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# Inapproximability proof of DSTLB and USTLB in planar graphs

Dimitri Watel      Marc-Antoine Weisser      Cédric Bentz

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This document proves the problem of finding a minimum cost Steiner Tree covering  $k$  terminals with at most  $p$  branching nodes (with outdegree greater than 1), in a directed or an undirected planar graph with  $n$  nodes, is hard to approximate within a better ratio than  $n$ , even when the parameter  $p$  is fixed.

## 1 Theorem

**Definition 1.** In a undirected (resp. directed) tree, a *branching node* is a node whose degree (resp. outdegree) is strictly greater than 2 (resp. 1).

**Problem 1. min- $(*, p)$ -USTLB:** Given an undirected graph  $G = (V, E)$  with  $n$  nodes and a non negative cost function  $\omega$  on its edges, an integer  $k$  and a set  $X \subset V$  of  $k$  terminals, determine, if it exists, a minimum cost tree  $T^*$  spanning all the nodes of  $X$  and containing at most  $p$  branching nodes.

**Problem 2. min- $(*, p)$ -DSTLB:** Given a directed graph  $G = (V, E)$  with  $n$  nodes and a non negative cost function  $\omega$  on its arcs, a node  $r$ , an integer  $k$  and a set  $X \subset V$  of  $k$  terminals, determine, if it exists, a minimum cost directed tree  $T^*$  rooted at  $r$ , spanning all the nodes of  $X$  and containing at most  $p$  branching nodes.

**Theorem 1.** *Let  $\epsilon < 1$  be a real number. If  $P \neq NP$ , the min- $(*, p)$ -DSTLB and the min- $(*, p)$ -USTLB problems in planar graphs with unit costs cannot be approximated within a factor of  $N^\epsilon$  where  $N$  is the number of nodes in the instance, even if there is a trivial feasible solution.*

## 2 Proof of the theorem

### 2.1 Reduction

We prove the theorem in the directed case. The proof is similar in the undirected case.

Finding a hamiltonian path starting at a specified node  $v$  in a 3-connected directed planar graph is a NP-Complete problem [1].

Let  $\mathcal{I} = (G = (V, A), v)$  be an instance of the hamiltonian path problem in a 3-connected directed planar graph  $G$ . The 3-connected property is used in Section 2.3. We construct a min- $(*, p)$ -DSTLB instance  $\mathcal{I}'_v = (G'_v, r, X, \omega)$  where  $G'_v$  is a directed planar graph.

The main idea is that  $G'_v$  is divided in three parts. An example is shown in Figure 1. Firstly, a graph  $G' = (V' = V \cup W, A')$  built from  $G$  where each arc of  $A$  is divided in two or more arcs. Secondly, a binary tree  $\mathcal{B}$  rooted at  $r$  with  $p$  branching nodes and  $p + 1$  leaves. We link one of the leaves of  $\mathcal{B}$  to  $v$  with an arc  $a_v$ . We define  $X$  as the leaves of  $\mathcal{B}$  and  $V$ . Finally, a graph  $H$  and an integer  $h$  which ensures the three following properties:

**Property 1.** Let  $n$ ,  $n_{G'}$  and  $n_H$  be the number of nodes in  $G$ ,  $G'$  and  $H$ .  $n_{G'} - n$  is no more than  $n^3$  and  $n_{G'} - n + n_H$  is no more than  $4 \cdot n^3 \cdot h$ .

**Property 2.** There exists an elementary path  $P$  in  $G' \cup H$  going through each node of  $G$  starting at  $v$ .

**Property 3.** Any elementary path in  $G' \cup H$  going through each node of  $G$  starting at  $v$  using a node of  $H$  not as endpoint contains at least  $h$  nodes of  $H$ .

Property 2 ensures the existence of a feasible solution. Properties 1, and 3 ensure an inapproximability gap, described in section 2.2. If  $h$  is long enough, Property 3 ensures that any node of  $H$  will not be allowed in any approximated solution. We will fix  $G'$ ,  $H$  and the value of the parameter  $h$  later.

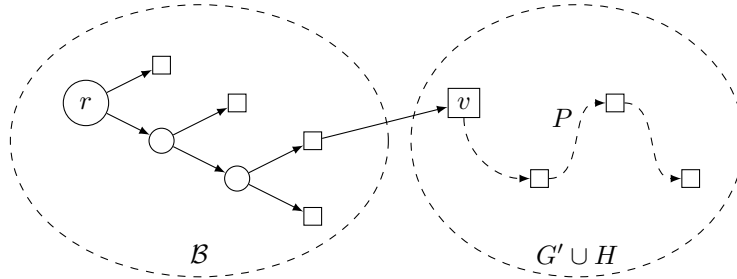


Figure 1: Example of reduction from a graph  $G$  with 4 nodes, and  $p = 3$ . Nodes of  $W (= V' \setminus V)$  and  $H$ , and arcs of  $G'$  and  $H$  do not appear on that figure.

The number of nodes  $\mathcal{N}$  in  $G'_v$  is  $n_{G'} + n_H + p + (p + 1)$ .

## 2.2 Inapproximability gap

In this part, we fix the parameter  $h$  and show the approximability hardness of  $(*, p)$ -DSTLB.

Let  $T^*$  be an optimal solution of  $\mathcal{I}'_v$ . It exists because  $\mathcal{B} \cup P \cup a_v$  is a feasible solution by Property 2. Let  $\epsilon < 1$ , and suppose it exists a polynomial  $\mathcal{N}^\epsilon$ -approximation algorithm for min- $(*, p)$ -DSTLB in a planar graph. We will

show that in that case, we could use this algorithm to decide whether  $G$  has a hamiltonian path starting at  $v$ .

If there exists a hamiltonian path in  $\mathcal{I}$  starting at  $v$ ,  $T^*$  contains at most  $n_{G'} + 2p + 1$  nodes (the  $n_{G'}$  nodes of  $G'$  and the  $2p + 1$  nodes of  $\mathcal{B}$ ), thus it contains at most  $n_{G'} + 2p$  arcs. So the approximate solution has a cost  $c_{\text{YES}} \leq (n_{G'} + 2p) \cdot \mathcal{N}^\epsilon$ .

We now discuss the case where there is no hamiltonian path starting at  $v$  in  $\mathcal{I}$ . Then, without  $H$ , we cannot build an elementary path going through each node of  $G$ .

**Lemma 1.** *Any feasible solution of  $\mathcal{I}'_v$  contains an elementary path going through each node of  $G$  starting at  $v$ .*

*Proof.* Let  $T$  be a feasible solution.  $T$  covers every leaf of  $\mathcal{B}$ , as a consequence it covers  $\mathcal{B}$  entirely. Because  $\mathcal{B}$  contains  $p$  branching nodes, all other terminals are covered with elementary paths connected to  $\mathcal{B}$ .  $T$  covers every nodes of  $G$  and  $a_v$  is the only arc linking  $\mathcal{B}$  to a node  $G$ . So  $T$  contains an elementary path going through each node of  $G$  starting at  $v$ .  $\square$

By Lemma 1, without  $H$ , we cannot build a feasible solution of  $\mathcal{I}'_v$ . So the approximate solution uses at least one node of  $H$ . On of those node is not an endpoint. Indeed, in this case, we can remove them to get a hamiltonian path in  $G$ . By Property 3, it uses at least  $h$  nodes of  $H$ . So it has a cost  $c_{\text{NO}} > h$ .

If  $c_{\text{NO}} > h > c_{\text{YES}}$ , then the approximation algorithm can decide whether there is a hamiltonian path starting at  $v$ .

**Lemma 2.** *Let  $h$  satisfies  $h = 5^{\frac{\epsilon}{1-\epsilon}} (2n^3 + 2 \cdot p + 1)^{\frac{1+\epsilon}{1-\epsilon}} + 1$ . Then  $c_{\text{NO}} > h > c_{\text{YES}}$ .*

*Proof.* Notice that  $h > 1$  for all  $\epsilon < 1$  and  $n \geq 1$ . Lines 9 and 13 are proven by Property 1.

$$\begin{aligned}
h &> 5^{\frac{\epsilon}{1-\epsilon}} (2n^3 + 2p + 1)^{\frac{1+\epsilon}{1-\epsilon}} & (1) \\
h^{1-\epsilon} &> 5^\epsilon (2n^3 + 2p + 1)^{1+\epsilon} & (2) \\
h &> 5^\epsilon (2n^3 + 2p + 1)^{1+\epsilon} h^\epsilon & (3) \\
h &> (2n^3 + 2p + 1)^{1+\epsilon} (5h)^\epsilon & (4) \\
h &> (2n^3 + 2p + 1)^{1+\epsilon} (1 + 4h)^\epsilon & (5) \\
h &> (2n^3 + 2p + 1) \cdot (1 + 4h)^\epsilon (2n^3 + 2p + 1)^\epsilon & (6) \\
h &> (2n^3 + 2p + 1) \cdot (1 + 4h)^\epsilon (n^3 + 2p + 1)^\epsilon & (7) \\
h &> (n^3 + n + 2p + 1) \cdot (1 + 4h)^\epsilon (n^3 + 2p + 1)^\epsilon & (8) \\
h &> (n_{G'} + 2p) \cdot (1 + 4h)^\epsilon (n^3 + 2p + 1)^\epsilon & (9) \\
h &> (n_{G'} + 2p) \cdot ((n^3 + 2p + 1) + 4 \cdot (n^3 + 2p + 1) \cdot h)^\epsilon & (10) \\
h &> (n_{G'} + 2p) \cdot ((n + 2p + 1) + 4 \cdot (n^3 + 2p + 1) \cdot h)^\epsilon & (11) \\
h &> (n_{G'} + 2p) \cdot ((n + 2p + 1) + 4 \cdot n^3 \cdot h)^\epsilon & (12) \\
h &> (n_{G'} + 2p) \cdot ((n + 2p + 1) + n_{G'} + n_H - n)^\epsilon & (13) \\
c_{\text{NO}} &> h > c_{\text{YES}} & (14)
\end{aligned}$$

□

As a consequence, if  $P \neq NP$ , such an algorithm does not exist.

## 2.3 Existence of $G'$ and $H$

In this section, we explain how to build the graphs  $G'$  and  $H$ .

### 2.3.1 Construction of $G' = (V \cup W, A')$

$G'$  is built from  $G$  where each arc of  $a$  is divided into several arcs of  $A'$  and nodes of  $W$ .

$G$  is a 3-connected planar graph. As a consequence, we can embed it in  $\mathbb{R}^2$  as a convex polygon such that  $v$  is on the outer face of  $G$ , using for instance the technique of [2]. For a node  $w \in V$ , we define its coordinates as  $x_w$  and  $y_w$ .

**Lemma 3.** *It exists an angle  $\alpha$  such that the rotation  $r_\alpha(G)$ , of angle  $\alpha$  and center  $v$ , rotates  $G$  so that each node  $w \in V$  has a unique  $x$ -coordinate  $x_w$  with  $x_v \leq x_w$  ( $v$  is 'on the left').*

*Proof.* Let  $\alpha_m$  and  $\alpha_M$  be two angles in  $[0; 2\pi]$  such that for each  $\alpha \in [\alpha_m; \alpha_M]$ ,  $r_\alpha(G)$  places  $v$  on the left.

If there is no angle where, after  $G$  rotates, each node  $w \in V$  has a unique  $x$ -coordinate  $x_w$ , for each  $\alpha \in [\alpha_m; \alpha_M]$ , there are two nodes  $(u, w)$  with  $x_u = x_w$  and  $y_u < y_w$ . There are at most  $n^2$  such couples. Let  $\alpha_i$ ,  $i \in [1..(n^2 + 1)]$ , be distinct angles in  $[\alpha_m; \alpha_M]$ , there are two distinct angles for which the same

couple of nodes  $(u, w)$  verified, after  $G$  rotates,  $x_u = x_w$  and  $y_u < y_w$ , which implies a contradiction.  $\square$

We then sort the list of nodes  $v_i$  by its  $x$  coordinate :  $x_v = x_{v_1} < x_{v_2} < x_{v_3} < \dots < x_{v_n}$ .

We define  $D_i$  for  $i \in [2..n]$  as the vertical strait lines of abscissa  $x_i = \frac{x_{v_{i-1}} + x_{v_i}}{2}$ . For each arc  $a = (t, u)$  of  $G$  crossing a line  $D_i$ , we add a node  $w$  to  $W$  at the intersection of  $a$  and  $D_i$  and replace  $a$  in  $A'$  by the two arcs  $(t, w)$  and  $(w, u)$ . An example is shown in Figure 2.

As no new arc cross,  $G'$  is planar.

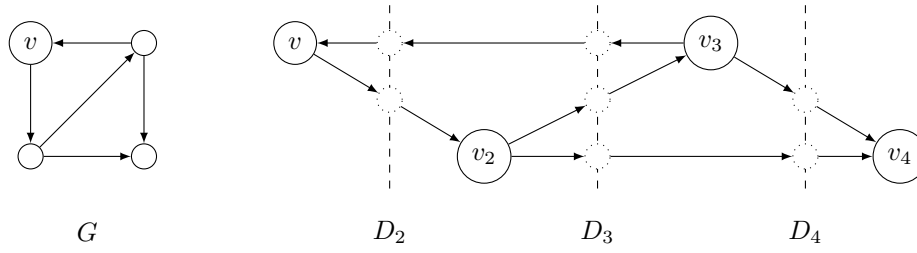


Figure 2: Example of graph  $G' = (V \cup W, A')$  built from a graph  $G$  with 4 nodes.  $W$  is the set of dashed nodes.

### 2.3.2 Construction of $H$

We first prove three intermediate lemmas :

**Lemma 4.** *Any arc of  $G'$  starting at a vertical strait line  $D_i$  goes to the left, or goes above, below, from or to  $v_i$ .*

*Proof.* Let  $a$  be an arc of  $G'$  crossing a vertical strait line  $D_i$  at a node  $u$ . If  $a$  goes to the left, the lemma is verified. Else, if  $a$  do not go above, below, from and to  $v_i$ , there is a node  $t \in V'$  with  $a = (u, t)$  or  $a = (t, u)$  and  $x_t \in [x_i, x_{v_i}[$ . If  $t \in V$ , by definition of  $D_i$ ,  $x_i > x_t$  which implies a contradiction. If  $t \in W$ , there is a strait line  $D_j$  with  $x_t = x_j \in [x_i, x_{v_i}[$  which also implies a contradiction.  $\square$

We can similarly prove the following lemma :

**Lemma 5.** *Any arc of  $G'$  going above, below, from or to  $v_i$  goes to the right of  $v_i$  or cross  $D_i$ .*

**Lemma 6.** *For each node  $v_i \in V$ ,  $i \in [2..n]$ , we can add to  $H$  a node  $v_{i,l}$  on  $D_i$  and an arc  $(v_{i,l}, v_i)$  such that the graph  $G' \cup H$  remains planar.*

*Proof.* Let  $a_m$  and  $a_M$  be respectively the lowest arc of  $G'$  going above  $v_i$  and the highest arc going below or to  $v_i$ , going from or to the left of  $v_i$ . An example is shown in figure 3.

If  $a_m$  and  $a_M$  do not exist, by Lemma 4, there is no arc crossing  $D_i$  going to or from the right (the graph is then disconnected). We can add  $v_{i,l}$  on  $D_i$  anywhere there is no node of  $W$ .

If only  $a_m$  exists, the arc cross  $D_i$  at a node  $t_m$  by Lemma 5. We can add  $v_{i,l}$  on  $D_i$  anywhere below  $t_m$  where there is no node of  $W$ .

If only  $a_M$  exists, the arc cross  $D_i$  at a node  $t_M$  by Lemma 5. We can add  $v_{i,l}$  on  $D_i$  anywhere above  $t_M$  where there is no node of  $W$ .

If  $a_m$  and  $a_M$  exists, they cannot cross at a point of abscissa  $x \in [x_i; x_{v_i}]$ . If they do, either  $G'$  is not planar which is not true, or  $G'$  contains a node  $t \in [x_i; x_{v_i}]$ . Like in the proof of lemma 1, this would imply a contradiction. So we can add  $v_{i,l}$  on  $D_i$  anywhere above  $t_M$  and below  $t_m$  where there is no node of  $W$ .  $\square$

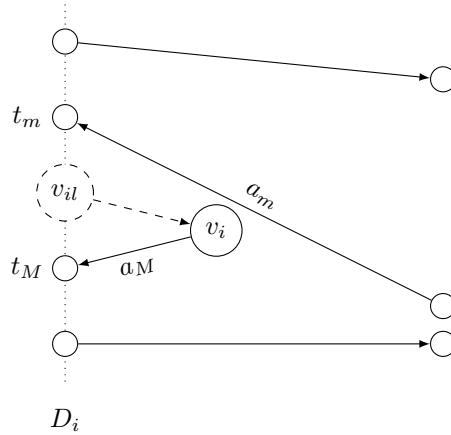


Figure 3: Example of insertion of  $v_{i,l}$

Similarly, for each node  $v_i \in V$ ,  $i \in [2..n]$  we can add to  $H$  a node  $v_{i,r}$  on  $D_{i+1}$  and an arc  $(v_i, v_{i,r})$  such that the graph  $G' \cup H$  remains planar.

Finally, for  $i \in [2..n]$ , we sort the nodes of abscissa  $x_i$  by increasing  $y$ -coordinate (those nodes are nodes of  $G'$  or nodes of  $H$ ). For each couple  $(u, t)$  of consecutive nodes we add to  $H$  a path of  $h$  nodes going from  $u$  to  $t$  and a path from  $t$  to  $u$  through the same  $h$  nodes. An example is shown in Figure 4.

**Lemma 7.**  $G'$ ,  $H$  and  $h$  verify Properties 1, 2 and 3.

*Proof.*  $n'_G - n$  and  $n_H$  are the number of arcs in  $W$  and  $H$ , in other words, the nodes of all the lines  $D_i$ . For each vertical line  $D_i$ , we create at most  $m$  nodes

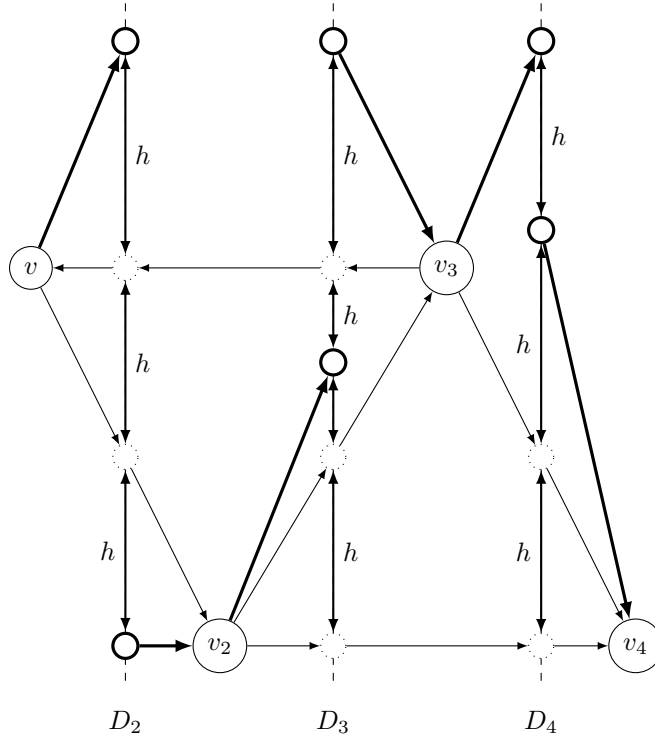


Figure 4: Example of graph  $G' \cup H$  built from a graph  $G$  with 4 nodes. Thick nodes are  $v_{i,r}$  and  $v_{i,l}$ . Dashed nodes are  $W$ . Each vertical *arc* is actually a path with  $h$  nodes.

of  $G'$  and  $2 + h \cdot (1 + m)$  nodes of  $H$ .

$$n'_G \leq (n - 1) \cdot m \quad (15)$$

$$n'_G \leq n^3 \quad (16)$$

$$n'_G + n_H - n \leq (n - 1) \cdot ((1 + m)(h + 1) + 1) \quad (17)$$

$$n'_G + n_H - n \leq n(1 + m)(h + 1) + (n - 1) - (1 + m)(h + 1) \quad (18)$$

Because  $n \leq m$  in a connected graph and  $h \geq 0$ , we now that  $(n - 1) < (1 + m)(h + 1)$ . Thus  $n'_G + n_H - n \leq n(1 + m)(h + 1) < n \cdot (2m) \cdot (2h) < 4n^3h$ . Property 1 is verified. If the graph  $G$  is not connected, there is no solution to the hamiltonian path problem.

The path  $P$  starting at  $v$ , going to  $v_{1,r}$ , from  $v_{i-1,r}$  to  $v_{i,l}$  through  $D_i$  and to  $v_i$  for  $i \in [2..n]$  goes through each node of  $G$ . Property 2 is verified.

Let  $P$  be an elementary path going through every nodes of  $G$  and one node of  $H$  not as endpoint. As only the nodes  $v_{i-1,r}$  and  $v_{i,l}$ ,  $i \in [2..n]$  are linked to a node of  $G$ . If  $P$  contains a node of  $H$ , it exists a node  $t$  and  $i \in [2..n]$  such



that  $t = v_{i-1,r}$  or  $t = v_{i,l}$  is in  $P$ . As  $t$  is linked to only one node not in  $D_i$ ,  $P$  goes out of  $D_i$  (or enters  $D_i$ ) through an other node of  $D_i$  and  $P$  contains at least  $h$  nodes of  $D_i$ . Property 3 is verified.  $\square$

## References

- [1] GAREY, M. R., JOHNSON, D. S., AND TARJAN, R. E. The planar hamiltonian circuit problem is np-complete. *SIAM Journal on Computing* 5, 4 (1976), 704–714.
- [2] PLESTENJAK, B. An algorithm for drawing planar graphs. *Software-Practice and Experience* 29, 11 (1999), 973–984.