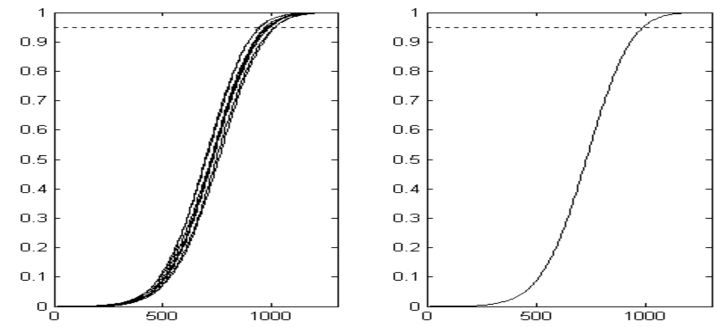
On the other side there are several reasons, both from a practical and mathematical point of view [1, 29], that suggest to keep separate the epistemic and aleatory uncertainties. The main advantage is that the separation of uncertainty permits to understand the causes of the uncertainty in the results. In this respect, a possible way to represent separately the uncertainty is to fix a given percentile of the cumulative distributions $F^{i_e}\left(TTF \middle| H_{\max}^{i_e}\right)$, and then build the probabilistic distribution of this percentile from the different realizations of the epistemic variables. Figure 13 reports the cumulative distribution of the β =0.95 percentile of $F^{i_e}\left(TTF \middle| H_{\max}^{i_e}\right)$.

For the interpretation of the obtained distribution it is possible to choose, also in this case, an additional level of confidence $1-\alpha$ connected with the epistemic uncertainty and then provide the confidence interval for the 95% percentile of the TTF given by the percentiles $(\alpha/2)$ and $(1-\alpha/2)$. For example, for a confidence $(1-\alpha)=0.95$ with respect to the epistemic uncertainty the 95% percentile of the TTF is between [939,1020].

5.2.3. Comparison of the results

Let us first compare the representation of the available uncertain information given by the probabilistic and possibilistic methods. Figure 9 shows that while the possibilistic representation of the uncertainty on $H_{\max}$ allows us to consider all the possible cumulative distributions lying between the believe and the plausibility measures of the set $A=[0,u]$, the probabilistic representation, Figure 12(b) and Figure 13, forces us to consider only one specific cumulative distribution, albeit conservatively the one containing as much uncertainty as possible (thanks to the transformation of Section 2.3). In this application characterized by a rather limited available information on the value of $H_{\max}$, the probabilistic density function tends for this reason to force information in the representation.

With respect to the uncertainty propagation and representation techniques, notice that the interpretation of the results obtained by all proposed methods, possibilistic and probabilistic, requires to fix the level of confidence β that the analyst wants in the estimation, i.e. the classical confidence level in statistics. Furthermore, the Ferson method requires an additional level of trust on the value of $H_{\max}$, i.e. the parameter $(1-\alpha)$ indicating a degree of certainty that the interval of values of $H_{\max}$ that are considered in the uncertainty propagation contains the true $H_{\max}$. Also the interpretation of the results obtained with the purely probabilistic method, maintaining separate the uncertainty, requires a supplementary parameter, representing the confidence level on the 95% percentile of the TTF.



(a) (b)

Figure 12: Results obtained with the pure probabilistic method: **(a)** 10 conditional cumulative probability distributions; **(b)** total cumulative probability distribution

Table 2 reports the confidence intervals obtained for the TTF by the different methods fixing the levels of confidence α and β to 0.95. Notice that the hybrid method combined with the Ferson aggregation method finds a larger confidence interval than the purely probabilistic method in the case in which the epistemic and aleatory uncertainties

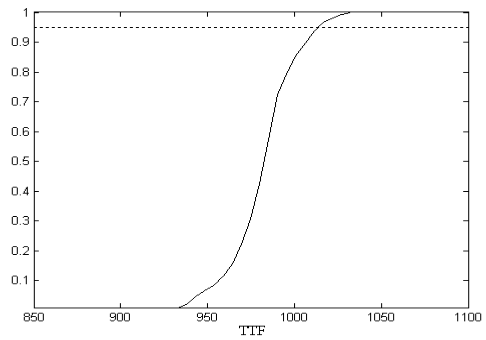


Figure 13: Cumulative distribution of the $\beta = 0.95$ percentile of the TTF

Table 2. Synthesis of the results

	β	$1-\alpha$	Confidence interval
Hybrid method and joint aggregation method	0.95	-	[962, 1015]
Hybrid method and Ferson aggregation method	0.95	0.95	[930, 1039]
Purely probabilistic method – merging aleatory and epistemic uncertainties	0.95	-	988
Purely probabilistic method – keeping aleatory and epistemic uncertainties separate	0.95	0.95	[939, 1020]

are kept separated. This is due to the different interpretation of the parameter $(1-\alpha)$ of the Ferson method and of the purely probabilistic method, the former indicating a degree of certainty that the interval of values of $H_{\max}$ that are considered in the uncertainty propagation contains the true $H_{\max}$, the latter being amenable of the classical interpretation of confidence level in statistics. Hence, fixing $(1-\alpha)$ in the probabilistic method does not imply a choice on the interval of values of the epistemic variable that will be propagated further in the analysis, whereas in the Ferson method it does not imply a choice on the probability that the obtained confidence interval of the success probability will contain its true value.

6. Conclusions

In this paper, the problems of the representation and propagation of uncertainty in prognostic models have been addressed by investigating a purely probabilistic method and a hybrid Monte Carlo and possibilistic method.

In the pure probabilistic approach, both epistemic and aleatory uncertainties are represented in terms of probability distributions and a double-randomization Monte Carlo simulation is carried out in which epistemic variables are sampled in the outer loop and aleatory variables are sampled in the inner loop, to propagate the uncertainty through the model. On the other side in the hybrid Monte Carlo and possibilistic method, Monte Carlo sampling of the random variables is repeatedly performed to process the aleatory behavior and possibilistic distribution analysis is carried out at each sampling to process the epistemic uncertainty in the epistemic variables.

The investigation of the two methods has been carried out with reference to a case study regarding the prediction of component reliability and time to failure, knowing at the current time the value of the degradation parameter of a component which is randomly degrading in time according to a stochastic fatigue crack growth model of literature.

With respect to the representation of the uncertainties affecting the parameters of the degradation process (epistemic uncertainty), possibilistic distributions seem to provide a more satisfactory characterization of the uncertainties than probability distributions in the common case of incomplete information or knowledge on the parameters values, whereas the randomness due to the inherent variability in the system behavior (aleatory uncertainty) is appropriately represented by probability distributions.

For what concerns the propagation of the uncertainties in the input variables and parameters to the output of the degradation model, the purely probabilistic method results in a family of cumulative distribution functions. The information contained in these distributions has been aggregated in two different ways: *i*) by merging the epistemic and aleatory uncertainties affecting the reliability or TTF distribution onto one single cumulative distribution *ii*) by maintaining them separated. Although in the former case it is possible to obtain a synthetic value of the reliability or of a given percentile of the TTF whereas in the latter case one can only build a probabilistic distribution of the reliability or of a percentile of the TTF from the different realizations of the epistemic variables, it seems more useful for the interpretation of the results to keep the uncertainties separated providing to the analyst the possibility of understanding the causes of the uncertainty. In this respect, notice that the epistemic uncertainty affecting the results, represented by the spread of the probabilistic distribution of the percentile, can be theoretically reduced by acquiring new information on the parameters of the degradation process.

The hybrid Monte Carlo - Possibilistic method has proven effective for jointly propagating the aleatory and epistemic uncertainties through the model. The possibilistic random distributions obtained by this method can be combined into a set of limiting cumulative distributions characterized by different degrees of confidence (Ferson Aggregation Method) or, accordingly to the Dempster-Shafer Theory, into Belief and Plausibility measures that can be interpreted as "rational averages" of the above mentioned limiting cumulative distributions (joint aggregation method). The interpretation of the results in the form of limiting cumulative distributions (Ferson aggregation method) requires the introduction of a degree of confidence directly connected with the confidence on the value of the epistemic parameters. On the contrary, the joint aggregation method provides average information on the limiting cumulative distributions without requiring to the analyst the definition of confidence levels, giving in this way a more synthetic, albeit less informative, representation of uncertainty in the prediction of the reliability or TTF.

Finally, the hybrid Monte Carlo - Possibilistic method for the prognosis of the reliability or TTF of a degrading component seems more appropriate in case of lack of information or knowledge on the degradation model parameters given that it does not force information into the model and it is able to joint-propagate epistemic and aleatory uncertainties.

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