

Faster Diffusion across an Irregular Boundary: Supplementary Materials

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SOLUTION FOR A SPHERE

The problem of heat propagation inside a sphere can be solved analytically due to the rotational symmetry. Assuming zero initial condition inside the sphere of radius R , $T(r, t = 0) = 0$ (for $r < R$), and the Dirichlet boundary condition at the boundary, $T(R, t) = 1$ (for $t > 0$), the Laplace-transformed heat density, $\tilde{T}(r, s)$, satisfies the following equation in d -dimensional space:

$$s\tilde{T}(r, s) - K \left[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} \right] \tilde{T}(r, s) = 0 \quad (r < R),$$

with the Dirichlet boundary condition $\tilde{T}(R, s) = 1/s$. Its solution is well-known (see, e.g., [1])

$$\tilde{T}(r, s) = \frac{1}{s} \left(\frac{r}{R} \right)^{1-d/2} \frac{I_{d/2-1}(r\sqrt{s/K})}{I_{d/2-1}(R\sqrt{s/K})},$$

where $I_\nu(z)$ is the modified Bessel function of the first kind. Integrating this density over the sphere, one gets the Laplace transform of the heat $Q(t)$

$$\tilde{Q}(s) = \frac{dV_d}{sR\sqrt{s/K}} \frac{I_{d/2}(R\sqrt{s/K})}{I_{d/2-1}(R\sqrt{s/K})}, \quad (1)$$

where V_d is the volume of the sphere of radius R in d dimensions. The short-time behavior of the transferred heat $Q(t)$ can be derived from the asymptotic behavior of its Laplace transform as $s \rightarrow \infty$, from which

$$Q(t) \simeq \frac{2}{\sqrt{\pi}} \frac{dV_d}{R} \sqrt{Kt} = \frac{2}{\sqrt{\pi}} S_d \sqrt{Kt} \quad (t \rightarrow 0), \quad (2)$$

where S_d is the surface area of the sphere.

On the other hand, one can define the position r_t of a given temperature level \mathcal{T} at time t implicitly, as a solution of the equation

$$T(r_t, t) = \mathcal{T}. \quad (3)$$

As previously, the short-time asymptotic behavior of the density $T(r, t)$ can be derived from the asymptotic behavior of $\tilde{T}(r, s)$ as $s \rightarrow \infty$:

$$\tilde{T}(r, s) \simeq \frac{1}{s} \left(\frac{r}{R} \right)^{(1-d)/2} \exp[-(R-r)\sqrt{s/K}],$$

from which

$$T(r, t) \simeq \left(\frac{r}{R} \right)^{(1-d)/2} \operatorname{erfc} \left(\frac{R-r}{2\sqrt{Kt}} \right) \quad (t \rightarrow 0),$$

where $\operatorname{erfc}(z)$ is the complementary error function. In this regime, one can search for an approximate solution r_t in the form

$$r_t = R - 2\beta\sqrt{Kt}, \quad (4)$$

from which one gets the approximate equation for the coefficient β (in the leading order in \sqrt{Kt}):

$$\operatorname{erfc}(\beta) = \mathcal{T}. \quad (5)$$

For a given temperature level T , this equation determines the constant β in the dependence of r_t on \sqrt{Kt} . The volume between this position and the boundary is then

$$S(t) = V_d(1 - (r_t/R)^d) \simeq 2\beta S_d \sqrt{Kt}. \quad (6)$$

One can see the qualitative agreement between the exact solution of the heat equation (Eq. 2) and its approximation by Minkowski sausage (Eq. 6). In order to make this agreement quantitative, one has to rescale the time by $\pi\beta^2$, i.e., to replace t by $t/(\pi\beta^2)$ in Eq. (6). On logarithmic scale, this rescaling corresponds to a shift of the curve $S(t)$ along the horizontal (t) axis. Only when $\pi\beta^2 = 1$ (i.e., $\mathcal{T} = \operatorname{erfc}(1/\sqrt{\pi}) \simeq 0.4249$), the Minkowski sausage at length \sqrt{Kt} provides the same short-time asymptotic behavior as the exact solution of the heat equation. In general, the rescaling factor and its dependence on the temperature \mathcal{T} are not known, and the approximation of the transferred heat $Q(t)$ by $S(t)$ remains qualitative.

Long-time behavior

It is instructive to analyze the long-time behavior for both $Q(t)$ and $S(t)$. Using the properties of Bessel functions, Eq. (1) can be rewritten as

$$\tilde{Q}(s) = V_d \left(\frac{1}{s} - \frac{I_{d/2+1}(R\sqrt{s/K})}{sI_{d/2-1}(R\sqrt{s/K})} \right).$$

The computation of the inverse Laplace transform yields

$$Q(t) = V_d \left(1 - \sum_{n=1}^{\infty} \frac{2J_{d/2+1}(\alpha_n)}{\alpha_n J_{d/2}(\alpha_n)} e^{-\alpha_n^2 Kt/R^2} \right), \quad (7)$$

where α_n are the positive roots of the function $J_{d/2-1}(z)$. As expected, the transferred heat $Q(t)$ exponentially approaches the constant (here, the volume V_d of the domain given that the imposed temperature and heat capacity were set to 1). The long-time asymptotic behavior is determined by the smallest positive root α_1 :

$$Q(t) \simeq V_d \left(1 - A_d e^{-\alpha_1^2 K t / R^2} \right), \quad (8)$$

with a numerical prefactor

$$A_d = \frac{2J_{d/2+1}(\alpha_1)}{\alpha_1 J_{d/2}(\alpha_1)}.$$

For instance, one has

d	1	2	3
α_1	$\pi/2$	2.4048...	π
A_d	$8/\pi^2$	0.6917...	$6/\pi^2$

Similarly, one gets an explicit solution for the temperature density for any $r > 0$

$$T(r, t) = 1 - \sum_{n=1}^{\infty} \frac{2J_{d/2-1}(r\alpha_n/R)}{\alpha_n J_{d/2}(\alpha_n)} e^{-\alpha_n^2 K t / R^2} \quad (9)$$

(the point $r = 0$ has to be considered separately). For any $0 < r < 1$, this function monotonously increases from 0 to 1 as the time goes from 0 to infinity. The solution r_t of Eq. (3) is uniquely determined and decreases from 1 to 0. For any prescribed temperature $0 < \mathcal{T} < 1$, the radius r_t of the isothermal line shrinks to 0 at finite time t_c (since $T(r, t) > \mathcal{T}$ for $t > t_c$ for any $r > 0$).

In the long-time limit, the asymptotic behavior is determined by the term with the smallest α_1 , from which the radius r_t of the isocolor line from Eq. (3) can be approximated as

$$J_{d/2-1}(r_t \alpha_1 / R) \approx (1 - \mathcal{T}) \frac{\alpha_1 J_{d/2}(\alpha_1)}{2} e^{\alpha_1^2 K t / R^2}.$$

For small r_t , one can expand the Bessel function to get an explicit (but approximate) dependence of r_t on t . For $d = 2$, one gets

$$\frac{r_t^2 \alpha_1^2}{4R^2} \approx 1 - (1 - \mathcal{T}) \frac{\alpha_1 J_{d/2}(\alpha_1)}{2} e^{\alpha_1^2 K t / R^2},$$

from which

$$S(t) \approx V_d \left(1 - \frac{4}{\alpha_1^2} \left[1 - (1 - \mathcal{T}) \frac{\alpha_1 J_{d/2}(\alpha_1)}{2} e^{\alpha_1^2 K t / R^2} \right] \right). \quad (10)$$

The critical time is obtained from $S(t_c) = V_d$:

$$t_c \approx \frac{R^2}{K \alpha_1^2} \ln \left(\frac{2}{\alpha_1 J_{d/2}(\alpha_1) (1 - \mathcal{T})} \right). \quad (11)$$

For $t > t_c$, the temperature $T(r, t)$ throughout the whole domain is greater than \mathcal{T} so that the corresponding isocolor line disappears.

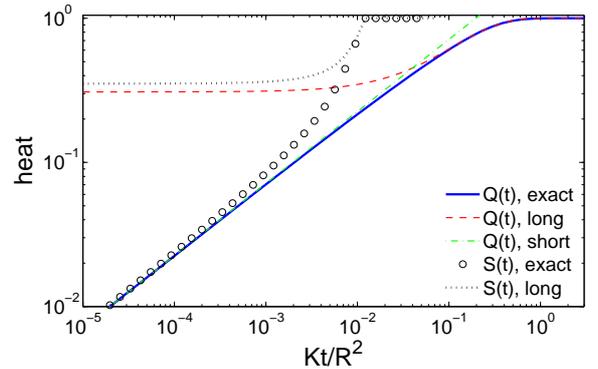


FIG. 1: The transferred heat $Q(t)/Q(\infty)$ from Eq. (7) (solid line), its short-time asymptotic from Eq. (2) (dash-dotted line) and long-time asymptotic from Eq. (8) (dashed lines) for the disk ($d = 2$) of radius R . The approximation by $S(t)$ is obtained by solving numerically the equation $T(r_t, t) = \mathcal{T}$ with $\mathcal{T} = 0.9$ (a.u) and is shown by circles. An approximate long-time solution (10) is shown by dotted line. One can see that the critical time t_c is accurately determined by Eq. (11). Note that both curves $S(t)$ were shifted along the horizontal axis by $\ln(\pi\beta^2)$ that corresponds to rescaling the time by $\pi\beta^2$ (which is equal to 0.0248 for $\mathcal{T} = 0.9$).

The comparison of Eqs. (8, 10) reveals significantly different behaviors for the transferred heat $Q(t)$ and its “approximation” by $S(t)$ at long times (Fig. 1). The former quantity shows an exponential approach to the limiting constant (here, V_d) which takes infinitely long time. In turn, $S(t)$ approaches the same limit V_d at a finite time t_c , after which there is no change in $S(t)$. The same feature was observed in the experiment for prefractal boundaries.

SOLUTION FOR A SQUARE

The solution of the heat equation inside a square of size L with a constant temperature on the boundary is well known [2, 3]:

$$T(x, y, t) = T_s - (T_s - T_0) u(x, t) u(y, t), \quad (12)$$

where

$$u(x, t) = \frac{4}{\pi} \sum_{n=0}^{\infty} e^{-K\pi^2(2n+1)^2 t / L^2} \frac{\sin(\pi(2n+1)x/L)}{2n+1}. \quad (13)$$

is the solution of the corresponding one-dimensional problem. The amount of heat transferred into the square

is

$$\begin{aligned}
 Q(t) &= c\delta \int_0^L dx \int_0^L dy (T(x, y, t) - T_0) \\
 &= c\delta(T_s - T_0)L^2 \left\{ 1 - \frac{64}{\pi^4} \left(\sum_{n=0}^{\infty} \frac{e^{-K\pi^2(2n+1)^2 t/L^2}}{(2n+1)^2} \right)^2 \right\}.
 \end{aligned}
 \tag{14}$$

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