On the stability and robustness of Stuart-Landau oscillators

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On the Stability and Robustness of Stuart-Landau Oscillators ⋆

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Abstract: The study of oscillations, from a dynamical-systems-theory viewpoint is a subject of interest in a variety of research domains ranging from physical sciences to engineering. One of the main motivations to study the behaviour of solutions of these complex systems lies in their role in modelling of collective behaviour, such as synchrony, which appears naturally in some biological systems but also in technological creations such as power grids. In particular, Stuart-Landau oscillators are used to model the so-called Andronov bifurcation, from oue equilibrium to a limit cycle. In this paper, we employ modern tools of stability theory to analyse the behaviour of solutions of Stuart-Landau forced and unforced oscillators. We establish suffcient conditions for global asymptotic and input-to-state stability with respect to sets.

Keywords: Synchronisation, oscillations, nonlinear systems, robust stability

1. INTRODUCTION

Generally speaking, an oscillation may be thought of as the *repetition* of a pattern; examples of oscillations in nature are endless: circadian rhythm, heart beating, neuron firing, breathing cycles, fireflies' lightening, *etc.* For the purpose of analysis of oscillating phenomena, as well as motivated by technology design, scientists and engineers have come up with a number of famous (mathematical) models of oscillators: the Lorenz system Lorenz (1963), the van der Pol system van der Pol (1920), the Lotka-Volterra equations Lotka (1910) *etc.* While the latter correspond to so-called self-sustained *autonomous* oscillators, certain *coupled* limit-cycle oscillators constitute mathematical models that allow to analyse *collective* behaviour. This plays an important role in physics, chemistry, biology, neuroscience, engineering, robotics Kamimura et al. (2003) and even computer animation Park et al. (2009). Hence, coupled nonlinear oscillators appear in various settings as *e.g.*, in gene regulatory networks Hasty et al. (2001), neuro-muscular regulation of movement and posture Kelso and Kay (1987); Haken et al. (1985a), electronic oscillator circuits Ramana et al. (2000) and Josephson-junction arrays Wiesenfeld et al. (1996) to name a few.

A particularly significant phenomenon, intrinsically linked to collective behaviour of oscillators, is *synchronisation.* Roughly speaking, this is the capability of (self-sustained) oscillators to coordinate their motion as a consequence of weak interaction, *e.g.*, to oscillate at the same frequency, with or without phase drift.

One of the pioneering schools in the formal study of synchronisation of oscillators is that of A. A. Andronov −see *e.g.*, Andronov et al. (1987, (2nd ed. 1959 in Russian). The so-called Andronov-Hopf bifurcation, which consists in the birth of a limit cycle out of an equilibrium point, is modelled by the equations of the same name (also known as Stuart-Landau oscillators). Their limit case, has become one of the most popular models of oscillators, in the control community, the so-called Kuramoto's model, which consists in a set of phase oscillators rotating at disordered intrinsic frequencies and with nonlinear couplings (the sine of their phase differences) Kuramoto (1975). This model is broadly used, for instance, in the analysis of power grids Dörfler et al. (2013) and neuronal activity. Indeed, neuronal synchrony is involved in many healthy brain functions but can also lead to pathological phenomena such as Parkinson disease or epilepsy, which are known to be linked to coherent neuronal hyper-activity. It is well accepted Tass (2007); Sarnthein et al. (2003) that appearance of such pathological brain rhythms is caused by the synchronisation in a large population of interacting neurons.

Whether they represent a collective behaviour or an isolated phenomenon, mathematical models of oscillators, in spite of their relative simplicity, capture fundamental characteristics of many *a priory* different systems with oscillatory behaviour. Whence the importance of studying oscillators' solutions. Indeed, the analysis of coupled oscillators is an area of active research not only in these application domains but also in dynamical systems Nicolis and Prigogine (1977); Jackson (1992) and automatic control Efimov (2014); Shafi et al. (2013); Pogromsky and Matveev (2013); see also some of the references therein.

One approach of analysis, in the case when some type of collective synchrony appears, consists in considering each oscillator as being forced by a weak time-dependent input from other oscillators in the network Izhikevich (2007). Motivated by such a qualitative consideration, analysis of the collective behaviour can be reduced to to that of a low-dimensional dynamics (see *e.g.*, Pyragas et al. (2004); Reddy et al. (2000);

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Tukhlina et al. (2007) and therefore on a macroscopic level, the collective dynamics can be viewed as a projection of a high-dimensional system describing the coupled oscillators onto a centre manifold corresponding to the synchronised motion.

At another level of abstraction, a mathematical concept that captures well the oscillatory behaviour is recurrence. The function $x : \mathbb{R} \to \mathbb{R}^n$ is called recurrent if for any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that for any $t \geq 0$ there exists $T(t, \varepsilon) \in (0, T_\varepsilon)$ such that

$$|x(t + T(t, \varepsilon)) - x(t)| < \varepsilon$$

In words, a recurrent trajectory keeps on passing arbitrarily close to any point and the time intervals between passages through a point and its ε-vicinity are not necessarily equal but their length cannot grow indefinitely.

The concept of recurrence clearly points at well-established notions in stability theory, in particular, stability of sets Yoshizawa (1966); Lin et al. (1996); Teel et al. (2002) and orbital stability. For instance, when studying the behaviour of a dynamic autonomous system with an oscillatory behaviour, one may want to know whether there exist closed orbits with the property that solutions starting away from them converge asymptotically to them or whether trajectories starting arbitrarily close to the orbits remain close to it forever after.

In this paper we analyse the behaviour of the Stuart-Landau oscillator from such a stability-theory viewpoint. The difficulty in the study of stability for Stuart-Landau oscillators is that the set of solutions of the differential equations is composed of two disjoint subsets: one closed orbit and one equilibrium point; the former being stable and attractive and the latter being anti-stable (unstable and repelling). Our contribution is twofold: first, we study the stability of the unforced oscillator and then, with respect to additive bounded disturbances. We employ modern tools of input to state stability, tailored for systems with disjoint sets of equilibria Angeli and Efimov (2013).

The behaviour of limit cycle oscillators under external disturbances was considered *e.g.*, in Mackey et al. (1990); Wieczorek (2011) where the effects of stochastic external signal were analysed or in Montbrió (2004) where effects of high-frequency external signals were considered. Beyond the analysis of oscillations' slutions, control of is also an important problem in many applications –see *e.g.*, Fradkov and Pogromsky (1998).

The rest of this paper is organised as follows. In the next section we discuss the model of the generalised Stuart-Landau oscillator, which is described in complex coordinates. In Section 2 we present our main results, before concluding with some remarks, in Section 3.

Notation. For a complex number, $z \in \mathbb{C}$, we use the common notation $z = z_R + \mathrm{i} z_I$ where $\mathrm{i} := \sqrt{-1}$ and z_{R}, $z_{\mathrm{I}} \in \mathbb{R}$ denote, respectively, the real and imaginary parts of z. We denote by $\bar{z}$ the complex conjugate of z, *i.e.*, $\bar{z} = z_{\mathrm{R}} - \mathrm{i} z_{\mathrm{I}}$. Correspondingly, for complex vectors $\boldsymbol{z} \in \mathbb{C}^N$, $\boldsymbol{z} = [z_1 \cdots z_N]^\top$ (where $^\top$ denotes the usual transpose operator) and complex matrices $M \in \mathbb{C}^{N \times P}$, $M = [m_{ij}]$, we denote by $\bar{\boldsymbol{z}}$ and $\bar{M}$, their respective complex conjugates, *i.e.*, $\bar{\boldsymbol{z}} = [\bar{z}_1 \cdots \bar{z}_N]^\top$ and $\bar{M} = [\bar{m}_{ij}]$. Finally, we denote by * the transpose conjugate operator for complex matrices and vectors hence, $\boldsymbol{z}^* = [\bar{z}_1 \cdots \bar{z}_N]$. Also, we use $|\cdot|$ to denote $|z| = \bar{z}z$ and $|\boldsymbol{z}| = \boldsymbol{z}^*\boldsymbol{z}$. For a closed set $\mathcal{A} \subset \mathbb{C}^n$ and $\boldsymbol{x} \in \mathbb{C}^n$, we define $|\boldsymbol{x}|_{\mathcal{A}} := \inf_{\boldsymbol{y} \in \mathcal{A}} |\boldsymbol{x} - \boldsymbol{y}|$.

2. THE GENERALIZED STUART-LANDAU OSCILLATOR

The Stuart-Landau equation, which represents a normal form of the Andronov-Hopf bifurcation, is given by

$$\dot{z} = -\nu |z|^2 z + \mu z \tag{1}$$

where $z \in \mathbb{C}$ denotes the state of the oscillator, ν, $\mu \in \mathbb{C}$ are parameters defined as $\nu = \nu_{\mathrm{R}} + \mathrm{i}\nu_{\mathrm{I}}$ and $\mu = \mu_{\mathrm{R}} + \mathrm{i}\mu_{\mathrm{I}}$. The real component of μ, μ_{R}, determines the distance from the Andronov-Hopf bifurcation. In the literature, the system (1) with $\mu_{\mathrm{R}} > 0$, is known as the Stuart-Landau oscillator Aoyagi (1995), Kentaro and Yasumasa (2008), Matthews et al. (1991). It is also known as the Andronov-Hopf oscillator Perko (2000). The Stuart-Landau equation is in normal form, which means that the limit cycle dynamics of many other oscillators can be transformed onto or can be approximated by the dynamics given by equation (1), Iooss and Adelmeyer (Jan 1999). We cite, for example, the papers Haken et al. (1985b), Tass and Haken (1996) where the van der Pol oscillator and the Haken-Kelso-Bunz (HKB) model in the neuro-physiological applications are approximated by the equations (2a) and (2b).

The analysis of oscillators (1) is well documented in the literature via, *e.g.*, Lyapunov-exponents methods (see *e.g.*, Kuznetsov (1998) and Perko (2000), for a detailed overview), or using the second Lyapunov method (see *e.g.*, Matthews and Strogatz (1990) and Pham and Slotine (2007)). Of particular interest in the study Stuart-Landau oscillators is the case when $\nu_{\mathrm{R}} > 0$ since otherwise, in the case that $\nu_{\mathrm{R}} < 0$, the solutions of the system may explode in finite time and if $\nu_{\mathrm{R}} = 0$, the oscillator becomes a simple first-order linear system. It is also clear that the origin is unstable if $\mu_{\mathrm{R}} > 0$. Its behaviour on the phase plane is illustrated in Figure 1

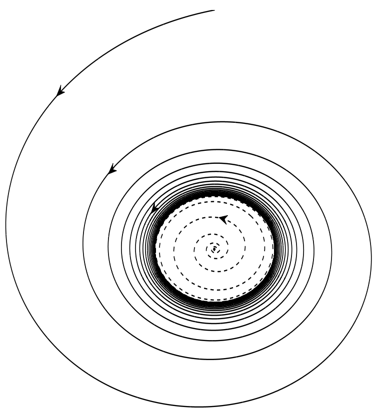

Fig. 1. Trajectories of the Stuart-Landau oscillator on the complex plane.

The behaviour of the system is more clearly illustrated in polar coordinates. That is, let $z = re^{i\varphi}$ then, the equations for the radial amplitude r and the angular variable φ can be decoupled into:

$$\dot{r} = \mu_{\mathrm{R}} r - \nu_{\mathrm{R}} r^3 \tag{2a}$$

$$\dot{\varphi} = \mu_{\mathrm{I}} - \nu_{\mathrm{I}} r^2. \tag{2b}$$

When $\mu_{\mathrm{R}} < 0$, Equation (2a) has only one stable fixed point at $r = 0$. Moreover, the latter is Lyapunov (globally exponentially) stable. However, if $\mu_{\mathrm{R}} > 0$, this equation has a stable fixed point $r = \sqrt{\frac{\mu_{\mathrm{R}}}{\nu_{\mathrm{R}}}}$, while $r = 0$ becomes unstable.

This implies, in this case, that the trajectories of the system converge to a circle of radius r, starting from initial conditions either inside or outside the circle. Thus, the latter is an attractor and the system (1) exhibits periodic oscillations. In this case, z represents the position of the oscillator in the complex plane and $z(t)$ has a stable limit cycle of the amplitude $|z| = \sqrt{\frac{\mu_\mathrm{R}}{\nu_\mathrm{R}}}$ on which it moves at its natural frequency. The bifurcation of the limit cycle from the origin that appears at the value $\mu_\mathrm{R} = 0$ is known in the literature as the Andronov-Hopf bifurcation. The curves

$$\Gamma_\alpha = \sqrt{\frac{\mu_\mathrm{R}}{\nu_\mathrm{R}}} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \tag{3}$$

define the limit cycle of the system.

Remark 1. In the analysis of (the solutions of) (1) we use some statements originally formulated for systems whose state space is Euclidean. In this regard, it is convenient to stress that, for a dynamical system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, with $\boldsymbol{x} \in \mathbb{C}^N$, one can define stability in the sense of Lyapunov similarly as for systems whose state-space is restricted to $\mathbb{R}^N$. Indeed, for a complex vector $\boldsymbol{x} = \boldsymbol{x}_\mathrm{R} + \mathrm{i}\boldsymbol{x}_\mathrm{I} \in \mathbb{C}^N$, we may define the vector $\tilde{\boldsymbol{x}} \in \mathbb{R}^{2N}$ as $\tilde{\boldsymbol{x}} := [\boldsymbol{x}_\mathrm{R}^\top \ \boldsymbol{x}_\mathrm{I}^\top]^\top$. Note that, in particular, $|\tilde{\boldsymbol{x}}|^2 = |\boldsymbol{x}|^2$. Then, provided that f admits the decomposition $f(\boldsymbol{x}) := f_\mathrm{R}(\boldsymbol{x}_\mathrm{R}, \boldsymbol{x}_\mathrm{I}) + \mathrm{i} f_\mathrm{I}(\boldsymbol{x}_\mathrm{R}, \boldsymbol{x}_\mathrm{I})$, we may re-express the dynamics of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ in a $2N$-dimensional Euclidean space, via

$$\begin{aligned} \dot{\boldsymbol{x}}_\mathrm{R} &= f_\mathrm{R}(\boldsymbol{x}_\mathrm{R}, \boldsymbol{x}_\mathrm{I}) \\ \dot{\boldsymbol{x}}_\mathrm{I} &= f_\mathrm{I}(\boldsymbol{x}_\mathrm{R}, \boldsymbol{x}_\mathrm{I}) \end{aligned}$$

and stability of the origin $\{\boldsymbol{x} = 0\} \subset \mathbb{C}^N$ is equivalent to the stability of $\{\tilde{\boldsymbol{x}} = 0\} \subset \mathbb{R}^{2N}$. Consequently, we may safely invoke statements originally formulated for systems on Euclidean spaces, to draw conclusions regarding stability of solutions of systems in the complex (hyper)plane.

Furthermore, note that the assumption that f admits the previous factorisation is a mild assumption that holds for (at least once) differentiable functions, in particular polynomials, the exponential function *etc.*

2.1 Stability of the unforced Stuart-Landau oscillator

As we have explained, the set

$$\mathcal{W} := \left\{ z \in \mathbb{C} : |z| = \sqrt{\frac{\mu_\mathrm{R}}{\nu_\mathrm{R}}} \right\} \bigcup \{z = 0\} \tag{4}$$

is invariant for the trajectories of the unforced oscillator (1). More precisely, the following theorem generalises a statement from Pham and Slotine (2007) concerning the case of real coefficients, *i.e.*, with $\nu_\mathrm{R} = 1$ and $\nu_\mathrm{I} = 0$.

Theorem 1. For the unforced Stuart-Landau oscillator, defined by Equation (1), the following statements hold true:

(1) if $\mu_R \leq 0$ then the origin $z \equiv 0$ is globally exponentially stable;

(2) if $\mu_R > 0$ then the limit cycle $\mathcal{W}_1 = \{ z \in \mathbb{C} : |z| = \sqrt{\mu_R/\nu_R} \}$ is almost globally asymptotically stable and the origin $\{z = 0\}$ is antistable[1]*. Moreover, in this case, the oscillation frequency on $\mathcal{W}_1$ is defined by*

$$\omega = \mu_I - \frac{\nu_I}{\nu_R}\mu_R.$$

[1] That is, the poles of the linearised system have all positive real parts.

Proof of Item 1. Global asymptotic stability of the origin $\{z = 0\}$ may be established using the Lyapunov function candidate $V(z) = |z|^2 = \bar{z}z$. Indeed, taking the derivative of V along trajectories of (1) we obtain

$$\begin{aligned} \dot{V}(z) &= \left[-\bar{\nu}|z|^2\bar{z} + \bar{\mu}\bar{z} \right] z + \bar{z}\left[-\nu|z|^2 z + \mu z \right] \\ &= -(\nu + \bar{\nu})|z|^4 + (\mu + \bar{\mu})|z|^2 \\ &= -2\nu_\mathrm{R}|z|^4 + 2\mu_\mathrm{R}|z|^2. \end{aligned}$$

Since $\mu_\mathrm{R} \leq 0$, we have $\dot{V}(z) \leq -|\mu_\mathrm{R}||z|^2$ for all $z \in \mathcal{C}$ and global exponential stability of the origin follows.

Proof of Item 2. Anti-stability of the origin follows trivially by evaluating the total derivative of $V(z) = |z|^2$ along the trajectories of Equation (1) linearised around the origin, *i.e.*, $\dot{z} = \mu z$. Indeed, locally, $\dot{V}(z) = \mu_\mathrm{R}|z|^2$ where $\mu_\mathrm{R} > 0$.

Next, to analyse the stability of the limit cycle $\mathcal{W}_1$, we introduce the Lyapunov function candidate

$$V(z) = \frac{1}{4\nu_\mathrm{R}}\left[|z|^2 - \alpha\right]^2, \tag{5}$$

where $\alpha = \mu_\mathrm{R}/\nu_\mathrm{R}$. Notice that $V(z) = 0$ for all $z \in \mathcal{W}_1$ and it is positive otherwise.

Evaluating the total derivative of V, along the solutions of (1), we get

$$\begin{aligned} \dot{V}(z) &= \frac{1}{2\nu_\mathrm{R}}\left[|z|^2 - \alpha\right]\left[\dot{\bar{z}}z + \bar{z}\dot{z}\right] \\ &= \frac{1}{2\nu_\mathrm{R}}\left[|z|^2 - \alpha\right]\left[(-\bar{\nu}|z|^2\bar{z} + \bar{\mu}\bar{z})z + \bar{z}(-\nu|z|^2 z + \mu z)\right] \end{aligned}$$

and, after regrouping the terms in the last bracket, we obtain

$$\begin{aligned} \dot{V}(z) &= \frac{1}{2\nu_\mathrm{R}}\left[|z|^2 - \alpha\right]\left[-(\nu + \bar{\nu})|z|^4 + (\mu + \bar{\mu})|z|^2\right] \\ &= \frac{1}{\nu_\mathrm{R}}\left[|z|^2 - \alpha\right]\left[-\nu_\mathrm{R}|z|^2 + \mu_\mathrm{R}\right]|z|^2 \\ &= -\left[|z|^2 - \alpha\right]^2|z|^2. \end{aligned}$$

We conclude that $\dot{V}$ is negative definite with respect to $\mathcal{W}_1$ that is, $\dot{V} < 0$ for all $z \notin \mathcal{W}_1$ and $\dot{V} = 0$ for all $z \in \mathcal{W}_1$. Since the origin is an antistable equilibrium point, $\mathcal{W}_1$ is almost globally asymptotically stable.

It also follows that $r \to \sqrt{\mu_\mathrm{R}/\nu_\mathrm{R}}$ hence, after Equation (2b) and $\omega = \dot{\varphi}$, we have $\omega \to \mu_\mathrm{I} - (\nu_\mathrm{I}\mu_\mathrm{R})/\nu_\mathrm{R}$. ■

2.2 Stability of the forced Stuart-Landau oscillator

For the case when $\mu_\mathrm{R} > 0$, in the previous section we proved that the Stuart-Landau oscillator without input, given by (1), presents a limit cycle which is almost globally asymptotically stable. Now, we analyse the stability and robustness of the solutions of a forced generalised Stuart-Landau oscillator, as defined by the equation

$$\dot{z} = -\nu|z|^2 z + \mu z + u \tag{6}$$

where $u \in \mathbb{C}$ is an input to the oscillator. That is, we analyse the input-to-state stability of this system, *i.e.*, stability with respect to external disturbances. Furthermore, the notion of *almost input-to-state stability*, introduced in Angeli (2001) (see also Angeli and Praly (2011)), applies to the case of an equilibrium point which is stable for all initial states except for a set of measure zero. For Stuart-Landau oscillators, for which there exists a disjoint invariant set, not consituted of disjoint equilibria, we use a recently developed refined tool for input-to-state stability with respect to decomposable invariant sets –see Angeli and Efimov (2013). For the sake of clarity we start by putting in context the essential technical tools that we use.

The mathematical setting The main advantage of the approach introduced in Angeli and Efimov (2013) is that it allows to analyse the robustness properties of the complex invariant sets without the use of tools involving manifolds and dimensionality arguments, while being applicable to the case when the invariant set is compact. For the sake of self-containedness, we briefly recall below the essential definitions and statements from Angeli and Efimov (2013) which are required for the robustness analysis of (6).

Consider a nonlinear system

$$\dot{x} = f(x, d), \tag{7}$$

where the map $f : M \times D \to T_x M$ is assumed to be of class $\mathcal{C}^1$, M is an n dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without boundary and D is a closed subset of $\mathbb{R}^m$ containing the origin.

Let $\mathcal{W}$ be a compact invariant set containing all α and ω limit sets of the unforced system

$$\dot{x} = f(x, 0)$$

and which admits a finite decomposition without cycles, *i.e.*,

$$\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i \tag{8}$$

where $\mathcal{W}_i$ denote non-empty disjoint compact sets which form a *filtration ordering* of $\mathcal{W}$. According to Angeli and Efimov (2013) cycles and filtration ordering are defined as follows. First, we introduce the "domains of attraction" and "repulsion" of a set Λ, respectively, as

$$W^s(\Lambda) := \{x_\circ \in M : |x(t, x_\circ, d)|_\Lambda \to 0 \text{ as } t \to +\infty\}$$
$$W^u(\Lambda) := \{x_\circ \in M : |x(t, x_\circ, d)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Then, for two subsets, $\Lambda \subset M$ and $\Gamma \subset M$, we define the relation $\Lambda \prec \Gamma$ as

$$\Lambda \prec \Gamma \quad \Leftrightarrow \quad W^s(\Lambda) \cap W^u(\Gamma) \neq \varnothing. \tag{9}$$

Based on these notations, we say that the decomposition $\mathcal{W}_1, \ldots, \mathcal{W}_k$ of $\mathcal{W}$ presents an r-cycle if there is an ordered r-tuple such that $\mathcal{W}_1 \prec \cdots \prec \mathcal{W}_r \prec \mathcal{W}_1$; a 1-cycle if for some i we have $[W^u(\Lambda_i) \cap W^s(\Lambda_i)] - \Lambda_i \neq \varnothing$. Finally, a filtration ordering is an ordered sequence of sets Λ_i such that $\Lambda_i \prec \Lambda_j$ for $i \leq j$.

For the case of the Stuart-Landau oscillator, we have the following. Firstly, $\mathcal{W} \subset \mathbb{C}$ defined in (4) is a compact invariant set which contains the α and ω limit sets of (6). This set admits the finite decomposition in compact sets:

$$\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2, \qquad \mathcal{W}_1 := \left\{ z \in \mathbb{C} : |z| = \sqrt{\frac{\mu_R}{\nu_R}} \right\}$$
$$\mathcal{W}_2 := \{z = 0\}.$$

Then, we have following for the system (6):

- $W^s(\mathcal{W}_1) = \{z_\circ \in \mathbb{C} : |z(t, z_\circ)|_{\mathcal{W}_1} \to 0 \text{ as } t \to +\infty\}$. This corresponds to the set of initial conditions generating trajectories which converge to the circumference $\mathcal{W}_1$. Since, according to Theorem 1, $\mathcal{W}_1$ is almost globally asymptotically stable, $W^s(\mathcal{W}_1) = \mathbb{C} - \{0\}$.
- $W^u(\mathcal{W}_1) = \{z_\circ \in \mathbb{C} : |z(t, z_\circ)|_{\mathcal{W}_1} \to 0 \text{ as } t \to -\infty\}$. This corresponds to the set of initial conditions generating trajectories that are repulsed away from the circle $\mathcal{W}_1$ hence, $W^u(\mathcal{W}_1) = \varnothing$ since $\mathcal{W}_1$ is almost globally attractive.
- $W^s(\mathcal{W}_2) = \{z_\circ \in \mathbb{C} : |z|_{\mathcal{W}_2} = |z(t, z_\circ)| \to 0 \text{ as } t \to +\infty\}$. This corresponds to the domain of attraction of the origin, however, we know from the proof of Theorem 1 that $\{0\}$ is antistable hence, $W^s(\mathcal{W}_2) = \varnothing$.
- $W^u(\mathcal{W}_2) = \{z_\circ \in \mathbb{C} : |z(t, z_\circ)|_{\mathcal{W}_2} \to 0 \text{ as } t \to -\infty\}$. This corresponds to the set of initial states generating trajectories which are repulsed away from the origin, hence, it corresponds to the disk whose boundary corresponds to $\mathcal{W}_1$, taken away the origin, *i.e.*, $W^u(\mathcal{W}_2) = \{z \in \mathbb{C} : 0 < |z| < \sqrt{\mu_R/\nu_R}\}$.

We conclude that $\mathcal{W}$ admits the filtration ordering $\mathcal{W}_1 \prec \mathcal{W}_2$ because $[\mathbb{C} - \{0\}] \cap W^u(\mathcal{W}_2) \neq \varnothing$ but it contains no 2-cycle because $\mathcal{W}_2 \not\prec \mathcal{W}_1$ since $W^s(\mathcal{W}_2) \cap W^u(\mathcal{W}_1) = \varnothing$. It contains no 1-cycle either because $[W^u(\mathcal{W}_1) \cap W^s(\mathcal{W}_1)] - \mathcal{W}_1 = \varnothing$ and $[W^u(\mathcal{W}_2) \cap W^s(\mathcal{W}_2)] - \mathcal{W}_2 = \varnothing$.

The previous characterisation of decomposable compact invariant sets constitutes a formal framework to establish conditions under which a perturbed system admits an input-to-stable stability Lyapunov function, as defined next.

Definition 2.1. Angeli and Efimov (2013). We say that a $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is an input-to-state-stability Lyapunov function for (7) if there exist $\mathcal{K}_\infty$ functions α_1, α_2, α and γ, and a non-negative real c such that

$$\alpha_1(|x|_\mathcal{W}) \leq V(x) \leq \alpha_2(|x|_\mathcal{W}) + c, \tag{10}$$

the function V is constant on each $\mathcal{W}_i$ and the following dissipation condition holds:

$$DV(x) f(x, d) \leq -\alpha(|x|_\mathcal{W}) + \gamma(|d|). \tag{11}$$

The following statement, which corresponds to a paraphrasis of (Angeli and Efimov, 2013, Theorem 1), serves to establish robust stability of Stuart-Landau oscillators (6). Indeed, as we show farther below, Stuart-Landau oscillators admit input-to-state stability Lyapunov functions.

Theorem 2. Consider the nonlinear system (7) and let $\mathcal{W}$ correspond to the union of disjoint compact invariant sets containing all α and ω and limit sets of the unforced system $\dot{x} = f(x, 0)$, such that $\mathcal{W}$ admits a filtration ordering without cycles. Then, the following are equivalent:

- *the system (7) possesses the asymptotic gain property, i.e., there exists $\eta \in \mathcal{K}_\infty$ such that, for all $x \in M$ and all measurable essentially bounded inputs d, the solutions of (7), with initial conditions $x_\circ$, are defined for all $t \geq 0$ and*
$$\limsup_{t \to +\infty} |x(t, x_\circ, d)|_\mathcal{W} \leq \eta(\|d\|_\infty) \tag{12}$$
where $\|d\|_\infty := \sup_{t \geq 0} |d(t)|$.
- *The system (7) admits an input-to-state stability Lyapunov function therefore, it is input to state stable with respect to the input u and the set $\mathcal{W}$.*

Robustness analysis of Stuart-Landau oscillator We are ready to apply the framework briefly recalled above to analysis of the system (6) which, as we have showed, possesses an invariant set decomposable in invariant compacts which admit a filtration ordering with no cycles. These compacts correspond to the (antistable) origin of the complex plane and the almost globally asymptotically stable circle of radius $\sqrt{\mu_R/\nu_R}$. According to Theorem 2, in order to establish input to state stability with respect to $\mathcal{W}$ it is sufficient and necessary to establish that the Stuart-Landau oscillator possesses the asymptotic gain property. To that end, we start by defining the norm $|\cdot|_{\mathcal{W}}$, as follows.

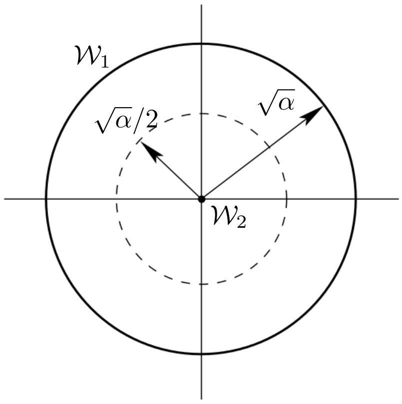

Fig. 2. Illustration of $|z|_{\mathcal{W}}$

$$|z|_{\mathcal{W}} = \begin{cases} \sqrt{3}\,|z| & \text{if } |z| \leq \sqrt{\alpha}/2, \\ \sqrt{\left|\,|z|^2 - \alpha\,\right|} & \text{if } |z| \geq \sqrt{\alpha}/2 \end{cases} \tag{13}$$
$$\alpha := \mu_R/\nu_R$$

The following result ensures that the system (6) possesses the asymptotic gain property, *i.e.*, asymptotically, the distance between the oscillator's trajectory and set $\mathcal{W}$ becomes proportional to the size of perturbations, $\|d\|_\infty$.

Theorem 3. Consider the system (6) with initial conditions $z_\circ \in \mathbb{C}$ and let the set $\mathcal{W}$ be defined by (4). Then, the system (6) has the asymptotic gain property, i.e.,

$$\limsup_{t\to+\infty} |z(t, z_\circ, u)|_{\mathcal{W}} \leq \eta\big(\|u\|_\infty\big). \tag{14}$$

Proof. It follows using the input-to-state-stability Lyapunov function candidate V defined in (5), which we used previously to prove almost global asymptotic stability for the system (6). One can show that this function satisfies the inequalities (10), (11). This is omitted here due to space constraints. ∎

3. CONCLUSIONS

We presented two results on the stability of disjoint sets of equilibria for Stuart-Landau oscillators. Our results are motivated by the study of collective behaviour of interconnected oscillators, in particular, by the capacity of these systems to exhibit synchrony. Current research is carried out in this direction.

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