Equidimensional triangularization of multidimensional linear systems

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***Abstract*— Based on the results obtained in [12] on the purity filtration of a finitely presented module associated with a multidimensional linear system, this paper aims at obtaining an equivalent block-triangular representation of the multidimensional linear system defined by equidimensional diagonal blocks. The multidimensional linear system can then be integrated in cascade by solving equidimensional homogeneous linear systems. Many multidimensional linear systems defined by under/overdetermined linear systems of partial differential equations can be explicitly solved by means of the PurityFiltration and `AbelianSystems` packages, but cannot be computed by classical computer algebra systems such as Maple. The results developed in this paper generalize those obtained in the literature on Monge parametrizations and on the classification of autonomous elements by their codimensions.**

I. Introduction

This paper is the continuation of [12]. We refer the reader to [12] for the notations and the results used in what follows.

We recall that D denotes a noetherian domain with a finite global dimension $\mathrm{gld}(D) = n$ and which satisfies

$$\forall\, i \in \mathbb{N} = \{0, 1, 2, \ldots\}, \quad \mathrm{ext}^i_D(\mathrm{ext}^{i+1}_D(M, D), D) = 0,$$

for all left D-modules M. For more details, see [5], [12]. In particular, these conditions hold for a *Auslander regular ring* D [4], [5] such as, for instance, the commutative polynomial ring $D = k[x_1, \ldots, x_n]$ in $x_1, \ldots, x_n$ with coefficients in a field k or the noncommutative polynomial rings $A_n(k)$ (resp., $B_n(k)$, $\hat{\mathcal{D}}_n(k)$ and $\mathcal{D}_n(k)$) of partial differential (PD) operators in $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, with coefficients in the ring $k[x_1, \ldots, x_n]$ (resp., the ring $k(x_1, \ldots, x_n)$ of rational functions, the ring $k[\![x_1, \ldots, x_n]\!]$ of formal power series, or the ring $\mathbb{R}\{x_1, \ldots, x_n\}$ of locally convergent power series), where k is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) [4].

The purpose of this paper is to apply Theorem 3.1 and Corollary 3.1 of [12] to the short exact sequences defined in (42) of [12] to determine a new finite presentation of the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ defined by a block-triangular matrix P formed by block-diagonal matrices defining finite presentations of the *pure* left D-modules $M/t(M)$, $\mathrm{coker}\,\gamma_{21}$ and $\mathrm{coker}\,\gamma_{32}$ defined in [12], and of the left D-module $\mathrm{ext}^3_D(N_{33}, D)$ of *grade* greater or equal to 3 (which is also pure when $\ker_D(R_3.) = 0$). For more details and definitions, we refer the reader to [12].

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II. Main results

Let us now precisely describe the left D-homomorphisms γ_{32} and γ_{21} and the left D-modules $\mathrm{coker}\,\gamma_{32}$ and $\mathrm{coker}\,\gamma_{21}$ defined in [12] (see (41) and (42)). Applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ to the commutative diagram defined in Fig. 3 of [12], we obtain the following commutative diagram formed by horizontal complexes:

$$\begin{array}{ccccc}
D^{1\times p_{-13}} & \xleftarrow{.R_{03}} & D^{1\times p_{03}} & \xleftarrow{.R_{13}} & D^{1\times p_{13}} \\
\downarrow .F_{-13} & & \downarrow .F_{03} & & \downarrow .F_{13} \\
D^{1\times p_{-12}} & \xleftarrow{.R_{02}} & D^{1\times p_{02}} & \xleftarrow{.R_{12}} & D^{1\times p_{12}} \\
\downarrow .F_{-12} & & \downarrow .F_{02} & & \| \\
D^{1\times p_{-11}} & \xleftarrow{.R_{01}} & D^{1\times p_{01}} & \xleftarrow{.R_{11}} & D^{1\times p_{11}}.
\end{array} \tag{1}$$

Using (34) of [12], the defect of exactness of the top (resp., middle, bottom) horizontal complex of (1) is $\mathrm{ext}^1_D(N_{13}, D)$ (resp., $\mathrm{ext}^1_D(N_{12}, D)$, $\mathrm{ext}^1_D(N_{11}, D)$). Let us introduce the canonical projections defined in (2). The commutative diagram (1) induces the two left D-homomorphisms:

$$\begin{array}{rcl}
\ker_D(.R_{03})/(D^{1\times p_{13}}\, R_{13}) & \xrightarrow{\alpha_{32}} & \ker_D(.R_{02})/(D^{1\times p_{12}}\, R_{12}) \\
\rho_3(\lambda) & \longmapsto & \rho_2(\lambda\, F_{03}),
\end{array} \tag{3}$$

$$\begin{array}{rcl}
\ker_D(.R_{02})/(D^{1\times p_{12}}\, R_{12}) & \xrightarrow{\alpha_{21}} & \ker_D(.R_{01})/(D^{1\times p_{11}}\, R_{11}) \\
\rho_2(\mu) & \longmapsto & \rho_1(\mu\, F_{02}).
\end{array} \tag{4}$$

Two chases in the commutative diagram (1) show that ρ_3 and ρ_2 are well-defined (see, e.g., [17]).

Let us now use Proposition 2.2 of [12] to get a finite presentation of the left D-modules $\mathrm{ext}^3_D(N_{33}, D)$, $\mathrm{ext}^2_D(N_{22}, D)$ and $\mathrm{ext}^1_D(N_{11}, D)$. Let $R'_{1k} \in D^{p_{0k}\times p'_{1k}}$ be such that $\ker_D(.R_{0k}) = D^{1\times p'_{1k}}\, R'_{1k}$ for $k = 1, 2, 3$. Since $D^{1\times p_{1k}}\, R_{1k} \subseteq D^{1\times p'_{1k}}\, R'_{1k}$, there exists $R''_{1k} \in D^{p_{1k}\times p'_{1k}}$ such that:

$$R_{1k} = R''_{1k}\, R'_{1k}. \tag{5}$$

Let $R'_{2k} \in D^{p'_{1k}\times p'_{2k}}$ be a matrix such that $\ker_D(.R'_{1k}) = D^{1\times p'_{2k}}\, R'_{2k}$. Then, using Proposition 2.2 of [12], we obtain the left D-homomorphism defined by (6), where

$$L_k \triangleq D^{1\times p'_{1k}}/(D^{1\times p_{1k}}\, R''_{1k} + D^{1\times p'_{2k}}\, R'_{2k}),$$

and $\rho'_k : D^{1\times p'_{1k}} \longrightarrow L_k$ is the canonical projection.

Since $R'_{1k}\, F_{0k}\, R_{0(k-1)} = R'_{1k}\, R_{0k}\, F_{-1k} = 0$, then

$$D^{1\times p'_{1k}}\, (R'_{1k}\, F_{0k}) \subseteq \ker_D(.R_{0(k-1)}) = D^{1\times p'_{1(k-1)}}\, R'_{1(k-1)},$$

and thus there exists $F'_{1k} \in D^{p'_{1k}\times p'_{1(k-1)}}$ such that:

$$\forall\, k = 2, 3, \quad R'_{1k}\, F_{0k} = F'_{1k}\, R'_{1(k-1)}. \tag{7}$$

$$
\begin{aligned}
\rho_3 &: \ker_D(.R_{03}) \longrightarrow \ker_D(.R_{03})/(D^{1\times p_{13}}\, R_{13}) \cong \mathrm{ext}^1_D(N_{13}, D) \cong \mathrm{ext}^3_D(N_{33}, D),\\
\rho_2 &: \ker_D(.R_{02}) \longrightarrow \ker_D(.R_{02})/(D^{1\times p_{12}}\, R_{12}) \cong \mathrm{ext}^1_D(N_{12}, D) \cong \mathrm{ext}^2_D(N_{22}, D),\\
\rho_1 &: \ker_D(.R_{01}) \longrightarrow \ker_D(.R_{01})/(D^{1\times p_{11}}\, R_{11}) \cong \mathrm{ext}^1_D(N_{11}, D) \cong t(M).
\end{aligned} \tag{2}
$$

$$
\begin{array}{rcl}
\chi_k : L_k \triangleq D^{1\times p'_{1k}}/(D^{1\times p_{1k}}\, R''_{1k} + D^{1\times p'_{2k}}\, R'_{2k}) & \longrightarrow & (D^{1\times p'_{1k}}\, R'_{1k})/(D^{1\times p_{1k}}\, R_{1k}) \cong \mathrm{ext}^1_D(N_{1k}, D),\\
\rho'_k(\lambda) & \longmapsto & \rho_k(\lambda\, R'_{1k}).
\end{array} \tag{6}
$$

Similarly, there exists $F'_{2k} \in D^{p'_{2k}\times p'_{2(k-1)}}$ such that:

$$\forall\, k = 2, 3, \quad R'_{2k}\, F'_{1k} = F'_{2k}\, R'_{2(k-1)}. \tag{8}$$

Thus, we obtain the commutative exact diagram (9).

Remark 2.1: If $R_{0k} = 0$, i.e., $\ker_D(R_{1k}.) = 0$, then applying the functor $\hom_D(\cdot\,, D)$ to the short exact sequence $0 \longrightarrow D^{p_{0k}} \xrightarrow{R_{1k}.} D^{p_{1k}} \xrightarrow{\kappa_{1k}} N_{1k} \longrightarrow 0$, we get the complex $0 \longleftarrow D^{1\times p_{0k}} \xleftarrow{.R_{1k}} D^{1\times p_{1k}}$, which yields $\ker_D(.R_{0k}) = D^{1\times p_{0k}}$, i.e.:

$$R'_{1k} = I_{p_{0k}}, \quad p'_{1k} = p_{0k}, \quad R'_{2k} = 0.$$

Let us now deduce two identities which will be used in what follows. Combining (30) of [12] for $k = 2$ with (5) for $k = 1$ and $k = 2$, and with (7) for $k = 2$, we obtain

$$R''_{11}\, R'_{11} = R_{11} = R_{12}\, F_{02} = R''_{12}\, R'_{12}\, F_{02} = R''_{12}\, F'_{12}\, R'_{11},$$

and thus $(R''_{11} - R''_{12}\, F'_{12})\, R'_{11} = 0$, i.e.,

$$D^{1\times p_{11}}\, (R''_{11} - R''_{12}\, F'_{12}) \subseteq \ker_D(.R'_{11}) = D^{1\times p'_{21}}\, R'_{21},$$

which proves the existence of $X_{12} \in D^{p_{11}\times p'_{21}}$ such that:

$$R''_{11} = R''_{12}\, F'_{12} + X_{12}\, R'_{21}. \tag{10}$$

Combining (31) of [12] for $k = 3$ with (5) for $k = 2$ and $k = 3$, and with (7) for $k = 3$, we obtain

$$
\begin{aligned}
F_{13}\, (R''_{12}\, R'_{12}) = F_{13}\, R_{12} &= R_{13}\, F_{03} = (R''_{13}\, R'_{13})\, F_{03}\\
&= R''_{13}\, F'_{13}\, R'_{12},
\end{aligned}
$$

and thus $(F_{13}\, R''_{12} - R''_{13}\, F'_{13})\, R'_{12} = 0$, i.e., $D^{1\times p_{13}}\, (F_{13}\, R''_{12} - R''_{13}\, F'_{13}) \subseteq \ker_D(.R'_{12}) = D^{1\times p'_{22}}\, R'_{22}$, which proves the existence of $X_{22} \in D^{p_{13}\times p'_{22}}$ such that:

$$F_{13}\, R''_{12} - R''_{13}\, F'_{13} = X_{22}\, R'_{22}. \tag{11}$$

Using $t(M) \cong \mathrm{ext}^1_D(N_{11}, D)$ (see (2) and [12]) and the isomorphisms χ_k's defined by (6), we get:

$$
\left\{
\begin{aligned}
L_1 &= D^{1\times p'_{11}}/(D^{1\times p_{11}}\, R''_{11} + D^{1\times p'_{21}}\, R'_{21})\\
&\cong \mathrm{ext}^1_D(N_{11}, D) \cong t(M),\\
L_2 &= D^{1\times p'_{12}}/(D^{1\times p_{12}}\, R''_{12} + D^{1\times p'_{22}}\, R'_{22})\\
&\cong \mathrm{ext}^2_D(N_{22}, D),\\
L_3 &= D^{1\times p'_{13}}/(D^{1\times p_{13}}\, R''_{13} + D^{1\times p'_{23}}\, R'_{23})\\
&\cong \mathrm{ext}^3_D(N_{33}, D).
\end{aligned}
\right. \tag{12}
$$

Then, we can define the left D-homomorphism

$$\overline{\alpha}_{32} = \chi_2^{-1} \circ \alpha_{32} \circ \chi_3 : L_3 \longrightarrow L_2,$$

where the χ_i's are defined by (6) and α_{32} is defined by (3). Using (7) for $k = 3$, we have

$$
\begin{aligned}
\overline{\alpha}_{32}(\rho'_3(\lambda)) = (\chi_2^{-1} \circ \alpha_{32})(\rho_3(\lambda\, R'_{13})) &= \chi_2^{-1}(\rho_2(\lambda\, R'_{13}\, F_{03}))\\
&= \chi_2^{-1}(\rho_2(\lambda\, F'_{13}\, R'_{12})) = \rho'_2(\lambda\, F'_{13}),
\end{aligned} \tag{13}
$$

for all $\lambda \in D^{1\times p'_{13}}$. Using (11) and (8) for $k = 3$, we get

$$
\begin{aligned}
\begin{pmatrix} R''_{13} \\ R'_{23} \end{pmatrix} F'_{13} &= \begin{pmatrix} F_{13}\, R''_{12} - X_{22}\, R'_{22} \\ F'_{23}\, R'_{22} \end{pmatrix}\\
&= \begin{pmatrix} F_{13} & -X_{22} \\ 0 & F'_{23} \end{pmatrix} \begin{pmatrix} R''_{12} \\ R'_{22} \end{pmatrix},
\end{aligned}
$$

which yields the commutative exact diagram (14).

Up to isomorphism, the short exact sequence

$$0 \longrightarrow \mathrm{ext}^3_D(N_{33}, D) \xrightarrow{\gamma_{32}} \mathrm{ext}^2_D(N_{22}, D) \longrightarrow \mathrm{coker}\, \gamma_{32} \longrightarrow 0$$

(see [12]) becomes the following short exact sequence:

$$0 \longrightarrow L_3 \xrightarrow{\overline{\alpha}_{32}} L_2 \xrightarrow{\theta_2} \mathrm{coker}\, \overline{\alpha}_{32} \longrightarrow 0. \tag{15}$$

Using 3 of Proposition 3.1 of [8], the left D-module $\mathrm{coker}\, \overline{\alpha}_{32}$ is then defined by:

$$\mathrm{coker}\, \overline{\alpha}_{32} = D^{1\times p'_{12}}/(D^{1\times p'_{13}}\, F'_{13} + D^{1\times p_{12}}\, R''_{12} + D^{1\times p'_{22}}\, R'_{22}).$$

Then, we can easily check that the commutative exact diagram (16) holds, where $\psi_2 : D^{1\times(p'_{13}+p_{12}+p'_{22})} \longrightarrow L_3$ is the left D-homomorphism defined by:

$$
\psi_2(e_i) = \begin{cases} \rho'_3(e_i) & i = 1, \ldots, p'_{13},\\ 0, & i = p'_{13} + 1, \ldots, p'_{13} + p_{12} + p'_{22}. \end{cases}
$$

Applying Theorem 3.1 of [12] to the short exact sequence (15) with the matrix

$$A = \begin{pmatrix} I_{p'_{13}} \\ 0 \\ 0 \end{pmatrix} \in D^{(p'_{13}+p_{12}+p'_{22})\times p'_{13}},$$

(see also Corollary 3.1 of [12]), we obtain the following characterization of the left D-module L_2 in terms of the presentations of $L_3 \cong \mathrm{ext}^3_D(N_{33}, D)$ and $\mathrm{coker}\, \overline{\alpha}_{32}$.

$$
\begin{array}{ccccccc}
D^{1\times p_{-13}} & \xleftarrow{.R_{03}} & D^{1\times p_{03}} & \xleftarrow{.R'_{13}} & D^{1\times p'_{13}} & \xleftarrow{.R'_{23}} & D^{1\times p'_{23}} \\
\downarrow .F_{-13} & & \downarrow .F_{03} & & \downarrow .F'_{13} & & \downarrow .F'_{23} \\
D^{1\times p_{-12}} & \xleftarrow{.R_{02}} & D^{1\times p_{02}} & \xleftarrow{.R'_{12}} & D^{1\times p'_{12}} & \xleftarrow{.R'_{22}} & D^{1\times p'_{22}} \\
\downarrow .F_{-12} & & \downarrow .F_{02} & & \downarrow .F'_{12} & & \downarrow .F'_{22} \\
D^{1\times p_{-11}} & \xleftarrow{.R_{01}} & D^{1\times p_{01}} & \xleftarrow{.R'_{11}} & D^{1\times p'_{11}} & \xleftarrow{.R'_{21}} & D^{1\times p'_{21}}.
\end{array} \tag{9}
$$

$$
\begin{array}{ccccccc}
D^{1\times(p_{13}+p'_{23})} & \xrightarrow{.(R''^T_{13} \quad R'^T_{23})^T} & D^{1\times p'_{13}} & \xrightarrow{\rho'_3} & L_3 & \longrightarrow & 0 \\
\downarrow .\begin{pmatrix} F_{13} & -X_{22} \\ 0 & F'_{23} \end{pmatrix} & & \downarrow .F'_{13} & & \downarrow \overline{\alpha}_{32} & & \\
D^{1\times(p_{12}+p'_{22})} & \xrightarrow{.(R''^T_{12} \quad R'^T_{22})^T} & D^{1\times p'_{12}} & \xrightarrow{\rho'_2} & L_2 & \longrightarrow & 0.
\end{array} \tag{14}
$$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & D^{1\times p_{12}}\, R''_{12} + D^{1\times p'_{22}}\, R'_{22} & & & & \\
 & & & & \downarrow & & & & \\
 & & D^{1\times(p'_{13}+p_{12}+p'_{22})} & \xrightarrow{.(F'^T_{13} \quad R''^T_{12} \quad R'^T_{22})^T} & D^{1\times p'_{12}} & \xrightarrow{\sigma_2} & \operatorname{coker}\overline{\alpha}_{32} & \longrightarrow & 0 \\
 & & \downarrow \psi_2 & & \downarrow \rho'_2 & & \| & & \\
0 & \longrightarrow & L_3 & \xrightarrow{\overline{\alpha}_{32}} & L_2 & \xrightarrow{\theta_2} & \operatorname{coker}\overline{\alpha}_{32} & \longrightarrow & 0. \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \tag{16}
$$

Proposition 2.1: With the previous notations, let

$$
Q_2 = \begin{pmatrix} R''_{12} \\ R'_{22} \end{pmatrix} \in D^{(p_{12}+p'_{22})\times p'_{12}},
$$

$$
P_2 = \begin{pmatrix} F'_{13} & -I_{p'_{13}} \\ R''_{12} & 0 \\ R'_{22} & 0 \\ 0 & R''_{13} \\ 0 & R'_{23} \end{pmatrix} \in D^{(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p'_{12}+p'_{13})},
$$

and let us consider the following left D-modules:

$$
\begin{cases} L_2 = D^{1\times p'_{12}}/(D^{1\times p_{12}}\, R''_{12} + D^{1\times p'_{22}}\, R'_{22}), \\ E_2 = D^{1\times(p'_{12}+p'_{13})}/(D^{1\times(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\, P_2). \end{cases}
$$

If $\varrho_2 : D^{1\times(p'_{12}+p'_{13})} \longrightarrow E_2$ is the canonical projection, then $E_2 \cong L_2$, where the left D-isomorphism is defined by:

$$
\begin{array}{rcl}
\phi_2 : L_2 & \longrightarrow & E_2 \\
\rho'_2(\mu) & \longmapsto & \varrho_2(\mu\,(I_{p'_{12}} \quad 0)), \\
\phi_2^{-1} : E_2 & \longrightarrow & L_2 \\
\varrho_2(\nu) & \longmapsto & \rho'_2(\nu\,(I^T_{p'_{12}} \quad F'^T_{13})^T).
\end{array} \tag{17}
$$

If $\mathcal{F}$ is a left D-module, then applying the functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the isomorphism $E_2 \cong L_2$, and using Malgrange's theorem (see Theorem 1.1 of [12]), we obtain:

$$
\ker_{\mathcal{F}}(Q_2.) \cong \ker_{\mathcal{F}}(P_2.).
$$

More precisely, using (17), we get the following corollary.

Corollary 2.1: If $\mathcal{F}$ is a left D-module, then

$$
\ker_{\mathcal{F}}(Q_2.) \cong \ker_{\mathcal{F}}(P_2.),
$$

and the following equivalence

$$
\begin{cases} R''_{12}\, \upsilon = 0, \\ R'_{22}\, \upsilon = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} F'_{13}\, \tau_2 - \tau_3 = 0, \\ R''_{12}\, \tau_2 = 0, \\ R'_{22}\, \tau_2 = 0, \\ R''_{13}\, \tau_3 = 0, \\ R'_{23}\, \tau_3 = 0, \end{cases}
$$

holds under the following invertible transformations:

$$
\begin{array}{rcl}
\delta : \ker_{\mathcal{F}}(P_2.) & \longrightarrow & \ker_{\mathcal{F}}(Q_2.) \\
\begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} & \longmapsto & \upsilon = \tau_2, \\
\delta^{-1} : \ker_{\mathcal{F}}(Q_2.) & \longrightarrow & \ker_{\mathcal{F}}(P_2.) \\
\upsilon & \longmapsto & \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{p'_{12}} \\ F'_{13} \end{pmatrix} \upsilon.
\end{array} \tag{18}
$$

Let us now introduce the left D-homomorphism

$$
\overline{\alpha}_{21} = \chi_1^{-1} \circ \alpha_{21} \circ \chi_2 : L_2 \longrightarrow L_1,
$$

where the χ_i's are defined by (6) and α_{21} is defined by (4). Then, using (7) for $k = 2$, for all $\mu \in D^{1\times p'_{12}}$, we get

$$
\begin{aligned}
\overline{\alpha}_{21}(\rho'_2(\mu)) &= (\chi_1^{-1} \circ \alpha_{21})(\rho_2(\mu\, R'_{12})) = \chi_1^{-1}(\rho_1(\mu\, R'_{12}\, F_{02})) \\
&= \chi_1^{-1}(\rho_1(\mu\, F'_{12}\, R'_{11})) = \rho'_1(\mu\, F'_{12}).
\end{aligned} \tag{19}
$$

Moreover, using (10) and (8) for $k=2$, we have

$$\begin{pmatrix} R''_{12} \\ R'_{22} \end{pmatrix} F'_{12} = \begin{pmatrix} R''_{11} - X_{12}\, R'_{21} \\ F'_{22}\, R'_{21} \end{pmatrix} = \begin{pmatrix} I_{p_{11}} & -X_{12} \\ 0 & F'_{22} \end{pmatrix} \begin{pmatrix} R''_{11} \\ R'_{21} \end{pmatrix},$$

which yields the commutative exact diagram (20). Up to isomorphism, the short exact sequence

$$0 \longrightarrow \mathrm{ext}^2_D(N_{22}, D) \xrightarrow{\gamma_{21}} t(M) \longrightarrow \mathrm{coker}\,\gamma_{21} \longrightarrow 0,$$

(see [12]) becomes the following short exact sequence

$$0 \longrightarrow L_2 \xrightarrow{\overline{\alpha}_{21}} L_1 \xrightarrow{\theta_1} \mathrm{coker}\,\overline{\alpha}_{21} \longrightarrow 0, \tag{21}$$

where, using 3 of Proposition 3.1 of [8], the left D-module $\mathrm{coker}\,\overline{\alpha}_{21}$ is defined by:

$$\mathrm{coker}\,\overline{\alpha}_{21} = D^{1\times p'_{11}}/(D^{1\times p'_{12}}\, F'_{12} + D^{1\times p_{11}}\, R''_{11} + D^{1\times p'_{21}}\, R'_{21}).$$

Using the left D-isomorphism $\phi_2^{-1} : E_2 \longrightarrow L_2$ defined by (17), the short exact sequence (21) yields the following short exact sequence

$$0 \longrightarrow E_2 \xrightarrow{\overline{\alpha}_{21}\circ \phi_2^{-1}} L_1 \xrightarrow{\theta_1} \mathrm{coker}\,\overline{\alpha}_{21} \longrightarrow 0, \tag{22}$$

where $\overline{\alpha}_{21}\circ\phi_2^{-1} : E_2 \longrightarrow L_1$ is defined by:

$$(\overline{\alpha}_{21}\circ\phi_2^{-1})(\varrho_2(\nu)) = \overline{\alpha}_{21}\left(\rho'_2\left(\nu \begin{pmatrix} I_{p'_{12}} \\ F'_{13} \end{pmatrix}\right)\right) = \rho'_1\left(\nu \begin{pmatrix} F'_{12} \\ F'_{13}\, F'_{12} \end{pmatrix}\right).$$

Now, we can check that the commutative exact diagram (23) holds, where $\psi_1 : D^{1\times(p'_{12}+p_{11}+p'_{21})} \longrightarrow E_2$ is the left D-homomorphism defined by

$$\psi_1(f_j) = \begin{cases} \varrho_2(f_j\, F), & j = 1, \ldots, p'_{12}, \\ 0, & j = p'_{12}+1, \ldots, p'_{12}+p_{11}+p'_{21}, \end{cases}$$

where $\{f_j\}_{j=1,\ldots,p'_{12}+p_{11}+p'_{21}}$ is the standard basis of $D^{1\times(p'_{12}+p_{11}+p'_{21})}$ and:

$$F = \begin{pmatrix} I_{p'_{12}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{(p'_{12}+p_{11}+p'_{21})\times(p'_{12}+p'_{13})}.$$

Applying Theorem 3.1 of [12] to the short exact sequence (22) with the matrix $A = F$ (see Corollary 3.1 of [12]), we obtain the following proposition.

Proposition 2.2: With the hypotheses of Proposition 2.1 and the previous notations, let

$$r = p'_{12} + p_{11} + p'_{21} + p'_{13} + p_{12} + p'_{22} + p_{13} + p'_{23},$$

and let us consider the following two matrices

$$P_1 = \begin{pmatrix} F'_{12} & -I_{p'_{12}} & 0 \\ R''_{11} & 0 & 0 \\ R'_{21} & 0 & 0 \\ 0 & F'_{13} & -I_{p'_{13}} \\ 0 & R''_{12} & 0 \\ 0 & R'_{22} & 0 \\ 0 & 0 & R''_{13} \\ 0 & 0 & R'_{23} \end{pmatrix} \in D^{r\times(p'_{11}+p'_{12}+p'_{13})},$$

$$Q_1 = \begin{pmatrix} R''_{11} \\ R'_{21} \end{pmatrix} \in D^{(p_{11}+p'_{21})\times p'_{11}},$$

and the following two finitely presented left D-modules:

$$\begin{cases} L_1 = D^{1\times p'_{11}}/(D^{1\times(p_{11}+p'_{21})}\, Q_1), \\ E_1 = D^{1\times(p'_{11}+p'_{12}+p'_{13})}/(D^{1\times r}\, P_1). \end{cases}$$

If $\varrho_1 : D^{1\times(p'_{11}+p'_{12}+p'_{13})} \longrightarrow E_1$ is the canonical projection, then $E_1 \cong L_1$, where the left D-isomorphism is defined by:

$$\begin{array}{rcl} \phi_1 : L_1 & \longrightarrow & E_1 \\ \rho'_1(\nu) & \longmapsto & \varrho_1(\nu\,(I_{p'_{11}} \quad 0 \quad 0)), \\ \phi_1^{-1} : E_1 & \longrightarrow & L_1 \\ \varrho_1(\lambda) & \longmapsto & \rho'_1\left(\lambda \begin{pmatrix} I_{p'_{11}} \\ F'_{12} \\ F'_{13}\, F'_{12} \end{pmatrix}\right). \end{array} \tag{24}$$

Finally, we have $L_1 \cong t(M)$ and:

$$\begin{array}{rcl} \vartheta : L_1 & \longrightarrow & t(M) \\ \rho'_1(\nu) & \longmapsto & \pi(\nu\, R'_{11}), \end{array} \qquad \begin{array}{rcl} \vartheta^{-1} : t(M) & \longrightarrow & L_1 \\ \pi(\nu\, R'_{11}) & \longmapsto & \rho'_1(\nu). \end{array}$$

If $\mathcal{F}$ is a left D-module, then applying the functor $\hom_D(\cdot, \mathcal{F})$ to the isomorphism $E_1 \cong L_1$, and using Malgrange's theorem (see Theorem 1.1 of [12]), we obtain:

$$\ker_{\mathcal{F}}(Q_1.) \cong \ker_{\mathcal{F}}(P_1.).$$

More precisely, using (24), we get the following corollary.

Corollary 2.2: If $\mathcal{F}$ is a left D-module, then we have

$$\ker_{\mathcal{F}}(Q_1.) \cong \ker_{\mathcal{F}}(P_1.),$$

and the following equivalence

$$\begin{cases} R''_{11}\,\theta = 0, \\ R'_{21}\,\theta = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} F'_{12}\,\tau_1 - \tau_2 = 0, \\ R''_{11}\,\tau_1 = 0, \\ R'_{21}\,\tau_1 = 0, \\ F'_{13}\,\tau_2 - \tau_3 = 0, \\ R''_{12}\,\tau_2 = 0, \\ R'_{22}\,\tau_2 = 0, \\ R''_{13}\,\tau_3 = 0, \\ R'_{23}\,\tau_3 = 0, \end{cases}$$

$$\begin{array}{cccccc} D^{1\times(p_{12}+p'_{22})} & \xrightarrow{.(R''^T_{12}\quad R'^T_{22})^T} & D^{1\times p'_{12}} & \xrightarrow{\rho'_2} & L_2 & \longrightarrow 0 \\ \downarrow .\begin{pmatrix} I_{p_{11}} & -X_{12}\\ 0 & F'_{22}\end{pmatrix} & & \downarrow .F'_{12} & & \downarrow \overline{\alpha}_{21} & \\ D^{1\times(p_{11}+p'_{21})} & \xrightarrow{.(R''^T_{11}\quad R'^T_{21})^T} & D^{1\times p'_{11}} & \xrightarrow{\rho'_1} & L_1 & \longrightarrow 0. \end{array} \tag{20}$$

$$\begin{array}{cccccccc} & & & & 0 & & & \\ & & & & \downarrow & & & \\ & & & & D^{1\times p_{11}}\, R''_{11} + D^{1\times p'_{21}}\, R'_{21} & & & \\ & & & & \downarrow & & & \\ & D^{1\times(p'_{12}+p_{11}+p'_{21})} & \xrightarrow{.(F'^T_{12}\quad R''^T_{11}\quad R'^T_{21})^T} & & D^{1\times p'_{11}} & \xrightarrow{\sigma_1} & \operatorname{coker}\overline{\alpha}_{21} & \longrightarrow 0 \\ & \downarrow \psi_1 & & & \downarrow \rho'_1 & & \| & \\ 0 \longrightarrow & E_2 & \xrightarrow{\overline{\alpha}_{21}\circ\phi_2^{-1}} & & L_1 & \xrightarrow{\theta_1} & \operatorname{coker}\overline{\alpha}_{21} & \longrightarrow 0. \\ & & & & \downarrow & & & \\ & & & & 0 & & & \end{array} \tag{23}$$

holds under the following invertible transformations:

$$\begin{array}{rcl} \varpi : \ker_{\mathcal{F}}(P_1.) & \longrightarrow & \ker_{\mathcal{F}}(Q_1.) \\ \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} & \longmapsto & \theta = \tau_1, \\ \varpi^{-1} : \ker_{\mathcal{F}}(Q_1.) & \longrightarrow & \ker_{\mathcal{F}}(P_1.) \\ \theta & \longmapsto & \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{p'_{12}} \\ F'_{12} \\ F'_{13}\, F'_{12} \end{pmatrix} \theta. \end{array} \tag{25}$$

Using Proposition 2.2, let $\vartheta \circ \phi_1^{-1} : E_1 \longrightarrow t(M)$ be the left D-isomorphism defined by:

$$(\vartheta \circ \phi_1^{-1})(\varrho_1(\lambda)) = \pi \left(\lambda \begin{pmatrix} R'_{11} \\ F'_{12}\, R'_{11} \\ F'_{13}\, F'_{12}\, R'_{11} \end{pmatrix} \right).$$

Then, the short exact sequence

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0$$

(see [12]) yields the following one:

$$0 \longrightarrow E_1 \xrightarrow{i\circ\vartheta\circ\phi_1^{-1}} M \xrightarrow{\rho} M/t(M) \longrightarrow 0. \tag{26}$$

Now, we can easily check that the commutative exact diagram (27) holds, where $\psi : D^{1\times p'_{11}} \longrightarrow E_1$ is defined by $\psi(g_k) = \varrho_1(g_k\,(I_{p'_{11}} \quad 0 \quad 0))$, and $\{g_k\}_{k=1,\dots,p'_{11}}$ is the standard basis of $D^{1\times p'_{11}}$. Then, we can apply Theorem 3.1 of [12] to the short exact sequence (26) with the matrix $A = (I_{p'_{11}} \quad 0 \quad 0) \in D^{p'_{11}\times(p'_{11}+p'_{12}+p'_{13})}$ (see Corollary 3.1 of [12]) and we get the following theorem.

Theorem 2.1: With the hypotheses of Proposition 2.1 and the previous notations, let

$$s = r + p'_{11} = p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23},$$

$P \in D^{s\times(p_{01}+p'_{11}+p'_{12}+p'_{13})}$ be defined by

$$P = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 \\ 0 & R''_{11} & 0 & 0 \\ 0 & R'_{21} & 0 & 0 \\ 0 & 0 & F'_{13} & -I_{p'_{13}} \\ 0 & 0 & R''_{12} & 0 \\ 0 & 0 & R'_{22} & 0 \\ 0 & 0 & 0 & R''_{13} \\ 0 & 0 & 0 & R'_{23} \end{pmatrix},$$

and let us consider the following left D-modules:

$$\begin{cases} M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11}), \\ E = D^{1\times(p_{01}+p'_{11}+p'_{12}+p'_{13})}/(D^{1\times s}\, P). \end{cases}$$

If $\varrho : D^{1\times(p_{01}+p'_{11}+p'_{12}+p'_{13})} \longrightarrow E$ is the canonical projection, then we have

$$E \cong M,$$

where the left D-isomorphism is defined by:

$$\begin{array}{rcl} \phi : M & \longrightarrow & E \\ \pi(\lambda) & \longmapsto & \varrho(\lambda\,(I_{p_{01}} \quad 0 \quad 0 \quad 0)), \\ \phi : E & \longrightarrow & M \\ \varrho(\epsilon) & \longmapsto & \pi\left(\epsilon \left(\begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}\, R'_{11} \\ F'_{13}\, F'_{12}\, R'_{11} \end{pmatrix}\right)\right). \end{array} \tag{28}$$

Using (10), we note that the row of P containing the matrix R''_{11} can be removed. We get the following corollary of Theorem 2.1.

Corollary 2.3: With the hypotheses of Proposition 2.1 and the previous notations, if

$$t = s - p_{11} = p'_{11}+p'_{12}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23},$$

$$
\begin{array}{ccccccc}
 & D^{1\times p'_{11}} & \xrightarrow{.R'_{11}} & D^{1\times p_{01}} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow 0\\
 & \downarrow \psi & & \downarrow \pi & & \| & \\
0 \longrightarrow & E_1 & \xrightarrow{i\circ\vartheta\circ\phi_1^{-1}} & M & \xrightarrow{\rho} & M/t(M) & \longrightarrow 0,
\end{array}
\tag{27}
$$

and $Q \in D^{t\times(p_{01}+p'_{11}+p'_{12}+p'_{13})}$ is the matrix defined by

$$
Q = \begin{pmatrix}
R'_{11} & -I_{p'_{11}} & 0 & 0\\
0 & F'_{12} & -I_{p'_{12}} & 0\\
0 & R'_{21} & 0 & 0\\
0 & 0 & F'_{13} & -I_{p'_{13}}\\
0 & 0 & R''_{12} & 0\\
0 & 0 & R'_{22} & 0\\
0 & 0 & 0 & R''_{13}\\
0 & 0 & 0 & R'_{23}
\end{pmatrix},
$$

then $M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11})$ is isomorphic to

$$
M \cong E = D^{1\times(p_{01}+p'_{11}+p'_{12}+p'_{13})}/(D^{1\times t}\, Q),
$$

where the isomorphism is defined by (28).

Corollary 2.3 were implemented by the author in the OREMODULES package [7] (Maple) called PURITYFILTRATION [13]. See also the recent `homalg` package [1] (GAP4) called `AbelianSystems` [3] developed by Barakat and the author. The `AbelianSystems` package is more efficient than the first algorithm implemented in `homalg` based on the computation of spectral sequences [2], [4], [5].

If $\mathcal{F}$ is a left D-module, then applying the functor $\hom_D(\,\cdot\,,\mathcal{F})$ to $M \cong E$, and using Malgrange's theorem, we obtain $\ker_{\mathcal{F}}(R_{11}.) \cong \ker_{\mathcal{F}}(P.) = \ker_{\mathcal{F}}(Q.)$. More precisely, using (28), we get the following corollary.

Corollary 2.4: If $\mathcal{F}$ is a left D-module, then we have

$$
\ker_{\mathcal{F}}(R_{11}.) \cong \ker_{\mathcal{F}}(P.) = \ker_{\mathcal{F}}(Q.),
$$

and the following system equivalence holds

$$
R_{11}\,\eta = 0 \quad \Leftrightarrow \quad \left\{\begin{array}{l}
R'_{11}\,\zeta - \tau_1 = 0,\\
F'_{12}\,\tau_1 - \tau_2 = 0,\\
R'_{21}\,\tau_1 = 0,\\
F'_{13}\,\tau_2 - \tau_3 = 0,\\
R''_{12}\,\tau_2 = 0,\\
R'_{22}\,\tau_2 = 0,\\
R''_{13}\,\tau_3 = 0,\\
R'_{23}\,\tau_3 = 0,
\end{array}\right.
\tag{29}
$$

under the following invertible transformations:

$$
\begin{array}{rcl}
\gamma : \ker_{\mathcal{F}}(Q.) & \longrightarrow & \ker_{\mathcal{F}}(R_{11}.)\\
\begin{pmatrix} \zeta\\ \tau_1\\ \tau_2\\ \tau_3 \end{pmatrix} & \longmapsto & \eta = \zeta,\\
\gamma^{-1} : \ker_{\mathcal{F}}(R_{11}.) & \longrightarrow & \ker_{\mathcal{F}}(Q.)\\
\eta & \longmapsto & \begin{pmatrix} \zeta\\ \tau_1\\ \tau_2\\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{p_{01}}\\ R'_{11}\\ F'_{12}\,R'_{11}\\ F'_{13}\,F'_{12}\,R'_{11} \end{pmatrix} \eta.
\end{array}
\tag{30}
$$

Definition 2.1 ([5], [10]): A ring D is a *Cohen-Macaulay ring* if D is a noetherian ring equipped with a *dimension function* $\dim_D(\,\cdot\,)$ [10] such that:

$$
\begin{aligned}
\mathrm{codim}_D(M) &\triangleq \dim_D(D) - \dim_D(M)\\
&= j_D(M) \triangleq \min\{i \geq 0 \mid \mathrm{ext}_D^i(M,D) \neq 0\}.
\end{aligned}
$$

Example 2.1: If k is a field (resp., a field of characteristic 0), then the ring $D = k[x_1,\ldots,x_n]$ (resp., $D = A_n(k)$, $B_n(k)$, $\hat{\mathcal{D}}_n(k)$, $\mathcal{D}_n(k)$) is a Cohen-Macaulay ring with $\dim_D(D) = n$ (resp., $\dim_D(D) = 2\,n$, $\dim_D(D) = n$, $\dim_D(D) = 2\,n$, $\dim_D(D) = 2\,n$) [4], [5].

Proposition 2.3 ([5])): If D is an Auslander regular ring, then $j_D(\mathrm{ext}_D^i(M,D)) \geq i$ for all $i \in \mathbb{N}$ and for all left D-module M.

If D is a Cohen-Macaulay ring, using Theorem 3.1 of [12] and Proposition 2.3, then the left D-modules $\mathrm{ext}_D^3(N_{33},D)$, $\mathrm{coker}\,\gamma_{32}$, $\mathrm{coker}\,\gamma_{21}$ and $M/t(M)$ are either 0 or satisfy:

$$
\left\{\begin{array}{l}
\dim_D(\mathrm{ext}_D^3(N_{33},D)) \leq \dim_D(D) - 3,\\
\dim_D(\mathrm{coker}\,\gamma_{32}) = \dim_D(D) - 2,\\
\dim_D(\mathrm{coker}\,\gamma_{21}) = \dim_D(D) - 1,\\
\dim_D(M/t(M)) = \dim_D(D).
\end{array}\right.
\tag{31}
$$

Remark 2.2: If $S_0 = R'_{11}$ and

$$
S_1 = \begin{pmatrix} F'_{12}\\ R''_{11}\\ R'_{21} \end{pmatrix}, \quad S'_1 = \begin{pmatrix} F'_{12}\\ R'_{21} \end{pmatrix},
$$

$$
S_2 = \begin{pmatrix} F'_{13}\\ R''_{12}\\ R'_{22} \end{pmatrix}, \quad S_3 = \begin{pmatrix} R''_{13}\\ R'_{23} \end{pmatrix},
$$

then using (31), we get:

1) The linear system $\ker_{\mathcal{F}}(S_3.) \cong \hom_D(L_3, \mathcal{F}) \cong \hom_D(\mathrm{ext}_D^3(N_{33}, D), \mathcal{F})$ is either 0 or has dimension $\leq \dim_D(D) - 3$.
2) The linear system $\ker_{\mathcal{F}}(S_2.) \cong \hom_D(\operatorname{coker} \overline{\alpha}_{32}, \mathcal{F}) \cong \hom_D(\operatorname{coker} \gamma_{32}, \mathcal{F})$ has dimension $\dim_D(D) - 2$ when it is non-trivial,
3) The linear system $\ker_{\mathcal{F}}(S_1.) = \ker_{\mathcal{F}}(S_1'.) \cong \hom_D(\operatorname{coker} \overline{\alpha}_{21}, \mathcal{F}) \cong \hom_D(\operatorname{coker} \gamma_{21}, \mathcal{F})$ is either 0 or has dimension $\dim_D(D) - 1$.
4) The linear system $\ker_{\mathcal{F}}(S_0.) \cong \hom_D(M/t(M), \mathcal{F})$ has dimension $\dim_D(D)$ when it is non-trivial.

If R_3 has full row rank, i.e., $\ker_D(.R_3) = 0$, then $N_{33} \cong \mathrm{ext}_D^3(M, D)$, and thus $\mathrm{ext}_D^3(N_{33}, D) \cong \mathrm{ext}_D^3(\mathrm{ext}_D^3(M, D), D)$, and Theorem 4.1 of [12] yields

$$\dim_D(\mathrm{ext}_D^3(M, D)) = \dim_D(D) - 3,$$

which shows that $\ker_{\mathcal{F}}(S_3.)$ is either 0 or has $\dim_D(D) - 3$.

If $\mathcal{F}$ is an injective left D-module, then the linear system $\ker_{\mathcal{F}}(R.) = \ker_{\mathcal{F}}(R_{11}.)$ can be obtained by integrating the linear system $\ker_{\mathcal{F}}(Q.)$, i.e., by integrating in cascade the linear system $\ker_{\mathcal{F}}(S_3.)$ of dimension less or equal to $\dim_D(D) - 3$, and then the inhomogeneous linear systems of dimension respectively $\dim_D(D) - 2$, $\dim_D(D) - 1$ and $\dim_D(D)$. Finally, $\ker_{\mathcal{F}}(R_{11}'.) = R_{01}\, \mathcal{F}^{p-_{11}}$ (see [12]).

Even if the size of the matrix Q is larger than the one of $R = R_{11}$, the matrix Q is generally more suitable for a fine study of the module properties of $M \cong E$ than R, i.e., for the study of the structural properties of the linear system $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q.)$. This new form is particularly interesting for the computation of *Monge parametrizations* [15], [16], [12] of the linear system $\ker_{\mathcal{F}}(R.)$. Many under/overdetermined linear PD systems $\ker_{\mathcal{F}}(R.)$, which cannot directly be integrated by computer algebra systems such as Maple, can be integrated by means of their equivalent forms $\ker_{\mathcal{F}}(Q.)$ using the PURITYFILTRATION [13], `AbelianSystems` [3] and `homalg` [2] packages.

Example 2.2: Let us consider the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module $M = D^{1\times 4}/(D^{1\times 6}\, R)$ finitely presented by:

$$R = \begin{pmatrix} 0 & -2\,\partial_1 & \partial_3 - 2\,\partial_2 - \partial_1 & -1 \\ 0 & \partial_3 - 2\,\partial_1 & 2\,\partial_2 - 3\,\partial_1 & 1 \\ \partial_3 & -6\,\partial_1 & -2\,\partial_2 - 5\,\partial_1 & -1 \\ 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 & 0 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 & 0 \\ \partial_1 & -\partial_1 & -2\,\partial_1 & 0 \end{pmatrix}.$$

Using the PURITYFILTRATION package [13], we obtain that $M \cong E = D^{1\times 11}/(D^{1\times 15}\, Q)$, where Q is defined by (32). Let $\mathcal{F} = C^{\infty}(\mathbb{R}^3)$ be the injective D-module (see [11] or Example 2.3 of [12]) and let us explicitly compute $\ker_{\mathcal{F}}(Q.)$. We first integrate the last diagonal block of Q, i.e., the 0-dimensional linear PD system $\ker_{\mathcal{F}}(R_{13}.)$:

$$\begin{cases} -\partial_2\, \tau_3 = 0, \\ -\partial_3\, \tau_3 = 0, \\ \partial_1\, \tau_3 = 0, \end{cases} \quad \Leftrightarrow \quad \tau_3 = c_1 \in \mathbb{R}.$$

Then, we integrate the inhomogeneous linear PD system in $\tau_2 = (\tau_{21} \quad \tau_{22} \quad \tau_{23})^T$ and τ_3, formed by the third triangular block of Q, namely:

$$\begin{cases} \tau_{23} - \tau_3 = 0, \\ \tau_{21} = 0, \\ -\tau_{21} + (4\,\partial_1 - \partial_3)\, \tau_{22} = 0, \\ \tau_{21} + (4\,\partial_1 - \partial_3)\, \tau_{22} + \partial_3\, \tau_{23} = 0, \\ (\partial_1 - \partial_2)\, \tau_{22} = 0, \end{cases}$$
$$\Leftrightarrow \begin{cases} \tau_{23} = \tau_3 = c_1, \\ \tau_{21} = 0, \\ (4\,\partial_1 - \partial_3)\, \tau_{22} = 0, \\ (\partial_1 - \partial_2)\, \tau_{22} = 0. \end{cases}$$

We obtain $\tau_{21} = 0$, $\tau_{22} = f_1(x_3 + \frac{1}{4}(x_1 + x_2))$, where f_1 is an arbitrary smooth function, and $\tau_{23} = c_1$, where c_1 is an arbitrary constant. Then, we have to integrate the inhomogeneous linear PD system in $\tau_1 = (\tau_{11} \quad \tau_{12} \quad \tau_{13})^T$ and τ_2 formed by the second triangular block of Q:

$$\begin{cases} -2\,\partial_1\, \tau_{12} + \tau_{13} - \tau_{21} = 0, \\ -\tau_{12} - \tau_{22} = 0, \\ \tau_{11} - \tau_{12} - \tau_{23} = 0, \end{cases}$$
$$\Leftrightarrow \begin{cases} \tau_{12} = -\tau_{22} = -f_1(x_3 + \frac{1}{4}(x_1 + x_2)), \\ \tau_{11} = -\tau_{22} + \tau_{23} = -f_1(x_3 + \frac{1}{4}(x_1 + x_2)) + c_1, \\ \tau_{13} = -2\,\partial_1\, \tau_{22} + \tau_{21} = -\frac{1}{2}\, \dot{f}_1(x_3 + \frac{1}{4}(x_1 + x_2)). \end{cases}$$

The entries of $\tau_1 = (\tau_{11} \quad \tau_{12} \quad \tau_{13})^T$ are 1-dimensional and not 2-dimensional. This result comes from the fact that the matrix S_1' (or S_1) defined in Remark 2.2 admits a left inverse over D, i.e., $\operatorname{coker} \overline{\alpha}_{21} = 0$, which yields $\ker_{\mathcal{F}}(S_1'.) \cong \hom_D(\operatorname{coker} \overline{\alpha}_{21}, \mathcal{F}) \cong \hom_D(\operatorname{coker} \gamma_{21}, \mathcal{F}) = 0$. Finally, we have to integrate the inhomogeneous linear PD system in $\zeta = (\zeta_1 \ \ldots \ \zeta_4)^T$ and τ_1, formed by the first triangular block of Q, namely:

$$\begin{cases} \zeta_1 - \zeta_3 - \tau_{11} = 0, \\ \zeta_2 + \zeta_3 - \tau_{12} = 0, \\ (\partial_1 - 2\,\partial_2 + \partial_3)\, \zeta_3 - \zeta_4 - \tau_{13} = 0, \end{cases}$$
$$\Leftrightarrow \begin{cases} \zeta_1 - \zeta_2 = -f_1(x_3 + \frac{1}{4}(x_1 + x_2)) + c_1, \\ \zeta_2 + \zeta_3 = -f_1(x_3 + \frac{1}{4}(x_1 + x_2)), \\ (\partial_1 - 2\,\partial_2 + \partial_3)\, \zeta_3 - \zeta_4 = -\frac{1}{2}\, \dot{f}_1(x_3 + \frac{1}{4}(x_1 + x_2)). \end{cases} \tag{33}$$

Using the fact that $\mathcal{F}$ is an injective D-module and $M/t(M) = D^{1\times 4}/(D^{1\times 3}\, R_{11}')$ is a torsion-free D-module, where R_{11}' is the matrix formed by the first 3 rows and the first 4 columns of Q, $\ker_{\mathcal{F}}(R_{11}'.)$ is then parametrized by $R_{01} = (1 \quad -1 \quad 1 \quad \partial_1 - 2\,\partial_2 + \partial_3)^T$, i.e., $\ker_{\mathcal{F}}(R_{11}'.) = R_{01}\, \mathcal{F}$ (see [6] or 1 of Corollary 2.1 of [12]). Moreover, the matrix R_{11}' admits the following right inverse

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$Q=\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_1-2\,\partial_2+\partial_3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\,\partial_1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4\,\partial_1-\partial_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4\,\partial_1-\partial_3 & \partial_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1-\partial_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 \end{pmatrix}. \quad (32)$$

which implies that $M/t(M)$ is a projective D-module [6], and thus $M/t(M)$ is a free D-module of rank 1 by the Quillen-Suslin theorem [6], [9], [17]. Hence, Remark 14 of [16] (see also Remark 8 of [15]) proves that the general $\mathcal{F}$-solution of (33) is defined by $\zeta = R_{01}\,\xi + X\,\tau_1$, i.e.,

$$\left\{\begin{array}{l} \zeta_1=\xi-f_1(x_3+\frac{1}{4}\,(x_1+x_2))+c_1, \\ \zeta_2=-\xi-f_1(x_3+\frac{1}{4}\,(x_1+x_2)), \\ \zeta_3=\xi, \\ \zeta_4=(\partial_1-2\,\partial_2+\partial_3)\,\xi+\frac{1}{2}\,\dot{f}_1(x_3+\frac{1}{4}\,(x_1+x_2)), \end{array}\right.$$

for all $\xi\in C^\infty(\mathbb{R}^3)$, all $f_1\in C^\infty(\mathbb{R})$ and all $c_1\in\mathbb{R}$. Using the D-isomorphism γ defined by (30), we finally get

$$\left\{\begin{array}{l} -2\,\partial_1\,\eta_2+\partial_3\,\eta_3-2\,\partial_2\,\eta_3-\partial_1\,\eta_3-\eta_4=0, \\ \partial_3\,\eta_2-2\,\partial_1\,\eta_2+2\,\partial_2\,\eta_3-3\,\partial_1\,\eta_3+\eta_4=0, \\ \partial_3\,\eta_1-6\,\partial_1\,\eta_2-2\,\partial_2\,\eta_3-5\,\partial_1\,\eta_3-\eta_4=0, \\ \partial_2\,\eta_2-\partial_1\,\eta_2+\partial_2\,\eta_3-\partial_1\,\eta_3=0, \\ \partial_2\,\eta_1-\partial_1\,\eta_2-\partial_2\,\eta_3-\partial_1\,\eta_3=0, \\ \partial_1\,\eta_1-\partial_1\,\eta_2-2\,\partial_1\,\eta_3=0, \end{array}\right.$$

$$\Leftrightarrow\left\{\begin{array}{l} \eta_1=\xi-f_1(x_3+\frac{1}{4}\,(x_1+x_2))+c_1, \\ \eta_2=-\xi-f_1(x_3+\frac{1}{4}\,(x_1+x_2)), \\ \eta_3=\xi, \\ \eta_4=(\partial_1-2\,\partial_2+\partial_3)\,\xi+\frac{1}{2}\,\dot{f}_1(x_3+\frac{1}{4}\,(x_1+x_2)), \end{array}\right.$$

where ξ (resp., f_1, c_1) is an arbitrary function of $C^\infty(\mathbb{R}^3)$ (resp., $C^\infty(\mathbb{R})$, constant).

Finally, using the regular patterns of the matrix P and of (28), we can easily generalize Theorem 2.1, Corollary 2.4 and Remark 2.2 when $\ker_D(.R_3)\neq 0$, i.e., for a finitely presented left D-module $M=D^{1\times p_0}/(D^{1\times p_1}\,R_1)$ defined by a longer finite free resolution of the form:

$$0\longleftarrow M\xleftarrow{\pi} D^{1\times p_0}\xleftarrow{.R_1} D^{1\times p_1}\xleftarrow{.R_2}\ \ldots\ \xleftarrow{.R_m} D^{1\times p_m}.$$

If $\ker_D(.R_m)=0$, then the corresponding generalization defines the purity filtration of the left D-module M [12].

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