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Abstract. MACBETH 2-additive is the generalization of the Choquet integral to the MACBETH approach, a MultiCriteria Decision Aid method. In the elicitation of a 2-additive capacity step, the inconsistencies of the preferential information, given by the Decision Maker on the set of binary alternatives, is tested by using the MOPI conditions. Since a 2-additive capacity is related to all binary alternatives, this inconsistencies checking can be more complex if the set of alternatives is very large. In this paper, we show that it is possible to limited the test of MOPI conditions to the only alternatives used in the preferential information.
Keywords: MCDA, Preference modeling, MOPI conditions, Choquet integral, MACBETH.

1 Introduction

Multiple Criteria Decision Aid (MCDA) aims at helping a decision maker (DM) in the representation of his preferences over a set of alternatives, on the basis of several criteria which are often contradictory. One possible model is the transitive decomposable one where an overall utility is determined for each option. In this category, we have the model based on Choquet integral, especially the 2-additive Choquet integral (Choquet integral w.r.t. a 2-additive) [6, 8, 14]. The 2-additive Choquet integral is defined w.r.t. a capacity (or nonadditive monotonic measure, or fuzzy measure), and can be viewed as a generalization of the arithmetic mean. Any interaction between two criteria can be represented and interpreted by a Choquet integral w.r.t. a 2-additive capacity, but not more complex interaction.

Usually the DM is supposed to be able to express his preference over the set of all alternatives X. Because this is not feasible in most of practical situations (the cardinality of X may be very large), the DM is asked to give, using pairwise comparisons, an ordinal information (a preferential information containing only a strict preference and an indifference relations) on a subset $X' \subseteq X$, called reference set. The set X' we use in this paper is the set of binary alternatives or binary actions denoted by $\mathcal{B}$. A binary action is an (fictitious) alternative representing a prototypical situation where on a given subset of at most two criteria,

the attributes reach a satisfactory level **1**, while on the remaining ones, they are at a neutral level (neither satisfactory nor unsatisfactory) **0**. The characterization theorem of the representation of an ordinal information by a 2-additive Choquet integral [13] is based on the MOPI property. The inconsistencies test of this condition is done on every subsets of three criteria.

We are interested in the following problem: *how to reduce the complexity of this test of inconsistencies when the number of criteria is large?* We propose here a simplification of the MOPI property based only on the binary alternatives related to the ordinal information.

After some basic notions given in the next section, we present in Section 3 our main result.

2 Basic concepts

Let us denote by $N = \{1, \ldots, n\}$ a finite set of n criteria and $X = X_1 \times \cdots \times X_n$ the set of actions (also called alternatives or options), where $X_1, \ldots, X_n$ represent the point of view or attributes. For all $i \in N$, the function $u_i : X_i \to \mathbb{R}$ is called a utility function. Given an element $x = (x_1, \ldots, x_n)$, we set $U(x) = (u_1(x_1), \ldots, u_n(x_n))$. For a subset A of N and actions x and y, the notation $z = (x_A, y_{N-A})$ means that z is defined by $z_i = x_i$ if $i \in A$, and $z_i = y_i$ otherwise.

2.1 Choquet integral w.r.t. a 2-additive capacity

The Choquet integral w.r.t. a 2-additive capacity [6], called for short a *2-additive Choquet integral*, is a particular case of the Choquet integral [8, 9, 14]. This integral generalizes the arithmetic mean and takes into account interactions between criteria. A 2-additive Choquet integral is based on a 2-additive capacity [4, 8] defined below and its Möbius transform [3, 7]:

Definition 1

1. *A capacity on N is a set function $\mu : 2^N \to [0, 1]$ such that:*
   (a) $\mu(\emptyset) = 0$
   (b) $\mu(N) = 1$
   (c) $\forall A, B \in 2^N,\ [A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ *(monotonicity).*
2. *The Möbius transform [3] of a capacity μ on N is a function $m : 2^N \to \mathbb{R}$ defined by:*
$$m(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K), \forall T \in 2^N. \quad (1)$$
   When m is given, it is possible to recover the original μ by the following expression:
$$\mu(T) := \sum_{K \subseteq T} m(K), \forall T \in 2^N. \quad (2)$$

For a capacity μ and its Möbius transform m, we use the following shorthand: $\mu_i := \mu(\{i\})$, $\mu_{ij} := \mu(\{i,j\})$, $m_i := m(\{i\})$, $m_{ij} := m(\{i,j\})$, for all $i,j \in N$, $i \neq j$. Whenever we use i and j together, it always means that they are different.

Definition 2 *A capacity μ on N is said to be 2-additive if*

- *For all subsets T of N such that $|T| > 2$, $m(T) = 0$;*
- *There exists a subset B of N such that $|B| = 2$ and $m(B) \neq 0$.*

The following important Lemma shows that a 2-additive capacity is entirely determined by the value of the capacity on the singletons $\{i\}$ and pairs $\{i,j\}$ of 2^N:

Lemma 1

1. *Let μ be a 2-additive capacity on N. We have for all $K \subseteq N$, $|K| \geq 2$,*

$$\mu(K) = \sum_{\{i,j\} \subseteq K} \mu_{ij} - (|K| - 2) \sum_{i \in K} \mu_i. \tag{3}$$

2. *If the coefficients μ_i and μ_{ij} are given for all $i,j \in N$, then the necessary and sufficient conditions that μ is a 2-additive capacity are:*

$$\sum_{\{i,j\} \subseteq N} \mu_{ij} - (n-2) \sum_{i \in N} \mu_i = 1 \tag{4}$$

$$\mu_i \geq 0, \ \forall i \in N \tag{5}$$

For all $A \subseteq N, |A| \geq 2, \forall k \in A$,

$$\sum_{i \in A \setminus \{k\}} (\mu_{ik} - \mu_i) \geq (|A| - 2)\mu_k. \tag{6}$$

Proof. See [6].

For an alternative $x := (x_1, ..., x_n) \in X$, the expression of the Choquet integral w.r.t. a capacity μ is given by:

$$C_\mu(U(x)) := \sum_{i=1}^{n} (u_{\tau(i)}(x_{\tau(i)}) - u_{\tau(i-1)}(x_{\tau(i-1)}))\ \mu(\{\tau(i), \ldots, \tau(n)\})$$

where τ is a permutation on N such that $u_{\tau(1)}(x_{\tau(1)}) \leq u_{\tau(2)}(x_{\tau(2)}) \leq \cdots \leq u_{\tau(n-1)}(x_{\tau(n-1)}) \leq u_{\tau(n)}(x_{\tau(n)})$, and $u_{\tau(0)}(x_{\tau(0)}) := 0$.

The 2-additive Choquet integral can be written also as follows [9]:

$$C_\mu(U(x)) = \sum_{i=1}^{n} v_i u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij} |u_i(x_i) - u_j(x_j)| \tag{7}$$

where $v_i = \sum_{K \subseteq N \setminus i} \frac{(n - |K| - 1)!|K|!}{n!} (\mu(K \cup i) - \mu(K))$ is the importance of criterion i corresponding to the Shapley value of μ [17] and $I_{ij} = \mu_{ij} - \mu_i - \mu_j$ is the interaction index between the two criteria i and j [6, 15].

2.2 Binary actions and relations

MCDA methods based on multiattribute utility theory, e.g, UTA [19], robust methods [1, 5, 11], require in practice a preferential information of the DM on a subset X_R of X because of the cardinality of X which can be very large. The set X_R is called reference subset and it is generally chosen by the DM. His choice may be guided by his knowledge about the problem addressed, his experience or his sensitivity to one or more particular alternatives, etc. This task is often difficult for the DM, especially when the alternatives are not known in advance, and sometimes his preferences on X_R are not sufficient to specify all the parameters of the model as interaction between criteria. For instance, in the problem of the design of a complex system for the protection of a strategic site [16], it is not easy for the DM to choose X_R himself because these systems are not known a priori. For these reasons, we suggest him to use as a reference subset a set of fictitious alternatives called binary actions defined below. We assume that the DM is able to identify for each criterion i two reference levels:

1. A reference level $\mathbf{1}_i$ in X_i which he considers as good and completely satisfying if he could obtain it on criterion i, even if more attractive elements could exist. This special element corresponds to the *satisficing level* in the theory of bounded rationality of Simon [18].
2. A reference level $\mathbf{0}_i$ in X_i which he considers neutral on i. The neutral level is the absence of attractiveness and repulsiveness. The existence of this neutral level has roots in psychology [20], and is used in bipolar models [21].

We set for convenience $u_i(\mathbf{1}_i) = 1$ and $u_i(\mathbf{0}_i) = 0$. Because the use of Choquet integral requires to ensure the commensurateness between criteria, the previous reference levels can be used in order to define the same scale on each criterion [10, 12]. More details about these reference levels can be found in [8, 9].

We call a *binary action or binary alternative*, an element of the set

$$\mathcal{B} = \{\mathbf{0}_N,\ (\mathbf{1}_i, \mathbf{0}_{N-i}),\ (\mathbf{1}_{ij}, \mathbf{0}_{N-ij}),\ i, j \in N, i \neq j\} \subseteq X$$

where

- $\mathbf{0}_N = (\mathbf{1}_\emptyset, \mathbf{0}_N) =: a_0$ is an action considered neutral on all criteria.
- $(\mathbf{1}_i, \mathbf{0}_{N-i}) =: a_i$ is an action considered satisfactory on criterion i and neutral on the other criteria.
- $(\mathbf{1}_{ij}, \mathbf{0}_{N-ij}) =: a_{ij}$ is an action considered satisfactory on criteria i and j and neutral on the other criteria.

Using the Choquet integral, we get the following consequences:

1. For any capacity μ,

$$C_\mu(U((\mathbf{1}_A, \mathbf{0}_{N-A}))) = \mu(A),\ \forall A \subseteq N. \tag{8}$$

2. Using Equation (2), we have for any 2-additive capacity μ:

$$C_\mu(U(a_0)) = 0 \tag{9}$$

$$C_\mu(U(a_i)) = \mu_i = v_i - \frac{1}{2} \sum_{k \in N,\ k \neq i} I_{ik} \tag{10}$$

$$C_\mu(U(a_{ij})) = \mu_{ij} = v_i + v_j - \frac{1}{2} \sum_{k \in N,\ k \notin \{i,j\}} (I_{ik} + I_{jk}) \tag{11}$$

With the arithmetic mean, we are able to compute the weights by using the reference subset $X_R = \{a_0, a_i, \forall i \in N\}$ (see MACBETH methodology [2]). For the 2-additive Choquet integral model, these alternatives are not sufficient to compute interaction between criteria, hence the elaboration of $\mathcal{B}$ by adding the alternatives a_{ij}. The Equations (10) and (11) show that the binary actions are directly related to the parameters of the 2-additive Choquet integral model. Therefore a preferential information on $\mathcal{B}$ given by the DM permits to determine entirely all the parameters of the model.

As shown by the previous equations (9),(10), (11) and Lemma 1, it should be sufficient to get some preferential information from the DM only on binary actions. To entirely determine the 2-additive capacity this information is expressed by the following relations:

- $P = \{(x, y) \in \mathcal{B} \times \mathcal{B} :$ DM strictly prefers x to $y\}$,
- $I = \{(x, y) \in \mathcal{B} \times \mathcal{B} :$ DM is indifferent between x and $y\}$.

The relation P is irreflexive and asymmetric while I is reflexive and symmetric.

Definition 3 *The ordinal information on $\mathcal{B}$ is the structure $\{P,\ I\}$.*

These two relations are completed by adding the relation M which models the natural relations of monotonicity between binary actions coming from the monotonicity conditions $\mu(\{i\}) \geq 0$ and $\mu(\{i, j\}) \geq \mu(\{i\})$ for a capacity μ. For $(x, y) \in \{(a_i, a_0), i \in N\} \cup \{(a_{ij}, a_i), i, j \in N, i \neq j\}$,

$$x\ M\ y \text{ if not}(x\ (P \cup I)\ y).$$

Example 1 *Mary wants to buy a digital camera for her next trip. To do this, she consults a website where she finds six propositions based on three criteria: resolution of the camera (expressed in million of pixels), price (expressed in euros) and zoom (expressed by a real number)*

Cameras	1 : *Resolution*	2 : *Price*	3 : *Zoom*
a : *Nikon*	6	150	5
b : *Sony*	7	180	5
c : *Panasonic*	10	155	4
d : *Casio*	12	175	5
e : *Olympus*	10	160	3
f : *Kodak*	8	165	4

Using our notations, we have $N = \{1, 2, 3\}$, $X_1 = [6, 12]$, $X_2 = [150, 180]$, $X_3 = [3, 5]$ *and* $X = X_1 \times X_2 \times X_3$.

	1 : *Resolution*	2 : *Price*	3 : *Zoom*
$Satisfactory\ level$	12	150	4
$Neutral\ level$	9	160	3.5

Based on these reference levels, the set of binary actions is $\mathcal{B} = \{a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}\}$, *where for instance the alternative* a_{12} *refers to a camera for which Mary is satisfied on resolution and price, but neutral on zoom. In order to make her choice, Mary gives also the following ordinal information:* $I = \{(a_{12}, a_3)\}$, $P = \{(a_{13}, a_1), (a_2, a_0)\}$. *Hence we have* $M = \{(a_1, a_0), (a_3, a_0), (a_{12}, a_1), (a_{12}, a_2), (a_{13}, a_3), (a_{23}, a_2), (a_{23}, a_3)\}$.

2.3 The representation of ordinal information by the Choquet integral

An ordinal information $\{P, I\}$ is said to be *representable by a 2-additive Choquet integral* if there exists a 2-additive capacity μ such that:

1. $\forall x, y \in \mathcal{B},\ x\ P\ y \Rightarrow C_\mu(U(x)) > C_\mu(U(y))$
2. $\forall x, y \in \mathcal{B},\ x\ I\ y \Rightarrow C_\mu(U(x)) = C_\mu(U(y))$.

A characterization of an ordinal information is given by Mayag et al. [13]. This result, presented below, is based on the following property called MOPI:

Definition 4 *[MOPI property]*

1. *For a binary relation* $\mathcal{R}$ *on* $\mathcal{B}$ *and* x, y *elements of* $\mathcal{B}$, $\{x_1, x_2, \cdots, x_p\} \subseteq \mathcal{B}$ *is a* path *of* $\mathcal{R}$ *from* x *to* y *if* $x = x_1\ \mathcal{R}\ x_2\ \mathcal{R} \cdots \mathcal{R}\ x_{p-1}\ \mathcal{R}\ x_p = y$. *A path of* $\mathcal{R}$ *from* x *to* x *is called a* cycle *of* $\mathcal{R}$.
   - *We denote* $x\ TC\ y$ *if there exists a path of* $(P \cup I \cup M)$ *from* x *to* y.
   - *A path* $\{x_1, x_2, ..., x_p\}$ *of* $(P \cup I \cup M)$ *is said to be a* strict path *from* x *to* y *if there exists* i *in* $\{1, ..., p-1\}$ *such that* $x_i\ P\ x_{i+1}$. *In this case, we will write* $x\ TC_P\ y$.
   - *We write* $x \sim y$ *if there exists a nonstrict cycle of* $(P \cup I \cup M)$ *(hence a cycle of* $(I \cup M)$*) containing* x *and* y.
2. *Let* $i, j, k \in N$, i *fixed. We call* Monotonicity of Preferential Information in $\{i, j, k\}$ w.r.t. i *the following property (denoted by* $(\{i,j,k\}$***,i)-MOPI****):*

$$\begin{cases} a_{ij} \sim a_i \\ a_{ik} \sim a_k \end{cases} \Rightarrow not(a_j\ TC_P\ a_0)$$
and
$$\begin{cases} a_{ij} \sim a_j \\ a_{ik} \sim a_k \end{cases} \Rightarrow not(a_i\ TC_P\ a_0)$$
and
$$\begin{cases} a_{ij} \sim a_j \\ a_{ik} \sim a_i \end{cases} \Rightarrow not(a_k\ TC_P\ a_0).$$

3. *We say that, the set* $\{i, j, k\}$ *satisfies the property of* MOnotonicity of Preferential Information *(MOPI) if* $\forall l \in \{i, j, k\}$, $(\{i, j, k\}, l)$*-MOPI is satisfied.*

Theorem 1 *An ordinal information* $\{P, I\}$ *is representable by a 2-additive Choquet integral on* $\mathcal{B}$ *if and only if the following two conditions are satisfied:*

1. $(P \cup I \cup M)$ *contains no strict cycle;*
2. *Any subset* K *of* N *such that* $|K| = 3$ *satisfies the MOPI property.*

Proof. See [13].

Using this characterization theorem, we deal with inconsistencies in the ordinal information [14]. But, the inconsistencies test of MOPI conditions requires to test them on all subsets of three criteria. Therefore, all the binary alternatives are used in the MOPI conditions test. If the number of elements of $\mathcal{B}$ is large ($n > 2$), it can be impossible to show to the DM a graph, where vertices are binary actions, for the explanation of inconsistencies. To solve this problem, we give an equivalent characterization of an ordinal information which concerns only the binary actions related the preferences $\{P, I\}$. This is done by extending the relation M to some couples (a_{ij}, a_0). Therefore, this new characterization theorem can be viewed as a reduction of complexity of inconsistencies test

3 Reduction of the complexity in the inconsistencies test of ordinal information

Let us consider the following sets:

$$\mathcal{B}' = \{a_0\} \cup \{x \in \mathcal{B} \mid \exists y \in \mathcal{B} \text{ such that } (x, y) \in (P \cup I) \text{ or } (y, x) \in (P \cup I)\}$$

$$M' = M \cup \{(a_{ij}, a_0) \mid a_{ij} \in \mathcal{B}', a_i \notin \mathcal{B}' \text{ et } a_j \notin \mathcal{B}'\}$$

$$(P \cup I \cup M')_{|_{\mathcal{B}'}} = \{(x, y) \in \mathcal{B}' \times \mathcal{B}' \mid (x, y) \in (P \cup I \cup M')\}$$

The set $\mathcal{B}'$ is the set of all binary actions related to the preferential information of the DM. The relation on M' on $\mathcal{B}$ is an extension of the monotonicity relation on $\mathcal{B}$. The restriction of the relation $(P \cup I \cup M')$ to the set $\mathcal{B}'$ corresponds to $(P \cup I \cup M')_{|_{\mathcal{B}'}}$.

The following result shows that, when it is possible to extend the monotonicity relation M to the set $\mathcal{B}'$, then the test of inconsistencies for the representation of ordinal information can be only limited to the elements of $\mathcal{B}'$.

Proposition 1 *Let be* $\{P, I\}$ *an ordinal information on* $\mathcal{B}$.

The ordinal information $\{P, I\}$ *is representable by a 2-additive Choquet integral if and only if the following two conditions are satisfied:*

1. $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ *contains no strict cycle;*
2. *Every subset* K *of* N *such that* $|K| = 3$ *satisfies the MOPI conditions reduced to* $\mathcal{B}'$ *(Only the elements of* $\mathcal{B}'$ *are concerned in this condition).*

Proof. See Section 3.1.

Example 2 $N = \{1, 2, 3, 4, 5, 6\}$, $P = \{(a_5, a_{12})\}$, $I = \{(a_3, a_5)\}$, $\mathcal{B}' = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{23}, a_{24}, a_{25}, a_{26}, a_{34}, a_{35}, a_{36}, a_{45}, a_{46}, a_{56}\}$, *we will have*

$$\mathcal{B}' = \{a_0, a_{12}, a_3, a_5\},$$

$$M' = M \cup \{(a_{12}, a_0)\},$$

$$(P \cup I \cup M')_{|_{\mathcal{B}'}} = \{(a_5, a_{12}), (a_3, a_5), (a_3, a_0), (a_5, a_0), (a_{12}, a_0)\}$$

Hence the inconsistencies test will be limited on $\mathcal{B}'$.

3.1 Proof of Proposition 1

Let be $\{P, I\}$ an ordinal information on $\mathcal{B}$. In this section, for all elements $x, y \in \mathcal{B}$, we denote by:

1. $x\ TC'\ y$ a path of $(P \cup I \cup M')$ from x to y.
2. $x\ TC'_{|_{\mathcal{B}'}}\ y$ a path of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ from x to y i.e. a path from x to y containing only the elements of $\mathcal{B}'$.
3. $x \sim' y$ if one of the two following conditions happens:
   (a) $x = y$;
   (b) there is a non strict cycle $(P \cup I \cup M')$ containing x and y.
4. $x \sim'_{|_{\mathcal{B}'}} y$ if one of the two following conditions happens:
   (a) $x = y$;
   (b) there is a non strict cycle of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ containing x and y.
5. $i \vee j$ represents one of the two elements i or j, $\forall i, j \in N, i \neq j$.

We will use the following lemmas in the proof of the result:

Lemma 2 *If* $(x_1, x_2, \ldots, x_p)$ *is a cycle of* $(P \cup I \cup M)$, *then every elements of* $\mathcal{B}'$ *of this cycle are contained in a cycle of* $(P \cup I \cup M')_{|_{\mathcal{B}'}}$.

Proof. For all x_l, elements of the cycle $(x_1, x_2, \ldots, x_p)$ which are not in $\mathcal{B}'$, there exists necessarily $i, j \in N$ such that $a_{ij}\ M\ a_i\ M\ a_0$ (see Figure 1) where $x_{l-1} = a_{ij}$, $x_l = a_i$ and $x_{l+1} = a_0$ ($x_0 = x_p$ and $x_{p+1} = x_1$). Therefore, We can cancel the element a_i of the cycle because the elements a_{ij} and a_0 can be related as follows:

- if $a_j \notin \mathcal{B}'$, we will have $a_{ij}\ M'\ a_0$;
- if $a_j \in \mathcal{B}'$, we will have $a_{ij}\ (P \cup I \cup M)\ a_j\ (P \cup I \cup M)\ a_0$. This element a_j, which is not necessarily an element of the cycle $(x_1, x_2, \ldots, x_p)$, will be an element of the new cycle of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$.

The cycle of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ obtained is then constituted by the elements of $(x_1, x_2, \ldots, x_p)$ belonging in $\mathcal{B}'$ and eventually the elements a_j coming from the cancelation of the elements a_i of $(x_1, x_2, \ldots, x_p)$ which are not in $\mathcal{B}'$.

Lemma 3 *Let* $i, j \in N$ *such that* $a_{ij} \sim a_{i \vee j}$. *We have the following results:*

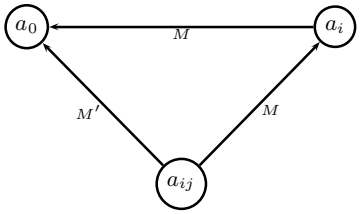

Fig. 1. Relation M' between a_{ij}, a_i and a_0

1. $a_{ij} \in \mathcal{B}'$;
2. *If* $a_{i\vee j} \notin \mathcal{B}'$ *then* $a_{ij} \ \sim'_{|_{\mathcal{B}'}} \ a_0$;
3. *If* $a_{i\vee j} \in \mathcal{B}'$ *then* $a_{ij} \ \sim'_{|_{\mathcal{B}'}} \ a_{i\vee j}$.

Proof.

1. If $a_{ij} \sim a_{i\vee j}$ then there exists $x \in \mathcal{B}$ such that $x \ (P \cup I \cup M) \ a_{ij}$. Using the definition of M, one may have $x \ M \ a_{ij}$. Hence $a_{ij} \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.
2. $a_{ij} \sim a_{i\vee j} \Rightarrow a_{ij} \ M \ a_{i\vee j} \ M \ a_0 \ TC \ a_{ij}$ because $a_{i\vee j} \notin \mathcal{B}'$. Using Lemma 2, a_{ij} and a_0 are contained in a cycle of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ i.e. $a_{ij} \ \sim'_{|_{\mathcal{B}'}} \ a_0$.
3. Since a_{ij} and $a_{i\vee j}$are in $\mathcal{B}'$, then using Lemma 2, they are contained in a cycle of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ i.e. $a_{ij} \ \sim'_{|_{\mathcal{B}'}} \ a_{i\vee j}$.

Lemma 4 *If* $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ *contains no strict cycle then* $(P \cup I \cup M)$ *contains no strict cycle.*

Proof. Let $(x_1, x_2, \ldots, x_p)$ a strict cycle of $(P \cup I \cup M)$. Using Lemma 2, all the elements of $(x_1, x_2, \ldots, x_p)$ belonging to $\mathcal{B}'$ are contained in a cycle $\mathcal{C}$ de $(P \cup I \cup M')_{|_{\mathcal{B}'}}$. Since $(x_1, x_2, \ldots, x_p)$ is a strict cycle of $(P \cup I \cup M)$, there exists $x_{i_o}, x_{i_o+1} \in \{x_1, x_2, \ldots, x_p\}$ such that $x_{i_o} \ P \ x_{i_o+1}$. Therefore $\mathcal{C}$ is a strict cycle of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ because $x_{i_o}, x_{i_o+1} \in \mathcal{B}'$, a contradiction with the hypothesis.

Lemma 5 *Let* $x \in \mathcal{B}$. *If* $x \ TC_P \ a_0$ *then for each strict path* $(P \cup I \cup M)$ *from* x *to* a_0, *there exists a strict path of* $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ *from* x *to* a_0.

Proof. If $x \notin \mathcal{B}'$ then we can only have $x \ M \ a_0$. Therefore we will not have $x \ TC_P \ a_0$, a contradiction. Hence we have $x \in \mathcal{B}'$.

Let $x \ (P \cup I \cup M) \ x_1 \ (P \cup I \cup M) \ldots \ x_p \ (P \cup I \cup M) \ a_0$ a strict path of $(P \cup I \cup M)$ from x to a_0. If there exists an element $y \notin \mathcal{B}'$ belonging to this path, then there necessarily exists $i, j \in N$ such that $y = a_i$ and $x \ TC_P \ a_{ij} \ M \ a_i \ M \ a_0$. So we can suppress the element y and have the path $x \ TC_P \ a_{ij} \ M' \ a_0$ if $a_j \notin \mathcal{B}'$ or the path $x \ TC_P \ a_{ij} \ (P \cup I \cup M) \ a_j \ (P \cup I \cup M) \ a_0$ if $a_j \in \mathcal{B}'$. If we suppress all the elements of $\mathcal{B} \setminus \mathcal{B}'$ like this, then we obtain a strict path of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ containing only elements of $\mathcal{B}'$.

Lemma 6 *Let us suppose that* $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ *contains no strict cycle.*

1. *If we have* $\begin{cases} a_{ij} \sim a_i \\ a_{ik} \sim a_k \end{cases}$ *and* $(a_j \; TC_P \; a_0)$ *then* a_i, a_k *and* a_j *are the elements of* $\mathcal{B}'$.
2. *If we have* $\begin{cases} a_{ij} \sim a_j \\ a_{ik} \sim a_i \end{cases}$ *and* $(a_k \; TC_P \; a_0)$ *then* a_i, a_j *and* a_k *are the elements* $\mathcal{B}'$.
3. *If we have* $\begin{cases} a_{ij} \sim a_j \\ a_{ik} \sim a_k \end{cases}$ *and* $(a_i \; TC_P \; a_0)$ *then* a_j, a_k *and* a_i *are the elements of* $\mathcal{B}'$.

Proof.

1. a_j is an element of $\mathcal{B}'$ using Lemma 5.
   - If $a_i \notin \mathcal{B}'$ then using Lemma 3 we have $a_{ij} \; \sim'_{|_{\mathcal{B}'}} \; a_0$. Since $a_j \; TC_P \; a_0$, then using Lemma 5, we have $a_j \; TC'_{P|_{\mathcal{B}'}} \; a_0$ a strict path from a_j to a_0. Hence, we will have $a_0 \; \sim'_{|_{\mathcal{B}'}} \; a_{ij} \; (P \cup I \cup M) \; a_j \; TC'_{P|_{\mathcal{B}'}} \; a_0$. Therefore we obtain un strict cycle of $(P \cup I \cup M')_{|_{\mathcal{B}'}}$, which is a contradiction with the hypothesis. Hence $a_i \in \mathcal{B}'$
   - If $a_k \notin \mathcal{B}'$ then using Lemma 3, $a_{ik} \; \sim'_{|_{\mathcal{B}'}} \; a_0$. Therefore, since $a_i \in \mathcal{B}'$ (using the previous point), we will have the following cycle $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ of
     $a_0 \; \sim'_{|_{\mathcal{B}'}} \; a_{ik} \; M \; a_i \; TC'_{|_{\mathcal{B}'}} \; a_{ij} \; (P \cup I \cup M) \; a_j \; TC'_{P|_{\mathcal{B}'}} \; a_0$.
     This cycle is strict because $a_j \; TC'_{P|_{\mathcal{B}'}} \; a_0$ is a strict path from a_j to a_0 using Lemma 5, a contradiction. Hence $a_k \in \mathcal{B}'$.
2. The proof of the two last points is similar to the first point.

Proof of the Proposition 1 :

Il is obvious that that $\{P, I\}$ is representable by a 2-additive Choquet integral then the two following conditions are satisfied:

- $(P \cup I \cup M')_{|_{\mathcal{B}'}}$ contains no strict cycle;
- Every subset K of N such that $|K| = 3$ satisfies the MOPI conditions reduced to $\mathcal{B}'$ (Only the elements of $\mathcal{B}'$ are concerned in this condition).

The converse of the proposition is a consequence of Lemmas 4 and 6.

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