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Inapproximability proof of DSTLB and USTLB in planar graphs

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This document proves the problem of finding a minimum cost Steiner Tree covering \( k \) terminals with at most \( p \) branching nodes (with outdegree greater than 1), in a directed or an undirected planar graph with \( n \) nodes, is hard to approximate within a better ratio than \( n \), even when the parameter \( p \) is fixed.

1 Theorem

Definition 1. In a undirected (resp. directed) tree, a branching node is a node whose degree (resp. outdegree) is strictly greater than 2 (resp. 1).

Problem 1. \textbf{min-}\((\cdot, p)\)-USTLB: Given an undirected graph \( G = (V, E) \) with \( n \) nodes and a non negative cost function \( \omega \) on its edges, an integer \( k \) and a set \( X \subseteq V \) of \( k \) terminals, determine, if it exists, a minimum cost tree \( T^* \) spanning all the nodes of \( X \) and containing at most \( p \) branching nodes.

Problem 2. \textbf{min-}\((\cdot, p)\)-DSTLB: Given a directed graph \( G = (V, E) \) with \( n \) nodes and a non negative cost function \( \omega \) on its arcs, a node \( r \), an integer \( k \) and a set \( X \subseteq V \) of \( k \) terminals, determine, if it exists, a minimum cost directed tree \( T^* \) rooted at \( r \), spanning all the nodes of \( X \) and containing at most \( p \) branching nodes.

Theorem 1. Let \( \epsilon < 1 \) be a real number. If \( P \neq NP \), the \textbf{min-}\((\cdot, p)\)-DSTLB and the \textbf{min-}\((\cdot, p)\)-USTLB problems in planar graphs with unit costs cannot be approximated within a factor of \( N^\epsilon \) where \( N \) is the number of nodes in the instance, even if there is a trivial feasible solution.

2 Proof of the theorem

2.1 Reduction

We prove the theorem in the directed case. The proof is similar in the undirected case.

Finding a hamiltonian path starting at a specified node \( v \) in a 3-connected directed planar graph is a NP-Complete problem [1].
Let $I = (G = (V, A), v)$ be an instance of the hamiltonian path problem in a 3-connected directed planar graph $G$. The 3-connected property is used in Section 2.3. We construct a min-$(\ast, p)$-DSTLB instance $I' = (G'_v, r, X, \omega)$ where $G'_v$ is a directed planar graph.

The main idea is that $G'_v$ is divided in three parts. An example is shown in Figure 1. Firstly, a graph $G' = (V' = V \cup W, A')$ built from $G$ where each arc of $A$ is divided in two or more arcs. Secondly, a binary tree $B$ rooted at $r$ with $p$ branching nodes and $p + 1$ leaves. We link one of the leaves of $B$ to $v$ with an arc $a_v$. We define $X$ as the leaves of $B$ and $V$. Finally, a graph $H$ and an integer $h$ which ensures the three following properties:

**Property 1.** Let $n$, $n_{G'}$ and $n_H$ be the number of nodes in $G$, $G'$ and $H$. $n_{G'} - n$ is no more than $n^3$ and $n_{G'} - n + n_H$ is no more than $4 \cdot n^3 \cdot h$.

**Property 2.** There exists an elementary path $P$ in $G' \cup H$ going through each node of $G$ starting at $v$.

**Property 3.** Any elementary path in $G' \cup H$ going through each node of $G$ starting at $v$ using a node of $H$ not as endpoint contains at least $h$ nodes of $H$.

Property 2 ensures the existence of a feasible solution. Properties 1, and 3 ensure an inapproximability gap, described in section 2.2. If $h$ is long enough, Property 3 ensures that any node of $H$ will not be allowed in any approximated solution. We will fix $G'$, $H$ and the value of the parameter $h$ later.

![Figure 1: Example of reduction from a graph $G$ with 4 nodes, and $p = 3$. Nodes of $W$ ($= V \setminus V$) and $H$, and arcs of $G'$ and $H$ do not appear on that figure.](image)

The number of nodes $N$ in $G'_v$ is $n_{G'} + n_H + p + (p + 1)$.

### 2.2 Inapproximability gap

In this part, we fix the parameter $h$ and show the approximability hardness of $(\ast, p)$-DSTLB.

Let $T^*$ be an optimal solution of $I'_v$. It exists because $B \cup P \cup a_v$ is a feasible solution by Property 2. Let $\epsilon < 1$, and suppose it exists a polynomial $N^\epsilon$-approximation algorithm for min-$(\ast, p)$-DSTLB in a planar graph. We will
show that in that case, we could use this algorithm to decide whether $G$ has a hamiltonian path starting at $v$.

If there exists a hamiltonian path in $I$ starting at $v$, $T^*$ contains at most $n_{G'} + 2p + 1$ nodes (the $n_{G'}$ nodes of $G'$ and the $2p + 1$ nodes of $B$), thus it contains at most $n_{G'} + 2p$ arcs. So the approximate solution has a cost $c_{YES} \leq (n_{G'} + 2p) \cdot N^\epsilon$.

We now discuss the case where there is no hamiltonian path starting at $v$ in $I$. Then, without $H$, we cannot build an elementary path going through each node of $G$.

**Lemma 1.** Any feasible solution of $I_v'$ contains an elementary path going through each node of $G$ starting at $v$.

**Proof.** Let $T$ be a feasible solution. $T$ covers every leaf of $B$, as a consequence it covers $B$ entirely. Because $B$ contains $p$ branching nodes, all other terminals are covered with elementary paths connected to $B$. $T$ covers every nodes of $G$ and $a_v$ is the only arc linking $B$ to a node $G$. So $T$ contains an elementary path going through each node of $G$ starting at $v$. \hfill \Box

By Lemma 1, without $H$, we cannot build a feasible solution of $I_v'$. So the approximate solution uses at least one node of $H$. On of those node is not an endpoint. Indeed, in this case, we can remove them to get a hamiltonian path in $G$. By Property 3, it uses at least $h$ nodes of $H$. So it has a cost $c_{NO} > h$.

If $c_{NO} > h > c_{YES}$, then the approximation algorithm can decide whether there is a hamiltonian path starting at $v$.

**Lemma 2.** Let $h$ satisfies $h = 5^{\frac{\epsilon}{1-\epsilon}}(2n^3 + 2p + 1)\frac{1}{1-\epsilon} + 1$. Then $c_{NO} > h > c_{YES}$.

**Proof.** Notice that $h > 1$ for all $\epsilon < 1$ and $n \geq 1$. Lines 9 and 13 are proven by Property 1.
\[ h > 5^{\frac{1}{1+\epsilon}} (2n^3 + 2p + 1)^{1+\epsilon} \]  
\[ h^{1-\epsilon} > 5' (2n^3 + 2p + 1)^{1+\epsilon} \]  
\[ h > 5' (2n^3 + 2p + 1)^{1+\epsilon} h^\epsilon \]  
\[ h > (2n^3 + 2p + 1)^{1+\epsilon} (5h)^\epsilon \]  
\[ h > (2n^3 + 2p + 1)^{1+\epsilon} (1 + 4h)^\epsilon \]  
\[ h > (2n^3 + 2p + 1) \cdot (1 + 4h)^\epsilon (2n^3 + 2p + 1)^\epsilon \]  
\[ h > (n^3 + n + 2p + 1) \cdot (1 + 4h)^\epsilon (n^3 + 2p + 1)^\epsilon \]  
\[ h > (ng' + 2p) \cdot (1 + 4h)^\epsilon (n^3 + 2p + 1)^\epsilon \]  
\[ h > (ng' + 2p) \cdot ((n^3 + 2p + 1) + 4 \cdot (n^3 + 2p + 1) \cdot h)^\epsilon \]  
\[ h > (ng' + 2p) \cdot ((n + 2p + 1) + 4 \cdot (n^3 + 2p + 1) \cdot h)^\epsilon \]  
\[ h > (ng' + 2p) \cdot ((n + 2p + 1) + 4 \cdot n^3 \cdot h)^\epsilon \]  
\[ h > (ng' + 2p) \cdot ((n + 2p + 1) + ng' + nh - n)^\epsilon \]  
\[ c_{NO} > h > c_{YES} \]  
\[ \square \]

As a consequence, if \( P \neq NP \), such an algorithm does not exist.

2.3 Existence of \( G' \) and \( H \)

In this section, we explain how to build the graphs \( G' \) and \( H \).

2.3.1 Construction of \( G' = (V \cup W, A') \)

\( G' \) is built from \( G \) where each arc of \( a \) is divided into several arcs of \( A' \) and nodes of \( W \).

\( G \) is a 3-connected planar graph. As a consequence, we can embed it in \( \mathbb{R}^2 \) as a convex polygon such that \( v \) is on the outer face of \( G \), using for instance the technique of [2]. For a node \( w \in V \), we define its coordinates as \( x_w \) and \( y_w \).

**Lemma 3.** It exists an angle \( \alpha \) such that the rotation \( r_\alpha(G) \), of angle \( \alpha \) and center \( v \), rotates \( G \) so that each node \( w \in V \) has a unique \( x \)-coordinate \( x_w \) with \( x_v \leq x_w \) (\( v \) is ’on the left’).

**Proof.** Let \( \alpha_m \) and \( \alpha_M \) be two angles in \([0; 2\pi]\) such that for each \( \alpha \in [\alpha_m; \alpha_M] \), \( r_\alpha(G) \) places \( v \) on the left.

If there is no angle where, after \( G \) rotates, each node \( w \in V \) has a unique \( x \)-coordinate \( x_w \), for each \( \alpha \in [\alpha_m; \alpha_M] \), there are two nodes \( (u, w) \) with \( x_u = x_w \) and \( y_u < y_w \). There are at most \( n^2 \) such couples. Let \( \alpha_i, i \in [1..(n^2 + 1)] \), be distinct angles in \([\alpha_m; \alpha_M] \), there are two distinct angles for which the same
couple of nodes \((u, w)\) verified, after \(G\) rotates, \(x_u = x_w\) and \(y_u < y_w\), which implies a contradiction.

We then sort the list of nodes \(v_i\) by its \(x\) coordinate: \(x_{v_1} < x_{v_2} < \ldots < x_{v_n}\).

We define \(D_i\) for \(i \in \{2, n\}\) as the vertical straight lines of abscissa \(x_i = x_{v_i-1} + x_{v_i}\). For each arc \(a = (t, u)\) of \(G\) crossing a line \(D_i\), we add a node \(w\) to \(W\) at the intersection of \(a\) and \(D_i\) and replace \(a\) in \(A'\) by the two arcs \((t, w)\) and \((w, u)\). An example is shown in Figure 2.

As no new arc cross, \(G'\) is planar.

![Figure 2: Example of graph \(G' = (V \cup W, A')\) built from a graph \(G\) with 4 nodes. \(W\) is the set of dashed nodes.](image)

### 2.3.2 Construction of \(H\)

We first prove three intermediate lemmas:

**Lemma 4.** Any arc of \(G'\) starting at a vertical straight line \(D_i\) goes to the left, or goes above, below, from or to \(v_i\).

**Proof.** Let \(a\) be an arc of \(G'\) crossing a vertical straight line \(D_i\) at a node \(u\). If \(a\) goes to the left, the lemma is verified. Else, if \(a\) do not go above, below, from and to \(v_i\), there is a node \(t \in V'\) with \(a = (u, t)\) or \(a = (t, u)\) and \(x_t \in [x_i, x_{v_i}]\). If \(t \in V\), by definition of \(D_i\), \(x_t > x_i\) which implies a contradiction. If \(t \in W\), there is a straight line \(D_j\) with \(x_t = x_j \in [x_i, x_{v_i}]\) which also implies a contradiction. □

We can similarly prove the following lemma:

**Lemma 5.** Any arc of \(G'\) going above, below, from or to \(v_i\) goes to the right of \(v_i\) or cross \(D_i\).

**Lemma 6.** For each node \(v_i \in V\), \(i \in \{2..n\}\), we can add to \(H\) a node \(v_{i,t}\) on \(D_i\) and an arc \((v_{i,t}, v_i)\) such that the graph \(G' \cup H\) remains planar.

**Proof.** Let \(a_m\) and \(a_M\) be respectively the lowest arc of \(G'\) going above \(v_i\) and the highest arc going below or to \(v_i\), going from or to the left of \(v_i\). An example is shown in figure 3.
If \( a_m \) and \( a_M \) do not exist, by Lemma 4, there is no arc crossing \( D_i \) going to or from the right (the graph is then disconnected). We can add \( v_{i,l} \) on \( D_i \) anywhere there is no node of \( W \).

If only \( a_m \) exists, the arc cross \( D_i \) at a node \( t_m \) by Lemma 5. We can add \( v_{i,l} \) on \( D_i \) anywhere below \( t_m \) where there is no node of \( W \).

If only \( a_M \) exists, the arc cross \( D_i \) at a node \( t_M \) by Lemma 5. We can add \( v_{i,l} \) on \( D_i \) anywhere above \( t_M \) where there is no node of \( W \).

If \( a_m \) and \( a_M \) exists, they cannot cross at a point of abscissa \( x \in [x_i; x_{vi}] \). If they do, either \( G' \) is not planar which is not true, or \( G' \) contains a node \( t \in [x_i; x_{vi}] \). Like in the proof of lemma 1, this would imply a contradiction. So we can add \( v_{i,l} \) on \( D_i \) anywhere above \( t_M \) and below \( t_m \) where there is no node of \( W \).

\[\text{Figure 3: Example of insertion of } v_{i,l}\]

Similarly, for each node \( v_i \in V, i \in [2..n] \) we can add to \( H \) a node \( v_{i,r} \) on \( D_{i+1} \) and an arc \( (v_i, v_{i,r}) \) such that the graph \( G' \cup H \) remains planar.

Finally, for \( i \in [2..n] \), we sort the nodes of abscissa \( x_i \) by increasing \( y \)-coordinate (those nodes are nodes of \( G' \) or nodes of \( H \)). For each couple \((u, t)\) of consecutive nodes we add to \( H \) a path of \( h \) nodes going from \( u \) to \( t \) and a path from \( t \) to \( u \) through the same \( h \) nodes. An example is shown in Figure 4.

**Lemma 7.** \( G', H \) and \( h \) verify Properties 1, 2 and 3.

**Proof.** \( n'_{G} - n \) and \( n_H \) are the number of arcs in \( W \) and \( H \), in other words, the nodes of all the lines \( D_i \). For each vertical line \( D_i \), we create at most \( m \) nodes
Figure 4: Example of graph $G' \cup H$ built from a graph $G$ with 4 nodes. Thick nodes are $v_{ir}$ and $v_{il}$. Dashed nodes are $W$. Each vertical arc is actually a path with $h$ nodes.

of $G'$ and $2 + h \cdot (1 + m)$ nodes of $H$.

$$n'_G \leq (n - 1) \cdot m$$

$$n'_G \leq n^3$$

$$n'_G + n_H - n \leq (n - 1) \cdot (1 + m)(h + 1) + 1$$

$$n'_G + n_H - n \leq n(1 + m)(h + 1) + (n - 1) - (1 + m)(h + 1)$$

Because $n \leq m$ in a connected graph and $h \geq 0$, we now that $(n - 1) < (1 + m)(h + 1)$. Thus $n'_G + n_H - n \leq n(1 + m)(h + 1) < n \cdot (2m) \cdot (2h) < 4n^3h$. Property 1 is verified. If the graph $G$ is not connected, there is no solution to the hamiltonian path problem.

The path $P$ starting at $v$, going to $v_{1,r}$, from $v_{i-1,r}$ to $v_{i,l}$ through $D_i$ and to $v_i$ for $i \in [2..n]$ goes through each node of $G$. Property 2 is verified.

Let $P$ be an elementary path going through every nodes of $G$ and one node of $H$ not as endpoint. As only the nodes $v_{i-1,r}$ and $v_{i,l}$, $i \in [2..n]$ are linked to a node of $G$. If $P$ contains a node of $H$, it exists a node $t$ and $i \in [2..n]$ such
that $t = v_{i-1,r}$ or $t = v_{i,l}$ is in $P$. As $t$ is linked to only one node not in $D_i$, $P$ goes out of $D_i$ (or enters $D_i$) through another node of $D_i$ and $P$ contains at least $h$ nodes of $D_i$. Property 3 is verified.

References
