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Abstract: We present preliminary results on synchronisation of nonlinear oscillators interconnected in heterogeneous networks that is, we assume that the systems' dynamic models are different, albeit of the same dimension. Under mild conditions, we show that the synchronisation errors may be diminished by increasing the interconnection gain. That is, we establish results on practical synchronisation. Although this problem has been studied in the literature, our approach is novel from an analytical perspective: the behaviour of the interconnected systems is determined by two main components, the stability of an averaged dynamics, relative to an attractor of what we call *emergent dynamics* and, secondly, the synchronisation of each individual oscillator relative to the emergent dynamics. Our framework is general, it covers as a particular case that of (set-point) consensus but also trajectory-tracking synchronisation and consensus over manifolds.

Keywords: Stability, synchronisation, networked systems, emergent dynamics.

1. INTRODUCTION

The collective behaviour of network-interconnected complex systems depends on some key factors, such as: the dynamics of the individual units, the interconnection among the nodes and the network structure. Network dynamics may be modelled via *ordinary* differential equations –*cf.* Pogromski and Nijmeijer (2001); Pogromski et al. (2002),

$$\dot{\mathbf{x}}_i = f_i(\mathbf{x}_i) + B\mathbf{u}_i, \qquad i \in \mathcal{I} := \{1, \ldots, N\} \tag{1a}$$

$$\mathbf{y}_i = C\mathbf{x}_i, \tag{1b}$$

where $\mathbf{x}_i \in \mathbb{R}^n$, $\mathbf{u}_i \in \mathbb{R}^m$ and $\mathbf{y}_i \in \mathbb{R}^m$ denote the state, the input and the output of the ith unit, respectively. Usually, graph theory is employed to describe the topological (structural) properties of networks; a network of N nodes is defined by its $N \times N$ adjacency matrix $D = [d_{ij}]$ whose (i, j) element, denoted by d_{ij}, specifies an interconnection between the ith and jth nodes. From a dynamical-systems viewpoint a general setting such as *e.g.*, in Blekhman et al. (1997); Nijmeijer and Rodríguez-Angeles (2003), synchronisation may be qualitatively measured by equating a functional of the trajectories to zero and measuring the distance of the latter to the synchronisation manifold. In the case of a network of identical nodes, *i.e.*, if $f_i = f_j$ for all $i, j \in \mathcal{I}$ this may be defined in the space of $\mathbf{x} := [\mathbf{x}_1^\top, \cdots, \mathbf{x}_N^\top]^\top$ as

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^{nN} : \mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N\} \tag{2}$$

Such stability problem may be approached in a number of ways, *e.g.*, using tools developed for semi-passive, incrementally passive or incrementally input-output stable systems –see Pogromski and Nijmeijer (2001); Pogromski et al. (1999); Jouffroy and Slotine (2004); Lohmiller and Slotine (2005); Scardovi et al. (2009); Franci et al. (2011). If the manifold $\mathcal{S}$ is stabilised one says that the networked units are synchronised.

In general, the nodes' interconnections depend on the strength of the coupling and on the nodes' state variables or on functions of the latter, *i.e.*, outputs which define the coupling terms. This may be nonlinear, as *e.g.*, in the case of the the well-known Kuramoto's oscillator –see Belykh et al. (2005); Corson et al. (2012). In this paper we consider a particular case of coupling which is known in the literature as *diffusive coupling*. We assume that all the units have inputs and outputs of the same dimension and that the coupling between the ith and jth units is defined as a weighted difference: $d_{ij}(\mathbf{y}_i - \mathbf{y}_j)$, where $\mathbf{y}_i$ and $\mathbf{y}_j$ are the outputs of the units i and j respectively, and $d_{ij} > 0$ is constant.

Thirdly, depending on whether the nodes are identical or not the network is respectively called *homogeneous* or *heterogeneous*. The behaviour of networks of systems with non-identical models is more complex due to the fact that the synchronisation manifold $\mathcal{S}$ does not necessarily exist. An alternative approach based on stability theory, is to address the synchronisation problem in a *practical* sense that is, to admit that, asymptotically, the differences between the units' motions are bounded and become smaller for larger values of the interconnection gain γ, but they do not necessarily vanish. This is the approach that we pursue here.

For the purpose of analysis we propose to analyse the behaviour of network-interconnected systems via two separate properties: the stability of what we call the *emergent dynamics* and the synchronisation errors of each of the units' motions, relative to an averaged system, also called "mean-field" system. This formalism covers the classical paradigm of consensus of a collection of integrators, in which case the emergent dynamics is null and the mean field trajectory corresponds to a weighted average of the nodes' trajectories. Moreover, for a balanced graph, we know that all units reach consensus and the steady-state value is an equilibrium point corresponding to the average of the initial conditions –see Ren et al. (2007). In our framework, the emergent dynamics possesses a stable attractor, in contrast to (the particular case of) an equilibrium point. Then, we say that the network presents dynamic consensus if there exists an attractor $\mathcal{A}$, in the phase-space of the emergent state, such that the trajectories of all units are attracted to $\mathcal{A}$ asymptotically and remain close to it. In the setting of heterogeneous networks, only *practical* synchronisation is achievable in general that is, the trajectories of all units converge to a neighbourhood of the attractor of the emergent dynamics and remain close to this neighbourhood.

In section 2 we present the network model, suitable for analysis; in Section 4 we present our main statements, whose proofs are provided in Panteley (2015). In Section 5 we present some illustrative simulation results, before concluding with some remarks in SectionPsec:concl.

2. NETWORKS OF HETEROGENEOUS SYSTEMS

2.1 System model

We consider a network composed of N heterogeneous diffusively coupled nonlinear dynamical systems in normal form:

$$\dot{\mathbf{y}}_i = f_i^1(\mathbf{y}_i, \mathbf{z}_i) + \mathbf{u}_i \tag{3a}$$

$$\dot{\mathbf{z}}_i = f_i^2(\mathbf{y}_i, \mathbf{z}_i) \tag{3b}$$

where $\mathbf{u}_i \in \mathbb{R}^m$ denote the inputs, $\mathbf{y}_i \in \mathbb{R}^m$ the outputs to be synchronised and the state $\mathbf{z}_i$ corresponds to that of the ith agent's zero-dynamics *i.e.*, $\dot{\mathbf{z}}_i = f_i^2(0, \mathbf{z}_i)$. The functions $f_i^1 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$, $f_i^2 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ are assumed to be locally Lipschitz. We consider that the network units are connected via *diffusive coupling*, *i.e.*, for the i-th unit the coupling is given by

$$\mathbf{u}_i = -\sigma \sum_{j=1}^{N} d_{ij}(\mathbf{y}_i - \mathbf{y}_j), \quad d_{ij} = d_{ji} \tag{4}$$

where $\sigma > 0$ corresponds to the coupling gain between the units and the individual interconnections weights, d_{ij} which define the so-called Laplacian matrix,

$$L = \begin{bmatrix} \sum_{i=2}^{N} d_{1i} & -d_{12} & \dots & -d_{1N} \\ -d_{21} & \sum_{i=1, i\neq 2}^{N} d_{2i} & \dots & -d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{N1} & -d_{N2} & \dots & \sum_{i=1}^{N-1} d_{Ni}. \end{bmatrix}. \tag{5}$$

By construction, all row sums of L are equal to zero and all its eigenvalues of are real, exactly one of which (say, λ_1) equal to zero, while others are positive, *i.e.*, $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$.

Next, let $\mathbf{y} = [\mathbf{y}_1^\top \ \cdots \ \mathbf{y}_N^\top]^\top$, $\mathbf{u} = [\mathbf{u}_1^\top \ \cdots \ \mathbf{u}_N^\top]^\top$, $\mathbf{x} = [\mathbf{x}_1^\top \ \cdots \ \mathbf{x}_N^\top]^\top$, and define $F : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ as

$$F(\mathbf{x}) = \begin{bmatrix} F_1(\mathbf{x}_1) \\ \vdots \\ F_N(\mathbf{x}_N) \end{bmatrix}, \quad F_i(\mathbf{x}_i) = \begin{bmatrix} f_i^1(\mathbf{y}_i, \mathbf{z}_i) \\ f_i^2(\mathbf{y}_i, \mathbf{z}_i) \end{bmatrix}_{i \in \mathcal{I}}. \tag{6}$$

With this notation, the diffusive coupling inputs $\mathbf{u}_i$, defined in (4), can be re-written in the compact form

$$\mathbf{u} = -\sigma[L \otimes I_m]\mathbf{y},$$

where the symbol $\otimes$ stands for the right Kronecker product. Then, the network dynamics becomes

$$\dot{\mathbf{x}} = F(\mathbf{x}) - \sigma[L \otimes E_m]\mathbf{y} \tag{7a}$$

$$\mathbf{y} = [I_N \otimes E_m^\top]\mathbf{x}, \tag{7b}$$

where $E_m^\top = [I_m, 0_{m\times(n-m)}]$. The qualitative analysis of the solutions to the latter equations is our main subject of study.

2.2 Dynamic consensus and practical synchronisation

We generalise the consensus paradigm by introducing what we call *dynamic consensus*. This property is achieved by the systems interconnected over a network if and only if their motions converge to one generated by what we call *emergent dynamics*. In the case that the Laplacian is symmetric, the emergent dynamics is naturally defined as the average of the units' drifts that is, via the functions $f_s^1 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$, $f_s^2 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$, defined as

$$f_s^1 := \frac{1}{N}\sum_{i=1}^{N} f_i^1(\mathbf{y}_e, \mathbf{z}_e), \quad f_s^2 := \frac{1}{N}\sum_{i=1}^{N} f_i^2(\mathbf{y}_e, \mathbf{z}_e) \tag{8}$$

hence, the emergent dynamics may be written in the compact form

$$\dot{\mathbf{x}}_e = f_s(\mathbf{x}_e) \qquad \mathbf{x}_e = [\mathbf{y}_e^\top \ \mathbf{z}_e^\top]^\top, \quad f_s := [f_s^{1\top} \ f_s^{2\top}]^\top. \tag{9}$$

For the sake of comparison, in the classical (set-point) consensus paradigm, all systems achieving consensus converge to a common equilibrium *point* that is, $f_s \equiv 0$ and $\mathbf{x}_e$ is constant. In the case of formation tracking *control*, Equation (9) can be seen as the reference dynamics to the formation.

Next, we introduce the *average* state (also called mean-field) and its corresponding dynamics. Let

$$\mathbf{x}_s = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i \tag{10}$$

which comprises an average output, $\mathbf{y}_s \in \mathbb{R}^m$, defined as $\mathbf{y}_s = E_m^\top \mathbf{x}_s$ and the state of the average zero dynamics, $\mathbf{z}_s \in \mathbb{R}^{n-m}$, that is, $\mathbf{x}_s = [\mathbf{y}_s^\top, \mathbf{z}_s^\top]^\top$. Now, by differentiating on both sides of (10) and after a direct computation in which we use (3), (4) and the fact that the sums of the elements of the Laplacian's rows equal to zero, *i.e.*,

$$\frac{1}{N}\sum_{i=1}^{N} -\sigma\big[d_{i1}(\mathbf{y}_i - \mathbf{y}_1) + \cdots + d_{iN}(\mathbf{y}_i - \mathbf{y}_N)\big] = 0,$$

we obtain

$$\dot{\mathbf{y}}_s = \frac{1}{N}\sum_{i=1}^{N} f_i^1(\mathbf{y}_i, \mathbf{z}_i), \quad \dot{\mathbf{z}}_s = \frac{1}{N}\sum_{i=1}^{N} f_i^2(\mathbf{y}_i, \mathbf{z}_i).$$

Next, in order to write the latter in terms of the average state $\mathbf{x}_s$, we use the functions f_s^1 and f_s^2 defined above so, after (8) the *average dynamics* may be expressed as

$$\dot{\mathbf{y}}_s = f_s^1(\mathbf{y}_s, \mathbf{z}_s) + \frac{1}{N}\sum_{i=1}^{N}\big[f_i^1(\mathbf{y}_i, \mathbf{z}_i) - f_i^1(\mathbf{y}_s, \mathbf{z}_s)\big],$$

$$\dot{\mathbf{z}}_s = f_s^2(\mathbf{y}_s, \mathbf{z}_s) + \frac{1}{N}\sum_{i=1}^{N}\big[f_i^2(\mathbf{y}_i, \mathbf{z}_i) - f_i^2(\mathbf{y}_s, \mathbf{z}_s)\big].$$

The latter equations may be regarded as composed of the nominal parts $\dot{\mathbf{y}}_s = f_s^1(\mathbf{y}_s, \mathbf{z}_s)$, $\dot{\mathbf{z}}_s = f_s^2(\mathbf{y}_s, \mathbf{z}_s)$

and the perturbation terms $\left[f_i^1(\mathbf{y}_i,\mathbf{z}_i) - f_i^1(\mathbf{y}_s,\mathbf{z}_s)\right]$ and $\left[f_i^2(\mathbf{y}_i,\mathbf{z}_i) - f_i^2(\mathbf{y}_s,\mathbf{z}_s)\right]$. The former functions correspond exactly to (9), only re-written with another state variable. In the case that dynamic consensus is achieved (that is, in the case of complete synchronisation) and the graph is balanced and connected, we have $(\mathbf{y}_i,\mathbf{z}_i) \to (\mathbf{y}_s,\mathbf{z}_s)$. The latter is possible only for homogeneous connected and balanced networks. In the case of a heterogeneous network, asymptotic synchronisation cannot be achieved in general hence, $\mathbf{y}_i \not\to \mathbf{y}_s$ and, consequently, the terms $\left[f_i^1(\mathbf{y}_i,\mathbf{z}_i) - f_i^1(\mathbf{y}_s,\mathbf{z}_s)\right]$ and $\left[f_i^2(\mathbf{y}_i,\mathbf{z}_i) - f_i^2(\mathbf{y}_s,\mathbf{z}_s)\right]$ remain.

Thus, from a dynamical systems viewpoint, the average dynamics may be considered as a perturbed variant of the emergent dynamics. Consequently, it appears natural to study the problem of dynamic consensus, recasted in that of *robust* stability analysis, in a broad sense hence, we introduce the manifold

$$\mathcal{S}_y = \{\mathbf{y} \in \mathbb{R}^{mN} : \mathbf{y}_1 - \mathbf{y}_s = \mathbf{y}_2 - \mathbf{y}_s = \cdots = \mathbf{y}_N - \mathbf{y}_s = 0\}. \tag{11}$$

For heterogeneous networks one may only aspire at establishing stability of the output synchronisation manifold $\mathcal{S}_y$ in a *practical* sense. The following definition covers that of practical stability used in Teel et al. (1999); Chaillet and Loria (2006), by considering a stability property with respect to sets.

Consider a parameterised system of differential equations

$$\dot{x} = f(x,\epsilon), \tag{12}$$

where $x \in R^n$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and ϵ is a scalar parameter such that $\epsilon \in (0,\epsilon_\circ]$ with $\epsilon_\circ < \infty$. Given a closed set $\mathcal{A}$, we define the norm $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} \|x - y\|$.

Definition 2.1. For the system (12), we say that the closed set $\mathcal{A} \subset \mathbb{R}^n$ is practically uniformly asymptotically stable if there exists a closed set $\mathcal{D}$ such that $\mathcal{A} \subset \mathcal{D} \subset \mathbb{R}^n$ and:

(1) the system is forward complete for all $x_\circ \in \mathcal{D}$;
(2) for any given $\delta > 0$ and $R > 0$, there exist $\epsilon^* \in (0,\epsilon_\circ]$ and a class $\mathcal{KL}$ function $\beta_{\delta R}$ such that, for all $\epsilon \in (0,\epsilon^*]$ and all $x_\circ \in \mathcal{D}$ such that $|x_\circ|_{\mathcal{A}} \le R$, we have
$$|x(t,x_\circ,\epsilon)|_{\mathcal{A}} \le \delta + \beta_{\delta R}\big(|x_\circ|_{\mathcal{A}},t\big).$$

3. NETWORK DYNAMICS

For the purpose of analysis we stress that the networked dynamical systems model (7) is equivalent, up to a coordinate transformation, to a set of equations composed of the average system dynamics with average state $\mathbf{x}_s$ and a synchronisation errors equation with state $\mathbf{e} = [\mathbf{e}_1^\top \ \ldots \ \mathbf{e}_N^\top]^\top$ where $\mathbf{e}_i = \mathbf{x}_i - \mathbf{x}_s$ for all $i \in \mathcal{I}$ –see Panteley (2015).

The states $\mathbf{x}_s$ and $\mathbf{e}$ are *intrinsic* to the network and not the product of an artifice with purely theoretical motivations –see Panteley (2015). Hence, the general synchronisation problem is recasted in the study of stability of the dynamics of $\mathbf{e}$ and $\mathbf{x}_s$. We proceed to derive the differential equations in terms of the average state $\mathbf{x}_s$ and the synchronisation errors $\mathbf{e}$.

3.1 Dynamics of the average unit

Using the network dynamics equations (7a), as well as (10), which is equivalent to

$$\mathbf{x}_s = \frac{1}{N}(\mathbf{1}^\top \otimes I_n)\mathbf{x}, \quad \mathbf{1} := [1\ 1 \cdots 1]^\top \tag{13}$$

we obtain

$$\dot{\mathbf{x}}_s = \frac{1}{N}(\mathbf{1}^\top \otimes I_n)F(\mathbf{x}) - \frac{1}{N}\sigma(\mathbf{1}^\top \otimes I_n)[L \otimes E_m]\mathbf{y}. \tag{14}$$

Now, using the property of the Kronecker product,

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \tag{15}$$

and in view of the identity $\mathbf{1}^\top L = 0$ we obtain

$$(\mathbf{1}^\top \otimes I_n)(L \otimes E_m) = (\mathbf{1}^\top L) \otimes (I_n E_m) = 0. \tag{16}$$

This reveals the important fact that the average dynamics, *i.e.*, the *functions* on the right-hand side of (14), are independent of the interconnections gain σ, even though the *solutions* $\mathbf{x}_s(t)$ are, certainly, affected by the synchronisation errors hence, by the coupling strength.

Now, using (6) and defining

$$f_s(\mathbf{x}_s) := \frac{1}{N}\sum_{i=1}^N F_i(\mathbf{x}_s) \tag{17}$$

we obtain

$$\dot{\mathbf{x}}_s = f_s(\mathbf{x}_s) + \frac{1}{N}\sum_{i=1}^N \Big[F_i(\mathbf{x}_i) - F_i(\mathbf{x}_s)\Big].$$

Therefore, defining

$$G_s(\mathbf{e},\mathbf{x}_s) := \frac{1}{N}\sum_{i=1}^N \Big[F_i(\mathbf{e}_i + \mathbf{x}_s) - F_i(\mathbf{x}_s)\Big], \tag{18}$$

we see that we may write the average dynamics in the compact form,

$$\dot{\mathbf{x}}_s = f_s(\mathbf{x}_s) + G_s(\mathbf{e},\mathbf{x}_s). \tag{19}$$

Furthermore, since the functions F_i, with $i \in \mathcal{I}$, are locally Lipschitz so is the function G_s and, moreover, there exists a continuous, positive, non-decreasing function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\|G_s(\mathbf{e},\mathbf{x}_s)\| \le k\big(\,\|\mathbf{e}\|, \|\mathbf{x}_s\|\,\big)\,\|\mathbf{e}\|\,. \tag{20}$$

In summary, the average dynamics is described by the equations (19), which may be regarded as the nominal system (9), which corresponds to the emergent dynamics, perturbed by the synchronisation error of the network.

3.2 Dynamics of the synchronisation errors

To study the effect of the synchronisation errors, $\mathbf{e}(t)$, on the emergent dynamics, we start by introducing the vectors

$$F_s(\mathbf{x}_s) := \big[F_1(\mathbf{x}_s)^\top \quad \cdots \quad F_N(\mathbf{x}_s)^\top\big]^\top \tag{21}$$

$$\tilde{F}(\mathbf{e},\mathbf{x}_s) = F(\mathbf{x}) - F_s(\mathbf{x}_s). \tag{22}$$

Then, differentiating on both sides of

$$\mathbf{e} = \mathbf{x} - (\mathbf{1} \otimes I_n)\mathbf{x}_s \tag{23}$$

and using (7a) and (19), we obtain

$$\begin{aligned}
\dot{\mathbf{e}} &= -\sigma[L \otimes E_m]\mathbf{y} + F(\mathbf{x}) - (\mathbf{1} \otimes I_n)\,[f_s(\mathbf{x}_s) + G_s(\mathbf{e},\mathbf{x}_s)] \\
&= -\sigma[L \otimes E_m]\mathbf{y} + [F(\mathbf{x}) - F_s(\mathbf{x}_s)] + F_s(\mathbf{x}_s) \\
&\quad -(\mathbf{1} \otimes I_n)\,[f_s(\mathbf{x}_s) + G_s(\mathbf{e},\mathbf{x}_s)] \\
&= -\sigma[L \otimes E_m]\mathbf{y} + \big[F_s(\mathbf{x}_s) - (\mathbf{1} \otimes I_n)f_s(\mathbf{x}_s)\big] \\
&\quad +\big[\tilde{F}(\mathbf{e},\mathbf{x}_s) - (\mathbf{1} \otimes I_n)G_s(\mathbf{e},\mathbf{x}_s)\big].
\end{aligned} \tag{24}$$

Next, let us introduce the output synchronisation errors $\mathbf{e}_{yi} = \mathbf{y}_i - \mathbf{y}_s$, $\mathbf{e}_y = [\mathbf{e}_{y1}^\top, \cdots, \mathbf{e}_{yN}^\top]^\top$, which may also be written as

$$\mathbf{e}_y = \mathbf{y} - \mathbf{1} \otimes \mathbf{y}_s, \tag{25}$$

and let us consider the first term and the two groups of bracketed terms on the right-hand side of (24), separately. For the term $\big(L \otimes E_m\big)\mathbf{y}$ we observe, from (25), that

$$[L \otimes E_m]\,\mathbf{y} = [L \otimes E_m]\,\big[\mathbf{e}_y + \mathbf{1} \otimes \mathbf{y}_s\big]$$

and we use (15) and the fact that $L\mathbf{1} = 0$ to obtain

$$[L \otimes E_m]\,\mathbf{y} = [L \otimes E_m]\,\mathbf{e}_y.$$

Secondly, concerning the first bracket on the right-hand side of (24) we observe that, in view of (17) and (21),

$$f_s(\mathbf{x}_s) = \frac{1}{N}(\mathbf{1}^\top \otimes I_n)F_s(\mathbf{x}_s)$$

therefore,

$$F_s(\mathbf{x}_s) - (\mathbf{1} \otimes I_n)f_s(\mathbf{x}_s) = F_s(\mathbf{x}_s) - \tfrac{1}{N}(\mathbf{1} \otimes I_n)(\mathbf{1}^\top \otimes I_n)F_s(\mathbf{x}_s).$$

Then, using (15) we see that

$$\frac{1}{N}(\mathbf{1} \otimes I_n)(\mathbf{1}^\top \otimes I_n) = \frac{1}{N}(\mathbf{1}\mathbf{1}^\top) \otimes I_n \tag{26}$$

so, introducing

$$P = I_{nN} - \frac{1}{N}(\mathbf{1}\mathbf{1}^\top) \otimes I_n,$$

we obtain

$$F_s(\mathbf{x}_s) - (\mathbf{1} \otimes I_n)f_s(\mathbf{x}_s) = PF_s(\mathbf{x}_s). \tag{27}$$

Finally, concerning the term $\tilde{F}(\mathbf{e}, \mathbf{x}_s) - (\mathbf{1} \otimes I_n)G_s(\mathbf{e}, \mathbf{x}_s)$ on the right-hand side of (24), we see that, by definition, $G(\mathbf{e}, \mathbf{x}_s) = \frac{1}{N}\big(\mathbf{1}^\top \otimes I_n\big)\tilde{F}(\mathbf{e}, \mathbf{x}_s)$ hence, from (26), we obtain

$$(\mathbf{1} \otimes I_n)G_s(\mathbf{e}, \mathbf{x}_s) = \frac{1}{N}\big[(\mathbf{1}\mathbf{1}^\top) \otimes I_n\big]\tilde{F}(\mathbf{e}, \mathbf{x}_s)$$

and

$$\tilde{F}(e, \mathbf{x}_s) - (\mathbf{1} \otimes I_n)G_s(\mathbf{e}, \mathbf{x}_s) = P\tilde{F}(\mathbf{e}, \mathbf{x}_s). \tag{28}$$

Using (27) and (28) in (24) we see that the latter may be expressed as

$$\dot{\mathbf{e}} = -\sigma\big[L \otimes E_m\big]\mathbf{e}_y + P\big[\tilde{F}(\mathbf{e}, \mathbf{x}_s) + F_s(\mathbf{x}_s)\big].$$

The utility of this equation is that it clearly exhibits three terms: a term linear in the output $\mathbf{e}_y$ which reflects the synchronisation effect of diffusive coupling between the nodes, the term $P\tilde{F}(\mathbf{e}, \mathbf{x}_s)$ which vanishes with the synchronisation errors, *i.e.*, if $\mathbf{e} = 0$, and the term

$$PF_s(\mathbf{x}_s) = \begin{bmatrix} F_1(\mathbf{x}_s) - \frac{1}{N}\sum_{i=1}^{N} F_i(\mathbf{x}_s) \\ \vdots \\ F_N(\mathbf{x}_s) - \frac{1}{N}\sum_{i=1}^{N} F_i(\mathbf{x}_s) \end{bmatrix} = \begin{bmatrix} F_1(\mathbf{x}_s) - f_s(\mathbf{x}_s) \\ \vdots \\ F_N(\mathbf{x}_s) - f_s(\mathbf{x}_s) \end{bmatrix}$$

which represents the variation between the dynamics of the individual units and the average unit. This term equals to zero when the nominal dynamics, f_i in (1a), of all the units are identical that is, in the case of a homogeneous network.

4. MAIN RESULTS

For the networked systems dynamics

$$\dot{\mathbf{x}}_s = f_s(\mathbf{x}_s) + G_s(\mathbf{e}, \mathbf{x}_s), \tag{29a}$$

$$\dot{\mathbf{e}} = -\sigma\big[L \otimes E_m\big]\mathbf{e}_y + P\big[\tilde{F}(\mathbf{e}, \mathbf{x}_s) + F_s(\mathbf{x}_s)\big] \tag{29b}$$

we make some statements on stability with respect to a compact attractor which is proper to the emergent dynamics and we establish conditions under which the average of the trajectories of the interconnected units remains close to this attractor. We formulate conditions that ensure practical global asymptotic stability of the manifold $\mathcal{S}_y$ –see (11). This implies practical state and output synchronisation of the network, respectively. Furthermore, we show that the upper bound on the state synchronisation error depends on the mismatches between the dynamics of the individual units of the network.

One of our main hypotheses is that the solutions are ultimately bounded in a compact "ball of radius B_x"; which holds for chaotic oscillators. Our second main assumption is that the zero-dynamics is convergent, uniformly in the passive outputs, in a *practical* sense:

A1. For any compact sets $\mathbb{B}_z \subset \mathbb{R}^{n-m}$, $\mathbb{B}_y \subset \mathbb{R}^m$, there exist continuously differentiable positive definite functions $V_{\circ i} : \mathbb{B}_z \to \mathbb{R}_+$ and constants $\bar{\alpha}_i$, $\beta_i > 0$, $i \in \mathcal{I}$, such that

$$\nabla V_{\circ i}^\top(\mathbf{z}_1 - \mathbf{z}_2)\left[f_i^2(\mathbf{y}, \mathbf{z}_1) - f_i^2(\mathbf{y}, \mathbf{z}_2)\right] \leq -\bar{\alpha}_i|\mathbf{z}_1 - \mathbf{z}_2|^2 + \beta_i \tag{30}$$

for all $\mathbf{z}_1$, $\mathbf{z}_2 \in \mathbb{B}_z$ and $\mathbf{y} \in \mathbb{B}_y$.

Theorem 1. (Output synchronisation). *Let the solutions of the system (7) be globally ultimately bounded. Then, the set $\mathcal{S}_y$ is practically uniformly globally asymptotically stable with $\epsilon = 1/\sigma$. If, moreover, Assumption* **A1** *holds, then there exists a function $\beta \in \mathcal{K}_\infty$ such that for any $\epsilon \geq 0$ and $R > 0$ there exist $B_R := \{\mathbf{x}_\circ : \|\mathbf{x}_\circ\| \leq R\}$, $T^* > 0$ and $\sigma^* > 0$ such that the solutions of (29b) with $\sigma = \sigma^*$ satisfy*

$$\|\mathbf{e}(t, \mathbf{x}_\circ)\| \leq \beta(\Delta_f), \qquad \forall\, t \geq T^*,\ \mathbf{x}_\circ \in B_R \tag{31}$$

where

$$\Delta_f = \max_{\|\mathbf{x}\| \leq B_x} \max_{k,i \in N} \left\{ \left\| f_k^2(\mathbf{x}_k) - f_i^2(\mathbf{x}_k) \right\| \right\}. \tag{32}$$

The proof of the theorem is provided in Panteley (2015). Roughly speaking, the first statement (synchronisation) follows from two properties of the networked system – namely, negative definiteness of the second smallest eigenvalue of the Laplacian metric L and global ultimate boundedness.

Some interesting corollaries, on state synchronisation –*cf.* Pogromski and Nijmeijer (2001); Pogromski et al. (2002), follow from Theorem 1. For instance, if the interconnections among the network units depend on the whole state, that is, if $\mathbf{y} = \mathbf{x}$.

Corollary 1. Consider the system (7). Let Assumption **A1** *hold and let $\mathbf{y} = \mathbf{x}$. Then, the system is forward complete and the set*

$$\mathcal{S}_x = \{\mathbf{x} \in \mathbb{R}^{nN} : \mathbf{x}_1 - \mathbf{x}_s = \mathbf{x}_2 - \mathbf{x}_s = \cdots = \mathbf{x}_N - \mathbf{x}_s = 0\}$$

is practically uniformly globally asymptotically stable with $\epsilon = 1/\sigma$.

The constant Δ_f represents the maximal possible mismatch between the dynamics of any individual unit and that of the averaged unit, on a ball of radius B_x. The more heterogeneous is the network, the bigger is the constant Δ_f. Conversely, in the case that all the zero dynamics of the units are identical, we have $\Delta_f = 0$.

Corollary 2. Consider the system (7) under Assumption **A1**. *Assume that the functions f_i^2, which define zero dynamics of the network units, are all identical i.e., $f_i^2(x) = f_j^2(x)$ for all $i, j \in \mathcal{I}$ and all $x \in \mathbb{R}^n$. Then the set $\mathcal{S}_x$ is practically uniformly globally asymptotically stable with $\epsilon = 1/\sigma$.*

4.1 On practical stability of the collective network behaviour

To analyse the behaviour of the average unit, whose dynamics is given by the equations (29a). We naturally assume that the nominal dynamics of average-unit (*i.e.*, with $\mathbf{e} = 0$) has a stable compact attractor $\mathcal{A}$ and we establish that the stability properties of this attractor are preserved under the network interconnection, albeit, slightly weakened. This assumption is notably satisfied by chaotic oscillators.

A2. For the system (9), there exists a compact invariant set $\mathcal{A} \subset \mathbb{R}^n$ which is asymptotically stable with a domain of attraction $\mathcal{D} \subset \mathbb{R}^n$. Moreover, we assume that there exists a continuously differentiable Lyapunov function $V_\mathcal{A} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_i \in \mathcal{K}_\infty$, $i \in \{1, \dots, 4\}$ such that for all $\mathbf{x}_e \in \mathcal{D}$ we have

$$\alpha_1(|\mathbf{x}_e|_\mathcal{A}) \leq V_\mathcal{A}(\mathbf{x}_e) \leq \alpha_2(|\mathbf{x}_e|_\mathcal{A}) \tag{33a}$$

$$\dot{V}_\mathcal{A}(\mathbf{x}_e) \leq -\alpha_3(|\mathbf{x}_e|_\mathcal{A}) \tag{33b}$$

$$\left\| \frac{\partial V_\mathcal{A}}{\partial \mathbf{x}_e} \right\| \leq \alpha_4(\|\mathbf{x}_e\|). \tag{33c}$$

The second part of the assumption (the existence of V) is purely technical whereas the first part is essential to analyse the emergent synchronised behaviour as well as the synchronisation properties of the network, recasted as a (robust) stability problem. The following statement applies to the general case of diffusively coupled networks.

Theorem 2. For the system (7) assume that the solutions are globally ultimately bounded and Assumptions **A1**, **A2** *hold. Then, there exist a non-decreasing function $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and, for any $R, \epsilon > 0$ there exists $T^* = T^*(R, \epsilon)$, such that for all $t \geq T^*$ and all $\mathbf{x}_\circ$ such that $|\mathbf{x}_\circ|_\mathcal{A} \leq R$,*

$$|\mathbf{x}_s(t, \mathbf{x}_\circ)|_\mathcal{A} \leq \gamma(\Delta_f, R) + \epsilon. \tag{34}$$

In the case that the network is state practically synchronised, it follows that the set $\mathcal{A}$ is practically stable for the network (7).

Corollary 3. Consider the system (7) under Assumption **A2**. *If the set $\mathcal{S}_x$ is practically uniformly globally asymptotically stable for this system, then the attractor $\mathcal{A}$ defined in Assumption* **A2** *is practically asymptotically stable for the average unit (19).*

5. EXAMPLE

To illustrate our theoretical findings we present a brief case-study of analysis of interconnected heterogeneous systems via diffusive coupling. We consider three chaotic oscillators, two of the well-known Lorenz type, with different parameters, and a Lü system. The dynamics equations are

LORENZ:
$$\begin{aligned} \dot{x}_i &= \gamma_i(y_i - x_i), \quad i = 1, 2 \\ \dot{y}_i &= r_i x_i - y_i - x_i z_i \\ \dot{z}_i &= x_i y_i - b_i z_i \end{aligned}$$

LÜ:
$$\begin{aligned} \dot{x}_3 &= -\frac{\alpha\beta}{\alpha + \beta} x_3 - 2 y_3 z_3 + \delta \\ \dot{y}_3 &= \alpha y_3 + x_3 z_3 \\ \dot{z}_3 &= \beta z_3 + x_3 y_3. \end{aligned}$$

A direct computation shows that the corresponding emergent dynamics for these systems is given by

$$\begin{aligned} 3\dot{x}_e &= -\big[\gamma_1 + \gamma_2 + \frac{\alpha\beta}{\alpha + \beta}\big] x_e + [\gamma_1 + \gamma_2 - 2 z_e] y_e + \delta, \\ 3\dot{y}_e &= [r_1 + r_2] x_e - [2 - \alpha] y_e \\ 3\dot{z}_e &= 3 x_e y_e + [\beta - b_1 - b_2] z_e. \end{aligned}$$

The values of the parameters of the three systems are fixed in order for them to exhibit a chaotic behaviour:

$\gamma_1 = 10$	$\gamma_2 = 16$	$\alpha = -10$
$r_1 = 45.6$	$r_2 = 99.96$	$\beta = -4$
$b_1 = 4$	$b_2 = 8/3$	$\delta = 10$

Since the three chaotic systems are *oscillators* their trajectories are globally ultimately bounded –see Figure 1. Moreover, as it may be appreciated from Figure 2, the solutions remain bounded under the diffusive coupling which, for this test, we defined to be:

$$\begin{aligned} u_1 &= -\sigma\big[d_{12}(x_1 - x_2) + d_{13}(x_1 - x_3)\big], \quad d_{12} = 2,\ d_{13} = 4, \\ u_2 &= -\sigma\big[d_{21}(x_2 - x_1) + d_{23}(x_2 - x_3)\big], \quad d_{23} = 3, \\ u_3 &= -\sigma\big[d_{31}(x_3 - x_1) + d_{32}(x_3 - x_2)\big]. \end{aligned}$$

That is, the zero dynamics with respect to the output $\mathbf{y}_i = x_i$ has dimension two. For each Lorenz system, the zero dynamics is practically convergent (Assumption **A1** holds), as it may be showed using the function

$$V(\mathbf{z}_i - \mathbf{z}_i') = \|\mathbf{z}_i - \mathbf{z}_i'\|^2, \qquad \mathbf{z}_i = [y_i\, z_i]^\top, \quad i \in \{1, 2\}.$$

whose total derivative yields

$$\dot{V}(\mathbf{z}_i - \mathbf{z}_i') \leq -2\min\{b_i, 2\} \|\mathbf{z}_i - \mathbf{z}_i'\|^2.$$

For the Lü system, we have, defining $\mathbf{z} = [y_3\, z_3]^\top$,

$$\begin{aligned} \dot{V}(\mathbf{z} - \mathbf{z}') \leq &-2\alpha|y_3 - y_3'|^2 - 2\beta|z_3 - z_3'|^2 \\ &+ 4|x_3(t)||y_3(t) - y_3'(t)||z_3(t) - z_3'(t)|. \end{aligned}$$

Convergence in a practical sense (Assumption **A1**) holds since the trajectories are ultimately bounded hence, so is the last term on the right-hand side of the previous inequality.

Simulation results for different values of the interconnection gain σ are showed in Figure 2; it may be appreciated that the the synchronisation errors $\mathbf{e}_y(t)$ diminish as the interconnection gain is increased. The phase portraits of the three oscillators and that of the average dynamics, are also showed for three different values of σ.

In Figure 3 we show the phase portrait of the average dynamics (29a), for different values of the interconnection gain, compared to that of the emergent dynamics (9). As it is appreciated, the solutions generated by the emergent dynamics converge to an equilibrium –approximately, the point (2.9, 29.43) that is, in this case the attractor $\mathcal{A}$, defined in Assumption **A2**, is a point in the phase space. The average dynamics possesses a double scroll attractor for "small" values of σ and it becomes a point in the space at short distance from $\mathcal{A}$, roughly for $\sigma > 15$. This illustrates that the emergent dynamics constitutes, to some extent, a good "estimate" of the network's collective behaviour.

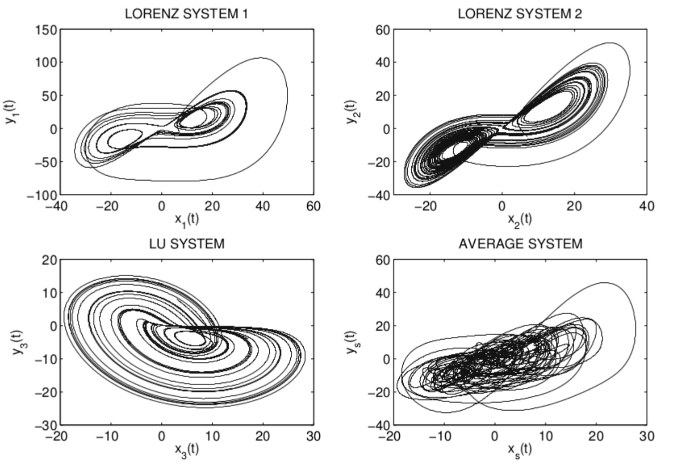

Fig. 1. Phase portraits of the three chaotic oscillators as well as that of the average dynamics, in the absence of interconnection, *i.e.*, with $\sigma = 0$.

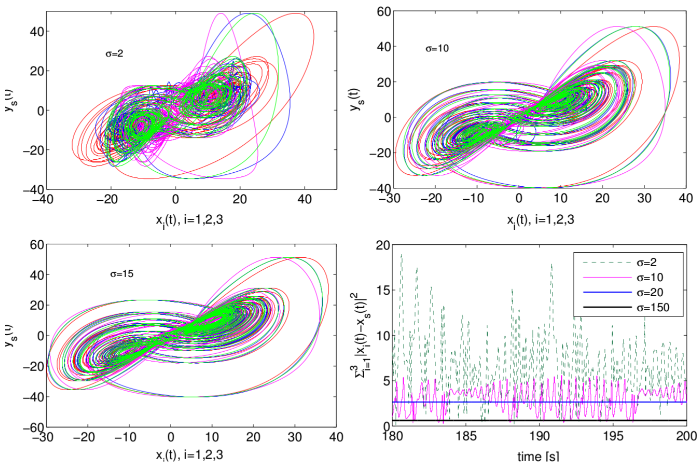

Fig. 2. First three plots: phase portraits of the three chaotic oscillators compared to that of the average unit, for different values of the interconnection gain σ. In all the plots the ordinates axes refer to $y_s(t)$. The lower-right plot depicts the synchronisation errors $\|\mathbf{e}_y(t)\|$.

The lower-right plot in Figure 3 depicts $\|\mathbf{x}_s(t) - \mathbf{x}_e(t)\| = |\mathbf{x}_s(t)|_{\mathcal{A}}$ which corresponds to the difference between the solutions $\mathbf{x}_s(t)$ of the average system (29a) and $\mathbf{x}_e(t)$, solution of the emergent dynamics $\dot{\mathbf{x}}_e = f_s(\mathbf{x}_e)$. Note that this difference diminishes as the interconnection gain increases however, it does not vanish as $\sigma \to \infty$ –see (34).

6. CONCLUSION

We have presented some preliminary but general statements on dynamic consensus and practical synchronisation of systems interconnected in heterogeneous networks. We establish the strongest property that may be achieved, that is practical synchronisation. Current work focuses on the use of this approach for controlled synchronisation.

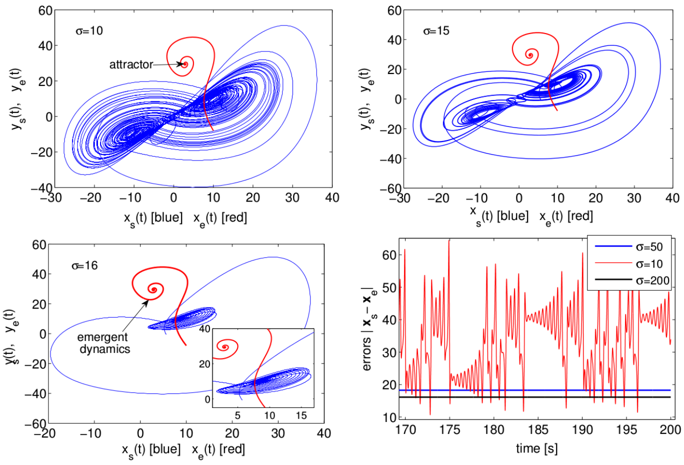

Fig. 3. x-y phase portraits corresponding to the emergent dynamics (9) and the average dynamics (29a) for different values of the interconnection gain σ. Bottom-right: $|\mathbf{x}_s(t)|_{\mathcal{A}}$.

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