Electrical Vehicles in the Smart Grid: A Mean Field Game Analysis

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Abstract

In this article, we investigate the competitive interaction between electrical vehicles or hybrid oil-electricity vehicles in a Cournot market consisting of electricity transactions to or from an underlying electricity distribution network. We provide a mean field game formulation for this competition, and introduce the set of fundamental differential equations ruling the behavior of the vehicles at the system equilibrium, namely the mean field equilibrium. This framework allows for a consistent analysis of the evolution of the sale-and-purchase price of electricity as well as of the instantaneous total demand. Simulations precisely quantify those parameters and suggest that following the charge and discharge policy at the equilibrium allows for a significant reduction of the daily electricity peak demand.

I. INTRODUCTION

Electrical vehicles (EV) and plug-in hybrid electrical vehicles (PHEV) have been recognized as natural components of future electricity distribution networks, known as smart grids [1], [2], [3]. As opposed to classical vehicles, EV and PHEV are equipped with batteries which can be charged or discharged by using a simple plug-in connector compatible with the local electricity distribution grid. Thus, EV and PHEV can be conceived as both energy consuming devices and mobile energy sources [4], [5], [6], [7]. In the former case, EV and PHEV can be seen as devices straining the energy demand of energy suppliers

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and, thus, adding a new constraint to reliably distribute the electricity. In the latter case, EV and PHEV can be used to store or even to transport the energy from one geographical area to another and then to increase the reliability of the energy supply in certain zones or time intervals. That is, either areas where the energy supply relies only non-easy predictable sources, e.g. wind turbines, solar panels, etc., or at time intervals where the demand naturally suddenly increases (rush hours).

Within this framework, it is therefore an important economical and social challenge to enforce charge and discharge policies to EV and PHEV in an optimal manner. Here, optimality must be interpreted in the sense of individual revenue obtained by the EV and PHEV owners when participating in the energy trades and also in terms of reliability of the energy supply process to the fixed consumers. In this paper, we consider that a way to improve reliability is to allow EV and PHEV to buy and sell energy to or from the smart grid, as in a classical Cournot competition [8]. Clearly, the price at which the energy is sold and bought depends on the existing demand in the grid and also on the demand and offer resulting from all the vehicles connected to the network. This competitive interaction resulting from the energy trade where each vehicle owner decides the amount of energy to be sold or bought, given a global price, can be analyzed using tools from dynamic game theory [9]. For instance, in [10], the coexistence of a number of PHEV groups aiming to sell part of their stored energy to the smart grid is studied using non-cooperative game theory [11]. An algorithm based on best response dynamics [11] is proposed to allow PHEV groups to reach a Nash equilibrium [12].

Nonetheless, in practical scenarios, the number of vehicles might be drastically large and, thus, elements from classical game theory might not necessarily bring enough insight about the global behavior of the market. To overcome this problem, in this paper, we study the energy trade when the number of vehicles tends to infinity and all vehicles are considered alike, following the paradigm of [13]. See also [14] for recent results. More precisely, we shall model this interaction as a mean field game [15], [16]. In contrast to games with a finite number of players, where each player follows the evolution of the state of the game and the actions taken by all the other players in order to maximize a given individual benefit, in the mean field game formulation, every player's action is driven not by the individual actions of each other player but by the collective (or mean) behavior of all players. Typically, the notion of (Nash) equilibrium in the context of mean field games is known as mean field equilibrium (MFE) [15], [16]. The MFE is found as the solution of a coupled system of stochastic partial differential equations (SPDE) which includes a (backward) Hamilton-Jacobi-Bellman (HJB) equation and a (forward) Fokker-Planck-Kolmogorov (FPK) equation.

The closest contribution to our specific problem setting is [17]. Therein, a mean field game approach

to the study of oil production in infinite-time horizon is developed. In [17], the selfish players are oil producers and the mean field variable is the oil selling price. In this article, we develop a similar framework to [17] but on a finite time horizon, applied to both EV and PHEV, with vehicle owners as the selfish players and electricity price as the mean field variable. In general, it is difficult to prove that the mean field solutions are well-defined limits of the finite game, see e.g. the discussions in [18]. Similarly, it is in general difficult to prove the existence and uniqueness of solutions of one equilibrium and, if so, to derive a numerical method that is provably able to converge to an equilibrium, see e.g. [19]. We will not try to prove any of these aspects here, our main target being rather to informally explore the potentials of the mean field game setting to the economical problem of optimal policies for EV and PHEV penetrations in the smart grid. Similar to [17], only the numerical results will convey a justification of the correct behavior of our method.

The reminder of this article unfolds as follows. In Section II, we describe the problem formulation in the case where only electrical vehicles interact with the energy market. Therein, the problem is formulated as a continuous time differential game with finite horizon. This formulation is then written under the form of a mean field game and the differential equations describing the mean field equilibrium are presented. In Section III, the same analysis presented for EV is carried out for the case of PHEV. In Section IV, we provide numerical simulations and derive conclusions for both scenarios. Finally, in Section V, we conclude this work.

II. PURELY ELECTRICAL VEHICLES

A. System Model

Consider a finite set $\mathcal{K} = \{1, \ldots, K\}$ of EV connected to an electricity distribution network. The average consumption rate of vehicle $k \in \mathcal{K}$ at time $t \in [0, T]$ is denoted by $g_t^{(k)}$. This consumption rate is measured in units of electricity per time. We assume the consumption function $g^{(k)} = \{g_t^{(k)}, 0 \le t \le T\}$ of EV k to be deterministic and known by EV k. The amount of energy stored in the battery of vehicle k at time t is denoted by $x_t^{(k)} \in [0, 1]$ evaluated in energy units. Here, $x_t^{(k)} = 0$ for an empty battery and $x_t^{(k)} = 1$ for a fully charged battery. We denote by $\alpha_t^{(k)}$ the energy provisioning rate of vehicle k at time t, that is, the rate at which vehicle k buys or sells its energy. We relate the variable $x_t^{(k)}$ to $g_t^{(k)}$ and $\alpha_t^{(k)}$ by the following transport equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x_t^{(k)} = \alpha_t^{(k)} - g_t^{(k)}. \tag{1}$$

In the following, we denote $\boldsymbol{x}_t = (x_t^{(1)}, \ldots, x_t^{(K)})$ and $\boldsymbol{\alpha}_t = (\alpha_t^{(1)}, \ldots, \alpha_t^{(K)})$ the sets of simultaneous

battery level and provisioning rates of all EV at time t, respectively. Consider now a predefined period $[0, T]$. We denote $\boldsymbol{x}^{(k)} = \{x_t^{(k)}, 0 \leq t \leq T\}$ and $\boldsymbol{\alpha}^{(k)} = \{\alpha_t^{(k)}, 0 \leq t \leq T\}$ the trajectories of the battery level and provisioning rates for EV k, respectively. We also denote $\boldsymbol{x} = \{\boldsymbol{x}_t, 0 \leq t \leq T\}$ and $\boldsymbol{\alpha} = \{\boldsymbol{\alpha}_t, 0 \leq t \leq T\}$ the joint trajectories of the battery levels and provisioning rates. We finally denote $\mathfrak{X}$ the set of all possible state trajectories and $\mathcal{A}_k$ the set of all possible controls $\boldsymbol{\alpha}^{(k)}$ of player k.

The price at which vehicles either sell or buy electricity at time t is evaluated by the function $p_t : \mathbb{R}^K \to \mathbb{R}$, $\boldsymbol{\alpha}_t \mapsto p_t(\boldsymbol{\alpha}_t)$. The time dependency of the price p_t models a realistic dynamic pricing policy accounting for the energy demand for other needs than EV battery loading. This function can be tuned to create incentives for EV to sell or buy energy at specific time periods. In addition to electricity price, other factors influence the selling and buying behavior of EV owners. We model these, for player k, by the following set of functions. The function $h_t^{(k)} : \mathbb{R} \to \mathbb{R}$, $\alpha \mapsto h_t^{(k)}(\alpha)$ models the (psychological) cost for player k to buy or sell electricity at rate α at time t. Indeed, EV owners are more likely to trade energy at some convenient time intervals, e.g. during nighttime when the EV is plugged to the distribution network. The function $f_t^{(k)} : [0, 1] \to \mathbb{R}$, $x \mapsto f_t^{(k)}(x)$ models the cost for vehicle k to possess only a fraction x of energy reserves at time t. For instance, during periods of high energy consumption, the interest of EV owners is to have as fully loaded batteries as possible. Finally, $\kappa^{(k)}[0, 1] \to \mathbb{R}$, $x \mapsto \kappa^{(k)}(x)$ models the cost for EV k to end the trade period $[0, T]$ with a fraction x of battery load. This function guarantees that EV owners do not sell all their battery content at the end of the trade. For simplicity, we will assume that the aforementioned functions are sufficiently regular for the following derivations to make sense. Specific realistic choices of functions are considered in Section IV.

The goal of EV k is to determine the consumption rates $\boldsymbol{\alpha}^{(k)}$ that minimizes its total cost over a time window $[0, T]$, given by

$$J_k\left(\boldsymbol{x}_0, \boldsymbol{\alpha}^{(k)}, \boldsymbol{\alpha}^{(-k)}\right) = \int_0^T \left(\alpha_t^{(k)} p_t(\boldsymbol{\alpha}_t) + h_t^{(k)}(\alpha_t^{(k)}) + f_t^{(k)}(x_t^{(k)})\right) \mathrm{d}t + \kappa^{(k)}(x_T^{(k)}) \tag{2}$$

for a given initial battery level vector $\boldsymbol{x}_0$ at time $t = 0$.

B. Classical Game Formulation

We model the energy trades resulting from the interactions among the electrical vehicles and the smart grid by a K-player continuous-time differential *game* of pre-specified fixed duration $T > 0$. Let $\mathcal{K}$, the set of EV, be the set of *players*. Player k controls the its *state trajectory* $\boldsymbol{x}^{(k)}$ through the control variable $\alpha_t^{(k)}$. The state trajectory $\boldsymbol{x}$ of the game is determined by the initial state $\boldsymbol{x}_0$ and by the transport equation

(1). The cost function of player k is defined by (2) and we assume that the information available for player k at time t is the actual price $p_t(\boldsymbol{\alpha}_t)$.

The objective for player k is to choose the control functions $\boldsymbol{\alpha}^{(k)} \in \mathcal{A}_k$ that minimize the cost J_k given the initial condition $x_0 \in \mathbb{R}_K$ and the instantaneous price $p_t(\boldsymbol{\alpha}_t)$. We denote $\boldsymbol{\alpha}^{(-k)} \in \mathcal{A}_1 \times \ldots \times \mathcal{A}_{k-1} \times \mathcal{A}_{k+1} \times \ldots \times \mathcal{A}_K$ the trajectories of actions of all players but k. Here, the interdependence between players appears through the sale and purchase electricity price; the control $\alpha_t^{(k)}$ depends on the price $p_t(\boldsymbol{\alpha}_t)$, which depends itself on all the other players' controls $\boldsymbol{\alpha}_t^{(-k)}$.

The formulation of the game is completed by restricting the deterministic function g_t and the set $\mathcal{A}_1 \times \ldots \times \mathcal{A}_K$ to lie in a subset of admissible consumption rates and control profiles $\boldsymbol{\alpha}_t$, in order for the solution of (1) to exist and be unique. This condition is necessary to ensure that the state trajectory x is unique and, thus, to define unequivocally the cost functions (2).

Finally, we interpret the notion of equilibrium of the game in the sense of Nash [20], [21] defined as follows,

Definition 1 (Nash Equilibrium (NE) [20]): The K-tuple of control functions $\boldsymbol{\alpha} = \left(\alpha^{(1)}, \ldots, \alpha^{(K)} \right) \in \mathcal{A}_1 \times \ldots \times \mathcal{A}_K$ is a Nash equilibrium if for all $k \in \mathcal{K}$ and for any admissible control $\alpha'^{(k)} \in \mathcal{A}_k$, it holds that

$$J_k \left(x_0, \alpha^{(k)}, \boldsymbol{\alpha}^{(-k)} \right) \leqslant J_k \left(x_0, \alpha'^{(k)}, \boldsymbol{\alpha}^{(-k)} \right), \tag{3}$$

for a given initial state x_0.

Our interest in the NE lies in the fact that, at a state of NE, all the EV are using a control policy which is optimal with respect to the control policy of all other EV. Otherwise stated, if the system is at equilibrium, the unilateral deviation in control of any player would lead to a higher cost and, thus, none of the players has an interest in unilaterally changing its control function. Nonetheless, analyzing the NE of such a game, where K is a large number, is a difficult problem. In fact, even if a NE exits, it would lead to solutions that are inherently difficult to exploit. In particular, it is clear that, under this formulation, any change in the battery level of a given player impacts all other players which must react as a consequence. We aim at reducing this complexity by adopting some additional, but reasonable, conditions.

C. Mean Field Game Formulation

The game described in the previous section can be remarkably simplified by considering a large population of symmetric players. We assume that the number of players K grows to infinity in a fluid

manner, in such a way that the set of players will now be considered continuous and that the individual cost functions become identical. This implies that the set of controls is now the same for all players. We will denote $\mathcal{A}_k = \mathcal{A}$. In this context, the impact of a particular control adopted by a single individual represents a negligible cost variation on all the other players' costs. Here, players are sensible to the distribution of all other players' states. We model this distribution by a limiting counting measure $F : [0, T] \times \mathcal{B} \to [0, 1]$,

$$F(t, B) = \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \delta_{\{x_t^{(k)} \in B\}}$$

where $\mathcal{B}$ is the Borel σ-field on $[0, 1]$. We assume F exists unequivocally and has a density $m(t, x) = \frac{\mathrm{d}}{\mathrm{d}x} F(t, x)$.

In order to relax the strong constraint $g_t^{(k)} \triangleq g_t$, now unrealistically identical for all users, the individual state of each player is assumed to be a noisy version of the mean state trajectory. That is, we define the evolution of the state $x_t^{(k)} \triangleq x_t$ by the following stochastic differential equation (SDE),

$$\mathrm{d}x_t = \alpha_t \mathrm{d}t - g_t \left(dt + \sigma_t \mathrm{d}W_t \right) + \mathrm{d}N_t. \tag{4}$$

Here, W_t is a Brownian motion and the term $\mathrm{d}W_t$ must be interpreted in the sense of Itô [22]. As such, the energy consumption becomes a random variable $g_t[\mathrm{d}t + \sigma_t \mathrm{d}W_t]$, at time t. The term $\mathrm{d}N_t$ is a reflective variable to ensure that x_t remains in $[0, 1]$. Now, under the assumption of independence between the different dimensions of the Brownian motion W_t, the analysis of the game reduces to the state trajectory analysis of a single player. The cost function of a given player can be obtained from (2) by dropping the player index and considering the density $m(t, \cdot)$, as follows,

$$J(\boldsymbol{x_0}, \boldsymbol{\alpha}, m_0) = \mathrm{E} \int_0^T \left(\alpha_t p_t(\boldsymbol{m}) + h_t(\alpha_t) + f_t(x_t) \right) \mathrm{d}t + \kappa(x_T), \tag{5}$$

where $\boldsymbol{m} \in \mathcal{M}$ denotes the trajectory of the probability distributions, i.e. $\boldsymbol{m} = \{m(t, \cdot), 0 \leq t \leq T\}$, and $\mathcal{M}$ is the set of all such trajectories. The initial state conditions for the considered player are $x_0 \in [0, 1]$ and $m_0 \triangleq m(0, \cdot)$, the initial density function. In this context, the energy sale and purchase price function writes $p_t : \mathcal{M} \to \mathbb{R}_+$, $\boldsymbol{m} \mapsto p_t(\boldsymbol{m})$. Indeed, the price is a function of the total instantaneous demand $\int_0^1 \alpha_t(x) m(t, x) \mathrm{d}x$. However, for computational ease, we will instead consider that prices are fixed not by the total consumption $\int_0^1 \alpha_t(x) m(t, x) \mathrm{d}x$ but by the *expected* consumption $g_t + \frac{\mathrm{d}}{\mathrm{d}t}(\int_0^1 x m(t, x) \mathrm{d}x)$, where both quantities only differ by an additional Brownian motion term when $\sigma_t > 0$. In practice, this suggests that the energy providers which set the instantaneous prices do not have the information on the instantaneous demand at time t but are able to track the density trajectory $\boldsymbol{m}$. We therefore define p_t as

$$p_t(\boldsymbol{m}) = D(t, \cdot)^{-1} \left(g_t + \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 x \, m(t, x) \mathrm{d}x \right),$$

where $D(t, p)$ is the total energy demand function at time t for a given price p, and the inverse is with respect to the operation of function composition. Under the above assumptions, the continuous time differential game discussed in Section II-B becomes a mean field game as introduced in [15], [16].

D. Mean Field Equilibrium

Our interest now is to transpose the notion of NE into the corresponding notion of equilibrium in the mean field game, namely the mean field equilibrium (MFE) [15], [16]. Based on Definition 1, a MFE is fully described a control $\boldsymbol{\alpha}^{\star} \in \mathcal{A}$ solution to the stochastic control problem,

$$v(0, x_0) = \inf_{\boldsymbol{\alpha} \in \mathcal{A}} \left[J\left(x_0, \boldsymbol{\alpha}, m_0\right) \right]$$

$$\mathrm{d}x_t = \alpha_t \mathrm{d}t - g_t \left[\mathrm{d}t + \sigma_t \mathrm{d}W_t\right] + \mathrm{d}N_t, \tag{6}$$

with initial state x_0 and m_0.

Let us define the partial cost function on the time interval $[u, T]$ as

$$v(u, x^\circ) = \inf_{\boldsymbol{\alpha} \in \mathcal{A}} \mathrm{E} \int_u^T \left(\alpha_t p_t(\boldsymbol{m}) + h_t(\alpha_t) + f_t(x_t) \right) \mathrm{d}t + \kappa(x_T)$$

where the eligible trajectories x_t are such that $x_u = x^\circ$. We will denote $\boldsymbol{v} = \{v(t, \cdot), 0 \leq t \leq T\}$ the trajectories of the partial cost function on the horizon $[0, T]$. The density measure $\boldsymbol{m}^\star$ obtained at the MFE and the associated optimal trajectories of the partial cost function $\boldsymbol{v}^\star$ at the MFE are solution of the following system of coupled stochastic partial differential equations (SPDE), for a given initial distribution $m_0 = m(0, \cdot)$ and an initial state x_0,

$$\partial_t v(t, x) = - \inf_{\alpha \in \mathbb{R}}, \left\{ \alpha \partial_x v(t, x) + C(\alpha, x, m^\star(t, x), t) \right\}$$

$$+ g_t \partial_x v(t, x) - \frac{1}{2} g_t^2 \sigma_t^2 \partial_{xx}^2 v(t, x) \tag{7}$$

$$\partial_t m(t, x) = -\partial_x \left[(\alpha_t^\star - g_t) m(t, x) \right] + \frac{1}{2} g_t^2 \sigma_t^2 \partial_{xx}^2 m(t, x). \tag{8}$$

Equation (7) is a backward HJB equation which determines the optimal trajectory $\boldsymbol{\alpha}^\star$ in the control problem (6), while Equation (8) is a forward FPK equation which determines the optimal density measure $\boldsymbol{m}^\star$.

We assume here that the cost $h(\alpha_t)$ for control is quadratic and reads

$$h(\alpha_t) = \frac{1}{2} h_t \alpha_t^2,$$

with $h_t > 0$ representing the unwillingness of the car owner to buy or sell energy at time t. This choice is seemingly non-natural as it implies that users are more willing to buy or sell small quantities rather than

large quantities of energy. Nonetheless, under the mean field game formulation, this has to be understood as the fact that, on average, only a limited population of users at time t is willing (or able) to buy energy. As such, intuitively, making the (psychological) cost of buying or selling energy larger for larger amounts of energy forces only part of the population to buy or sell. As for the particular choice of a quadratic cost rather than any other cost function, it is convenient for calculus mostly.

Under this assumption, solving

$$\inf_{\alpha \in \mathbb{R}}, \{\alpha \partial_x v(t,x) + C(\alpha, x, m^\star(t,x), t)\}$$

for all t, it is immediate to see that the optimal trajectory $\boldsymbol{\alpha}^\star$ is explicitly given by

$$\alpha_t^\star = -\frac{1}{h_t}\left[\partial_x v(t,x) + p_t(\boldsymbol{m}^\star)\right], \tag{9}$$

possibly subject to some boundary conditions to ensure that $x_t \in [0,1]$ at all times. In the remainder of the article, we will assume this condition always met, so that at no time we will consider EV owners with completely full or completely empty batteries.

The HJB equation now becomes

$$0 = \partial_t v(t,x) - \left(\frac{1}{h_t}\left[\partial_x v(t,x) + p_t(m^\star)\right] + g_t\right)\partial_x v(t,x)$$
$$- \frac{p_t(m^\star)}{h_t}\left[\partial_x v(t,x) + p_t(m^\star)\right] + f_t(x)$$
$$+ \frac{1}{2h_t}\left[\partial_x v(t,x) + p_t(m^\star)\right]^2 + \frac{1}{2}\sigma_t^2 g_t^2 \partial_{xx}^2 v(t,x),$$

which can be simplified as

$$\partial_t v(t,x) = \frac{1}{2h_t}(\partial_x v(t,x) + p_t(m^\star))^2 + g_t \partial_x v(t,x)$$
$$- f(t,x) - \frac{1}{2}\sigma_t^2 g_t^2 \partial_{xx}^2 v(t,x)$$

and the FPK equation is

$$\partial_t m(t,x) = \left(\frac{1}{h_t}\left[\partial_x v^\star(t,x) + p_t(\boldsymbol{m}^\star)\right] + g_t\right)\partial_x m(t,x)$$
$$+ \frac{1}{h_t}\partial_{xx}^2 v^\star(t,x)m(t,x) + \frac{1}{2}g_t^2\sigma_t^2 \partial_{xx}^2 m(t,x).$$

This defines the two fundamental differential equations to be solved, either explicitly or numerically, for determining the MFE.

In the next section, we improve the EV framework by turning the purely electrical vehicles into PHEV, introducing therefore the possibility for players to select between two alternative sources of energy.

III. PLUG-IN HYBRID VEHICLES

A. System Model

In this section, we consider that vehicles in the set $\mathcal{K}$ are PHEV. A PHEV can operate both with an electrical energy source and an alternative energy source, for instance oil. Thus, a PHEV interacts with the electricity distribution grid by buying and selling electricity with an elastic price. In the case of oil, it can be only bought at a fixed price, which is a natural assumption on a daily or even weekly basis. We describe the energy reserves of PHEV k by the two-dimensional vector $\boldsymbol{z}_t^{(k)} = (z_{1,t}^{(k)}, z_{2,t}^{(k)})^{\mathsf{T}} \in [0,1]^2$, where $z_{1,t}$ is the amount of energy stored in the batteries and $z_{2,t}$ the level of the oil reserve. We denote the provision rates of electricity and oil of PHEV k by $\mu_{1,t}^{(k)} \in \mathbb{R}$ and $\mu_{2,t}^{(k)} \in \mathbb{R}$, respectively. In addition, we denote $\beta^{(k)} : \mathbb{R}_+ \times [0,1]^2 \to [0,1]$, $(t, \boldsymbol{z}) \mapsto \beta^{(k)}(t, \boldsymbol{z})$ the function that determines the relative proportion of energy drawn from the batteries of PHEV k at time t. Typically, taking $\beta^{(k)}(t, \boldsymbol{z}) = z_1/(z_1 + z_2)$, with $\boldsymbol{z} = (z_1, z_2)^{\mathsf{T}}$, translates a policy where energy is consumed indistinctly of the energy source. Note that, depending on the typical distances covered by PHEV owners at time t (e.g. weekdays against weekends), $\beta^{(k)}(t, \boldsymbol{z}_t)$ may explicitly depend on t. Alternatively, we may have considered $\beta^{(k)}(t, \boldsymbol{z}_t)$ an additional control variable which can be set optimally by the car owner depending on the status of the energy market. Nonetheless, for simplicity of analysis, we do not consider this scenario here. We relate the variables $\boldsymbol{z}_t$, $\boldsymbol{\mu}_t$ and $\beta^{(k)}$ by the following two-dimensional transport equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{z}_t = \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} - \begin{bmatrix} \beta(t, \boldsymbol{z}_t) \\ 1 - \beta(t, \boldsymbol{z}_t) \end{bmatrix} g_t. \tag{10}$$

We define the cost of PHEV k in the time window $[0, T]$ as,

$$L_k\left(\boldsymbol{z}_0, \boldsymbol{\mu}^{(k)}, \boldsymbol{\mu}^{(-k)}\right) = \int_0^T \left(\boldsymbol{r}_t\left(\boldsymbol{\mu}_t^{(1)}, \ldots, \boldsymbol{\mu}_t^{(K)}\right) + q_t^{(k)}(\boldsymbol{\mu}_t^{(k)}) + s_t^{(k)}(\boldsymbol{z}_t^{(k)})\right) \mathrm{d}t + \xi^{(k)}(\boldsymbol{z}_T^{(k)}), \tag{11}$$

for a given initial state $\boldsymbol{z}_0 \in [0,1]^{2K}$, i.e. the energy reserves of all PHEV, where $\boldsymbol{\mu} = (\boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(K)})$, with $\boldsymbol{\mu}^{(k)} = \{\boldsymbol{\mu}_t^{(k)} = (\mu_{1,t}^{(k)}, \mu_{2,t}^{(k)}), 0 \le t \le T\}$.

Here, $\boldsymbol{r}_t : \mathbb{R}^{2K} \to \mathbb{R}^2$, $\boldsymbol{\mu} \mapsto \boldsymbol{r}_t(\boldsymbol{\mu}^{(1)}, \ldots, \boldsymbol{\mu}^{(K)}) = (r_{1,t}(\mu_1^{(1)}, \ldots, \mu_1^{(K)}), r_{2,t}(\mu_2^{(1)}, \ldots, \mu_2^{(K)}))$ evaluates the instantaneous prices $r_{1,t}$ of electricity and $r_{2,t}$ of oil, given the controls $\boldsymbol{\mu}^{(k)} = (\mu_1^{(k)}, \mu_2^{(k)})$. In particular, we assume here that the price for oil is fixed, given by $r_{2,t}(\mu_2^{(1)}, \ldots, \mu_2^{(K)}) = r_2$. Note that in this case the trajectory of the state $\boldsymbol{x} = \{\boldsymbol{x}_t = (x_t^{(1)}, \ldots, x_t^{(N)}), 0 \le t \le T\}$ is determined by the initial state $\boldsymbol{x}_0 = (x_0^{(1)}, \ldots, x_0^{(K)})$ and the transport equation (10).

The function $q_t^{(k)} : \mathbb{R}^2 \to \mathbb{R}$, $\boldsymbol{\mu} \mapsto q_t^{(k)}(\boldsymbol{\mu})$ evaluates the psychological cost of trading a quantity μ_1 of electricity and a quantity μ_2 of oil at time t, where $\boldsymbol{\mu} = (\mu_1, \mu_2)^{\mathsf{T}}$. The function $s_t^{(k)} : [0,1]^2 \to \mathbb{R}$,

$z \mapsto s_t^{(k)}(z)$ denotes the cost for PHEV k to be in state z at time t. Finally, $\xi^{(k)} : [0,1]^2 \to \mathbb{R}$, $z \mapsto \xi^{(k)}(z)$ is the cost for PHEV k to be in state z at time T. These are the analogous to the functions $h_t^{(k)}$, $f_t^{(k)}$, and $\kappa^{(k)}$ in (2), respectively.

In the following, we formulate the finite-number of players differential game.

B. Classical Game Formulation

The interaction between all PHEV is modeled by a K-player continuous-time stochastic differential game of pre-specified fixed duration $T > 0$. As for the case of EV, the set of players is the set $\mathcal{K}$ of PHEV and the objective for player k is to choose the control trajectory $\boldsymbol{\mu}^{(k)} = \{\boldsymbol{\mu}_t^{(k)}, 0 \leq t \leq T\}$ that minimizes the cost L_k in (11), given the initial conditions z_0 and the control functions $\boldsymbol{\mu}^{(-k)}$ adopted by all the other players. We denote the set of all possible controls $\boldsymbol{\mu}^{(k)}$ of player k over the time period $[0,T]$ by $\mathcal{U}_k$. The state trajectory z is determined by z_0 and by the evolution function (10). The set of state trajectories $z = (z^{(1)}, \ldots, z^{(K)})$, with $z^{(k)} = \{z_t^{(k)}, 0 \leq t \leq T\}$, is denoted by $\mathcal{Z}$. As in the previous case, the information available for each player k at time t is the initial state z_0 and the actual instantaneous price $p_t(\boldsymbol{m})$. Finally, we impose the necessary conditions for (10) to have a unique solution and thus we ensure that the resulting game is well formulated.

C. Mean Field Game Formulation

In this section, we proceed similarly to Section II-C. We use here a finite-game counting measure $F^{(K)} : \mathbb{R}_+ \times \mathcal{B} \times \mathcal{B} \to [0,1]$ of the form

$$F^{(K)}(t, \mathcal{B}_1, \mathcal{B}_2) = \frac{1}{K} \sum_{k=1}^{K} \delta_{\left\{z_{1,t}^{(k)} \in \mathcal{B}_1, \, z_{2,t}^{(k)} \in \mathcal{B}_2\right\}}, \tag{12}$$

and we assume that it admits a limiting measure F with density $m(t, \cdot, \cdot)$, i.e.

$$\lim_{K \to \infty} F^{(K)}(t, B_1, B_2) = F(t, B_1, B_2) \tag{13}$$

$$= \int_{B_1} \int_{B_2} m(t, z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2. \tag{14}$$

The function $m(t, z_1, z_2)$ evaluates the density of players lying in state $z = (z_1, z_2)^{\mathsf{T}}$. As previously, the individual state of each player is assumed to be a noisy version of the deterministic state trajectory in (10) determined by the following SDE,

$$\mathrm{d}z_t = \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} \mathrm{d}t - \begin{bmatrix} \beta(t, z_t) \\ 1 - \beta(t, z_t) \end{bmatrix} g_t \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sigma_t \mathrm{d}\boldsymbol{W}_t \right) + \mathrm{d}\boldsymbol{N}_t$$

for a given initial state $\boldsymbol{z}_0$. In particular, $\boldsymbol{W}_t = (W_{1,t}, W_{2,t})^{\mathsf{T}}$ is a two-dimensional Brownian motion with independent components. Similar to previously, σ_t determines the variance of the noise at time t and the term $\mathrm{d}N_t$ is a reflective term in order to guarantee that $\boldsymbol{z}_t \in [0,1]^2$. As in the EV case, the analysis of the game reduces to the analysis of the behavior of a single player. The cost function $L^{(k)} \triangleq L$ is now assumed identical to all players and is given by

$$L(\boldsymbol{z}_0, \boldsymbol{\mu}, m_0) = \mathrm{E} \int_0^T (\boldsymbol{r}_t(\boldsymbol{m}) + q_t(\boldsymbol{\mu}_t) + s_t(\boldsymbol{z}_t)) \, \mathrm{d}t + \xi(\boldsymbol{z}_T), \tag{15}$$

where $\boldsymbol{m} \in \mathcal{M}$ is the trajectory of the probability distributions, that is, $\boldsymbol{m} = \{m(t, \cdot, \cdot), 0 \le t \le T\}$ and $\mathcal{M}$ is the set of all possible trajectories $\boldsymbol{m}$. The initial state conditions are $\boldsymbol{z}_0 \in [0,1]^2$ and $m(0, \cdot, \cdot)$. The price for electricity is given by the function $r_{1,t} : \mathcal{M} \to \mathbb{R}_+$, with

$$r_{1,t}(\boldsymbol{m}) = D(t, \cdot)^{-1} \left(g_t \int_0^1 \int_0^1 \beta(t, z_1, z_2) m(t, z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 + \frac{d}{dt} \int_0^1 \int_0^1 z_1 m(t, z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 \right). \tag{16}$$

The price for oil is still constant, given by $r_{2,t} = r_2$.

The next section is dedicated to determining the MFE for this game.

D. Mean Field Analysis

Under the above game formulation, the optimal control problem which represents the equilibrium of the game formulates as

$$u(0, \boldsymbol{z}_0) = \inf_{\boldsymbol{\mu} \in \mathcal{A}} L(\boldsymbol{z}_0, \boldsymbol{\mu}, m_0)$$

$$\mathrm{d}\boldsymbol{z}_t = \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} \mathrm{d}t - \begin{bmatrix} \beta(t, \boldsymbol{z}_t) \\ 1 - \beta(t, \boldsymbol{z}_t) \end{bmatrix} g_t \left[\mathrm{d}t + \sigma_t \mathrm{d}\boldsymbol{W}_t \right] + \mathrm{d}\boldsymbol{N}_t. \tag{17}$$

We introduce the value function

$$v(u, \boldsymbol{z}^{\circ}) = \inf_{\boldsymbol{\mu} \in \mathcal{A}} \mathrm{E} \int_u^T (\boldsymbol{r}_t(\boldsymbol{m}) + q_t(\boldsymbol{\mu}_t) + s_t(\boldsymbol{z}_t)) \, \mathrm{d}t + \xi(\boldsymbol{z}_T) \tag{18}$$

with $\boldsymbol{z}_u = \boldsymbol{z}^{\circ}$.

As in the EV case, we consider the cost function as quadratic, that is,

$$q_t(\boldsymbol{\mu}_t) = \frac{1}{2} q_{1,t}(\mu_{1,t})^2 + \frac{1}{2} q_{2,t}(\mu_{2,t})^2,$$

with $(q_{1,t}, q_{2,t}) \in \mathbb{R}^2$.

The HJB equation is here given by

$$
\begin{aligned}
-\partial_t v(t,x,y) \;=\; & \inf_{\boldsymbol{\mu}_t \in \mathbb{R}^2} \{ \mu_{1,t}\, r_{1,t}\left(\boldsymbol{m}^\star\right) + \mu_{2,t} r_2 + q_t\left(\boldsymbol{\mu}_t\right) \\
& + \left(\mu_{1,t} - g_t \beta(t,x,y)\,\right)\partial_x v(t,x,y) \\
& + \left(\mu_{2,t} + g_t(\beta\left(t,x,y\right) - 1)\right)\partial_y v(t,x,y)\} \\
& + f_t(x,y) + \frac{1}{2}\sigma_t^2\, g_t^2\,[\,(\,\beta(t,x,y)\,)^2 \partial_{xx}^2 v(t,x,y) \\
& + 2\beta(t,x,y)(1 - \beta(t,x,y))\partial_{x,y}^2 v(t,x,y) \\
& + (1 - \beta(t,x,y))^2 \partial_{yy}^2 v(t,x,y)\,]\,,
\end{aligned}
\tag{19}
$$

where $\boldsymbol{m}^\star$ is solution to the FPK equation

$$
\begin{aligned}
\partial_t m\left(t,x,y\right) \;=\; & -\partial_x \left[(\mu_{1,t}^\star - \beta\left(t,x,y\right) g_t)m\left(t,x,y\right)\right] \\
& -\partial_y \left[(\mu_{2,t}^\star + (\beta\left(t,x,y\right) - 1)g_t)m\left(t,x,y\right)\right] \\
& + \frac{1}{2}g_t^2 \sigma_t^2 \left[\beta\left(t,x,y\right)^2 \partial_{xx}^2 m\left(t,x,y\right) + (1 - \beta\left(t,x,y\right))^2 \partial_{yy}^2 m\left(t,x,y\right)\right. \\
& \left. + 2\beta\left(t,x,y\right)(1 - \beta\left(t,x,y\right))\partial_{xy}^2 m\left(t,x,y\right)\right],
\end{aligned}
$$

where $\boldsymbol{\mu}_t^\star = (\mu_{1,t}^\star, \mu_{2,t}^\star)$, the cost minimizing control, is

$$
\mu_{1,t}^\star \;=\; -\frac{1}{q_{1,t}}(r_{1,t}\left(\boldsymbol{m}^\star\right) + \partial_x v(t,x,y))
\tag{20}
$$

$$
\mu_{2,t}^\star \;=\; -\frac{1}{q_{2,t}}\left(r_2 + \partial_y v(t,x,y)\right).
\tag{21}
$$

Assuming $\sigma_t = 0$, we obtain more compact forms. In particular, after substitution of the expression of $\boldsymbol{\mu}_t^\star$, the HJB equation becomes

$$
\begin{aligned}
\partial_t v\left(t,x,y\right) \;=\; & \frac{1}{2q_{1,t}}\left(\partial_x v\left(t,x,y\right) + r_{1,t}(\boldsymbol{m}^\star)\right)^2 \\
& + \frac{1}{2q_{2,t}}\left(\partial_y v\left(t,x,y\right) + r_2\right)^2 \\
& + g_t \beta\left(t,x,y\right) \partial_x v\left(t,x,y\right) \\
& + g_t(1 - \beta\left(t,x,y\right))\partial_y v\left(t,x,y\right) - f_t\left(x,y\right).
\end{aligned}
\tag{22}
$$

where $\boldsymbol{m}^\star$ is solution to

$$
\begin{aligned}
\partial_t m\left(t,x,y\right) \;=\; & \left[\frac{1}{q_{1,t}}(\partial_{xx} v^\star) + \frac{1}{q_{2,t}}(\partial_{yy} v^\star) + g_t\left[\partial_x \beta\left(t,x,y\right) - \partial_y \beta\left(t,x,y\right)\right]\right] m\left(t,x,y\right) \\
& + \left[\frac{1}{q_{1,t}}(r_{1,t}(\boldsymbol{m}) + \partial_x v^\star) + \beta\left(t,x,y\right) g_t\right]\partial_x m\left(t,x,y\right) \\
& + \left[\frac{1}{q_{2,t}}(r_2 + \partial_y v^\star) + (1 - \beta\left(t,x,y\right))g_t\right]\partial_y m\left(t,x,y\right)
\end{aligned}
\tag{23}
$$

with $v^\star$ the solution to (22).

In particular, for $\beta(t, z_t) = \frac{z_{1,t}}{z_{1,t}+z_{2,t}}$, which we will use in Section IV, we have that

$$[\partial_x \beta(t, x, y) - \partial_y \beta(t, x, y)] = \frac{1}{z_{1,t} + z_{2,t}}. \tag{24}$$

IV. SIMULATIONS

In this section, we provide simulation results for the electrical vehicle schemes developed in Section II and Section III.

A. EV analysis

We first consider the scenario of Section II. We assume a realistic three-day scenario ($t = 0$ at midnight the first day and $t = T = 1$ seventy-two hours later) where players have an average consumption rate that depends on specific periods of the days. The scenario is typical of a Friday to Sunday energy consumption, with higher overall electricity consumption on Friday and different patterns of car usage on Friday than on Saturday and Sunday. Since it is difficult to provide a universal system parametrization, we will take arbitrary scalings in the energy consumption functions. Some insights on the impact of different scalings will be provided.

The car electricity consumption function g_t is depicted in Figure 1, where we see in particular that consumption is higher on Friday and with a peak around 5pm, while consumption is lower on weekend days with different peak times. The variance σ_t^2 on the consumption is taken equal to 0.01 at all time, ensuring a standard deviation of the order of 10%. The demand function $D(t, p)$ is such that the price p is a quadratic function of the total electricity demand from both electrical vehicles and other electricity services. Specifically, we take here

$$p_t = \left(\left[g_t + \frac{d}{dt} \int x m(t, x) dx \right]^+ + d_t \right)^2$$

where d_t stands for the demand of electricity in services other than electrical cars, with $[x]^+ = \max(x, 0)$. We therefore assume that this demand is deterministic and is not altered by price evolution, which is unrealistic to some extent but helps appreciating the impact of EV consumption on prices. The function d_t is depicted in dashed line in Figure 4, up to a constant corresponding to the total average EV consumption; that is, the dashed line represents the total electricity consumption if EV consumption were distributed equally in time. For simplicity of understanding, we assume $h_t = 30$ constant; that is, we do not consider that the car owners have any particular incentive to charge or discharge at some specific time periods.[1]

---

[1]Note that the determination of a correct h_t is highly subjective and is better kept constant for the sake of interpretation.

We take $f(t, x) = (1 - x)^2$ to impose consumers to keep a certain level of electricity in their batteries, and the boundary condition $\kappa(x) = (1 - x)^2$ in order to avoid large sales at the last minute. The initial condition on $m(0, \cdot)$ is a triangle distribution m_0 centered at 0.5 and with support $[0.3, 0.7]$. The boundary conditions on m and v are such that $\partial_x m(0, \cdot) = \partial_x m(1, \cdot) = \partial_x v(0, \cdot) = \partial_x v(1, \cdot) = 0$ in order to force the energy content to lie in $[0, 1]$.

To solve the system of equations (7), (8) in (m, v), we proceed by solving sequentially the HJB and FPK equations using a simple fixed-point algorithm until convergence. We do not ensure here that this algorithm does converge, neither do we ensure that the solution obtained is the solution sought for. Using a finite difference method on a sampling of 144 points in the time axis (every 30min) and of 100 points in the battery level axis, the above scheme leads to the density evolution $\boldsymbol{m}^\star$ depicted in Figure 2. A few observations can be already made from this figure. We easily observe daily sequences of increases and decreases of the average battery levels. We see in particular that during nighttime, the battery levels increase, indicating that energy is purchased at time and consumed during daytime. It is interesting to note that, due to the small variance σ_t^2 that was chosen, the overall tendency is for $m^\star(t, \cdot)$ to concentrate into a single mass when $t \to 1$. This is a usual phenomenon which determines the steady state if time were to continue with constant values for all time-dependent system parameters.

From the expression of $\boldsymbol{m}^\star$, $\boldsymbol{v}^\star$, and the equations derived in Section II, it is now possible to obtain much information about the system. In particular, it is interesting to follow the electricity bought or sold by electrical vehicles at all time, that is the quantity

$$g_t + \frac{d}{dt} \int x m^\star(t, x) dx$$

or the overall electricity consumption in the market given by

$$g_t + \frac{d}{dt} \int x m^\star(t, x) dx + d_t$$

and the price $p_t(\boldsymbol{m}^\star)$ defined here as

$$p_t(\boldsymbol{m}^\star) = \left(\left[g_t + \frac{d}{dt} \int x m^\star(t, x) dx \right]^+ + d_t \right)^2 .$$

This is depicted in Figure 3, Figure 4 and in Figure 5, respectively.

We see first in Figure 3 that the peaks of electricity bought by electrical vehicles take place during the night where the overall demand is low, while they are at their lowest during peak demand periods. This is a natural outcome of the fact that prices are high during peak demand periods. However, we also see that the difference of amplitude between lowest and highest purchases is not large. This is due to
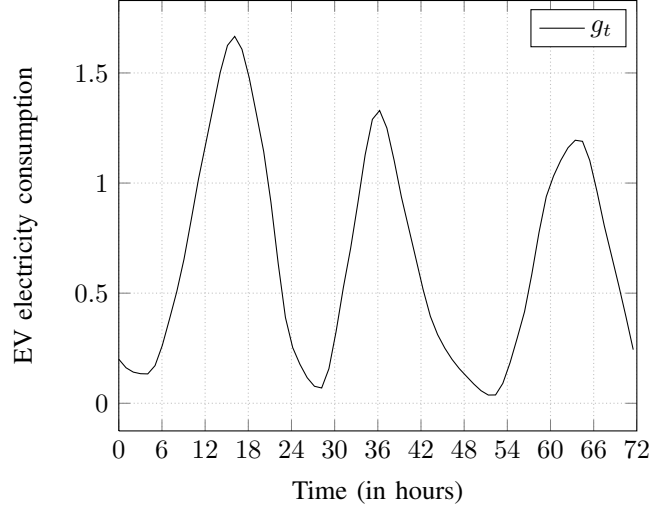
Fig. 1. Mean energy consumption g_t of EV as a function of time.

the fact that, while prices are high in peak demand periods, the EV owners still have a strong incentive not to find their batteries empty, driving them to keep buying electricity at peak periods. This behaviour can be hindered by relaxing the constraint $f(t, x)$.

Of more interest is Figure 4, where the differences between electricity consumption with or without incentives on EV behaviour. This figure depicts in dashed line the overall energy consumption if the EV purchases were equally distributed in the three-day period (that is, with no incentive), and in plain line the overall consumption under our current assumptions. It is seen here that the price incentives on electricity purchases produces a much expected peak demand reduction in the critical day periods, and a simultaneous increase of consumption during low consumption periods. Note importantly that our analysis does not consider changes in d_t when the price for electricity changes; only the part of electricity reserved for EV drives prices which in turn drive the EV behaviour, which is a natural assumption if different price conditions are applied to EV and other services. The price evolution is depicted in Figure 5, where it is seen in this setting that the price is mostly driven by the function d_t.

B. PHEV analysis

In this second section, we wish to analyze the behavior of hybrid vehicles as described in Section III. Since solving three-dimensional differential equations is time-consuming, we only provide results for the time scale discretized in 12 samples and for the "spatial" scales discretized both in 16 samples. For each differential equation, the resolution is performed by iterating the resolution of the two-dimensional
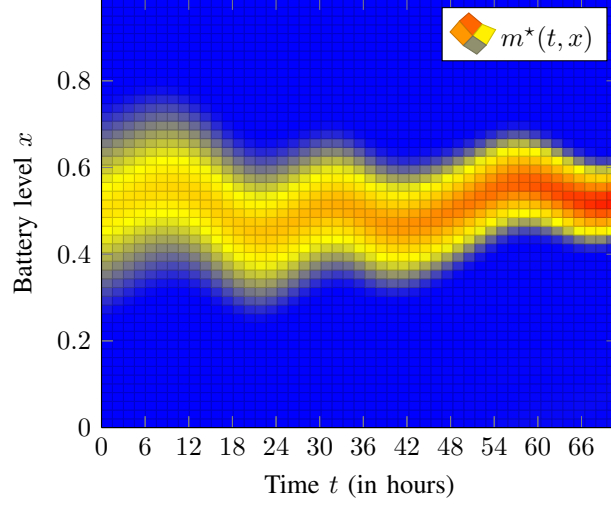
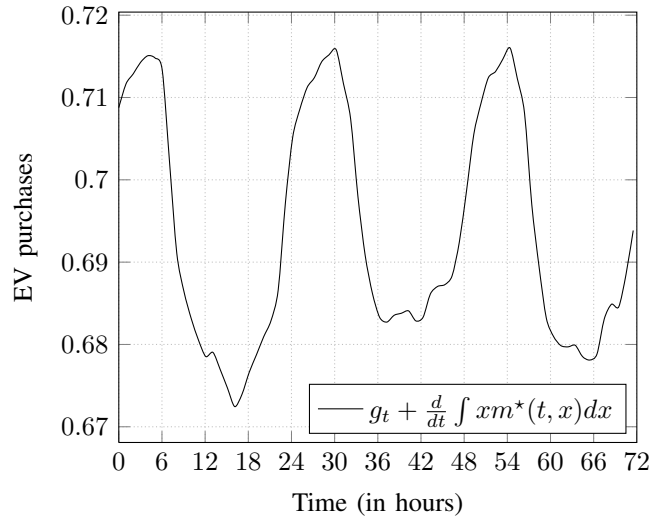Fig. 2. Density solution $m^\star(t, x)$ as a function of the time t and the battery level x.

Fig. 3. Electricity purchased by EV as a function of the time t.

differential equations along time and electricity scales for each fixed oil tank level, and time and oil scales for each fixed battery level. Then the system of HJB and FPK differential equations is solved by further iterating a fixed point algorithm as in the previous section. For simplicity of interpretation, we consider here a time-independent scenario where both $g_t = 0.2$ and $(q_{1,t}, q_{2,t}) = (125, 125)$ are
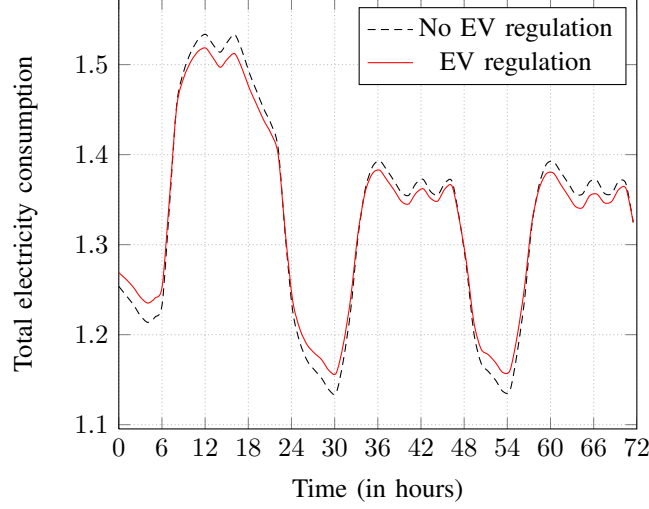
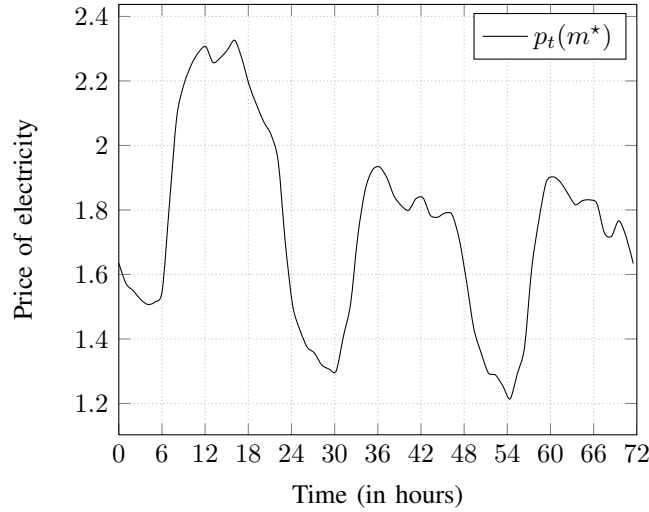Fig. 4. Total electricity consumption with or without EV regulation as a function of the time t.

Fig. 5. Evolution of the price $p_t(\boldsymbol{m}^\star)$ as a function of the time t.

constant with time.[2] We take the electricity price to be $r_{1,t} = (D(t, r_{1,t}))^+ + 0.5$, where now the demand is solely due to the electricity being bought by PHEV; that is, we do not consider other sources of electricity consumption in order to focus on the oil/electricity interaction solely. The oil price is set to $r_{2,t} = r_2 = 0.7$. This is a natural choice as it is expected that an approximate quantity $g_t = 0.2$ will

[2]Such a large value for the entries of h_t is motivated by faster algorithm convergence reasons, although it inhibits as a counterpart fast variations of m along time.

be asked for at any time to cover for the energy consumed, hence a price for electricity $r_{1,t} \simeq 0.7$. We impose a constraint $s_t(\boldsymbol{z}) = 20(2 - z_1 - z_2)^2$, where $\boldsymbol{z} = (z_1, z_2)^\mathsf{T}$. The relative consumption β of oil and electricity is proportional to the total quantity of energy, that is $\beta(t, \boldsymbol{z}) = z_1/(z_1 + z_2)$ and therefore $1 - \beta(t, \boldsymbol{z}) = z_2/(z_1 + z_2)$. We take $\sigma_t = 0$ for simplicity. The boundary constraints are identical to those in the previous section. As for the terminal constraint on v, it imposes that $v(T, \boldsymbol{z}) = \xi(\boldsymbol{z}) = (10(2 - (z_1 + z_2)))^2$.

We consider the scenario where $m(0, \cdot)$ is a (properly truncated and scaled) Gaussian distribution with mean $(0.4, 0.6)^\mathsf{T}$ and covariance $0.02 I_2$, with I_2 the 2×2 identity matrix. That is, we assume that, initially, most vehicles have more oil than electricity. This is depicted in Figure 6. We then let the system evolve freely under the above set of constraints. It is natural to guess that the overall behavior is a decrease of either or both quantities of oil and electricity to zero if the prices are too high, or an increase of either or both quantities to one, if the prices are more reasonable. What is interesting to observe is the trajectory jointly followed by the players. The resulting final distribution $m^\star(T, \cdot)$ is depicted in Figure 7. What we observe in the aforementioned conditions is that the initial distribution has shifted towards an increase of both electricity and oil levels, with a stronger increase of the mean battery level. Another observation is that the distribution tends to stretch along the $z_1 = z_2$ diagonal in the figure, translating the fact that oil and electricity are seen almost as equivalent goods due to the loosely constraining energy cost policy.

Among the different further analyses, in Figure 8, we consider a section of the distribution of the optimal transaction policy $\mu_{1,t}^\star$ and $\mu_{2,t}^\star$ at time $t = 0^+$, for $z_{2,t} = 0.5$ and $z_{2,t} = 0.9$ (we remind that both $\mu_{1,t}^\star$ and $\mu_{2,t}^\star$ are functions of t, $z_{1,t}$ and $z_{2,t}$). That is, we observe the initial behavior of players with half-filled oil tanks and almost completely filled oil tanks. It is seen that, for users with a very low level of electricity, buying electricity is an appealing choice. This can be interpreted by the fact that, as few players are in strong need for energy, it is possible to acquire a large quantity of electricity at a reasonable price. Those players with low reserves of electricity are the main beneficiaries. For users with already a reasonable level of electricity though, electricity and oil are seen as equivalent goods. As a matter of fact, our results also show that, at time $t = 0^+$, the price of electricity equals $r_{1,t} = 0.706 \simeq r_2$. That is, the players with low electricity levels draw as much of the electricity overhead (compared to oil) as is needed to reach an equilibrium price with oil. Now, it is also observed that, for users with large quantities of oil, electricity becomes a compelling purchase in order to further increase the total quantity of energy (since f imposes $z_{1,t} + z_{2,t}$ to be close to 2), hence a larger incentive for buying electricity when the battery level is not large. When both battery and tank levels are alike, we see that the quantity of electricity purchased is the same as the quantity of oil purchased.
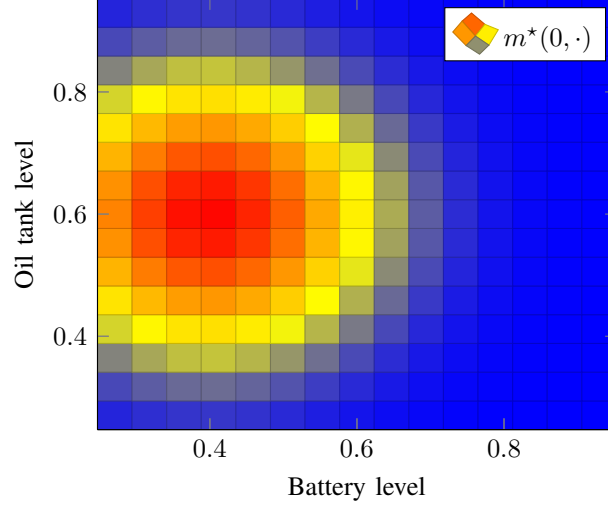
Fig. 6. Initial distribution $m(0, \cdot)$ at time $t = 0$, as a function of both levels of battery and oil tank.
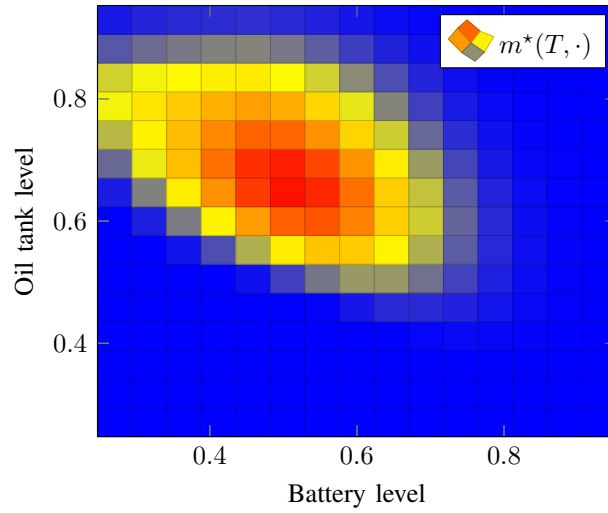

Fig. 7. Final distribution $m(T, \cdot)$ as a function of both levels of battery and oil tank.

Obviously, from the very generic settings of both EV and PHEV problems, many more scenarios can be carried out so to evaluate the actual impact of the EV and PHEV on realistic smart grid scenarios. The simulations above and their interpretations only provide a framework of understanding of fully rational vehicle owner's behavior, which needs be reported to real-life conditions with extreme care.
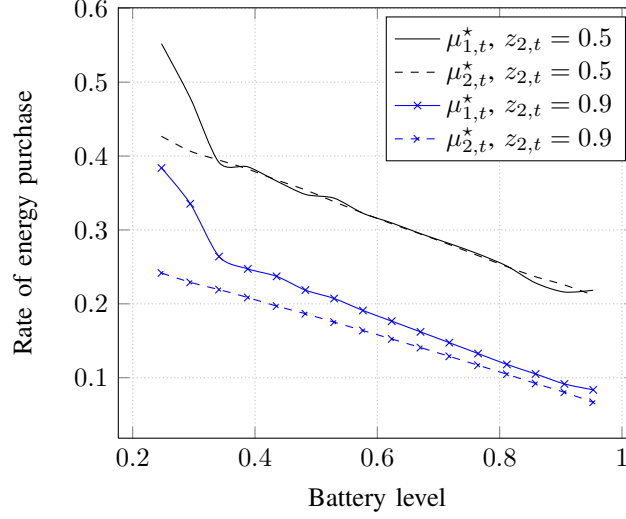
Fig. 8. Optimal transactions at time $t = 0^+$ for players with different oil and battery levels.

V. CONCLUSION

In this article, we proposed a game theoretical framework to model the behavior of electrical vehicle and hybrid electricity-oil vehicle owners aiming at selfishly minimizing their operating cost. As the number of selfish players is large, and players are assumed alike, we then turned the problem into a mean field game, for which we obtain the fundamental differential equations describing the mean field equilibrium of the game. Using numerical methods, we drew conclusions which give new insights on the way to optimize the electrical vehicle penetration in the future smart grid.

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