



HAL
open science

Inapproximability proof of DSTLB and USTLB in planar graphs

Dimitri Watel, Marc-Antoine Weisser, Cédric Bentz

► **To cite this version:**

Dimitri Watel, Marc-Antoine Weisser, Cédric Bentz. Inapproximability proof of DSTLB and USTLB in planar graphs. [Research Report] Supélec. 2013. <hal-00793424v2>

HAL Id: hal-00793424

<https://centralesupelec.hal.science/hal-00793424v2>

Submitted on 25 Feb 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Inapproximability proof of DSTLB and USTLB in planar graphs

Dimitri Watel Marc-Antoine Weisser Cédric Bentz

February 25, 2013

This document proves the problem of finding a minimum cost Steiner Tree covering k terminals with at most p branching nodes (with outdegree greater than 1), in a directed or an undirected planar graph with n nodes, is hard to approximate within a better ratio than n , even when the parameter p is fixed.

1 Theorem

Definition 1. In a undirected (resp. directed) tree, a *branching node* is a node whose degree (resp. outdegree) is strictly greater than 2 (resp. 1).

Problem 1. min- $(*, p)$ -USTLB: Given an undirected graph $G = (V, E)$ with n nodes and a non negative cost function ω on its edges, an integer k and a set $X \subset V$ of k terminals, determine, if it exists, a minimum cost tree T^* spanning all the nodes of X and containing at most p branching nodes.

Problem 2. min- $(*, p)$ -DSTLB: Given a directed graph $G = (V, E)$ with n nodes and a non negative cost function ω on its arcs, a node r , an integer k and a set $X \subset V$ of k terminals, determine, if it exists, a minimum cost directed tree T^* rooted at r , spanning all the nodes of X and containing at most p branching nodes.

Theorem 1. *Let $\epsilon < 1$ be a real number. If $P \neq NP$, the min- $(*, p)$ -DSTLB and the min- $(*, p)$ -USTLB problems in planar graphs with unit costs cannot be approximated within a factor of N^ϵ where N is the number of nodes in the instance, even if there is a trivial feasible solution.*

2 Proof of the theorem

2.1 Reduction

We prove the theorem in the directed case. The proof is similar in the undirected case.

Finding a hamiltonian path starting at a specified node v in a 3-connected directed planar graph is a NP-Complete problem [1].

Let $\mathcal{I} = (G = (V, A), v)$ be an instance of the hamiltonian path problem in a 3-connected directed planar graph G . The 3-connected property is used in Section 2.3. We construct a min- $(*, p)$ -DSTLB instance $\mathcal{I}'_v = (G'_v, r, X, \omega)$ where G'_v is a directed planar graph.

The main idea is that G'_v is divided in three parts. An example is shown in Figure 1. Firstly, a graph $G' = (V' = V \cup W, A')$ built from G where each arc of A is divided in two or more arcs. Secondly, a binary tree \mathcal{B} rooted at r with p branching nodes and $p + 1$ leaves. We link one of the leaves of \mathcal{B} to v with an arc a_v . We define X as the leaves of \mathcal{B} and V . Finally, a graph H and an integer h which ensures the three following properties:

Property 1. Let n , $n_{G'}$ and n_H be the number of nodes in G , G' and H . $n_{G'} - n$ is no more than n^3 and $n_{G'} - n + n_H$ is no more than $4 \cdot n^3 \cdot h$.

Property 2. There exists an elementary path P in $G' \cup H$ going through each node of G starting at v .

Property 3. Any elementary path in $G' \cup H$ going through each node of G starting at v using a node of H not as endpoint contains at least h nodes of H .

Property 2 ensures the existence of a feasible solution. Properties 1, and 3 ensure an inapproximability gap, described in section 2.2. If h is long enough, Property 3 ensures that any node of H will not be allowed in any approximated solution. We will fix G' , H and the value of the parameter h later.

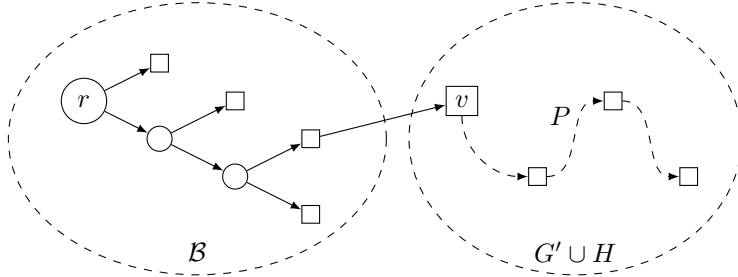


Figure 1: Example of reduction from a graph G with 4 nodes, and $p = 3$. Nodes of $W (= V' \setminus V)$ and H , and arcs of G' and H do not appear on that figure.

The number of nodes \mathcal{N} in G'_v is $n_{G'} + n_H + p + (p + 1)$.

2.2 Inapproximability gap

In this part, we fix the parameter h and show the approximability hardness of $(*, p)$ -DSTLB.

Let T^* be an optimal solution of \mathcal{I}'_v . It exists because $\mathcal{B} \cup P \cup a_v$ is a feasible solution by Property 2. Let $\epsilon < 1$, and suppose it exists a polynomial \mathcal{N}^ϵ -approximation algorithm for min- $(*, p)$ -DSTLB in a planar graph. We will

show that in that case, we could use this algorithm to decide whether G has a hamiltonian path starting at v .

If there exists a hamiltonian path in \mathcal{I} starting at v , T^* contains at most $n_{G'} + 2p + 1$ nodes (the $n_{G'}$ nodes of G' and the $2p + 1$ nodes of \mathcal{B}), thus it contains at most $n_{G'} + 2p$ arcs. So the approximate solution has a cost $c_{\text{YES}} \leq (n_{G'} + 2p) \cdot \mathcal{N}^\epsilon$.

We now discuss the case where there is no hamiltonian path starting at v in \mathcal{I} . Then, without H , we cannot build an elementary path going through each node of G .

Lemma 1. *Any feasible solution of \mathcal{I}'_v contains an elementary path going through each node of G starting at v .*

Proof. Let T be a feasible solution. T covers every leaf of \mathcal{B} , as a consequence it covers \mathcal{B} entirely. Because \mathcal{B} contains p branching nodes, all other terminals are covered with elementary paths connected to \mathcal{B} . T covers every nodes of G and a_v is the only arc linking \mathcal{B} to a node G . So T contains an elementary path going through each node of G starting at v . \square

By Lemma 1, without H , we cannot build a feasible solution of \mathcal{I}'_v . So the approximate solution uses at least one node of H . On of those node is not an endpoint. Indeed, in this case, we can remove them to get a hamiltonian path in G . By Property 3, it uses at least h nodes of H . So it has a cost $c_{\text{NO}} > h$.

If $c_{\text{NO}} > h > c_{\text{YES}}$, then the approximation algorithm can decide whether there is a hamiltonian path starting at v .

Lemma 2. *Let h satisfies $h = 5^{\frac{\epsilon}{1-\epsilon}} (2n^3 + 2 \cdot p + 1)^{\frac{1+\epsilon}{1-\epsilon}} + 1$. Then $c_{\text{NO}} > h > c_{\text{YES}}$.*

Proof. Notice that $h > 1$ for all $\epsilon < 1$ and $n \geq 1$. Lines 9 and 13 are proven by Property 1.

$$\begin{aligned}
h &> 5^{\frac{\epsilon}{1-\epsilon}}(2n^3 + 2p + 1)^{\frac{1+\epsilon}{1-\epsilon}} & (1) \\
h^{1-\epsilon} &> 5^\epsilon(2n^3 + 2p + 1)^{1+\epsilon} & (2) \\
h &> 5^\epsilon(2n^3 + 2p + 1)^{1+\epsilon}h^\epsilon & (3) \\
h &> (2n^3 + 2p + 1)^{1+\epsilon}(5h)^\epsilon & (4) \\
h &> (2n^3 + 2p + 1)^{1+\epsilon}(1 + 4h)^\epsilon & (5) \\
h &> (2n^3 + 2p + 1) \cdot (1 + 4h)^\epsilon(2n^3 + 2p + 1)^\epsilon & (6) \\
h &> (2n^3 + 2p + 1) \cdot (1 + 4h)^\epsilon(n^3 + 2p + 1)^\epsilon & (7) \\
h &> (n^3 + n + 2p + 1) \cdot (1 + 4h)^\epsilon(n^3 + 2p + 1)^\epsilon & (8) \\
h &> (n_{G'} + 2p) \cdot (1 + 4h)^\epsilon(n^3 + 2p + 1)^\epsilon & (9) \\
h &> (n_{G'} + 2p) \cdot ((n^3 + 2p + 1) + 4 \cdot (n^3 + 2p + 1) \cdot h)^\epsilon & (10) \\
h &> (n_{G'} + 2p) \cdot ((n + 2p + 1) + 4 \cdot (n^3 + 2p + 1) \cdot h)^\epsilon & (11) \\
h &> (n_{G'} + 2p) \cdot ((n + 2p + 1) + 4 \cdot n^3 \cdot h)^\epsilon & (12) \\
h &> (n_{G'} + 2p) \cdot ((n + 2p + 1) + n_{G'} + n_H - n)^\epsilon & (13) \\
c_{\text{NO}} &> h > c_{\text{YES}} & (14)
\end{aligned}$$

□

As a consequence, if $P \neq NP$, such an algorithm does not exist.

2.3 Existence of G' and H

In this section, we explain how to build the graphs G' and H .

2.3.1 Construction of $G' = (V \cup W, A')$

G' is built from G where each arc of a is divided into several arcs of A' and nodes of W .

G is a 3-connected planar graph. As a consequence, we can embed it in \mathbb{R}^2 as a convex polygon such that v is on the outer face of G , using for instance the technique of [2]. For a node $w \in V$, we define its coordinates as x_w and y_w .

Lemma 3. *It exists an angle α such that the rotation $r_\alpha(G)$, of angle α and center v , rotates G so that each node $w \in V$ has a unique x -coordinate x_w with $x_v \leq x_w$ (v is 'on the left').*

Proof. Let α_m and α_M be two angles in $[0; 2\pi]$ such that for each $\alpha \in [\alpha_m; \alpha_M]$, $r_\alpha(G)$ places v on the left.

If there is no angle where, after G rotates, each node $w \in V$ has a unique x -coordinate x_w , for each $\alpha \in [\alpha_m; \alpha_M]$, there are two nodes (u, w) with $x_u = x_w$ and $y_u < y_w$. There are at most n^2 such couples. Let α_i , $i \in [1..(n^2 + 1)]$, be distinct angles in $[\alpha_m; \alpha_M]$, there are two distinct angles for which the same

couple of nodes (u, w) verified, after G rotates, $x_u = x_w$ and $y_u < y_w$, which implies a contradiction. \square

We then sort the list of nodes v_i by its x coordinate : $x_v = x_{v_1} < x_{v_2} < x_{v_3} < \dots < x_{v_n}$.

We define D_i for $i \in [2..n]$ as the vertical strait lines of abscissa $x_i = \frac{x_{v_{i-1}} + x_{v_i}}{2}$. For each arc $a = (t, u)$ of G crossing a line D_i , we add a node w to W at the intersection of a and D_i and replace a in A' by the two arcs (t, w) and (w, u) . An example is shown in Figure 2.

As no new arc cross, G' is planar.

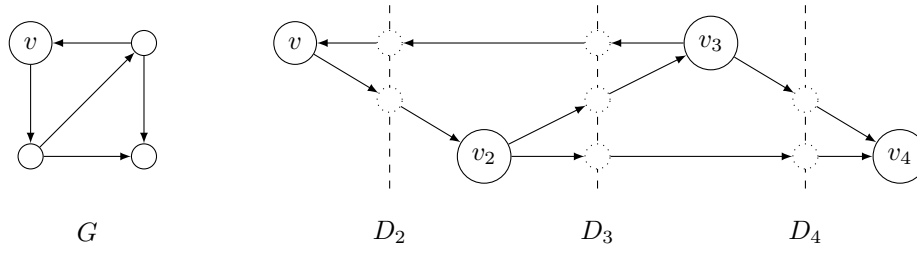


Figure 2: Example of graph $G' = (V \cup W, A')$ built from a graph G with 4 nodes. W is the set of dashed nodes.

2.3.2 Construction of H

We first prove three intermediate lemmas :

Lemma 4. *Any arc of G' starting at a vertical strait line D_i goes to the left, or goes above, below, from or to v_i .*

Proof. Let a be an arc of G' crossing a vertical strait line D_i at a node u . If a goes to the left, the lemma is verified. Else, if a do not go above, below, from and to v_i , there is a node $t \in V'$ with $a = (u, t)$ or $a = (t, u)$ and $x_t \in [x_i, x_{v_i}[$. If $t \in V$, by definition of D_i , $x_i > x_t$ which implies a contradiction. If $t \in W$, there is a strait line D_j with $x_t = x_j \in [x_i, x_{v_i}[$ which also implies a contradiction. \square

We can similarly prove the following lemma :

Lemma 5. *Any arc of G' going above, below, from or to v_i goes to the right of v_i or cross D_i .*

Lemma 6. *For each node $v_i \in V$, $i \in [2..n]$, we can add to H a node $v_{i,l}$ on D_i and an arc $(v_{i,l}, v_i)$ such that the graph $G' \cup H$ remains planar.*

Proof. Let a_m and a_M be respectively the lowest arc of G' going above v_i and the highest arc going below or to v_i , going from or to the left of v_i . An example is shown in figure 3.

If a_m and a_M do not exist, by Lemma 4, there is no arc crossing D_i going to or from the right (the graph is then disconnected). We can add $v_{i,l}$ on D_i anywhere there is no node of W .

If only a_m exists, the arc cross D_i at a node t_m by Lemma 5. We can add $v_{i,l}$ on D_i anywhere below t_m where there is no node of W .

If only a_M exists, the arc cross D_i at a node t_M by Lemma 5. We can add $v_{i,l}$ on D_i anywhere above t_M where there is no node of W .

If a_m and a_M exists, they cannot cross at a point of abscissa $x \in [x_i; x_{v_i}]$. If they do, either G' is not planar which is not true, or G' contains a node $t \in [x_i; x_{v_i}]$. Like in the proof of lemma 1, this would imply a contradiction. So we can add $v_{i,l}$ on D_i anywhere above t_M and below t_m where there is no node of W . \square

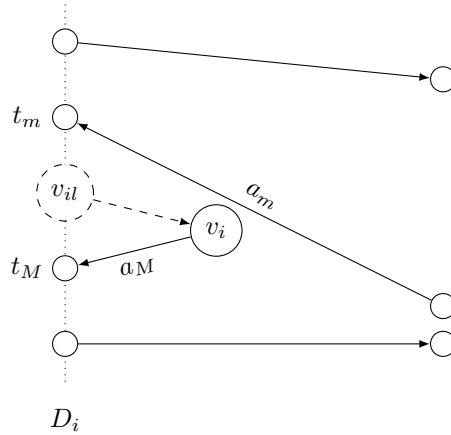


Figure 3: Example of insertion of $v_{i,l}$

Similarly, for each node $v_i \in V$, $i \in [2..n]$ we can add to H a node $v_{i,r}$ on D_{i+1} and an arc $(v_i, v_{i,r})$ such that the graph $G' \cup H$ remains planar.

Finally, for $i \in [2..n]$, we sort the nodes of abscissa x_i by increasing y -coordinate (those nodes are nodes of G' or nodes of H). For each couple (u, t) of consecutive nodes we add to H a path of h nodes going from u to t and a path from t to u through the same h nodes. An example is shown in Figure 4.

Lemma 7. G' , H and h verify Properties 1, 2 and 3.

Proof. $n'_G - n$ and n_H are the number of arcs in W and H , in other words, the nodes of all the lines D_i . For each vertical line D_i , we create at most m nodes

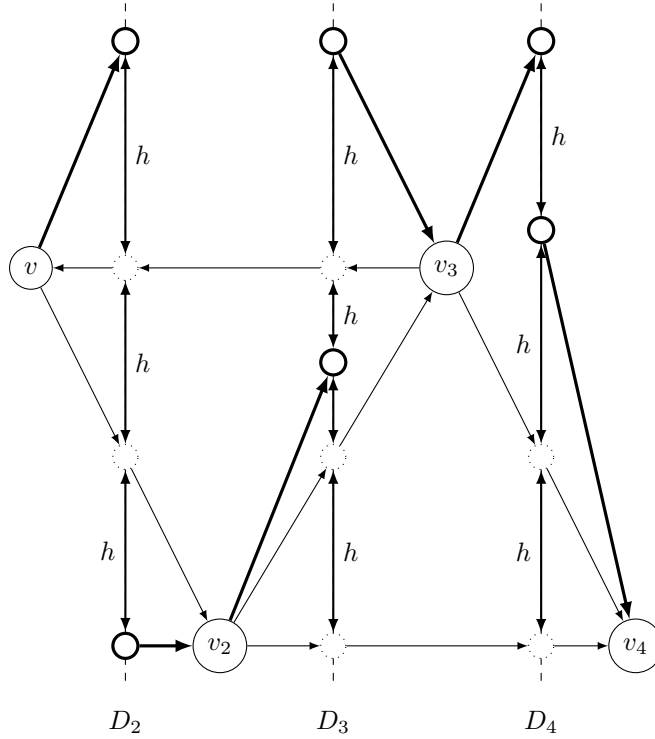


Figure 4: Example of graph $G' \cup H$ built from a graph G with 4 nodes. Thick nodes are v_{ir} and v_{il} . Dashed nodes are W . Each vertical *arc* is actually a path with h nodes.

of G' and $2 + h \cdot (1 + m)$ nodes of H .

$$n'_G \leq (n - 1) \cdot m \quad (15)$$

$$n'_G \leq n^3 \quad (16)$$

$$n'_G + n_H - n \leq (n - 1) \cdot ((1 + m)(h + 1) + 1) \quad (17)$$

$$n'_G + n_H - n \leq n(1 + m)(h + 1) + (n - 1) - (1 + m)(h + 1) \quad (18)$$

Because $n \leq m$ in a connected graph and $h \geq 0$, we now that $(n - 1) < (1 + m)(h + 1)$. Thus $n'_G + n_H - n \leq n(1 + m)(h + 1) < n \cdot (2m) \cdot (2h) < 4n^3h$. Property 1 is verified. If the graph G is not connected, there is no solution to the hamiltonian path problem.

The path P starting at v , going to $v_{1,r}$, from $v_{i-1,r}$ to $v_{i,l}$ through D_i and to v_i for $i \in [2..n]$ goes through each node of G . Property 2 is verified.

Let P be an elementary path going through every nodes of G and one node of H not as endpoint. As only the nodes $v_{i-1,r}$ and $v_{i,l}$, $i \in [2..n]$ are linked to a node of G . If P contains a node of H , it exists a node t and $i \in [2..n]$ such

that $t = v_{i-1,r}$ or $t = v_{i,l}$ is in P . As t is linked to only one node not in D_i , P goes out of D_i (or enters D_i) through an other node of D_i and P contains at least h nodes of D_i . Property 3 is verified. \square

References

- [1] GAREY, M. R., JOHNSON, D. S., AND TARJAN, R. E. The planar hamiltonian circuit problem is np-complete. *SIAM Journal on Computing* 5, 4 (1976), 704–714.
- [2] PLESTENJAK, B. An algorithm for drawing planar graphs. *Software-Practice and Experience* 29, 11 (1999), 973–984.