Robustness under saturated feedback: Strong iISS for a class of nonlinear systems

Rémi Azouit, Antoine Chaillet, Luca Greco

Robustness under saturated feedback: Strong iSS for a class of nonlinear systems*

Rémi Azouit, Antoine Chaillet[1] and Luca Greco

***Abstract*— This note proposes sufficient conditions under which a nonlinear system can be made Strongly iISS in the presence of actuator saturation. This property, recently proposed as a compromise between the strength of ISS and the generality of iISS, ensures boundedness of all solutions provided that the disturbance magnitude is below a certain threshold. We also show that, under a growth rate condition, the bounded feedback law proposed by Lin and Sontag for the stabilization of the disturbance-free system based on the knowledge of a control Lyapunov function, ensures Strong iISS in the presence of perturbations. We illustrate our findings on the angular velocity control of a spacecraft with limited-power thrusters.**

I. Introduction

Stabilization of dynamical systems in the presence of actuators saturation has been the object of a wide literature, specifically during the last two decades. It is well known that a necessary condition to stabilize a linear time-invariant (LTI) plant by saturated feedback is that the internal dynamics has no pole with positive real part [22]. A lot of effort has been made in order to propose bounded stabilizing feedback for particular classes of systems whose internal dynamics exhibits no exponential instability. For LTI systems having no eigenvalues with positive real part, it has been shown in [20] that stabilization by bounded output feedback can be achieved if the system is both detectable and stabilizable (which are also necessary requirements). For neutrally stable systems (meaning LTI systems whose internal dynamics exhibits no unbounded solutions), it has been shown that stabilization can be achieved using a saturated linear static feedback [7].

Nonetheless, it is known that some classes of systems, although having no poles with positive real parts, cannot be stabilized by saturated linear static state-feedback; this class includes chains of three or more integrators [6], [24]. Nested saturations [25] and neural networks architectures [23] have been proposed to stabilize such systems.

Stabilization by bounded control has also proved useful for nonlinear dynamics, especially in the context of systems in feedforward form [15], [26] or by relying on the so-called "universal constructions" [12].

Beyond stabilization, it is often desirable to ensure some robustness properties in order to cope, for instance, with parameter uncertainty, measurement noise or exogenous disturbances. To this aim, explicit estimates of $\mathcal{L}_p$ input/output gains have been obtained for neutrally stable systems based on a saturated linear static feedback [13]. Another natural candidate for the evaluation of robustness to exogenous inputs is the framework of input-to-state stability (ISS, [16], [19]) and its weaker variant integral ISS (iISS, [18]). In [1], a saturated linear state-feedback is proposed that ensures ISS with respect to sufficiently small disturbances despite parameter uncertainty for systems of dimension smaller than or equal to three, as well as feedforward systems. Other approaches to guarantee ISS and iISS with bounded control rely on the aforementioned "universal constructions" [10].

ISS ensures in particular a bounded response to any bounded disturbance. Intuitively, one may expect that bounded controls fail in general at guaranteeing the solutions' boundedness if the applied disturbance is too large. At first sight, for these systems, nothing more than ISS with respect to small inputs can be established, thus providing no information on the system's behavior for larger inputs. In this note, we provide sufficient conditions under which a more interesting property, namely Strong iISS, can be achieved by saturated feedback. This property, introduced in [4], not only guarantees ISS with respect to small inputs but also iISS. In particular, it ensures a bounded response to any disturbance whose amplitude is below a given threshold, but also the existence of solutions at all times even for disturbances above that threshold as well as the convergence of the state to the origin in response to any vanishing disturbance.

We start by formulating the problem and motivating it through an example (Section II). Our main results are presented in Section III: we provide a sufficient condition under which Strong iISS is achieved by saturated feedback and highlight their link with existing "universal constructions" that would guarantee global

*This work has received funding from the EU FP7 under grant agreement n257462 NoE HYCON2 and from the iCODE institute, research project of the Idex Paris-Saclay.

R. Azouit is with Univ. Paris Sud 11 - ENS Cachan, Master ATSI, Supélec, 3, rue Joliot-Curie, 91192, Gif-sur-Yvette, France, remi.azouit@lss.supelec.fr

A. Chaillet is with Univ. Paris-Sud 11-L2S-Supélec, same address, antoine.chaillet@supelec.fr

L. Greco is with Univ. Paris-Sud 11-L2S, same address, luca.greco@lss.supelec.fr

[1]Corresponding author. Tel.: +33169851383.

asymptotic stability of the disturbance-free system [12]. We illustrate our findings through the stabilization of the Euler equations of a rotating spacecraft (Section IV). All proofs are provided in Section V. Conclusive remarks are given in Section VI.

Notation. For a nondecreasing continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\gamma(\infty) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ denotes the quantity $\lim_{s\to+\infty} \gamma(s)$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{PD}$ if it is continuous and positive definite. It is of class $\mathcal{K}$ if, in addition, it is increasing. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\alpha(\infty) = \infty$. $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if, given any fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$ and, given any fixed $s \geq 0$, $\beta(s, \cdot)$ is continuous, nonincreasing and tends to zero as its argument tends to infinity. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a positive integer p, $\mathcal{U}^p$ denotes the set of all measurable locally essentially bounded functions $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$. For a given $d \in \mathcal{U}^p$, $\|d\| := \text{ess sup}_{t\geq 0} |d(t)|$. Given a constant $R > 0$, we let $\mathcal{U}^p_{<R}$ denote the set $\{d \in \mathcal{U}^p : \|d\| < R\}$. $\text{sat} : \mathbb{R}^n \to \mathbb{R}^n$ is the vector saturation function defined as $\text{sat}(x) = (\sigma(x_1), \ldots, \sigma(x_n))^T$, where $\sigma(s) := \min\{1; |s|\}\text{sign}(s)$ for each $s \in \mathbb{R}$. A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a *storage function* if it is continuously differentiable and satisfies $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. A storage function is said to be *proper* if, in addition, $\lim_{|x|\to\infty} V(x) = \infty$. Given a storage function V and a vector field f, $L_f V(x) := \frac{\partial V(x)}{\partial x} f(x)$.

II. Problem statement

Consider a nonlinear system of the form $\dot{x} = f(x, u, d)$, where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^p$ the exogenous disturbance and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ denotes a locally Lipschitz function satisfying $f(0, 0, 0) = 0$. If the system is stabilized through a static state feedback of the form $u = k(x)$, the system takes the form

$$\dot{x} = \tilde{f}(x, d)\,, \tag{1}$$

where $\tilde{f}(x, d) := f(x, k(x), d)$. Given $x_0 \in \mathbb{R}^n$ and an input signal $d \in \mathcal{U}^m$, the solution of (1) starting at x_0 at time $t = 0$ is referred to as $x(\cdot, x_0, d)$ (or simply $x(\cdot)$ when the context is clear) on the time domain where it is defined.

Consider the case where the state feedback is nominally designed to ensure input-to-state stability (ISS, [16], [19]) of the closed-loop system (1). Such a control law may be designed using existing techniques from the literature, such as [17], [9], [27], [11], [14]. Then, a natural question is: what can be said about the robustness of the system (1) in the presence of actuator saturation? Intuitively, we can expect that the applied control input $u = \text{sat}(k(x))$ will fail at guaranteeing a bounded state in response to any bounded disturbance, thus compromising ISS. Nonetheless a weaker robustness property, namely iISS [18], can reasonably be expected.

Definition 1: [iISS] The system (1) is said to be *integral input-to-state stable* if there exist a class $\mathcal{KL}$ function β and class $\mathcal{K}_\infty$ functions μ_1, μ_2 such that, for all $x_0 \in \mathbb{R}^n$ and all $d \in \mathcal{U}^p$, its solution satisfies, for all $t \geq 0$,

$$|x(t; x_0, d)| \leq \beta(|x_0|, t) + \mu_1\left(\int_0^t \mu_2(|d(s)|)ds\right).$$

Unfortunately, even if iISS systems prove robust with respect to classes of inputs with finite energy (in particular, $\int_0^\infty \mu_2(|d(s)|)ds < \infty$ implies $x(t) \to 0$ as $t \to \infty$), they can run unbounded in the presence of arbitrary small constant and even vanishing inputs [4]. Generically, we may expect a bounded state property at most for disturbances whose amplitude is below a given threshold. That is, we could consider systems which are ISS with respect to small inputs.

Definition 2: [ISS wrt small inputs] The system (1) is said to be *input-to-state stable with respect to small inputs* if there exist a constant $R > 0$ (referred to as an *input threshold*), a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function μ such that, for all $x_0 \in \mathbb{R}^n$ and all $d \in \mathcal{U}^p$, its solution satisfies, for all $t \geq 0$,

$$\|d\| < R \Rightarrow |x(t; x_0, d)| \leq \beta(|x_0|, t) + \mu(\|d\|)\,.$$

In the case when $R = +\infty$, we recover the classical definition of ISS [16], [19]. However, given a finite R, no guarantee on the behavior of the system can be given when the disturbance magnitude overpasses the threshold R. The very solution of the system **may fail to exist** if $\|d\| \geq R$. Hence, a good candidate to evaluate the robustness to exogenous disturbances of systems with saturated feedback seems to be the Strong iISS, recently introduced in [4].

Definition 3: [Strong iISS] The system (1) is said to be *Strongly iISS* if it is both ISS with respect to small inputs and iISS. In other words, there exist $R > 0$, $\beta \in \mathcal{KL}$ and $\mu_1, \mu_2, \mu \in \mathcal{K}_\infty$ such that, for all $d \in \mathcal{U}^p$, all $x_0 \in \mathbb{R}^n$ and all $t \geq 0$, its solution satisfies the following two properties:

$$|x(t)| \leq \beta(|x_0|, t) + \mu_1\left(\int_0^t \mu_2(|d(s)|)ds\right) \tag{2}$$

$$\|d\| < R \Rightarrow |x(t)| \leq \beta(|x_0|, t) + \mu(\|d\|)\,. \tag{3}$$

The constant R is then called an *input threshold*.

Nonetheless, the link between Strong iISS and systems with saturated control is not straightforward. For instance, elementary considerations convince that not every nominal ISS-stabilizing feedback ensures Strong iISS once saturated. Indeed, saturated feedback may be insufficient to compensate for unbounded sources of instability, thus compromising even the internal stability of the plant. An illustrative example of this is the LTI system $\dot{x} = x+u+d$: while the state feedback $u = -2x$ clearly guarantees ISS, its saturated version $u = -\text{sat}(2x)$ generates unbounded trajectories even in the absence of exogenous disturbances ($d = 0$). Further hypotheses are thus needed.

Since iISS, and consequently Strong iISS, imply global asymptotic stability in the absence of disturbance (this property will be called 0-GAS in the rest of the article), only systems that can be globally stabilized by saturated feedback can be expected to yield Strong iISS in the presence of actuator saturation. In other words, Strong iISS stabilization through saturated feedback faces all the challenges of global asymptotic stabilization by bounded control. In particular, for LTI systems, necessary requirements include stabilizability and the absence of eigenvalues with positive real part.

Based on these observations, a more reasonable hypothesis would be that any ISS-stabilizing nominal feedback that ensures 0-GAS when saturated, also ensures Strong iISS. The following example shows that this conjecture is not true in general.

Example 1: Consider the scalar system

$$\dot{x} = x^2 u + x^3 d. \tag{4}$$

The nominal feedback law $u = k(x) = -x^5$ makes the closed-loop system ISS. Indeed, the total derivative of the storage function $V(x) = x^2/2$ reads

$$\dot{V}(x) = -x^8 + x^4 d \leq -x^8 + x^4|d| \leq -\frac{x^8}{2} + \frac{d^2}{2},$$

which guarantees ISS by its classical Lyapunov characterization [21]. In the presence of actuator saturation, the closed-loop system becomes $\dot{x} = -x^2\text{sat}(x^5) + x^3 d$. Clearly, this system remains 0-GAS. Nonetheless it can be seen that, given any constant input $d^* > 0$, any solution starting from $x_0 \geq \max\{1; 2/d^*\}$ grows unbounded (and even presents finite escape times). This can be formally proven by noticing that, for such initial conditions, the solutions of (4) satisfy $\dot{x}(t) \geq x(t)^3 d^*/2$ for all $t \geq 0$. The system $\dot{y} = y^3 d^*/2$ being not forward complete, the comparison lemma shows that (4) is not forward complete either with the considered saturated feedback. Since d^* can be picked arbitrarily small, this fact contradicts the bounded-input bounded-state property for sufficiently small inputs. We conclude that the system in not Strong iISS in the presence of actuator saturation.

The above example highlights the necessity to conduct a more careful study on how the Strong iISS property may be ensured by saturated feedback.

III. Main results

In this note, we focus on input-affine systems:

$$\dot{x} = f(x) + g(x)u + h(x)d \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $d \in \mathbb{R}^p$ is the perturbation. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ are assumed locally Lipschitz and zero at zero. In the presence of actuator saturation, the system reads

$$\dot{x} = f(x) + g(x)\text{sat}(u) + h(x)d. \tag{6}$$

We stress that, if each control entries u_i saturates at a value $\bar{u}_i \neq 1$, considering $\tilde{u} := (u_1/\bar{u}_1, \ldots, u_n/\bar{u}_n)^T$ as the new control allows to fit the framework (6).

A. Sufficient conditions for Strong iISS

We start by stating an analysis result, which provides sufficient conditions under which a saturated control guarantees Strong iISS to (6). This result will serve as a basis to design such feedback control laws in the next subsections. This first result relies on the following two assumptions, involving a proper storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a locally Lispschitz state feedback $k : \mathbb{R}^n \to \mathbb{R}^m$.

Assumption 1: There exists a class $\mathcal{K}$ function γ such that, for all $x \neq 0$,

$$L_f V(x) + L_g V(x) k(x) < 0 \tag{7}$$

$$L_h V \neq 0 \;\Rightarrow\; \left|\frac{L_f V + L_g V k(x)}{L_h V}\right| \geq \gamma(|x|). \tag{8}$$

Assumption 1 contains two ingredients. First, (7) guarantees the 0-GAS of (5) in closed loop with $u = k(x)$. Second, as we will see in the proof of Theorem 1 below, the combination of (7) and (8) ensure ISS with respect to small inputs.

Assumption 2: It holds that

$$\limsup_{x\to\infty} \frac{|L_h V(x)|}{1 + V(x)} < +\infty. \tag{9}$$

Assumption 2 essentially guarantees that the perturbation does not yield finite escape times that a saturated feedback would not be able to tackle. This can be seen by observing that, by the continuity of the function $x \mapsto \frac{|L_h V(x)|}{1+V(x)}$, (9) ensures that

$$\frac{|L_h V(x)|}{1 + V(x)} \leq K, \quad \forall x \in \mathbb{R}^n, \tag{10}$$

for some positive constant K. Consequently, all the terms induced by the perturbation term in the total derivative of V is at most linear in $V(x)$. This, combined with Assumption 1, constitutes a sufficient condition for forward completeness [2].

Based on these assumptions, we can state the following result.

Lemma 1: Assume that there exists a proper storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a function $\gamma \in \mathcal{K}$, and a locally Lipschitz state feedback $k : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying Assumptions 1 and 2. Then the control law $u = k(x)$ makes the saturated-actuator system (6) Strong iISS with input threshold $R = \gamma(\infty)$.

The proof is rather straightforward, but is provided in Section V-A for the sake of completeness.

It is worth stressing that Lemma 1 would not hold if either Assumption 1 or 2 was removed. Assumption 2 alone does not provide any information on the system in the absence of exogenous perturbations. Moreover, it can easily be seen that the system (4) of Example 1 satisfies Assumption 1, but fails at fulfilling Assumption 2. It results as a non Strongly iISS system, as the system is not forward complete in the presence of arbitrary small, and even vanishing, perturbations.

B. Strong iISS stabilization by "universal" constructions

Although Lemma 1 gives some hints on when a saturated feedback yields Strong iISS, it does not provide any constructive way to design the corresponding control law. In this section, we rely on the so-called "universal" construction of Arstein's theorem to construct a bounded static state feedback ensuring Strong iISS for the closed-loop system.

This "universal" construction relies on the knowledge of a *control Lyapunov function* (CLF), whose definition is recalled below [12].

Definition 4: [CLF] A smooth proper storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a *control Lyapunov function with controls in the unit ball* for the (disturbance-free) system $\dot{x} = f(x) + g(x)u$ if it satisfies:

$$\inf_{|u|<1} \{L_f V(x) + L_g V(x)u\} < 0, \quad \forall x \neq 0.$$

In other words, a CLF with controls in the unit ball is a smooth storage function whose total derivative can be pointwisely assigned to a negative value for each non-zero state by a control value whose amplitude is lower than 1.

We may also require that this pointwise assignment be achievable by arbitrarily small control values, provided that the state is sufficiently close to the origin: this property is referred to as the *small control property* (SCP).

Definition 5: [SCP] A CLF $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to satisfy the *small control property* for the (disturbance-free) system $\dot{x} = f(x) + g(x)u$ if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $|x| < \varepsilon$ and $x \neq 0$, then there exists $|u| < \delta$ such that $L_f V(x) + L_g V(x)u < 0$.

We stress that, unlike other robust CLF proposed in the literature [5], [27], [11], [10], the above definitions are given for disturbance-free systems: the goal here is to provide a growth restriction on the function h so that the bounded control law proposed in [12] ensures Strong iISS for (6).

More precisely, the main contribution of the work [12] is to propose an explicit continuous state feedback law, smooth out of the origin and of amplitude smaller than 1, that globally asymptotically stabilizes the system $\dot{x} = f(x) + g(x)u$. This state-feedback law reads $k(x) = \kappa(L_f V(x), |L_g V(x)|^2) L_g V(x)^T$, where κ is defined, for each $(a, b) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$, as

$$\kappa(a,b) := \begin{cases} -\frac{a+\sqrt{a^2+b^2}}{b(1+\sqrt{1+b})} & \text{if } b > 0 \\ 0 & \text{if } b = 0. \end{cases} \tag{11}$$

The following result, proved in Section V-B, states that this control law may be used as such to make the system (6) Strongly iISS provided a growth restriction on the function h.

Theorem 1: Let V be a CLF with controls in the unit ball, satisfying the SCP for the disturbance-free system $\dot{x} = f(x) + g(x)u$. Assume that there exists $\alpha \in \mathcal{K}$ such that, for all $x \neq 0$,

$$|L_f V(x)| + \frac{|L_g V(x)|^2}{\sqrt{1 + |L_g V(x)|^2}} > \alpha(|x|) \, |L_h V(x)| \, . \tag{12}$$

Assume also that

$$L_f V(x) > 0 \quad \Rightarrow \quad \liminf_{|x|\to\infty} |L_g V(x)| > 0 \tag{13}$$

and

$$\limsup_{|x|\to\infty} \frac{|L_g V(x)|}{|L_f V(x)|} \neq 1. \tag{14}$$

Then, under Assumption 2, the feedback law proposed in [12], namely

$$u = k(x) = \kappa(L_f V(x), |L_g V(x)|^2) L_g V(x)^T, \tag{15}$$

where κ is defined in (11), is continuous on $\mathbb{R}^n$, smooth on $\mathbb{R}^n \setminus \{0\}$, has norm smaller than 1 and makes the saturated-actuation system (6) Strongly iISS.

Condition (12) expresses a growth rate limitation on the term $h(x)$ through which the perturbation acts on the system. Condition (13) is fairly intuitive: in the presence of a non-vanishing perturbation term, the control field needs to be non-vanishing as well in order to compensate these disturbances and ensure the state boundedness (hence Strong iISS). Condition (14) is far less intuitive, and is actually of a purely technical nature: it results from the particular choice of the control law proposed in [12], and is needed to ensure that the dissipation rate of V along the solutions of the disturbance-free system is not only positive, but can also be lower bounded by a $\mathcal{K}$ function (in other words, it does not become arbitrarily small for large values of the state). We may expect that slight modifications of the control law (11) may be used to address systems that do not fulfill the requirement (14), but this goes beyond the scope of the paper.

Remark 1: When the assumptions of Theorem 1 hold with a $\mathcal{K}_\infty$ function α, it can be seen along the proof that the system (6) in closed loop with the same static feedback $k(x)$ results ISS. This observation complements the results in the literature that rely on the notion of ISS-CLF [11], [10], [27].

C. iISS stabilization by "universal" constructions

The assumptions of Theorem 1 can be considerably relaxed if only iISS is needed. We state this fact in the following corollary.

Corollary 1: Let V be a CLF with controls in the unit ball satisfying the SCP for the disturbance-free system $\dot{x} = f(x) + g(x)u$ and let Assumption 2 hold. Then the static state feedback law (15) proposed in [12] makes the system (6) iISS.

The proof of this result is provided in Section V-C. Similarly to Theorem 1, it provides a growth rate limitation on the term $h(x)$ in such a way that the control law originally proposed in [12] for disturbance-free systems, also yields a robustness property to exogenous disturbances. We stress, however, that the robustness property ensured by Corollary 1 (namely, iISS) is much weaker than that guaranteed by Theorem 1 (namely, Strong iISS) as it implies neither solutions' boundedness in response to sufficiently small disturbances nor state convergence in response to a vanishing perturbation.

IV. EXAMPLE: SPACECRAFT VELOCITY CONTROL

We now provide an illustration of the results in this paper by considering the control of a rotating spacecraft, through limited-thrust actuators. Letting $x := (x_1, x_2, x_3)^T$ denote its the angular velocity and $u := (u_1, u_2, u_3)^T$ the control torques, the dynamics under concern is ruled by the following equations [8, Exercise 4.4]:

$$J_1\dot{x_1} = (J_2 - J_3)x_2x_3 + u_1 + d_1 \tag{16a}$$
$$J_2\dot{x_2} = (J_3 - J_1)x_3x_1 + u_2 + d_2 \tag{16b}$$
$$J_3\dot{x_3} = (J_1 - J_2)x_1x_2 + u_3 + d_3, \tag{16c}$$

where $d := (d_1, d_2, d_3)^T$ represents exogenous perturbations (e.g. actuation errors). We consider as a nominal control law proportional state feedback

$$u = k(x) := (-k_1x_1, -k_2x_2, -k_3x_3)^T,$$

where k_1, k_2, k_3 denote positive gains. It can easily be shown that this nominal control law makes the system (16) ISS. We claim that, in the presence of limited thrust (namely $|u_i| \leq \bar{u}$ for each $i \in \{1, 2, 3\}$), the system results Strongly iISS with input threshold $R = \bar{u}/\sqrt{3}$. Indeed, in the presence of such saturating thrusters, the dynamics reads

$$J_1\dot{x_1} = (J_2 - J_3)x_2x_3 - \bar{u}\mathrm{sat}(\tilde{k}_1x_1) + d_1 \tag{17a}$$
$$J_2\dot{x_2} = (J_3 - J_1)x_3x_1 - \bar{u}\mathrm{sat}(\tilde{k}_2x_2) + d_2 \tag{17b}$$
$$J_3\dot{x_3} = (J_1 - J_2)x_1x_2 - \bar{u}\mathrm{sat}(\tilde{k}_3x_3) + d_3, \tag{17c}$$

where $\tilde{k}_i := k_i/\bar{u}$ for each $i \in \{1, 2, 3\}$. We use the Lyapunov function $V(x) = \frac{1}{2}x^TPx$, where $P := \mathrm{diag}(J_1, J_2, J_3)$. Using the notation of (6), straightforward computations lead to $L_fV(x) = 0$, $L_gV(x) = -(\bar{u}x_1, \bar{u}x_2, \bar{u}x_3)$ and $L_hV(x) = (x_1, x_2, x_3)$. Consequently, noticing that $|L_hV(x)| \neq 0$ for all $x \neq 0$, it holds that

$$\frac{L_fV + L_gV\mathrm{sat}(k(x)/\bar{u})}{|L_hV|} = -\frac{x_1\mathrm{sat}(\tilde{k}_1x_1) + x_2\mathrm{sat}(\tilde{k}_2x_2) + x_3\mathrm{sat}(\tilde{k}_3x_3)}{|x|}\bar{u},$$

which is clearly a negative definite function, thus establishing (7). Moreover, letting $|x|_\infty := \max\{|x_1|, |x_2|, |x_3|\}$ and $\underline{k} := \min\{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3\}$, it holds that

$$\frac{x_1\mathrm{sat}(\tilde{k}_1x_1) + x_2\mathrm{sat}(\tilde{k}_2x_2) + x_3\mathrm{sat}(\tilde{k}_3x_3)}{|x|} \geq \frac{|x|_\infty \mathrm{sat}(\underline{k}\,|x|_\infty)}{|x|}.$$

Observing that $|x| \leq |x|_\infty \sqrt{3}$, we obtain that

$$\frac{L_fV + L_gV\mathrm{sat}(k(x)/\bar{u})}{|L_hV|} \leq -\frac{\bar{u}}{\sqrt{3}}\mathrm{sat}(\underline{k}\,|x|/\sqrt{3}),$$

which makes (8) fulfilled with $\gamma(s) = \mathrm{sat}(\underline{k}s/\sqrt{3})\bar{u}/\sqrt{3}$. Thus, Assumption 1 is satisfied. Assumption 2 being trivially satisfied in this case (since $|L_hV| = |x|$), we conclude from Lemma 1 that, as claimed, the system is Strongly iISS with input threshold $\gamma(\infty) = \bar{u}/\sqrt{3}$.

We note finally that the above function V is clearly a CLF with controls in the unit ball for (16) and that it satisfies the SCP. It can also be seen that (12), (13) and (14) hold for this system. Theorem 1 can thus be invoked to design a bounded continuous feedback law of the form $u = \kappa(0, |L_gV|^2)L_gV^T$, where κ is given in (11). The resulting control law is however slightly more involved than the feedback $u = \mathrm{sat}(k(x)/\bar{u})$ applied above.

V. PROOFS

A. Proof of Lemma 1

The total derivative of V along the solutions of (6) in closed loop with $u = k(x)$ reads

$$\begin{aligned}\dot{V} &= L_f V(x) + L_g V(x)\text{sat}(k(x)) + L_h V(x) d \\ &\leq L_f V(x) + L_g V(x)\text{sat}(k(x)) + |L_h V(x)|\,|d|\,.\end{aligned}$$

In view of (8) in Assumption 1 it holds that, for all $x \neq 0$,

$$\left.\begin{array}{l} L_h V(x) \neq 0 \\ |d| \leq \gamma(|x|) \end{array}\right\} \quad \Rightarrow \quad \dot{V} < 0.$$

Moreover, from (7), it holds that, for all $x \neq 0$,

$$L_h V(x) = 0 \quad \Rightarrow \quad \dot{V} < 0.$$

Therefore, noticing that $\dot{V} = 0$ for $x = 0$, there exists a $\mathcal{PD}$ function ρ such that, for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^p$,

$$|d| \leq \gamma(|x|) \quad \Rightarrow \quad \dot{V} < -\rho(|x|). \qquad (18)$$

We now rely on the following result.

Proposition 1: Assume that there exist a proper storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a class $\mathcal{K}$ function γ and a $\mathcal{PD}$ function ν such that, for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^p$,

$$|d| \leq \gamma(|x|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}(x) f(x, d) \leq -\nu(|x|).$$

Then (1) is ISS with respect to inputs $d \in \mathcal{U}^p_{<\gamma(\infty)}$.

The proof of this proposition follows from typical manipulations on ISS Lyapunov functions and is therefore omitted. We stress that, in the case when $\gamma \in \mathcal{K}_\infty$, we recover a classical ISS characterization [21].

Invoking Proposition 1 we conclude from (18) that, under the control law $u = k(x)$, the system (6) is ISS with respect to all $d \in \mathcal{U}^p_{<\gamma(\infty)}$. Consequently, there is only left to prove that this system is iISS. To this aim, let $W(x) := \ln(1 + V(x))$. Noticing that Assumption 1 ensures the existence of a $\mathcal{PD}$ function $\tilde{\rho}$ such that $L_f V(x) + L_g V(x)\text{sat}(k(x)) \leq -\tilde{\rho}(|x|)$ for all $x \in \mathbb{R}^n$, we get that

$$\begin{aligned}\dot{W} &= \frac{L_f V(x) + L_g V(x)\text{sat}(k(x)) + L_h V(x) d}{1 + V(x)} \\ &\leq -\frac{\tilde{\rho}(|x|)}{1 + V(x)} + \frac{|L_h V(x)|}{1 + V(x)}|d|.\end{aligned}$$

In view of (10), which is ensured by Assumption 2, we obtain that

$$\dot{W} \leq -\frac{\tilde{\rho}(|x|)}{1 + V(x)} + K|d|.$$

Noting that $x \mapsto \frac{\tilde{\rho}(|x|)}{1+V(x)}$ is a continuous positive definite function we can thus deduce by [3] that the system is iISS. We conclude that the system is indeed Strongly iISS with input threshold $\gamma(\infty)$.

B. Proof of Theorem 1

For notation simplicity, let $a(x) := L_f V(x)$ and $b(x) := |L_g V(x)|^2$. The smoothness and magnitude properties of the feedback law $k(x)$ are established in [12]. Moreover, since $|k(x)| \leq 1$, it holds that

$$\begin{aligned}L_f V + L_g V \text{sat}(k(x)) &= L_f V + L_g V k(x) \\ &= a(x) + b(x)\kappa(a(x), b(x)).\end{aligned}$$

In view of Lemma 1, since Assumption 2 is supposed to be fulfilled, only Assumption 1 needs to be checked. Since the feedback law proposed in [12] ensures $a(x) + b(x)\kappa(a(x), b(x)) < 0$ for all $x \neq 0$, all we need to show is (8), that is

$$|a(x) + b(x)\kappa(a(x), b(x))| > \tilde{\alpha}(|x|)\,|L_h V(x)|\,, \qquad (19)$$

for some $\mathcal{K}$ function $\tilde{\alpha}$. Pick any $x \neq 0$. We consider 4 cases.

Case 1: $b(x) = 0$. Then (12) implies that $|a(x)| > \alpha(|x|)\,|L_h V(x)|$. Consequently

$$|a(x) + b(x)\kappa(a(x), b(x))| = |a(x)| > \alpha(|x|)\,|L_h V(x)|\,. \qquad (20)$$

Case 2: $b(x) \neq 0$ and $a(x) \leq 0$. Then, omitting the x-dependency in the notation, it holds that

$$\begin{aligned}|a + b\kappa(a, b)| &= \left|\frac{a\sqrt{1+b} - \sqrt{a^2 + b^2}}{1 + \sqrt{1+b}}\right| \\ &= \frac{\sqrt{1+b}}{1 + \sqrt{1+b}}\left|a - \sqrt{\frac{a^2 + b^2}{1+b}}\right| \\ &> \frac{1}{2}\left|a - \sqrt{\frac{a^2 + b^2}{1+b}}\right|, \qquad (21)\end{aligned}$$

where the last bound comes from the fact that the function $b \mapsto \sqrt{1+b}/(1 + \sqrt{1+b})$ is greater than $1/2$ over $\mathbb{R}_{>0}$. Recalling that $a < 0$ we have

$$\begin{aligned}\frac{1}{2}\left|a - \sqrt{\frac{a^2 + b^2}{1+b}}\right| &= \frac{1}{2}\left(|a| + \sqrt{\frac{a^2 + b^2}{1+b}}\right) \\ &> \frac{1}{2}\left(|a| + \frac{b}{\sqrt{1+b}}\right).\end{aligned}$$

We thus obtain from (12) that

$$|a + b\kappa(a, b)| > \frac{\alpha(|x|)}{2}\,|L_h V(x)|\,. \qquad (22)$$

Case 3: $b(x) \geq 1$ and $a(x) > 0$. First notice that (21) is still valid under these assumptions. Consequently, we have that

$$\begin{aligned}|a + b\kappa(a, b)| &> \frac{1}{2}\left|a - \sqrt{\frac{a^2 + b^2}{1+b}}\right| \\ &= \frac{1}{2}\sqrt{\frac{a^2 + b^2}{1+b}}\left|\sqrt{\frac{a^2 + a^2 b}{a^2 + b^2}} - 1\right| \\ &> \frac{b}{2\sqrt{1+b}}\left|\sqrt{\frac{1+b}{1 + b^2/a^2}} - 1\right|. \qquad (23)\end{aligned}$$

Now, the fact that V is a CLF with u constrained in the unit ball guarantees that $a(x) < \sqrt{b(x)}$ for all $x \neq 0$ (see Definition 4). Since $a(x) > 0$ and $b(x) \geq 1$, we get that

$$\begin{aligned}\frac{3b}{\sqrt{1+b}} &= 2\sqrt{b}\sqrt{\frac{b}{1+b}} + \frac{b}{\sqrt{1+b}} \\ &> 2a\sqrt{\frac{b}{1+b}} + \frac{b}{\sqrt{1+b}} > a + \frac{b}{\sqrt{1+b}},\end{aligned}$$

where the last bound comes from the fact that $\sqrt{b/(1+b)} > 1/2$ whenever $b \geq 1$. It follows from (12) that

$$\frac{3b(x)}{\sqrt{1+b(x)}} > \alpha(|x|)\,|L_h V(x)|\,.$$

We therefore get from (23) that

$$|a + b\kappa(a,b)| > \frac{\alpha(|x|)\,|L_h V(x)|}{6}\left|\sqrt{\frac{1+b(x)}{1+\frac{b(x)^2}{a(x)^2}}} - 1\right|. \tag{24}$$

We claim that there exists $\alpha' \in \mathcal{K}$ such that

$$\left|\sqrt{\frac{1+b(x)}{1+b(x)^2/a(x)^2}} - 1\right| \geq \alpha'(|x|). \tag{25}$$

To see this first notice that, since $a(x) < \sqrt{b(x)}$, the above function is never zero on $\mathbb{R}^n \setminus \{0\}$. Moreover, condition (14) together with the fact that $0 < a(x) < \sqrt{b(x)}$ ensures that $\limsup_{|x|\to\infty} b(x)/a(x)^2 > 1$. Consequently:

$$\liminf_{|x|\to\infty}\left|\sqrt{\frac{1+b(x)}{1+b(x)^2/a(x)^2}} - 1\right| > 0.$$

We conclude that there indeed exists a function $\alpha' \in \mathcal{K}$ satisfying (25). Recalling that the product of two $\mathcal{K}$ functions is itself a $\mathcal{K}$ function, we conclude that the function $\alpha''(\cdot) := \frac{1}{6}\alpha(\cdot)\alpha'(\cdot)$ is of class $\mathcal{K}$ and we get from (24) and (25) that

$$|a(x) + b(x)\kappa(a(x), b(x))| > \alpha''(|x|)\,|L_h V(x)|\,. \tag{26}$$

<u>Case 4: $b(x) \neq 0$, $b(x) \leq 1$ and $a(x) > 0$.</u> Recalling that $a(x) < \sqrt{b(x)}$, we have that

$$\begin{aligned}a + \frac{b}{\sqrt{1+b}} &< \sqrt{b} + \frac{b}{\sqrt{1+b}} = \frac{\sqrt{b+b^2}+b}{\sqrt{1+b}} \\ &\leq \frac{\sqrt{2b}+b}{\sqrt{1+b}} \leq \frac{(\sqrt{2}+1)\sqrt{b}}{\sqrt{1+b}} < \frac{3\sqrt{b}}{\sqrt{1+b}}.\end{aligned}$$

The lower bound (23) being still valid, it follows from (12) that

$$|a + b\kappa(a,b)| > \frac{\alpha(|x|)\,|L_h V(x)|}{6}\sqrt{b}\left|\sqrt{\frac{1+b}{1+\frac{b^2}{a^2}}} - 1\right|. \tag{27}$$

Moreover, the assumption (13) implies the existence of a class $\mathcal{K}$ function μ such that $\sqrt{b} = |L_g V(x)| \geq \mu(|x|)$. Following a similar reasoning as in Case 3, we obtain that:

$$|a(x) + b(x)\kappa(a(x), b(x))| > \mu'(|x|)\,|L_h V(x)|\,, \tag{28}$$

where μ' denotes some $\mathcal{K}$ function.

Combining (20), (22), (26) and (28), we conclude that (19) holds. The conclusion then follows from the application of Lemma 1.

C. *Proof of Corollary 1*

Since V is a CLF with controls in the unit ball satisfying the SCP for the disturbance-free system $\dot{x} = f(x) + g(x)u$, it was shown in [12] that the control law (15) satisfies $L_f V(x) + L_g V(x)k(x) < 0$ for all $x \neq 0$. Equivalently, there exists a continuous positive definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $L_f V(x) + L_g V(x)k(x) \leq -\rho(|x|)$ for all $x \in \mathbb{R}^n$. Consequently, the derivative of $W := \ln(1+V)$ along the solutions of (6) satisfies

$$\begin{aligned}\dot{W} &= \frac{\dot{V}}{1+V(x)} \\ &= \frac{1}{1+V(x)}\left(L_f V(x) + L_g V(x)k(x) + L_h V(x)d\right) \\ &\leq -\frac{\rho(|x|)}{1+V(x)} + \frac{|L_h V(x)|}{1+V(x)}\,|d|\,.\end{aligned}$$

As already stressed, Assumption 2 implies the existence of a constant $K > 0$ such that (10) holds. Consequently:

$$\dot{W} \leq -\frac{\rho(|x|)}{1+V(x)} + K\,|d|\,.$$

The conclusion then follows by the classical Lyapunov characterization of iISS [3] after noticing that $x \mapsto \rho(|x|)/(1+V(x))$ is a positive definite function.

VI. Conclusion

After having presented a sufficient condition for Strong iISS, we have shown that Lin and Sontag's bounded control law for disturbance-free systems also ensures Strong iISS provided that a condition on the growth rate of the input term is satisfied. We have also shown that iISS can be obtained under relaxed conditions. Finally, we have illustrated the applicability of our results on a spacecraft control example.

References

[1] D. Angeli, Y. Chitour, and L. Marconi. Robust stabilization via saturated feedback. *IEEE Trans. on Automat. Contr.*, 50(12):1997–2014, January 2005.

[2] D. Angeli and E.D. Sontag. Forward completeness, unboundedness observability, and their Lyapunov characterizations. *Syst. & Contr. Letters*, 38:209–217, 1999.

[3] D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input to state stability. *IEEE Trans. on Automat. Contr.*, 45:1082–1097, 2000.

[4] A. Chaillet, D. Angeli, and H. Ito. Strong iISS: combination of iISS and ISS with respect to small inputs. In *Proc. IEEE Conf. on Decision and Control*, Hawaii, USA, Dec. 2012.

[5] R.A. Freeman and P.V. Kokotović. *Robust Nonlinear control design: State-space and Lyapunov control techniques*. Birkhäuser, Boston, 1996.

[6] A.T. Fuller. In-the-large stability of relay and saturating control systems with linear controllers. *International Journal of Control*, 10(4):457–480, 1969.

[7] P.O. Gutman and P. Hagander. A new design of constrained controllers for linear systems. *IEEE Trans. on Automat. Contr.*, 30(1):22–33, 1985.

[8] H. Khalil. *Nonlinear systems*. Prentice Hall, 3rd ed., New York, 2002.

[9] M. Krstic and Z.-H. Li. Inverse optimal design of Input-to-State Stabilizing nonlinear controllers. *IEEE Trans. on Automat. Contr.*, 43(3):336–350, 1998.

[10] D. Liberzon. ISS and integral-ISS disturbance attenuation with bounded controls. *Proc. 41st. IEEE Conf. Decision Contr.*, 3:2501–2506, 2002.

[11] D. Liberzon, E.D. Sontag, and Y. Wang. Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation. *Syst. & Contr. Letters*, 46:111–127, 2002.

[12] Y. Lin and E.D. Sontag. A universal formula for stabilization with bounded controls. *Syst. & Contr. Letters*, 16(6):393–397, 1991.

[13] W. Liu, Y. Chitour, and E.D. Sontag. On finite-gain stabilizability of linear systems subject to input saturation. *SIAM Journal on Control and Optimization*, 34(4):1190–1219, 1996.

[14] M. Malisoff, L. Rifford, and E.D. Sontag. Global asymptotic controllability implies Input-to-State stabilization. *SIAM J. Control Optim.*, 42:2221–2238, 2004.

[15] F. Mazenc and L. Praly. Adding integrators, saturated controls and global asymptotic stabilization of feedforward systems. *IEEE Trans. on Automat. Contr.*, 41(11):1559–1579, 1996.

[16] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. on Automat. Contr.*, 34(4):435–443, 1989.

[17] E.D. Sontag. Further facts about Input-to-State stabilization. *IEEE Trans. on Automat. Contr.*, 35:473–476, 1990.

[18] E.D. Sontag. Comments on integral variants of ISS. *Syst. & Contr. Letters*, 34:93–100, 1998.

[19] E.D. Sontag. *Input to state stability: Basic concepts and results*, chapter in Nonlinear and Optimal Control Theory, pages 163–220. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2006. P. Nistri and G. Stefani eds.

[20] E.D. Sontag and H.J. Sussmann. Nonlinear output feedback design for linear systems with saturating controls. In *Proc. 29th. IEEE Conf. Decision Contr.*, pages 3414–3416, Honolulu, HI, 1990.

[21] E.D. Sontag and Y. Wang. On characterizations of the Input-to-State Stability property. *Syst. & Contr. Letters*, 24:351–359, 1995.

[22] H. Sussmann, E.D. Sontag, and Y. Yang. A general result on the stabilization of linear systems using bounded controls. *IEEE Trans. on Automat. Contr.*, 39(12):2411–2425, 1994.

[23] H.J. Sussmann, E.D. Sontag, and Y. Yang. A general result on the stabilization of linear systems using bounded controls. *Proc. of the 32nd IEEE Conference on Decision and Control*, pages 1802–1807, 1993.

[24] H.J. Sussmann and Y. Yang. On the stabilizability of multiple integrators by means of bounded feedback controls. In *Proc. 30th. IEEE Conf. Decision Contr.*, Brighton, UK, December 1991.

[25] A.R. Teel. Global stabilization and restricted tracking for multiple integrators with bounded controls. *Syst. & Contr. Letters*, 18(3):165–171, 1992.

[26] A.R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Trans. on Automat. Contr.*, 40:1256–1270, 1996.

[27] A.R. Teel and L. Praly. On assigning the derivative of a disturbance attenuation Control Lyapunov Function. *Math. of Cont., Sign. and Syst.*, 13:95–124, 2000.