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# On the convergence of Maronna's $M$ -estimators of scatter

Yacine Chitour, Romain Couillet *Member, IEEE*, and Frédéric Pascal *Senior Member, IEEE*

**Abstract**—In this paper, we propose an alternative proof for the uniqueness of Maronna's  $M$ -estimator of scatter [1] for  $N$  vector observations  $\mathbf{y}_1, \dots, \mathbf{y}_N \in \mathbf{R}^m$  under a mild constraint of linear independence of any subset of  $m$  of these vectors. This entails in particular almost sure uniqueness for random vectors  $\mathbf{y}_i$  with a density as long as  $N > m$ . This approach allows to establish further relations that demonstrate that a properly normalized Tyler's  $M$ -estimator of scatter [2] can be considered as a limit of Maronna's  $M$ -estimator. More precisely, the contribution is to show that each  $M$ -estimator, verifying some mild conditions, converges towards a particular Tyler's  $M$ -estimator. These results find important implications in recent works on the large dimensional (random matrix) regime of robust  $M$ -estimation.

## I. INTRODUCTION

Subsequent to Huber's introduction of robust statistics in [3], Maronna proposed in [1] a class of robust estimates for scatter matrices defined as the solution of an implicit equation. In [1], the existence and uniqueness of such a solution are proved, under conditions involving both the ratio  $c_N := m/N$  of the population dimension  $m$  and the sample size  $N$ , and the parametrization of the estimate. This constraint was largely relaxed in [4], [5]. With the recent renewed interest in robust  $M$ -estimation under the random matrix regime  $N, m \rightarrow \infty$  with  $c_N \rightarrow c_\infty \in (0, 1)$  [6]–[9], alternative proofs of existence and uniqueness have appeared motivated by this assumption of large  $m$ . While Maronna's original results are valid for any (well-behaved) set of samples satisfying the condition on  $c_N$ , the results in e.g. [6] are expressed in probabilistic terms and are only valid for all large  $m, N$ .

Based on the ideas from [10]–[12], the present article proposes an alternative proof to [4] to show existence and uniqueness for all well-behaved set of samples with a known location parameter and for any  $c_N \in (0, 1)$ . More importantly, by a proper parametrization of the weight function appearing in Maronna's estimator, we prove that some sequences of Maronna's  $M$ -estimators converge to

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a unique Tyler's distribution-free  $M$ -estimator of scatter [2]. This result is a novel property of the Tyler's  $M$ -estimators, rigorously proved in this work. This completes the recent result (Theorem 1 of [13]) stating that the Tyler's  $M$ -estimator is the Maximum Likelihood estimator (MLE) of the scatter for various complex elliptically symmetric (CES) distributions as well as for the angular central Gaussian (ACG) distributions [14].

The paper is organized as follows: Section II presents our main results as well as Monte-Carlo simulations that corroborate our theoretical claims, the proofs of which are provided in Section III. Section IV draws some conclusions and perspectives of this work.

## II. NOTATIONS AND STATEMENT OF THE RESULTS

Let  $\mathbf{R}_+$  (resp.  $\mathbf{R}_+^*$ ) be the (resp. strictly) positive real line. We use  $M_m(\mathbf{R})$  and  $\text{Sym}_m$  to denote the vector space of  $m \times m$  matrices with real entries and the linear subspace of  $M_m(\mathbf{R})$  made of the symmetric matrices, respectively. We also use  $\text{Sym}_m^+$  and  $\text{PSD}_m$  to denote the non trivial cones in  $M_m(\mathbf{R})$  of the non negative symmetric matrices and of the symmetric positive definite matrices, respectively. Also,  $(\cdot)^T$  stands for the transpose,  $\text{Tr}(\cdot)$  and  $\det(\cdot)$  for the trace and the determinant. On  $M_m(\mathbf{R})$ , we use the inner product defined by the Frobenius norm  $\|\mathbf{A}\| = \sqrt{\text{Tr}(\mathbf{A}\mathbf{A}^T)}$ . We also use  $\leq$  to denote the partial order on  $\text{Sym}_m$  and  $\mathbf{I}_m$  the  $m \times m$  identity matrix. Functions of two non negative real variables  $(t, x)$  will be considered. If  $f$  is such a function, we use  $f_t, f_x, f_{tx}, \dots$  to denote (when defined) the partial derivatives of  $f$  with respect to  $t$  and/or  $x$ .

**Definition II.1** A family  $(\mathbf{y}_i)_{1 \leq i \leq N}$  of vectors in  $\mathbf{R}^m$  is admissible if

- (C1) for  $1 \leq i \leq N$ ,  $\|\mathbf{y}_i\| = 1$ ;
- (C2) the vectors in any subset of size  $m$  of  $\{\mathbf{y}_1, \dots, \mathbf{y}_N\}$  are linearly independent

This definition straightforwardly implies that if  $(\mathbf{y}_i)_{1 \leq i \leq N}$  is an admissible family of vectors in  $\mathbf{R}^m$  and if  $m$  vectors (say  $\mathbf{y}_1, \dots, \mathbf{y}_m$  which are then linearly independent by (C2) are fixed, for  $m+1 \leq l \leq N$ , we can write  $\mathbf{y}_l = \sum_{j=1}^m \gamma_{lj} \mathbf{y}_j$ . Then,  $\gamma_{lj} \neq 0$  for every  $1 \leq j \leq m$  and  $m+1 \leq l \leq N$ .

Let us now consider maps  $u : (\mathbf{R}_+^*)^2 \rightarrow \mathbf{R}_+$  of class  $C^1$  satisfying:

- (U1)  $u(t, \cdot)$  is strictly decreasing;
- (U2) for every  $t > 0$ ,  $v(t, x) := x \mapsto xu(t, x)$  is increasing on  $\mathbf{R}_+$  and  $l_t := \sup_{x \geq 0} v(t, x) > m$ ;

We furthermore define, for every  $x > 0$ ,  $u(0, x) = \frac{m}{x}$ . Note that, by continuity of  $u$ ,  $\forall x > 0$ ,  $\lim_{t \rightarrow 0^+} v(t, x) = m$ . Also, according to (U1) and (U2), for each  $t, x > 0$ ,

$$v(t, x) = m + tv_1(x) + tw(t, x), \quad (1)$$

with  $v_1(\cdot) := v_t(0, \cdot)$  and  $\forall x > 0$ ,  $\lim_{t \rightarrow 0} w(t, x) = 0$ . By a simple computation, one has that  $v_1$  is a nondecreasing function on  $\mathbf{R}_+^*$ .

For further use, we introduce the following additional notation. Let  $x_t > 0$  be the unique positive number such that,  $\forall t > 0$ ,  $v(t, x_t) = x_t u(t, x_t) = m$ .

We further consider the following assumption

$$(U3) \quad \begin{cases} v_x := dv/dx > 0 \\ v_1 \text{ is increasing} \\ 0 < \liminf_{t \rightarrow 0} x_t \leq \limsup_{t \rightarrow 0} x_t < \infty. \end{cases}$$

If the latter occurs and  $u$  is of class  $C^2$ , then  $w(t, x) = tw_1(x) + o(t)$ , with  $w_1(\cdot) := w_t(0, \cdot)$  continuous on  $(\mathbf{R}_+^*)^2$ , the convergence in (U2) is uniform in  $x$  on any compact of  $\mathbf{R}_+^*$  and  $x_t$  converges to the unique solution  $x_0$  of  $v_1(x) = 0$ .

We use  $\bar{u}(t, x)$  to denote the particular function

$$\bar{u}(t, x) = \frac{m(1+t)}{x+t} \quad (2)$$

which is analytic on every compact of  $(\mathbf{R}_+^*)^2 \setminus \{(0, 0)\}$ . Moreover,  $\bar{l}_t = m(1+t)$ ,  $\bar{v}_1(x) = m(1 - \frac{1}{x})$  and  $\bar{w}(t, x) = \frac{-mt}{t+x}$ .

The objective of the work is to study the solutions of the equation given, for all  $t > 0$ , by

$$(Eq)_t \quad \mathbf{M} = \frac{1}{N} \sum_{i=1}^N u(t, \mathbf{y}_i^T \mathbf{M}^{-1} \mathbf{y}_i) \mathbf{y}_i \mathbf{y}_i^T.$$

and to characterize them in the limit where  $t \rightarrow 0$ . Taking into account our definitions, if a solution to  $(Eq)_t$  exists, it must belong to  $\text{PSD}_m$ .

Remark that the condition M of [4] also imposes a ‘‘strictly’’ increasing  $v$  which excludes e.g. the Huber  $M$ -estimator.

To state our results, we need to consider the set of solutions of the equation  $(Eq)_0$  (that defines the Tyler’s  $M$ -estimator) given by

$$(Eq)_0 \quad \mathbf{M} = \frac{m}{N} \sum_{i=1}^N \frac{1}{\mathbf{y}_i^T \mathbf{M}^{-1} \mathbf{y}_i} \mathbf{y}_i \mathbf{y}_i^T.$$

Recall from [10] that the set of solutions of  $(Eq)_0$  is the half-line  $\mathbf{R}_+^* \mathbf{P}$  in  $\text{PSD}_m$ , where  $\mathbf{P}$  is the unique

solution of  $(Eq)_0$  with  $\text{Tr}(\mathbf{P}) = m$ .

Our main result is the following theorem.

**Theorem II.2** *Let  $(\mathbf{y}_i)_{1 \leq i \leq N}$  be an admissible family of vectors in  $\mathbf{R}^m$  and  $u : (\mathbf{R}_+^*)^2 \setminus \{(0, 0)\} \rightarrow \mathbf{R}_+$  be a  $C^1$  function verifying (U1)–(U2). Then,*

- (A)  $\forall t > 0$ ,  $(Eq)_t$  admits a unique solution,  $\mathbf{M}(t)$ .
- (B) *If, furthermore,  $u$  is  $C^2$  and satisfies (U3), then the mapping  $t \mapsto \mathbf{M}(t)$  is continuous and  $\lim_{t \rightarrow 0} \mathbf{M}(t) = \mathbf{M}_0$  the solution of  $(Eq)_0$  given by  $\mathbf{M}_0 = \xi_u \mathbf{P}$  with  $\xi_u > 0$  unique solution to*

$$\sum_{i=1}^N v_1 \left( \frac{\mathbf{y}_i^T \mathbf{P}^{-1} \mathbf{y}_i}{\xi} \right) = 0. \quad (3)$$

*In particular, for  $u = \bar{u}$ ,  $\mathbf{M}_0 = \mathbf{P}$ , i.e.,  $\xi_{\bar{u}} = 1$ .*

*Proof of Theorem II.2.* The proof is postponed in the next section.  $\blacksquare$

### Remark II.3

1) The interest of Theorem II.2, in addition to providing an alternative proof for the existence and uniqueness, lies in the convergence of all  $M$ -estimators to a Tyler’s  $M$ -estimator. This limit can be different (by a scale factor) from one  $M$ -estimator to another. While this result was expected, this paper rigorously proves it.

2) Moreover, the theorem provides a way of understanding why the Tyler’s estimator is the outmost robust<sup>1</sup>  $M$ -estimator. Indeed, considering a ML approach, the weight function  $u(t, x)$  is derived from the observations probability density function (PDF) and in such a case,  $t \rightarrow 0$  means that the underlying distribution becomes more and more heavy-tailed. For instance, considering  $t$  as the exponent parameter of a Generalized Gaussian distribution or of a W-distribution, the smaller the value of  $t > 0$  is, the heavier-tailed is the distribution. This is also the case for the degree of freedom of a Student-t distribution or the shape parameters of a K-distribution or of a Compound-Gaussian with inverse Gaussian texture (see [14] for more details). In all these cases, the MLEs satisfy the assumptions of Theorem II.2 (at least for small values of  $t$ ) and should be more robust when the distribution is heavier-tailed. To summarize, this result theoretically motivates the use of the Tyler’s estimator, since it will perform similarly as MLEs in heavy-tailed distribution contexts.

<sup>1</sup>Here the robustness has to be understood as the classical property considered in the robust estimation theory literature, see e.g. [15]

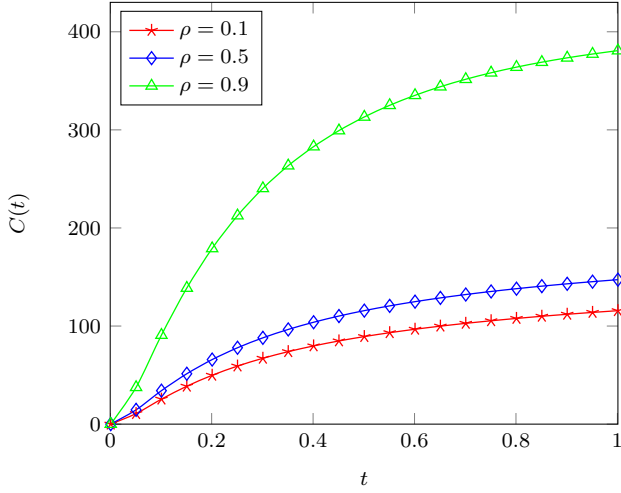


Fig. 1. Convergence of  $M(t)$  towards  $M_0$  when  $t \rightarrow 0$  for  $N = m + 1 = 51$ . The criterion used is the MSE:  $C(t) = E[\|M(t) - M_0\|_F^2]$ .

To illustrate Theorem II.2, Figure 1 presents the mean square error  $C(t) \triangleq E[\|M(t) - M_0\|_F^2]$  between Tyler's  $M$ -estimator and the Student-t MLE versus the parameter  $t$ , called the degree of freedom of the multivariate Student-t distribution [14], defined through the weight function  $u(t, x) = \frac{m+t}{t+x}$ . We take here  $N = m + 1 = 51$ . The data are zero-mean Gaussian distributed with Toeplitz covariance matrix, the  $(i, j)$  entry of which is equal to  $\rho^{|i-j|}$ , for some  $\rho \in (0, 1)$ . As proved in Theorem II.2, Item (A) is illustrated in the case where  $N = m + 1$  while Item (B) is illustrated for the Student-t MLE for different population covariance matrices.

### III. PROOF OF THEOREM II.2

The strategy of the proof is as follows: for every  $t > 0$ , we first build a positive functional  $H(t, \cdot)$  over  $\text{PSD}_m$  whose critical points (if any) are exactly the solutions of  $(\text{Eq})_t$ . To establish the existence of such critical points, we show that  $H(t, \cdot)$  is uniformly bounded and tends to zero at the boundary of  $\text{PSD}_m$ . To obtain uniqueness, we show that solutions of  $(\text{Eq})_t$  are all local strict maxima of  $H(t, \cdot)$  and conclude by applying the mountain pass theorem (cf. [16]). This gives Item (A). Item (B) is then obtained using the implicit function theorem and some limiting arguments.

For  $t > 0$ , we define the function

$$h : \mathbf{R}_+^* \times \mathbf{R}_+ \rightarrow \mathbf{R}_+^* \\ (t, x) \mapsto e^{-\frac{1}{m} \int_{x_t}^x u(t, y) dy}. \quad (4)$$

Then  $-\frac{h_x}{h} = \frac{u}{m}$  and  $h(t, x_t) = 1$ . Set  $h(0, x) = \frac{1}{x}$  for  $x > 0$  and  $g : \mathbf{R}_+^* \times \mathbf{R}_+ \rightarrow \mathbf{R}_+^*$  with  $g(t, x) = xh(t, x)$ .

In the case where  $u = \bar{u}$ ,  $\forall (t, x) \in (\mathbf{R}_+)^2 \setminus \{(0, 0)\}$ ,  $x_t \equiv 1$ ,  $\bar{h}(t, x) = \left(\frac{1+t}{x+t}\right)^{1+t}$ ,  $\bar{g}(t, x) = x \left(\frac{1+t}{x+t}\right)^{1+t}$ .

Then, define the functional  $H(t, \cdot)$  as

$$H : \mathbf{R}_+^* \times \text{PSD}_n \rightarrow \mathbf{R}_+^* \\ (t, \mathbf{M}) \mapsto \frac{\prod_{i=1}^N h(t, \mathbf{y}_i^T \mathbf{M}^{-1} \mathbf{y}_i)^m}{(\det \mathbf{M})^N} \quad (5)$$

as well as the functional considered in [10]

$$B : \text{PSD}_n \rightarrow \mathbf{R}_+^* \\ \mathbf{M} \mapsto \frac{\prod_{i=1}^N h(0, \mathbf{y}_i^T \mathbf{M}^{-1} \mathbf{y}_i)^m}{(\det \mathbf{M})^N}. \quad (6)$$

**Lemma III.1** For  $t > 0$  and  $\mathbf{M} \in \text{PSD}_m$ , one has  $-\mathbf{M}H_x(t, \mathbf{M})\mathbf{M}/N H(t, \mathbf{M}) = \mathbf{M} - \frac{1}{N} \sum_{i=1}^N u(t, \mathbf{y}_i^T \mathbf{M}^{-1} \mathbf{y}_i)$ , with  $H_x(t, \mathbf{M})$  the gradient of  $H(t, \cdot)$ . In particular,  $\mathbf{M}$  is a solution of  $(\text{Eq})_t$  if and only if  $\mathbf{M}$  is a critical point of  $H(t, \cdot)$ .

**Lemma III.2**  $\forall t > 0, \mathbf{M} \in \text{PSD}_m, H(t, \mathbf{M}) \leq B(\mathbf{M})$ . As a consequence,  $\lim_{\mathbf{M} \rightarrow \partial \text{PSD}_m} H(t, \mathbf{M}) = 0$ , so that  $H(t, \cdot)$  admits critical points.

*Proof of Lemma III.2.* An immediate calculus yields that  $x \mapsto g(t, x)$  reaches its maximum 1 at  $x = x_t$ . As a consequence, for  $t > 0, \mathbf{M} \in \text{PSD}_m, H(t, \mathbf{M}) \leq B(\mathbf{M})$ . Moreover,  $\lim_{x \rightarrow 0, \infty} g(t, x) = \lim_{x \rightarrow 0, \infty} xh(t, x) = 0$ . For the limit at  $x = 0$ , this is obvious. For  $x \rightarrow \infty$ , note that  $\ln(g(t, x)) = \frac{1}{m} \int_{x_t}^x \frac{m-yu(t, y)}{y} dy$  and, since  $m - l_t < 0$ , it is equivalent to  $(m - l_t) \ln(x)$  as  $x \rightarrow \infty$ . Consider now a sequence  $(\mathbf{M}_k)_{k \geq 0}$  in  $\text{PSD}_m$  converging to  $\partial \text{PSD}_m$ . For  $k \geq 0$ , set  $\mathbf{M}_k = \rho_k \mathbf{N}_k$  with  $\rho_k = \|\mathbf{M}_k\|$  and  $\mathbf{N}_k = \frac{\mathbf{M}_k}{\rho_k}$ . Note that  $\partial \text{PSD}_m$  is made of matrices either non invertible or with norm going to infinity. Therefore, up to subsequences, either (i)  $(\mathbf{N}_k)_{k \geq 0}$  converges itself to  $\partial \text{PSD}_m$  or (b) the sequence  $(\rho_k)_{k \geq 0}$  converges to zero or infinity and there exists  $\exists \alpha > 0, \forall k \geq 0, \mathbf{N}_k \geq \alpha \mathbf{I}_m$ . If Case (i) occurs, then  $\forall k \geq 0, H(t, \mathbf{M}_k) \leq B(\mathbf{N}_k)$ , which tends to zero as  $k \rightarrow \infty$  (cf. [10]). In Case (ii),

$$H(t, \mathbf{M}_k) = \frac{\prod_{i=1}^N h(t, x_{i,k})^m}{\rho_k^N \det(\mathbf{N}_k)^N} = B(\mathbf{N}_k) \prod_{i=1}^N g(t, x_{i,k})^m$$

where  $x_{i,k} = \mathbf{y}_i^T \mathbf{N}_k^{-1} \mathbf{y}_i / \rho_k$ . As  $k \rightarrow \infty, x_{i,k}$  tends either to zero or infinity and we conclude. For  $t > 0, H(t, \cdot)$  is uniformly bounded over  $\text{PSD}_m$  since  $B(\cdot)$  is. So  $H(t, \cdot)$  has a global maximum which must belong to  $\text{PSD}_m$  since  $H(t, \mathbf{M}) \rightarrow 0$  as  $\mathbf{M}$  tends to the boundary of  $\text{PSD}_m$ . So  $H(t, \cdot)$  admits critical points. ■

**Lemma III.3** Let  $t > 0$ . Then all critical points of  $H(t, \cdot)$  are local strict maxima.

*Proof of Lemma III.3.* We show that, if  $\mathbf{M}$  is a critical point then the Hessian of  $H(t, \cdot)$  at  $\mathbf{M}$  is a negative definite quadratic form implying that  $\mathbf{M}$  is a local strict maximum of  $H(t, \cdot)$ . Let  $\mathbf{M} \in \text{PSD}_m$  be a critical point of  $H(t, \cdot)$ . Then, one gets that for every  $\mathbf{Q} \in \text{Sym}_m$ ,

$$\langle \mathbf{Q}, \text{Hess}_{\mathbf{M}}(\mathbf{Q}) \rangle = -NH(t, \mathbf{M}) \left[ \langle \mathbf{Q}, \mathbf{M}^{-1} \mathbf{Q} \mathbf{M}^{-1} \rangle + \frac{1}{N} \sum_{i=1}^N u_x(t, \mathbf{y}_i^T \mathbf{M}^{-1} \mathbf{y}_i) (\mathbf{y}_i^T \mathbf{M}^{-1} \mathbf{Q} \mathbf{M}^{-1} \mathbf{y}_i)^2 \right].$$

Let  $\mathbf{R} := \mathbf{M}^{-1/2} \mathbf{Q} \mathbf{M}^{-1/2}$  and  $\mathbf{d}_i := \mathbf{M}^{-1/2} \mathbf{y}_i$ , one has

$$-\frac{\langle \mathbf{Q}, \text{Hess}_{\mathbf{M}}(\mathbf{Q}) \rangle}{NH(t, \mathbf{M})} = \|\mathbf{R}\|^2 + \frac{1}{N} \sum_{i=1}^N u_x(t, \|\mathbf{d}_i\|^2) (\mathbf{d}_i^T \mathbf{R} \mathbf{d}_i)^2. \quad (7)$$

Recall that  $\mathbf{M}$  is a critical point of  $H(t, \cdot)$  and thus a solution of  $(\text{Eq})_t$ , i.e.,

$$\mathbf{I}_m = \frac{1}{N} \sum_{i=1}^N u(t, \|\mathbf{d}_i\|^2) \mathbf{d}_i \mathbf{d}_i^T. \quad (8)$$

Multiplying (8) by  $\mathbf{R}$  on both left and right, taking the trace and plugging the result into (7) gives

$$(7) = \frac{1}{N} \sum_{i=1}^N u(t, \|\mathbf{d}_i\|^2) \|\mathbf{R} \mathbf{d}_i\|^2 + u_x(t, \|\mathbf{d}_i\|^2) (\mathbf{d}_i^T \mathbf{R} \mathbf{d}_i)^2.$$

Let  $I_{\mathbf{Q}} = \{i \in \{1, \dots, N\}, \mathbf{R} \mathbf{d}_i \neq 0\}$ . Then

$$(7) = \frac{1}{N} \sum_{i \in I_{\mathbf{Q}}} \|\mathbf{R} \mathbf{d}_i\|^2 [u(t, \|\mathbf{d}_i\|^2) + \|\mathbf{d}_i\|^2 u_x(t, \|\mathbf{d}_i\|^2) r_i]$$

where  $r_i := (\mathbf{d}_i^T \mathbf{R} \mathbf{d}_i / [\|\mathbf{d}_i\| \|\mathbf{R} \mathbf{d}_i\|])^2$ . Using  $0 \leq r_i \leq 1$  (by Cauchy-Schwarz's inequality) and  $u_x \leq 0$  (since  $u$  is of class  $C^1$  and verifies (U1)), we have  $r_i u_x(\cdot, \cdot) \geq u_x(\cdot, \cdot)$ . Then, recalling that  $v(t, x) = xu(t, x)$ ,

$$(7) \geq \frac{1}{N} \sum_{i \in I_{\mathbf{Q}}} \|\mathbf{R} \mathbf{d}_i\|^2 v_x(t, \|\mathbf{d}_i\|^2) \geq 0.$$

Moreover, if  $\mathbf{Q} \neq 0$ ,  $I_{\mathbf{Q}} \neq \emptyset$  and there exists  $\bar{i}$  such that  $v_x(t, \|\mathbf{d}_{\bar{i}}\|^2) > 0$ . Therefore  $\langle \mathbf{Q}, \text{Hess}_{\mathbf{M}}(\mathbf{Q}) \rangle < 0$ , i.e.,  $\text{Hess}_{\mathbf{M}}$  is negative definite, concluding the proof. ■

**Lemma III.4** *Let  $t > 0$ . Then  $(\text{Eq})_t$  admits a unique solution,  $\mathbf{M}(t)$ , the unique strict maximum of  $H(t, \cdot)$ .*

*Proof of Lemma III.4.* We reason by contradiction assuming  $H(t, \cdot)$  admits at least two local strict maxima. Applying the mountain-pass theorem [16] to the functional  $1/H(t, \cdot)$  which tends to infinity in the vicinity of  $\partial \text{PSD}_m$ , we obtain the existence of a saddle point of  $\mathbf{F}$  in  $\text{PSD}_m$  which is contradictory to Lemma III.3. ■

We next prove that  $\mathbf{M}(t)$  is uniformly bounded in  $\text{PSD}_m$  as  $t \rightarrow 0$ , i.e.

**Lemma III.5** *There exists  $0 < a \leq b$  and  $t_0 > 0$  such that, for every  $t \in (0, t_0)$ ,  $a\mathbf{I}_m \leq \mathbf{M}(t) \leq b\mathbf{I}_m$ .*

*Proof of Lemma III.5.* Let  $\mathbf{P}$  be the unique matrix of  $\text{PSD}_m$  satisfying  $B(\mathbf{P}) = \max_{\mathbf{M} \in \text{PSD}_m} B(\mathbf{M})$  and  $\text{Tr}(\mathbf{M}) = m$ . Then, for every  $t > 0$ ,  $H(t, \mathbf{P}) \leq H(t, \mathbf{M}(t))$  and  $B(\mathbf{M}(t)) \leq B(\mathbf{P})$ . Multiplying both inequalities, after simplifications, we get  $\prod_{i=1}^N g(t, \mathbf{y}_i^T \mathbf{P}^{-1} \mathbf{y}_i) \leq \prod_{i=1}^N g(t, \mathbf{y}_i^T \mathbf{M}(t)^{-1} \mathbf{y}_i) \leq 1$ , with  $\prod_{i=1}^N g(t, \mathbf{y}_i^T \mathbf{P}^{-1} \mathbf{y}_i) \rightarrow 1$  as  $t \rightarrow 0$ . So there exists  $t_0 > 0$  such that, for every  $t \in (0, t_0)$  and  $1 \leq i \leq N$ ,  $1/2 \leq g(t, \mathbf{y}_i^T \mathbf{M}(t)^{-1} \mathbf{y}_i)$ , and, since (U3) holds true, there exists  $0 < a \leq b$  s.t. for every  $t \in (0, t_0)$  and  $1 \leq i \leq N$ ,  $a \leq \mathbf{y}_i^T \mathbf{M}(t)^{-1} \mathbf{y}_i \leq b$ . This implies that, for every  $t \in (0, t_0)$  and  $1 \leq i \leq N$ ,  $u(t, b) \leq u(t, \mathbf{y}_i^T \mathbf{M}(t)^{-1} \mathbf{y}_i) \leq u(t, a)$ , hence  $u(t, b)\mathbf{C} \leq \mathbf{M}(t) \leq u(t, a)\mathbf{C}$  with  $\mathbf{C} := \frac{m}{N} \sum_{i=1}^N \mathbf{y}_i \mathbf{y}_i^T$ . One concludes easily. ■

**Lemma III.6** *Under the conditions of Theorem II.2,  $\lim_{t \rightarrow 0} \mathbf{M}(t) = \mathbf{M}_0$  solution of  $(\text{Eq})_0$  given by  $\mathbf{M}_0 = \xi_u \mathbf{P}$ , where  $\xi_u > 0$  is the unique solution of (3).*

*Proof of Lemma III.6.* Since  $\mathbf{M}(\cdot)$  is uniformly bounded in  $\text{PSD}_m$  as  $t \rightarrow 0$ , its accumulation points still belong to  $\text{PSD}_m$  and are necessarily of the form  $\mu \mathbf{P}$  where  $\mu > 0$  and  $\mathbf{P}$  is the solution of  $(\text{Eq})_0$  with trace  $m$ . Taking the trace in (8), one gets  $m = \frac{1}{N} \sum_{i=1}^N v(t, \|\mathbf{d}_i(t)\|^2)$ , where  $\mathbf{d}_i(t) = \mathbf{M}(t)^{-1/2} \mathbf{y}_i$  for  $1 \leq i \leq N$ . Using (1) and (U3), one deduces that, for every  $t > 0$ ,  $\sum_{i=1}^N v_1(\|\mathbf{d}_i(t)\|^2) + t \sum_{i=1}^N w_1(\|\mathbf{d}_i(t)\|^2) + o(t) = 0$ . Consider an accumulation point  $\mu \mathbf{P}$  of  $\mathbf{M}(\cdot)$  as  $t \rightarrow 0$ . Then, up to a subsequence,  $\lim_{t \rightarrow 0} \mathbf{M}(t) = \mu \mathbf{P}$  and, for  $1 \leq i \leq N$ ,  $\lim_{t \rightarrow 0} \mathbf{d}_i(t) = \mathbf{P}^{-1/2} \mathbf{y}_i / \sqrt{\mu}$ . According to (U3), the second sum in the previous equation tends to zero as  $t \rightarrow 0$  and we are left with  $\sum_{i=1}^N v_1(\mathbf{y}_i^T \mathbf{P}^{-1} \mathbf{y}_i / \mu) = 0$ . Since the left-hand side of the latter defines a decreasing function of  $\mu$ , it has a unique solution denoted  $\xi_u > 0$ , which concludes the proof since  $\mathbf{M}(\cdot)$  admits a unique accumulation point as  $t \rightarrow 0$ . ■

## IV. CONCLUSIONS

In this paper, an alternative proof for existence and uniqueness for the Maronna's  $M$ -estimators is provided. More importantly, using this particular approach leads to draw some connections between Maronna's and Tyler's estimators by expressing (properly scaled) Tyler's estimator in terms of a limit of a class of Maronna's estimators. This result may also find interest in studies of Tyler's  $M$ -estimator in the large random matrix regime.

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