Source Correlation in Randomly Excited Complex Media

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Abstract—When dealing with modal representations of the Green's function of a complex medium, the modal coefficients are often assimilated to random variables, where statistical independence is justified on heuristic arguments supported by the complexity of the propagation in multiple-scattering scenarios. This letter addresses this assumption when the randomness originates from an uncertain positioning of the sources, proving under what conditions the modal coefficients can be regarded as uncorrelated, showing that this special condition should not be taken for granted, even in complex media.

Index Terms—Random media, correlation, Green's function, stochastic fields, uncertain systems.

I. INTRODUCTION

Complex propagation media are of fundamental importance in a number of practical configurations, e.g., wave propagation through indoor environments, the generation of random field distributions in multi-modal waveguides and cavities, etc. In the framework of the present paper, we shall regard a medium as complex as soon as a wave propagating through it is systematically subjected to a large number of scattering interactions, leading to loss of coherence and typically depolarization phenomena [1], [2].

Statistical models are typically proposed to reproduce the behavior of the electromagnetic fields thus generated, e.g., by representing the fields as continuous stochastic processes [2], [3], [4], [5] or by representing the propagation through a superposition of contributions modulated by random coefficients [6], [7], [8], [9]. The rationale for approximating the fields as random variables is not only based on the complexity of the wave propagation, but also on the existence of further randomizing processes, such as the presence of moving scatterers or even in the case of a static medium where the sources are randomly positioned, e.g., in the case of channel fading [10].

In the case of subspace (or spectral) representations, a generic field $\boldsymbol{F}(\boldsymbol{r})$ can be expanded over a basis formed by the orthogonal functions $\boldsymbol{g}_n(\boldsymbol{r})$

$$\boldsymbol{F}(\boldsymbol{r}) = \sum_{n=1}^{N} \alpha_n \boldsymbol{g}_n(\boldsymbol{r}), \tag{1}$$

where the α_n are treated as random variables. Choosing the type of probability description for the α_n is far from trivial, but it is a common practice to regard them as Gaussian independent and identically distributed, for obvious simplifying reasons [11], [8], [7], [12], [13], [14]. As a result, (1) gives the

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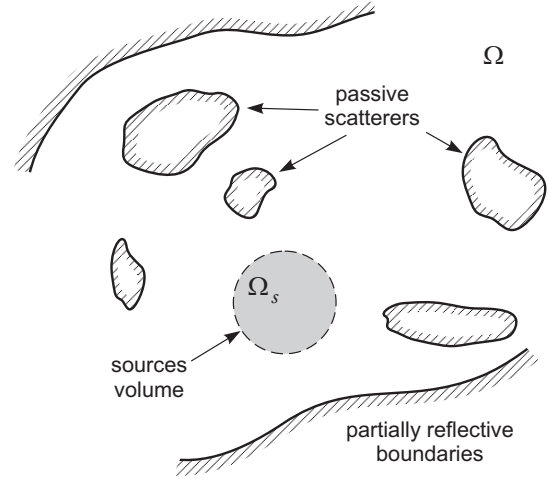

Fig. 1: Schematic representation of a medium occupying a region Ω, supporting a complex propagation of electromagnetic waves, due to multiple passive scatterers and reflective boundaries. The sources modeling the transmitter (or receiver) are bound to the volume Ω_s.

impression that N degrees of freedom are equally accessible in the medium.

In this letter we provide a formal proof that whenever the randomness is due to a non-deterministic position/orientation of the sources, the coefficients should not be regarded as independent, even in the case of strongly scattering media.

II. GREEN'S FUNCTIONS AND MODAL COVARIANCES

The electric field excited by an electric current distribution $\boldsymbol{J}(\boldsymbol{r})$ under a harmonic steady-state at the frequency ω can be expressed as a convolution integral involving the electric-electric dyadic Green's function of the medium

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{\Omega} \underline{\boldsymbol{G}}_{\text{ee}}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}') \mathrm{d}^3 \boldsymbol{r}', \tag{2}$$

where Ω is the region of space occupied by the medium under consideration (Fig. 1). Clearly, other types of Green's function can be considered when dealing with magnetic fields and/or magnetic sources. In a general way, these dyadic functions can always be expanded into a spectral representation of the type [15]

$$\underline{\boldsymbol{G}}_{\text{ee}}(\boldsymbol{r}, \boldsymbol{r}') = \sum_{n=1}^{\infty} \frac{\boldsymbol{e}_n(\boldsymbol{r}) \boldsymbol{e}_n(\boldsymbol{r}')}{k^2 - k_n^2}, \tag{3}$$

with $\boldsymbol{r}, \boldsymbol{r}' \in \Omega$, $k = k(\omega)$ the propagation constant related to the frequency by a generic dispersion law, and where $\boldsymbol{e}_n(\boldsymbol{r})$

and k_n are respectively the eigenfunctions and eigenvalues of Helmholtz equation

$$\nabla^2 e_n(r) + k_n^2 e_n(r) = 0 \qquad (4)$$

solved with boundary conditions specific to the medium, applied over $r \in \partial\Omega$. These eigensolutions will be assumed to be normalized in such a way that an orthonormality relationship holds

$$\int_\Omega e_i^\dagger(r) \cdot e_j(r) \mathrm{d}^3 r = \delta_{ij}, \qquad (5)$$

where δ_{ij} is Kronecker's delta and † stands for the Hermitian transpose. This kind of discrete expansion pertains to bounded media (e.g., cavities, waveguides), large collections of scatterers (e.g. fog and colloidal suspensions at optical frequencies) or quasi-periodic structures. Although free-space-like media can be considered by taking the limit of the summation in (3) to an integral, the case of a discrete set of normal modes shall be considered, with no loss of generality.

Inserting (3) into (2) yields

$$E(r) = \sum_{n=1}^\infty \frac{\gamma_n e_n(r)}{k^2 - k_n^2}, \qquad (6)$$

with γ_n the modal coefficients obtained by projecting the current distribution of the sources over the eigensolutions $e_n(r)$ by means of the Hilbert inner product

$$\gamma_n = \int_\Omega e_n(r) \cdot J(r) \mathrm{d}^3 r. \qquad (7)$$

In spite of the difficulties in predicting the normal modes $e_n(r)$, the most important information is arguably conveyed by the modal coefficients γ_n. As a matter of fact, the subspace representation (3) implies the possibility of independently exciting each of the degrees of freedom represented by each normal mode, an interesting feature in any domain concerned with the existence of rich multiple-scattering environments, from diversity communications to the statistics of the field generated within a random medium. The modal coefficients γ_n thus provide a direct measure of the degree of excitation of each of these potential degrees of freedom by means of the applied sources.

Due to the intrinsical difficulty of knowing beforehand the eigenfunctions $e_n(r)$, a statistical approach is often adopted, by approximating the modal coefficients γ_n to behave as random variables, described by a specific probability distribution law. Among the various hypothesis required in this respect, it is often assumed that the modal coefficients be independent and identically distributed, i.e., a covariance matrix with elements

$$\sigma_{ij}^2 = \langle \gamma_i \bar{\gamma}_j \rangle = \sigma_\gamma^2 \delta_{ij}, \qquad (8)$$

where $\langle \cdot \rangle$ is the ensemble average operator, the overbar representing the complex conjugate and σ_γ^2 the variance of the modal coefficients, i.e., their average power.

This paper addresses this common assumption in the case where the randomness is justified by a random positioning for the source's volume Ω_s. In general, for a given medium, with a set of normal modes $e_n(r)$, the modal coefficients would behave as random variables as soon as the position and the orientation of the sources can be modeled as random variables. Under these circumstances, the covariances (8) can be written as

$$\sigma_{ij}^2 = \int_\Omega \mathrm{d}^3 r \ e_i^\dagger(r) \cdot \int_\Omega \mathrm{d}^3 r' \ \underline{C}(r, r') \cdot e_j(r), \qquad (9)$$

where $\underline{C}(r, r')$ will be referred to as the coupling dyad, defined as

$$\underline{C}(r, r') = \left\langle J(r) J^\dagger(r') \right\rangle. \qquad (10)$$

Therefore, the coupling dyad operates as a kernel weighting in (5).

III. NECESSARY CONDITIONS FOR UNCORRELATED MODES

A necessary condition for (8) to hold can be readily derived by looking at the inner integral in (9) as a spatial filtering [16]: if the dyadic coupling function $\underline{C}(r, r')$ did not modify the spatial distributions $e_j(r)$, then (9) would coincide with the orthogonality relationship (5) existing between normal modes, directly implying uncorrelated modal coefficients. The general representation in (9) does not allow to push this idea further, as the double dependence of the coupling dyad on r and r' corresponds to a spatial-variant filtering; in other words, (9) is not equivalent to a convolution integral. This would be the case only if

$$\underline{C}(r, r') = \underline{C}(r - r'). \qquad (11)$$

It will be shown in the next Section that this property is automatically satisfied as soon as the source position and orientation are totally random with no a priori information. At the same time, it will be shown that these same conditions are needed if the coupling dyad is to be isotropic and non-polarized, a necessary condition proved at the end of this Section.

Condition (11) allows applying the convolution theorem in the reciprocal space k. To this effect, we need to introduce the Fourier transform $\tilde{e}_j(k)$ of a modal distribution $e_j(r)$ applied to the variable r

$$\tilde{e}_j(k) = \int_\Omega e_j(r) \mathrm{e}^{\mathrm{j} k \cdot r} \mathrm{d}^3 r. \qquad (12)$$

Hence, $\sigma_{ij}^2 \propto \delta_{ij}$ as soon as the Fourier transform $\underline{\tilde{C}}(k)$ of the coupling dyad is such that

$$\underline{\tilde{C}}(k) \cdot \tilde{e}_j(k) \propto \tilde{e}_j(k). \qquad (13)$$

Recalling that the normal modes are but steady-state solutions to the source-less Helmholtz equation (4), their spatial spectrum can be assumed to be essentially made up of propagative plane waves, i.e.,

$$\tilde{e}_i(k) = \delta(k - k_0) \mathring{e}_i(\hat{k}), \qquad (14)$$

where $\mathring{e}_i(\hat{k})$ is the angular spectrum [17] and k_0 is the wavenumber associated with the background material in Ω. This assumption holds as long as the mean free-path between two scattering events is larger than one wavelength, in order to avoid evanescent-wave couplings [2]. Therefore

$$\int_\Omega \underline{C}(r - r') \cdot e_i(r) \mathrm{d}^3 r' = \int \underline{\tilde{C}}(k) \cdot \tilde{e}_i(k) \mathrm{e}^{-\mathrm{j} k \cdot r} \mathrm{d}^3 k =$$
$$= \int_{4\pi} k_0^2 \underline{\tilde{C}}(k_0 \hat{k}) \cdot \mathring{e}_i(\hat{k}) \mathrm{e}^{-\mathrm{j} k_0 \hat{k} \cdot r} \mathrm{d}^2 \hat{k}, \qquad (15)$$
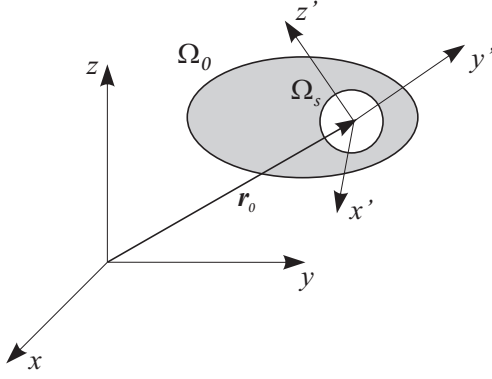
Fig. 2: Sources volume at a random position $\boldsymbol{r}_0$ bound to the region Ω_0. The reference system denoted by primed axis indicates the rotation operation $\underline{\boldsymbol{R}}_{\hat{q}}$ applied to the original orientation of the sources volume Ω_s.

whence the condition (13) is satisfied only if

$$\underline{\tilde{\boldsymbol{C}}}(k_0\hat{\boldsymbol{k}}) = C_0\underline{\boldsymbol{I}}, \tag{16}$$

with $\underline{\boldsymbol{I}}$ the identity dyad. Hence, the spatial spectrum of the coupling dyad needs to be: 1) constant over the surface $k = k_0$, i.e., isotropic along any direction and 2) non-polarizing, i.e., proportional to the identity dyad. These two conditions correspond to a spherically symmetric coupling dyad in the spatial domain. Under this double condition, (8) is immediately satisfied. These two conditions cannot be parted from (11), which is a fundamental requirement to these results. In other words, the coupling dyad needs to be position-invariant, isotropic in direction and non polarized.

IV. ISOTROPIC, POSITION-INVARIANT, NON-POLARIZED COUPLING DYADS

These requirements are shown in this Section to imply a specific random orientation and positioning of the source volume Ω_s. We first consider the case of an elementary electric current

$$\boldsymbol{J}(\boldsymbol{r}) = \underline{\boldsymbol{R}}_{\hat{q}} \cdot \hat{\boldsymbol{p}}\,\delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{17}$$

with $\hat{\boldsymbol{p}}$ the polarization of the elementary source, $\boldsymbol{r}_0$ its position and its orientation modified by the rotation operator $\underline{\boldsymbol{R}}_{\hat{q}}$, with $\hat{\boldsymbol{q}}$ the unit vector pointing to the direction of the source. We will assume the position and orientation of this source to be random and independent, assuming them as not causally related by any underlying deterministic process. It is therefore possible to split the ensemble average in (10) into two averages,

$$\underline{\boldsymbol{C}}(\boldsymbol{r}, \boldsymbol{r}') = \langle \underline{\boldsymbol{R}}_{\hat{q}} \cdot \hat{\boldsymbol{p}}\,\hat{\boldsymbol{p}}^\dagger \cdot \underline{\boldsymbol{R}}_{\hat{q}}^\dagger \rangle_{\hat{q}} \langle \delta(\boldsymbol{r} - \boldsymbol{r}_0)\delta(\boldsymbol{r}' - \boldsymbol{r}_0)\rangle_{\boldsymbol{r}_0}. \tag{18}$$

For a completely random orientation, with no preferential direction, $\hat{\boldsymbol{q}} \in \mathcal{U}(4\pi)$, i.e., $\hat{\boldsymbol{q}}$ can cover with uniform probability 4π steradian, hence a probability density function $f(\hat{\boldsymbol{q}}) = 1/4\pi$. The first average can be computed by observing that the vector $\underline{\boldsymbol{R}}_{\hat{q}} \cdot \hat{\boldsymbol{p}}$ resulting from the random rotation inherits the probability distribution of the rotation dyad. Hence, the first average, hereafter referred to as $\underline{\boldsymbol{C}}_{\hat{q}}$, is just the covariance

matrix of the three orthogonal components of $\hat{\boldsymbol{q}}$. Due to the initial assumption of a source oriented with equal probability along 4π steradian, the diagonal elements of $\underline{\boldsymbol{C}}_{\hat{q}}$ are bound to be identical. It is therefore sufficient to consider the case of the component of $\hat{\boldsymbol{q}}$ along the $\hat{\boldsymbol{z}}$ direction,

$$(\underline{\boldsymbol{C}}_{\hat{q}})_{ii} = \langle (\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{z}})^2 \rangle_{\hat{q}} = \frac{1}{3}, \tag{19}$$

whereas the off-diagonal elements of $\underline{\boldsymbol{C}}_{\hat{q}}$ are equal to zero, yielding $\underline{\boldsymbol{C}}_{\hat{q}} = \underline{\boldsymbol{I}}/3$. At the same time, the average on $\boldsymbol{r}_0 \in \Omega_0$, referred to as $\underline{\boldsymbol{C}}_{\boldsymbol{r}_0}$, yields $\underline{\boldsymbol{C}}_{\boldsymbol{r}_0} = \delta(\boldsymbol{r} - \boldsymbol{r}')/V_{\Omega_0}$, with V_{Ω_0} the volume covered by the random vector $\boldsymbol{r}_0$, i.e.,

$$\underline{\boldsymbol{C}}(\boldsymbol{r}, \boldsymbol{r}') = \frac{\underline{\boldsymbol{I}}}{3V_{\Omega_0}}\delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{20}$$

Whence, an elementary source arbitrarily polarized and randomly oriented and positioned satisfies the conditions put forward in the previous Section, thus ensuring uncorrelated coefficients for the modal expansion (3). In any other case, a probability density function $f(\hat{\boldsymbol{q}}; \boldsymbol{\Theta})$ depending on a set of parameters $\boldsymbol{\Theta}$ would be required, thus leading to results also dependent on $\boldsymbol{\Theta}$. An example is provided by Von Mises distributions, where a preferential direction is associated to a higher probability; as a result, the coupling dyad would present an anisotropic behavior (e.g., a full $\underline{\boldsymbol{C}}_{\hat{q}}$), with (9) becoming

$$\sigma_{ij}^2 = \frac{1}{V_{\Omega_0}} \int_\Omega \mathrm{d}^3\boldsymbol{r} \; \boldsymbol{e}_i^\dagger(\boldsymbol{r}) \cdot \underline{\boldsymbol{C}}_{\hat{q}} \cdot \boldsymbol{e}_j(\boldsymbol{r}), \tag{21}$$

thus altering the orthonormality condition (5) .

In the more general case of an extended source, it is no more possible to operate a factorization as done in (18), because of the distributed nature of $\boldsymbol{J}(\boldsymbol{r})$. We shall first consider the averaging obtained by a random rotation around the origin, which is equivalent to rotating the doublet of vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ while keeping the source region Ω_s centered at the origin. This operation can be resumed by a random rotation of $\boldsymbol{r}$, where it is regarded as a random vector $\boldsymbol{r} \rightsquigarrow \underline{\boldsymbol{R}}_{\hat{q}} \cdot \boldsymbol{r}$, which is equivalent to the substitution $\hat{\boldsymbol{r}} \rightsquigarrow \hat{\boldsymbol{q}}$. We define $\boldsymbol{r}'$ as

$$\boldsymbol{r}' = \boldsymbol{r} + \Delta r\underline{\boldsymbol{R}}_{\Delta \boldsymbol{r}} \cdot \hat{\boldsymbol{r}}, \tag{22}$$

where $\underline{\boldsymbol{R}}_{\Delta \boldsymbol{r}}$ represents the rotation operation linking the direction of $\boldsymbol{r}$ to $\Delta \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r}$. Hence

$$\left\langle \boldsymbol{J}(\boldsymbol{r})\boldsymbol{J}^\dagger(\boldsymbol{r}') \right\rangle_{\hat{q}} = \left\langle \hat{\boldsymbol{p}}(r\hat{\boldsymbol{q}})\hat{\boldsymbol{p}}^\dagger(r\hat{\boldsymbol{q}} + \Delta r\underline{\boldsymbol{R}}_{\Delta \boldsymbol{r}} \cdot \hat{\boldsymbol{q}}) \right.$$
$$\left. J(r\hat{\boldsymbol{q}})\bar{J}(r\hat{\boldsymbol{q}} + \Delta r\underline{\boldsymbol{R}}_{\Delta \boldsymbol{r}} \cdot \hat{\boldsymbol{q}}) \right\rangle_{\hat{q}}. \tag{23}$$

Introducing the rotation operator $\underline{\boldsymbol{R}}_{\hat{p}}$ linking the polarizations at $\boldsymbol{r}$ and $\boldsymbol{r}'$, the right-hand side of the above expression becomes

$$\left\langle \hat{\boldsymbol{p}}(r\hat{\boldsymbol{q}})\hat{\boldsymbol{p}}^\dagger(r\hat{\boldsymbol{q}}) \right\rangle_{\hat{q}} \cdot \underline{\boldsymbol{R}}_{\hat{p}}^\dagger(\boldsymbol{r}, \boldsymbol{r}')\left\langle J(r\hat{\boldsymbol{q}})\bar{J}(r\hat{\boldsymbol{q}} + \Delta r\underline{\boldsymbol{R}}_{\Delta \boldsymbol{r}} \cdot \hat{\boldsymbol{q}}) \right\rangle_{\hat{q}}. \tag{24}$$

In most practical configurations, sources are polarized in such a way that $\hat{\boldsymbol{p}}(\boldsymbol{r})$ is independent of $\boldsymbol{r}$, implying $\underline{\boldsymbol{R}}_{\hat{p}} \equiv \underline{\boldsymbol{I}}$. Recalling (20)

$$\left\langle \boldsymbol{J}(\boldsymbol{r})\boldsymbol{J}^\dagger(\boldsymbol{r}') \right\rangle_{\hat{q}} = \underline{\boldsymbol{C}}_{\hat{q}}(r, \Delta r)$$
$$= \underline{\boldsymbol{I}}\left\langle J(r\hat{\boldsymbol{q}})\bar{J}(r\hat{\boldsymbol{q}} + \Delta r\underline{\boldsymbol{R}}_{\Delta \boldsymbol{r}} \cdot \hat{\boldsymbol{q}}) \right\rangle_{\hat{q}}, \tag{25}$$

as long as $\hat{q} \in \mathcal{U}(4\pi)$. In any other case, the resulting correlation would be at least partially polarized, and thus dependent on the orientation of the source, i.e., non isotropic. This point was shown to be a necessary condition for uncorrelated modal coefficients in Section III.

If the source position were deterministic, then the resulting coupling dyad could not possibly be position-invariant, as clear from (25), still dependent on r. Conversely, we shall consider a random position r_0 spanning a region Ω_0, as depicted in Fig. 2, around the nominal position $\bar{r}_0$, a sort of barycenter of the region Ω_0. Once the positions r, r' at which the coupling dyad is evaluated are chosen, a random displacement of the sources by an offset r_0 is equivalent to applying a random offset $r_0 - \bar{r}_0$ to the vector r appearing in (25), resulting in a spatial averaging of the coupling dyad over Ω_0.

In the case $\Omega_0 \supseteq \Omega_s$, due to the rotational invariance of $\underline{C}_{\hat{q}}(r, \Delta r)$, the overall coupling dyad can be computed as

$$
\begin{aligned}
\underline{C}(\Delta r) &= \left\langle \underline{C}_{\hat{q}}(r, \Delta r) \right\rangle_{r_0} \\
&= \frac{\underline{I}}{3} \frac{1}{V_{\Omega_0}} \int_{\Omega_0} \mathrm{d}^3 r \int_{4\pi} \mathrm{d}^2 \hat{q} J(r\hat{q}) \bar{J}(r\hat{q} + \Delta r \underline{R}_{\Delta r} \cdot \hat{q}),
\end{aligned} \tag{26}
$$

where V_{Ω_0} is the volume of Ω_0.

The choice of Ω_0 only affects the absolute value of (26), if $\Omega_0 \supseteq \Omega_s$, since $\underline{C}_{\hat{q}}$ is identically equal to zero outside Ω_s. It is therefore convenient to chose a spherical volume Ω_0 with a radius R_0, yielding

$$
\begin{aligned}
\underline{C}(\Delta r) &= \frac{4\pi \underline{I}}{3 V_{\Omega_0}} \int_0^{R_0} \mathrm{d}r \int_{4\pi} \mathrm{d}^2 \hat{q} J(r\hat{q}) \bar{J}(r\hat{q} + \Delta r \underline{R}_{\Delta r} \cdot \hat{q}) \\
&= \frac{4\pi \underline{I}}{3 V_{\Omega_0}} \int_{\Omega_s} \mathrm{d}^3 r J(r) \bar{J}(r + \Delta r \underline{R}_{\Delta r} \cdot \hat{r})
\end{aligned} \tag{27}
$$

having exploited the spherical symmetry of $\underline{C}_{\hat{q}}$. One could wonder about the eventual relationship between (27) and the spatial auto-correlation function of $J(r)$

$$
\Phi_J(\Delta r) = \int_{\Omega_s} \mathrm{d}^3 r J(r) \bar{J}(r + \Delta r). \tag{28}
$$

Although this parallel could seem straightforward, (27) clearly shows that this is not the case, since the auto-correlation function only involves an integration carried out over r, while keeping $\Delta r = r - r'$ fixed. In fact, the averaging due to the random orientation of Ω_s submits Δr to the same operation, which cannot be expressed through any composite function of the auto-correlation function. It appears from (27) that the coupling dyad is rather a homogenization of the current distribution, resulting in a direct dependence only on Δr. This is indeed the condition required in Section III, while the non-polarization provided by the identity dyad and the isotropy of (27) complete the set of conditions leading to uncorrelated modal coefficients.

In any other case, either with a non-uniform probability of orientation, i.e., involving a preferential direction, or with a probability of positioning covering only part of the sources region Ω_s, the resulting coupling dyad would break at least one of the three conditions set out in the previous Section. As a result, the modal coefficients will be statistically correlated, thus reducing the actual number of degrees of freedom potentially provided by the medium, on average. A quantitative evaluation of (9) in this case will generally require a numerical approach, so that it is hardly possible to make any broad prediction, particularly because of the strong sensitivity that can be expected on the details of the configuration of interest.

V. CONCLUSIONS

This letter has introduced the concept of a coupling dyad linking the covariances of the modal coefficients in the spectral expansion of a generic Green's dyadic function to the orthogonality relationship existing between normal modes. Necessary conditions for perfectly uncorrelated modal coefficients were demonstrated, leading to the conclusion that in more general configurations the effective number of degrees of freedom available may be smaller than assumed. In other words, no equivalence between the number of available normal modes and the number of degrees of freedom should be taken for granted.

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