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# Stability Analysis by means of Discrete Abstraction. Application to Voltage Stability of Distributed Generators <sup>\*</sup>

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**Abstract:** This paper proposes a stability analysis method for discrete-time piecewise affine systems based on the reachability study of the equivalent discrete abstraction. The finite discrete abstraction eases reachability analysis but does not allow to conclude on system instability. To be able to conclude, this abstraction is refined by bisimulation calculation until the system is found to be unstable or the discrete abstraction is found to be equivalent to the system. This method allows to conclude both on the stability and the instability of the system, avoiding the undecidable situations. This approach is illustrated on voltage stability of a distribution feeder hosting distributed generators equipped with piecewise affine control laws.

*Keywords:* Hybrid systems, Transition systems, Nonlinear systems, Piecewise linear analysis, Electric power systems, Reactive power, Voltage stability.

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## 1. INTRODUCTION

Nowadays, many engineering systems can be represented as hybrid systems as they combine continuous dynamics – e.g.: dynamics of a physical system – and discrete dynamics – e.g.: of a computer based system such as controllers. Piecewise affine systems arise often as components such as dead-band, saturation, relays or hysteresis are widely used in control. Piecewise affine hybrid systems were introduced by Sontag (1981) as discrete automata with continuous-time affine dynamics in each mode.

To study stability of such systems, trajectories are analyzed looking for particular ones switching between several modes and thus leading to unstable behavior. This problem is generally  $\mathcal{NP}$ -complete or undecidable (Blondel and Tsitsiklis, 1999). Thus, only practical stability tests have been developed to overcome this difficulty. Johansson and Rantzer (1998) propose a method to construct a piecewise continuous Lyapunov function to prove stability of piecewise linear systems. By solving linear matrix inequalities, Rantzer and Johansson (2000) construct piecewise

quadratic Lyapunov functions to investigate stability of a continuous piecewise affine system.

Considering discrete-time piecewise affine systems, transitions between non-adjacent regions of the state-space can occur. Ferrari-Trecate et al. (2002) take it into account and construct Lyapunov functions by solving linear matrix inequalities to obtain sufficient conditions for discrete-time system stability. Cavichioli Gonzaga et al. (2011) propose a Lur’e type Lyapunov function to study stability of discrete-time switched systems interconnected with a switched nonlinearity. All these approaches provide only sufficient conditions for stability and so do not allow to conclude on system instability.

In this paper, we propose to study the stability of a discrete-time piecewise affine system using bisimulation relations. This notion has been introduced by Milner (1989) for continuous transition systems. Later work has extended this notion to hybrid systems (Haghverdi et al., 2005). In the present paper, the discrete abstraction of a discrete-time piecewise affine system is constructed. Then, the equivalent representation is obtained by bisimulation algorithms (Alur et al., 2000). With this equivalent finite discrete abstraction, reachability analysis can be conducted easily and allows to conclude on both stability and instability of the hybrid system. The main contribution of

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this work is to conclude on system instability and in this case to be able to identify dangerous operating regions i.e. involved in – or leading to – cycles.

First of all, a discrete-time piecewise affine autonomous system is described and modeled as a hybrid transition system in Section 2. Then, Section 3 details the construction of an equivalent discrete abstraction obtained with bisimulation algorithms and the analysis of its stability. In Section 4, this method is applied to a real case study: stability of a distribution feeder hosting a distributed generator equipped with a dead-band control law is discussed. Conclusions and perspectives are presented in Section 5.

## 2. HYBRID SYSTEM MODELING

The goal of this study is to analyze the stability of a system modeled as an autonomous, discrete-time system with piecewise affine dynamics. Let  $x_k \in \mathcal{X}$  be the discrete-time vector of state-space variables of this system at time  $t = kT_S$ , where  $T_S$  is the sample time and  $\mathcal{X}$  be the state-space. Let  $n_R \in \mathbb{N}$  be the number of affine operating regions of the piecewise affine dynamics. Each operating mode is associated to a domain of validity  $(\mathcal{S}_i)_{i \in \mathbb{I}}$  defined as a polyhedron:

$$\mathcal{S}_i = \{x_k \in \mathcal{X} \mid K_i x_k \leq L_i\} \quad \forall i \in \mathbb{I}, \quad (1)$$

where  $\mathbb{I} = \{1, \dots, n_R\}$ . As the system is piecewise affine, all the polyhedra  $(\mathcal{S}_i)_{i \in \mathbb{I}}$  are disjoint. In each of these polyhedra, the system behavior is locally described by an affine dynamical equation:

$$x_{k+1} = A_i x_k + b_i \quad \forall x_k \in \mathcal{S}_i \quad \forall i \in \mathbb{I}. \quad (2)$$

This description of the system leads to construct a hybrid transition system  $S_H$  as defined by Alur et al. (2000). This system  $S_H$  can be defined from a set  $\Omega$  of discrete states  $(q_i)_{i \in \mathbb{I}}$  corresponding to the  $n_R$  operating modes. The invariant function of  $S_H$  is defined such as:

$$\text{Inv} : \Omega \rightarrow 2^{\mathcal{X}} \\ q_i \mapsto \text{Inv}(q_i). \quad (3)$$

This function maps the domain of validity  $\mathcal{S}_i \subset \mathcal{X}$  to the corresponding discrete state  $q_i \in \Omega$ , thus:

$$\text{Inv}(q_i) = \{x \in \mathcal{X} \mid K_i x \leq L_i\}. \quad (4)$$

The functions  $A$  and  $B$  map the discrete-time dynamical matrices of a given operating mode  $i \in \mathbb{I}$  to the corresponding discrete state  $q_i \in \Omega$ :

$$A : \Omega \rightarrow \mathcal{A} \quad \text{and} \quad B : \Omega \rightarrow \mathcal{B} \\ q_i \mapsto A(q_i) \quad \text{and} \quad q_i \mapsto B(q_i), \quad (5)$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are the sets of dynamical matrices  $(A_i)_{i \in \mathbb{I}}$  and  $(b_i)_{i \in \mathbb{I}}$  defined in (2). So the hybrid transition system corresponding to the studied system can be defined by the following tuple:

$$S_H = \langle \Omega, \text{Inv}, A, B \rangle \quad (6)$$

To study the stability of  $S_H$ , its trajectories have to be analyzed. The trajectory starting from  $x_0 \in \mathcal{X}$  is the set of state vectors  $x_k \in \mathcal{X}$  and corresponding discrete states  $q_{i_k} \in \Omega$  at each time step. It can be noted:

$$(q_{i_0}, x_0) \rightarrow \dots \rightarrow (q_{i_k}, x_k) \rightarrow (q_{i_{k+1}}, x_{k+1}) \rightarrow \dots \quad (7)$$

where, for all  $k \in \mathbb{N}$ :

$$\begin{cases} i_k \in \mathbb{I} \text{ such as } x_k \in \text{Inv}(q_{i_k}) \\ x_{k+1} = A(q_{i_k})x_k + B(q_{i_k}) \end{cases}. \quad (8)$$

It can be noticed that a trajectory is finite if it exists  $n \in \mathbb{N}$  such as  $(q_{i_n}, x_n) = (q_{i_{n+1}}, x_{n+1})$ . Otherwise, the trajectory

is infinite. The goal is to study the stability of the hybrid system  $S_H$ . Thus, the stability of every possible trajectory has to be analyzed.

First of all, let us define several notions useful to study stability such as successors and predecessors. To each state  $x \in \mathcal{X}$ ,  $\text{Post}_{S_H}(x)$  defines the successor of  $x$ , i.e. the state of  $\mathcal{X}$  reachable from  $x$  in one time step:

$$\text{Post}_{S_H}(x) = A(q)x + B(q), \text{ with } q \in \Omega : x \in \text{Inv}(q). \quad (9)$$

By extension, the successor of a polyhedron  $d \subseteq \mathcal{X}$  can be defined as follows:

$$\text{Post}_{S_H}(d) = \bigcup_{x \in d} \text{Post}_{S_H}(x). \quad (10)$$

Let  $\text{Post}_{S_H}^n(x)$  define, for all  $n \in \mathbb{N}$ , the state of  $\mathcal{X}$  reachable from  $x$  in  $n$  time steps.

Similarly, the predecessor of  $x \in \mathcal{X}$  can be defined:

$$\text{Pre}_{S_H}(x) = \{x' \in \mathcal{X} \mid x = A(q)x' + B(q), \\ \text{with } q \in \Omega, x' \in \text{Inv}(q)\} \quad (11)$$

By extension, the predecessor of a polyhedron  $d \subseteq \mathcal{X}$  is defined in a similar way as (10).

Based on this, the system  $S_H$  is considered to be stable if for all initial conditions  $x_0 \in \mathcal{X}$ , none of the trajectories of  $S_H$  ends into a cycle. This can be formalized as follows:

$$\forall x \in \mathcal{X}, \exists n \in \mathbb{N} : \forall k \geq n, \text{Post}_{S_H}^k(x) \in \text{Inv}(q_{i_n}). \quad (12)$$

A hybrid transition system satisfying this property does not present cycles in between several discrete states. Then, stability of trajectories remaining in the same discrete state after a given number of time steps can be easily studied as their dynamics are affine.

As this hybrid transition system is defined over a continuous state-space  $\mathcal{X}$ , it presents an infinite number of trajectories. Thus stability analysis is difficult to conduct on  $S_H$ . To overcome this difficulty, a finite discrete abstraction  $S_D$  of the hybrid transition system  $S_H$  is constructed below.

## 3. DISCRETE ABSTRACTION AND STABILITY ANALYSIS

To analyze the stability of the hybrid system  $S_H$  in a finite number of steps, a finite discrete system is abstracted from  $S_H$  while preserving stability properties (Alur et al., 2000).

### 3.1 Discrete Transition System

Let  $P_a$  be a partition of the state-space  $\mathcal{X}$  and  $\Phi : P_a(\mathcal{X}) \rightarrow \mathfrak{M}$  a bijection associating to each polyhedron  $d \subset P_a(\mathcal{X})$  a discrete state  $m \in \mathfrak{M}$ . The discrete transition function  $T : \mathfrak{M}^2 \rightarrow \{0, 1\}$  is defined, for all  $(m, m') \in \mathfrak{M}^2$  by:

$$T(m, m') = 1 \Leftrightarrow \exists x \in \Phi^{-1}(m) : \text{Post}_{S_H}(x) \in \Phi^{-1}(m') \quad (13)$$

Then, a discrete transition system  $S_D$  can be constructed as follows:

$$S_D = \langle \mathfrak{M}, T \rangle. \quad (14)$$

A discrete transition system is usually represented as a finite graph with discrete states as nodes and discrete transitions as edges (Alur et al., 2000). Thanks to the finite number of states of  $S_D$ , the stability analysis in the sense of (12) is easier to conduct on a finite discrete transition system than on a hybrid transition system.

### 3.2 Discrete Abstraction Definition

To create a partition  $P_a$  of the state-space  $\mathcal{X}$  corresponding to the  $n_R$  operating modes of the hybrid system defined by (1) and (2),  $P_a(\mathcal{X})$  is chosen as follows:

$$P_a(\mathcal{X}) = \bigcup_{i \in \mathbb{I}} \text{Inv}(q_i). \quad (15)$$

The discrete state associated to each polyhedron of  $P_a(\mathcal{X})$  is defined by the function  $\Phi$ :

$$\begin{aligned} \Phi : P_a(\mathcal{X}) &\rightarrow \mathfrak{M} \\ \text{Inv}(q_i) &\mapsto m_i, \end{aligned} \quad (16)$$

where  $\mathfrak{M}$  is the set of discrete states  $(m_i)_{i \in \mathbb{I}}$ . The discrete transition function  $T : \mathfrak{M}^2 \rightarrow \{0, 1\}$  is defined by (13). With these definitions, the discrete abstraction  $S_D$  of the hybrid transition system  $S_H$  can be constructed following (14).

To test the stability of  $S_H$ , the trajectories of  $S_D$  are studied. By construction, all trajectories of  $S_H$  correspond to a trajectory of  $S_D$ . However, among all possible discrete trajectories, some of them may not correspond to any trajectory of  $S_H$ . This is why stability of  $S_D$  implies stability of  $S_H$  but the existence of cycling trajectories in  $S_D$  makes it impossible to conclude on the hybrid system stability. To have an equivalent relationship between the stability of  $S_H$  and  $S_D$ , all discrete states  $m_i \in \mathfrak{M}$  must satisfy:

$$\exists m_j \in \mathfrak{M} : \forall x \in \Phi^{-1}(m_i), \text{Post}_{S_H}(x) \in \Phi^{-1}(m_j) \quad (17)$$

It can be noticed that this is equivalent to say that all discrete states  $m_i \in \mathfrak{M}$  must define a single transition. In this way, all discrete trajectories are equivalent to trajectories of  $S_H$  so the existence of a cycle in the discrete abstraction is equivalent to the existence of a cycle in  $S_H$ . The discrete abstraction satisfying (17) is called the bisimulation of the hybrid transition system  $S_H$  (Kanellakis and Smolka, 1990) and denoted by  $S_D^B$ . If the initial finite discrete abstraction  $S_D$  is stable, then there is no need to construct  $S_D^B$  but, if not,  $S_D^B$  is needed to conclude on the instability of  $S_H$ .

### 3.3 Bisimulation Calculation

The aim is to obtain a set  $\mathfrak{M}^B$  of discrete states constructed from a partition  $P_a^B(\mathcal{X})$  such that all discrete states of  $\mathfrak{M}^B$  satisfy (17).

*Initialization.*  $P_a(\mathcal{X})$ ,  $\Phi$ ,  $\mathfrak{M}$ ,  $T$ , and  $S_D$  – as defined in the previous subsection – are used as initial discrete abstraction of the calculation. From here, they are denoted  $P_a^0(\mathcal{X})$ ,  $\Phi^0$ ,  $\mathfrak{M}^0$ ,  $T^0$ , and  $S_D^0$ . Assuming that  $\mathfrak{M}^0$  does not satisfy (17), and that it exists at least one cycling trajectory of  $S_D^0$ , the partition  $P_a^0(\mathcal{X})$  has to be iteratively refined in order to be able to build a set of discrete states satisfying (17).

*Iterations.* Let us note  $P_a^n$  the partition of  $\mathcal{X}$  obtained from  $P_a^0$  after  $n$  iterations of the bisimulation calculation. Similarly,  $\Phi^n$ ,  $\mathfrak{M}^n$ , and  $T^n$  are defined from  $\Phi^0$ ,  $\mathfrak{M}^0$ , and  $T^0$ . At each iteration, every domain  $d \subset P_a^n(\mathcal{X})$  corresponding to a discrete state  $m = \Phi^n(d)$  having several successors is divided into sub-domains such that each sub-domain leads to a single domain in one time step.

Let  $d_i^n \subset P_a^n(\mathcal{X})$  be a domain which corresponding discrete state  $m_i^n = \Phi^n(d_i^n) \in \mathfrak{M}^n$  has several possible destinations and  $S_{d_i^n}$  be the set of successors of  $d_i^n$ . Let  $d_j^n = \Phi^{n-1}(m_j^n)$  be a domain of  $S_{d_i^n}$ . Then,  $d_1 = d_i^n \cap \text{Pre}_{S_H}(d_j^n)$  represents the domain of  $d_i^n$  leading to  $d_j^n$  in one time step. It can be noticed that if  $d_1 = \emptyset$  then  $d_j^n$  is not a direct successor of  $d_i^n$ . On the other hand, if  $d_1 = d_i^n$ , then  $d_j^n$  is the only successor of  $d_i^n$ . Otherwise,  $d_i^n$  has several successors including  $d_j^n$ . Then, the remaining part  $d_2 = d_i^n - d_1$  of  $d_i^n$  is divided according to its successors.

At the end,  $d_i^n$  has been divided into as many sub-domains as  $d_i^n$  successors. It can be noticed that, by construction, the sub-domains of  $d_i^n$  satisfy (17). This calculation is done for every discrete state  $m^n \in \mathfrak{M}^n$  and thus  $P_a^{n+1}(\mathcal{X})$  is constructed. Then  $\Phi^{n+1}$  and  $\mathfrak{M}^{n+1}$  are also constructed. Note that  $T^{n+1}$  is not constructed from (13) but from  $T^n$  at each division of a domain as shown in Algorithm 1. Then,  $S_D^{n+1} = \langle \mathfrak{M}^{n+1}, T^{n+1} \rangle$  is constructed.

To conclude, an iteration of the bisimulation algorithm can be formulated as follows:

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#### Algorithm 1 Bisimulation Iteration

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 $P_a^{n+1} = P_a^n$ 
for all  $d_i^n \subset P_a^n(\mathcal{X})$  do
   $S_{d_i^n} \leftarrow \{d \subset P_a^n(\mathcal{X}) \mid T^n(\Phi^n(d_i^n), \Phi^n(d)) = 1\}$ 
  for  $j = 1 : \text{card}(S_{d_i^n})$  do
     $d_j \leftarrow d_i^n \cap \text{Pre}_{S_H}(S_{d_i^n}(j))$ 
     $T^{n+1}(\Phi^{n+1}(d_j), \Phi^{n+1}(S_{d_i^n}(j))) \leftarrow 1$ 
    for all  $d_k^n \subset P_a^n(\mathcal{X})$  do
      if  $T^n(\Phi^n(d_k^n), \Phi^n(d_i^n)) = 1$  then
         $T^{n+1}(\Phi^{n+1}(d_k^n), \Phi^{n+1}(d_i^n)) \leftarrow 1$ 
      end if
    end for
  end for
end for
 $n \leftarrow n + 1$ 

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It can be noticed that  $\text{Pre}_{S_H}(d_j^n)$  with  $d_j^n \in P_a^n(\mathcal{X})$  is never explicitly computed. Indeed, the algorithm calculates sub-domains under the form of  $d_i^n \cap \text{Pre}_{S_H}(d_j^n)$  which can be written as a polyhedron as in (3):

$$\{x \in \mathcal{X} \mid K_i x \leq L_i \text{ and } K_j(A_i x + b_i) \leq L_j\}. \quad (18)$$

So every sub-domain is easy to construct as it is defined by linear constraints.

If  $\mathfrak{M}^{n+1}$  satisfies (17), then the bisimulation calculation has converged. Let  $P_a^B$  be the partition of  $\mathcal{X}$  obtained after convergence. Similarly,  $\Phi^B$ ,  $\mathfrak{M}^B$ ,  $T^B$ , and  $S_D^B$  are defined.

*Discussion on convergence.* The convergence of this calculation is not guaranteed (Alur et al., 2000). However, even if the calculation does not converge, useful information can be obtained out of it. For instance, at each iteration, some discrete trajectories which do not correspond to a trajectory of the hybrid system are removed from the list of possible trajectories. If the discrete abstraction at the  $i$ -th iteration is found to be stable, then it can be concluded that  $S_H$  is also stable without continuing bisimulation calculation.

Moreover, in a partition before the convergence of the bisimulation algorithm, if a trajectory leads only to discrete states satisfying (17) – i.e. corresponding to a single discrete transition – then, none of the discrete states involved in this trajectory will be divided in any further iteration of the calculation. Thus, these discrete states belong to  $\mathfrak{M}^B$  and if this discrete trajectory defines a cycle, it can be concluded before convergence that  $S_H$  is unstable.

To conclude, bisimulation algorithm allows to construct the finite equivalent discrete abstraction  $S_D^B = \langle \mathfrak{M}^B, T^B \rangle$ . Its stability is equivalent to hybrid system stability as every discrete trajectory corresponds to a trajectory of  $S_H$ . In the next section, this method is illustrated on a real case study.

#### 4. APPLICATION TO A POWER SYSTEMS REAL CASE STUDY

##### 4.1 An example of voltage stability issue in power systems

With the increasing number of Distributed Generators (DGs), distribution networks have seen their voltage increase along distribution feeders hosting generation (Hadjisaid et al., 1999). To be able to maintain the voltage within specified limits while avoiding network reinforcement costs, Duval et al. (2009) proposes the use of a local control law of DGs reactive power ( $Q$ ) as a function of their voltage ( $U$ ). The role of  $Q(U)$  laws is to dynamically mitigate local voltage rise or drop by changing the DG reactive power set-point according to the measure of voltage at the DG node.

In the literature, several strategies of reactive power management have been studied. For instance, Duval et al. (2009) come up with a three-level  $Q(U)$  law (Fig. 1 (a)). A voltage droop  $Q(U)$  law (Fig. 1 (b)) and a dead-band  $Q(U)$  law (Fig. 1 (c)) are proposed by Witkowski et al. (2013). Fig. 1 (d) presents a voltage droop  $Q(U)$  law with hysteresis as proposed by Dugan et al. (2013).

Some of the proposed  $Q(U)$  laws have already been experimented. For instance, Witkowski et al. (2013), Beauné et al. (2014) and Esslinger and Witzmann (2013) present the results of field demonstrations. These studies have raised concerns about the negative impact of such laws on voltage stability. In order to generalize these empirical results, a formal analysis of voltage stability based on the proposed approach is carried out below.

##### 4.2 Real case study description

To illustrate the application of the methodology to the voltage stability study, the modeling of a real 124 nodes distribution feeder of the ERDF (“Électricité Réseau Distribution France”) network is presented. This feeder hosts about 300 kW of consumption and a single DG: a wind farm of 5 MW. This DG is considered to be equipped with one of the  $Q(U)$  laws presented in Fig. 1. Here, a dead-band  $Q(U)$  law is considered (Fig. 1 (c)).

The goal is to analyze the possible existence of cycling trajectories causing voltage instability. To do so, the system is modeled as an autonomous, discrete-time system with

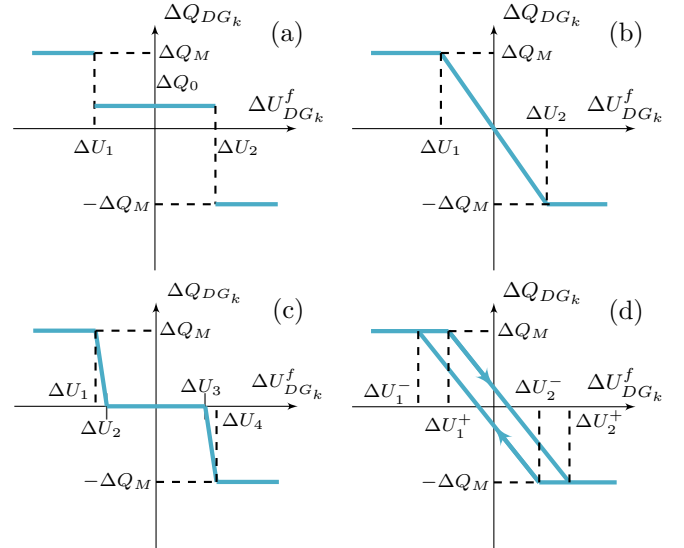


Fig. 1. Examples of reactive power strategies proposed in the literature: (a) three-level  $Q(U)$  law, (b) voltage droop  $Q(U)$  law, (c) dead-band  $Q(U)$  law, and (d) hysteresis voltage droop  $Q(U)$  law

piecewise affine dynamics (Fig. 2). To analyze voltage stability, the dynamics considered have a time constant about a few seconds (Kundur et al., 1994). Faster phenomena such as network line dynamics or power electronics transients are considered in steady-state and so are modeled by gains.

Network lines are represented by (Borghetti et al., 2010):

$$\Delta U_{DG_k} = K_Q \Delta Q_{DG_k} + \Delta U_{d_k}, \quad (19)$$

where  $K_Q$  represents the sensitivity coefficient of DG voltage magnitude deviation ( $\Delta U_{DG_k}$ ) with respect to DG reactive power deviation ( $\Delta Q_{DG_k}$ ). The impact of the uncontrolled variables variations – such as consumption, DG active power, etc. – are represented as disturbance magnitude variations ( $\Delta U_{d_k}$ ).

The DG behavior is considered to be equivalent to its control law behavior. The control law is composed of a filter to measure voltage variations and a dead-band  $Q(U)$  law. The measurement filter is modeled by a generic discrete-time state-space model ( $A_f, B_f, C_f, D_f$ ) as shown in Fig. 2. The state vector of the filter is  $x_k^f \in \mathcal{X}^f \subset \mathbb{R}^{n_x}$  where  $n_x$  is the dimension of this state-space vector. The input of the filter is the DG voltage magnitude deviation  $\Delta U_{DG_k} \in \mathcal{U} \subset \mathbb{R}$ . Its output is the filtered voltage deviation  $\Delta U_{DG_k}^f \in \mathcal{U}^f \subseteq \mathbb{R}$ . The filter is described by the following state-space equations:

$$\begin{cases} x_{k+1}^f = A_f x_k^f + B_f \Delta U_{DG_k} \\ \Delta U_{DG_k}^f = C_f x_k^f + D_f \Delta U_{DG_k} \end{cases} \quad (20)$$

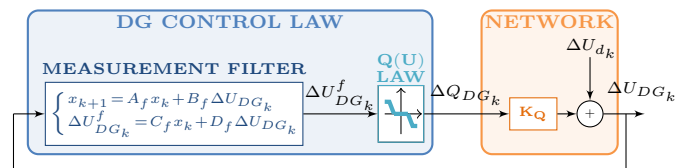


Fig. 2. Proposed model for the voltage stability study

The  $Q(U)$  law can be represented by a function from  $\mathcal{U}^f \rightarrow \mathbb{R}$  which associates to a filtered voltage deviation ( $\Delta U_{DG_k}^f$ ) a variation of the reactive power set-point ( $\Delta Q_{DG_k}$ ). This function is not linear but piecewise affine as it can be seen in Fig. 2. It presents five regions of affine operations:  $n_R = 5$ . The piecewise affine function is described as follows:

$$\Delta Q_{DG_k} = \alpha_i \Delta U_{DG_k}^f + \beta_i \text{ if } \Delta U_i^- \leq \Delta U_{DG_k}^f < \Delta U_i^+, \quad (21)$$

In the particular case of the dead-band  $Q(U)$  law shown in Fig. 1 (c), it can be noticed that:

$$\begin{cases} \alpha_1 = \alpha_3 = \alpha_5 = 0 \\ \alpha_2 = \alpha_4 \end{cases} \quad \text{and} \quad \begin{cases} \beta_1 = -\beta_2 = Q_M \\ \beta_3 = 0 \end{cases}, \quad (22)$$

and:

$$\begin{cases} \Delta U_i^+ = \Delta U_{i+1}^- = \Delta U_i \quad \forall i \in \{1, n_R - 1\} \\ \Delta U_1^- = \Delta U_{DG_{min}}^f \\ \Delta U_{n_R}^+ = \Delta U_{DG_{max}}^f \end{cases}, \quad (23)$$

with  $\Delta U_{DG_{min}}^f$  and  $\Delta U_{DG_{max}}^f$  such as:

$$\mathcal{U}^f = \left[ \Delta U_{DG_{min}}^f, \Delta U_{DG_{max}}^f \right]. \quad (24)$$

To represent both the filter state ( $x_k^f \in \mathcal{X}^f$ ) and the network influence ( $\Delta U_{d_k} \in \mathcal{U}_d \subset \mathbb{R}$ ), an extended discrete time state-space representation is chosen introducing a new state vector  $x_k \in \mathcal{X} \subset \mathbb{R}^{n_x+1}$  as follows:

$$x_k = \begin{bmatrix} x_k^f \\ \Delta U_{d_k} \end{bmatrix}. \quad (25)$$

Please note that network dynamics are neglected in this paper i.e. for all  $k \in \mathbb{N}$ ,  $\Delta U_{d_k} = \Delta U_d$ .

### 4.3 Hybrid Transition System

In each mode  $i \in \mathbb{I}$ ,  $x_k$  has a discrete-time affine dynamics that can be expressed from the equations describing the model shown in Fig. 2 i.e. by (19), (20) and (21):

$$\begin{cases} x_{k+1}^f = A_f x_k^f + B_f \Delta U_{DG_k} \\ \Delta U_{DG_k}^f = C_f x_k^f + D_f \Delta U_{DG_k} \\ \Delta Q_{DG_k} = \alpha_i \Delta U_{DG_k}^f + \beta_i \\ \Delta U_{DG_k} = K_Q \Delta Q_{DG_k} + \Delta U_{d_k} \\ \Delta U_{d_k} = \Delta U_d \end{cases} \quad (26)$$

So, from (26), the affine dynamics of the extended state-space vector in the  $i$ -th mode can be written as follows:

$$x_{k+1} = A_i x_k + b_i, \quad (27)$$

with:

$$A_i = \begin{bmatrix} A_f + \frac{B_f K_Q \alpha_i C_f}{1 - K_Q \alpha_i D_f} & \frac{B_f}{1 - K_Q \alpha_i D_f} \\ 0 & 1 \end{bmatrix}, \quad (28)$$

$$b_i = \begin{bmatrix} \frac{B_f K_Q \beta_i}{1 - K_Q \alpha_i D_f} \\ 0 \end{bmatrix}$$

This dynamical equation is associated with a given polyhedron  $d_i \subset \mathcal{X}$  defined by:

$$\begin{cases} \Delta U_i^- \leq \Delta U_{DG_k}^f < \Delta U_i^+ \\ \Delta U_d^f \in \mathcal{U}_d \\ \tilde{x}_k \in \mathcal{X} \end{cases} \Leftrightarrow K_i x_k \leq L_i, \quad (29)$$

To conclude, the system is modeled by a set  $\Omega$  of five discrete states ( $q_i$ ) $_{i \in \mathbb{I}}$ , an invariant function  $Inv : \Omega \rightarrow 2^{\mathcal{X}}$

associating to each  $q_i$  a polyhedron  $Inv(q_i)$  as defined by (4) and dynamical function  $A$  and  $B$  defined by (5). So the system can be modeled as a hybrid transition system  $S_H$  defined by the following tuple:

$$S_H = \langle \Omega, Inv, A, B \rangle \quad (30)$$

### 4.4 Discrete Abstraction and stability analysis

To simplify the calculations and be able to represent the polyhedra in the 2-dimensional plane ( $\Delta U_{DG_k}, \Delta U_d$ ), the measurement filter chosen is a pure delay. Its state-space model is given by:

$$\begin{cases} x_{k+1}^f = \Delta U_{DG_k} \\ \Delta U_{DG_k}^f = x_k^f \end{cases}. \quad (31)$$

*Initial Discrete Automaton.* From the hybrid transition system, a discrete abstraction can be constructed. The initial partition  $P_a^0$  of the state-space  $\mathcal{X}$  is defined as expressed in (15) with the invariant function defined by (4). In this case study, the partition  $P_a^0(\mathcal{X})$  is composed of five polytopes as shown in Fig. 3 (a). Discrete states ( $m_i^0$ ) $_{i \in \mathbb{I}}$  are defined by (16). The transition function  $T^0$  is defined by (13). The initial discrete abstraction of  $S_H$  is defined as follows:

$$S_D^0 = \langle \mathfrak{M}^0, T^0 \rangle. \quad (32)$$

The corresponding discrete automaton is shown in Fig. 4.

As explained previously, to conclude on system stability, the discrete abstraction has to be stable or to be equivalent

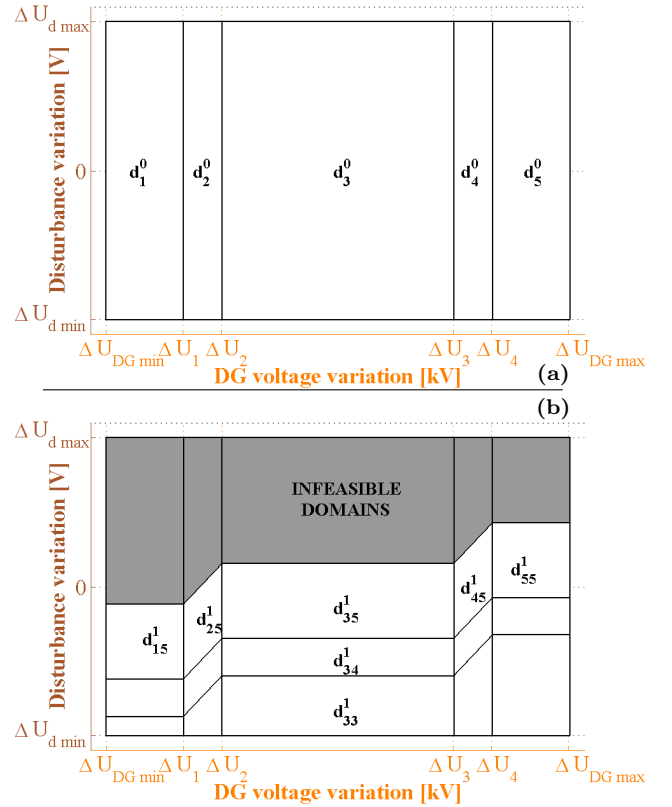


Fig. 3. (a) Initial discrete abstraction and (b) discrete abstraction after one iteration of the bisimulation calculation

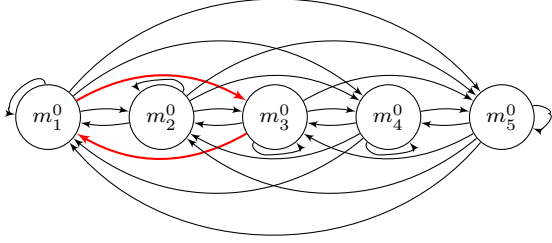


Fig. 4. Discrete automaton associated with the initial discrete abstraction

to a  $S_H$ . Here, transitions are defined by (13) and so they are not equivalent to transitions of  $S_H$ . In addition, it can be seen in Fig. 4 that several cycles exist in the discrete automaton so  $S_D^0$  is not stable. Thus no conclusion on  $S_H$  stability can be drawn for the study of  $S_D^0$ . Bisimulation calculation is needed to refine discrete trajectories to the one corresponding to trajectories of  $S_H$ .

*Bisimulation Calculation First Iteration.* Fig. 3 (b) represents the partition of the state-space after the first iteration of the bisimulation calculation. As it can be noticed in Fig. 3 (b), some regions of the state-space are infeasible. A given state  $x_k$  of the shaded area of Fig. 3 (b) leads to – after the application of the corresponding dynamics – a state  $x_{k+1}$  outside the state-space  $\mathcal{X}$ . This corresponds to a voltage outside of the circuit breaker tolerance range so the circuit will be switched off by circuit breakers. Thus, these regions are considered infeasible.

At the end of the first iteration, the continuous state-space has been divided into 15 subsets. It means that 10 of the 25 initial discrete transitions were not corresponding to a continuous transition. Some discrete states are still associated to several transitions as their destinations have been divided. More iterations of the bisimulation calculation are needed to obtain a discrete abstraction equivalent to the continuous transition system.

*Reachability analysis.* The partition  $P_a^3(\mathcal{X})$  obtained after three iterations is presented in Fig. 5 (a). The crosshatched area corresponds to the only domain having several successors. Thus, trajectories which do not include one of these domains can be analyzed without further calculations. Moreover, in this particular case, the crosshatched domain does not have any predecessor. Thus, the only way to reach it is that the initial condition belongs to it. For these reasons, every other domain will not be divided in any further iteration of the calculation, so they belong to  $P_a^B(\mathcal{X})$ . The study of these domains reveals the existence of three cycles (colored domains of Fig.5). For instance, reachability analysis evidenced that  $d_1^B$  is the sub-domain of  $d_3^0$  which leads entirely to  $d_2^B$  and  $d_2^B$  is the sub-domain of  $d_4^0$  which leads entirely to  $d_1^B$ . So every initial condition leading to a point in  $d_1^B$  at a given time-step  $t = kT_S$  leads at  $t = (k+1)T_S$  to a point in  $d_2^B$ . Then, at  $t = (k+2)T_S$  the operating point will be in  $d_1^B$  and thus the trajectory will describe a cycle. So, without reaching convergence, it can be concluded that the continuous transition system is unstable.

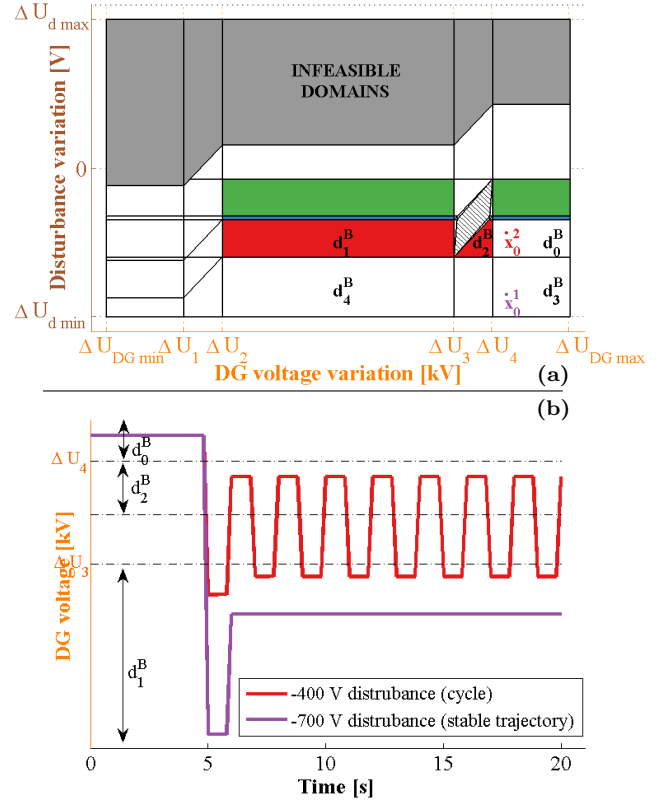


Fig. 5. (a) Results of the stability analysis in the state vector plane and (b) verification with dynamical analysis

It can be seen in Fig.5(a) that cycles can appear for  $\Delta U_d \in [-600V, -70V]$ . An empirical study has been performed for  $\Delta U_d = -700V$  (Fig 5 (b)). The initial conditions of the trajectory starting at  $x_0^0$  correspond to a DG voltage of 21.3kV and a disturbance of  $-700V$ . According to Fig. 5 (a),  $x_0^1$  belongs to  $d_3^B$  which is not a "dangerous" domain i.e. does not lead to a cycle. Indeed,  $d_3^B$  has been found to lead entirely to  $d_4^B$  which only successor is itself. This is confirmed by the empirical study (Fig. 5 (b)) as the trajectory from  $x_0^1$  is found to be stable.

This observation is not enough to conclude on the system stability. Indeed, formal analysis (Fig. 5 (a)) evidenced cycling trajectories. For instance, after a -400 V disturbance step, the initial conditions  $x_0^2$  corresponds to the domain  $d_0^B$  (Fig. 5 (a)) which is a "dangerous" domain as it leads to an unstable operating region ( $d_1^B$ ). As a matter of fact, the empirical study proved that the voltage oscillates in between  $d_1^B$  and  $d_2^B$  as shown in fig. 5 (b).

To conclude, the system can appear to be stable after empirical studies but a formal analysis is necessary to generalize as it detects unsafe operating regions.

## 5. CONCLUSION

A method to analyze piecewise affine hybrid transition system has been proposed. This approach is based on the construction of a discrete abstraction to study the stability of the hybrid system. This abstraction is iteratively refined in order to limit discrete trajectories to the ones equivalent to trajectories of the hybrid system. Then stability and

instability of the hybrid system can be proved easily and unstable operating regions, if there exist, can be evidenced. It can be noted that unstable operations can be identified after only few iterations of the bisimulation calculation.

This approach has been illustrated on the stability study of a distribution feeder hosting a single distributed generator equipped with a dead-band  $Q(U)$  law. The analysis proved instability of the real case study and evidenced dangerous operating regions.

Further developments of the proposed analytic method should be conducted in order to analyze the interactions between several systems having different dynamics while keeping acceptable computation load. Furthermore, future work will focus on the validation of this approach on several power systems case studies. In particular, a parametric study illustrating the impact of network parameters on system stability should be carried out on various network architectures.

The main interest of the proposed approach is to extend it to study solutions in case of instability. Indeed, in the case study, a measurement filter could have been designed in order to ensure system stability. Thus, further work will focus on developing a filter design tool based on this analytic method in order to solve instability issues.

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