Strong iISS for neutrally stable systems by saturated linear state feedback
Antoine Chaillet, Yacine Chitour

To cite this version:
Antoine Chaillet, Yacine Chitour. Strong iISS for neutrally stable systems by saturated linear state feedback. 34th IEEE Chinese Control Conference, Jul 2015, Hangzhou, China. pp.459 - 463, 10.1109/ChiCC.2015.7259680 . hal-01264971

HAL Id: hal-01264971
https://centralesupelec.hal.science/hal-01264971
Submitted on 12 Apr 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Strong iISS for neutrally stable systems by saturated linear state feedback

Antoine CHAILLET, Yacine CHITOUR
1. L2S - Univ. Paris Sud - Supelec. 3, rue Joliot-Curie. 91192 Gif sur Yvette, France
E-mail: antoine.chaillet@supelec.fr, yacine.chitour@iss.supelec.fr

Abstract: In this note, we propose a linear state-feedback that ensures Strong iISS to any neutrally stable system affected by actuator saturation. This robustness property was recently proposed as a compromise between the strength of input-to-state stability (ISS) and the generality of integral input-to-state stability (iISS). It ensures not only that the system is globally asymptotically stable in the absence of disturbances, but also that trajectories are bounded in response to any perturbation whose amplitude is below a certain threshold. It also guarantees that the state converges to the origin in response to any vanishing disturbance. Here Strong iISS is proven with respect to any additive disturbance acting outside the saturation, without any matching requirement.

Key Words: Saturated feedback, LTI systems, robustness

1 Introduction

It is well known that a necessary and sufficient condition for the stabilizability of a linear time-invariant (LTI) plant by saturated feedback is that its internal dynamics has no pole with positive real part [17]. Several works in the literature have proposed bounded stabilizing feedback for particular classes of systems whose internal dynamics exhibits no exponential instability. For neutrally stable systems (meaning LTI systems whose internal dynamics exhibits no unbounded solutions), it is also known that stabilization can be achieved using a saturated linear static feedback. Nonetheless, some classes of systems, although having no poles with positive real parts, cannot be stabilized by saturated linear static state-feedback; this class includes chains of three or more integrators [6, 18]. Several control strategies, by state or output feedback, have been proposed to stabilize such systems, including nested saturations [19] and linear combinations of saturated linear functions [16]. For LTI systems having no eigenvalues with positive real parts, it has been shown in [15] that global stabilization by bounded output feedback can be achieved if and only if the system is both detectable and stabilizable.

Beyond stabilization, it is often desirable to ensure some robustness properties in order to cope, for instance, with parameter uncertainty, measurement noise or exogenous disturbances. $L_p$-stabilization with respect to disturbances acting inside the saturation was achieved in [11] based on the low- and high-gain control law introduced in [10]. This robust stabilization has been extended to disturbances acting outside the saturation in [20] for chains of integrators under matching conditions. Also, explicit estimates of $L_p$ input/output gains have been obtained for neutrally stable systems based on a saturated linear static feedback in [9].

Another natural candidate for the evaluation of robustness to exogenous inputs is the framework of input-to-state stability (ISS, [12, 14]) and its weaker variant integral ISS (iISS, [13]). In [1], a saturated linear state-feedback was proposed to ensure ISS with respect to sufficiently small disturbances despite parameter uncertainty for systems of dimension smaller than or equal to three, as well as feedforward systems. ISS of neutrally stable systems with respect to disturbances acting outside the saturation have been proposed in [2] under matching conditions. Other approaches guarantee ISS and iISS with bounded control to nonlinear systems based on Arstein’s “universal constructions” [3, 8].

Among other robustness features, ISS ensures a bounded response to any bounded disturbance. Intuitively, one may expect that bounded controls fail in general at guaranteeing the solutions’ boundedness if the disturbance acts outside the saturation with a too large amplitude (unless matching conditions are met between the saturated actuator and the disturbance: see e.g. [2, 20]). At first sight, for these systems, nothing more than ISS with respect to small inputs can be established, thus providing no information on the system’s behavior for larger inputs. In this note, we provide sufficient conditions under which a more interesting property, namely Strong iISS, can be achieved by saturated feedback. This property, introduced in [4], not only guarantees ISS with respect to small inputs but also iISS. In particular, it ensures global asymptotic stability in the absence of disturbances, a bounded response to any disturbance whose amplitude is below a given threshold, but also the existence of solutions at all times even for disturbances above that threshold. It also guarantees that the system converges to zero in response to any vanishing disturbance, and it is known to be preserved under cascade interconnection. More details on this property can be found in [4, 5].

In this paper, we show that any LTI system with neutrally stable internal dynamics can be made Strongly iISS by a simple saturated linear state feedback. Strong iISS is established with respect to disturbances acting outside the saturation, and no matching condition between the actuation and the disturbances is assumed. We provide an explicit estimate of the maximum disturbance amplitude that can be tolerated without compromising solutions’ boundedness. This estimate is confronted to numerical observations in the example of an harmonic oscillator robust stabilization.
2 Notation and problem statement

2.1 Notation

A function \( \alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is of class \( PD \) if it is continuous and positive definite. It is of class \( K \) if, in addition, it is increasing. It is of class \( K_{\infty} \) if it is of class \( K \) and \( \lim_{s \to +\infty} \alpha(s) = \infty \). Let \( \beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) belongs to class \( KL \) if, given any fixed \( t \geq 0 \), \( \beta(\cdot, t) \in K \) and, given any fixed \( s \geq 0 \), \( \beta(s, \cdot) \) is continuous, nonincreasing and tends to zero as its argument tends to infinity. Given \( x \in \mathbb{R}^n \), \( |x| \) denotes its Euclidean norm. Given a positive integer \( p \), \( U^p \) denotes the set of all measurable locally essentially bounded functions \( d : \mathbb{R}_{\geq 0} \to \mathbb{R}^p \). For a given \( d \in U^p \), \( \|d\| := \text{ess sup}_{t \geq 0} |d(t)| \). Given a constant \( R > 0 \), we let \( U_{< R} \) denote the set \( \{ d \in U^p : \|d\| < R \} \).

2.2 Considered class of systems

We recall that a neutrally stable matrix is any matrix \( A \) such that the solutions of \( \dot{x} = Ax \) are bounded. This is equivalent to requiring that there exists a symmetric positive definite matrix \( P \) such that \( A^T P + P A \) is a negative semi-definite matrix. This paper focuses on the class of neutrally stable systems affected by actuator saturation and additive exogenous disturbances:

\[
\dot{x} = Ax - B\sigma(u) + d, \tag{1}
\]

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control input, and \( d \in \mathbb{R}^m \) is the disturbance. No matching condition on the disturbance \( d \) is assumed: it can affect all directions of \( \mathbb{R}^n \), not only those defined by the \( B \) matrix. The function \( \sigma : \mathbb{R}^m \to \mathbb{R}^m \) is assumed to belong to the class \( S^m \), as defined below.

Definition 1 (\( S \) and \( S^m \) functions, [9]) \( S \) denotes the set of all locally Lipschitz bounded function \( S : \mathbb{R} \to \mathbb{R} \) satisfying: \( S(0) = 0 \), \( S(s) > 0 \) for all \( s \neq 0 \), \( \lim_{s \to +\infty} S(s)/s > 0 \), and \( \lim_{s \to -\infty} |S(s)| > 0 \). A vector function \( \sigma : \mathbb{R}^m \to \mathbb{R}^m \) is said to be in the class \( S^m \) if it reads \( \sigma(u) = (S_1(u_1), \ldots, S_m(u_m))^T \), for all \( u \in \mathbb{R}^m \), where \( S_i \in S \) for each \( i \in \{1, \ldots, m\} \).

We stress that the class \( S \) does not include, but is limited to, “usual” saturation functions such as sigmoids, arctan, tanh or sign function \( s \mapsto \text{sign}(s) \min\{\ell, |s|\} \) with \( \ell > 0 \). For more details about the functions included in this class, the reader is invited to refer to [9].

We state the following useful property of \( S^m \)-functions, which is a straightforward \( m \)-dimensional extension of [9, Remark 2].

Fact 1 If \( \sigma \in S^m \), then there exist constants \( a, b, K > 0 \) and a measurable diagonal-matrix valued function \( \tau : \mathbb{R}^m \to \mathbb{R}^{m \times m} \) such that, for all \( u \in \mathbb{R}^m \),

\[
a I \leq \tau(u) \leq b I, \quad |\sigma(u) - \tau(u) u| \leq Ku|\sigma(u)|. \tag{2}
\]

2.3 The Strong iISS property

As already stressed, the purpose of this paper is to robustly stabilize the system (1) by linear state feedback. We rely on the formalism of iISS introduced by Sontag in [12]. We recall that a system is ISS if there exists \( \beta \in KL \) and \( \gamma \in K_{\infty} \) such that its solutions satisfy \( |x(t)| \leq \beta(|x_0|, t) + \gamma(||d||) \) at all times. In particular, ISS ensures boundedness of solutions for any bounded disturbance, and a vanishing state in response to any vanishing perturbation; see [14] for a survey.

Without matching requirements between the disturbance and the actuation, it is hopeless to try to make (1) ISS. Indeed, due to the bounded nature of \( \sigma \), we can always pick a sufficiently large disturbance \( d \) that makes solutions diverge.

An alternative robustness property could be the integral input-to-state stability (iISS, [13]). Instead of considering the impact of the amplitude of the disturbance \( d \), iISS evaluates the effect of the input energy on the solutions behavior. More precisely, a system is iISS if there exist \( \beta \in KL \) and \( \mu_1, \mu_2, \mu \in K_{\infty} \) such that its solutions satisfy:

\[
|x(t)| \leq \beta(|x_0|, t) + \mu_1 \int_0^t \mu_2(||d(s)||)ds
\]

at all times. Like ISS, iISS ensures global asymptotic stability (GAS) of the disturbance-free system. However, it is much weaker robustness property than ISS, as some iISS systems can be destabilized by arbitrarily small, and even vanishing, disturbances. In the present context, it seems that better robustness properties can be achieved. We believe that a better candidate to evaluate the robustness of (1) to exogenous disturbances is the Strong iISS, recently introduced in [4].

Definition 2 (Strong iISS, [4]) The system \( \dot{x} = f(x, d) \) is said to be Strongly iISS if it is both ISS wrt small inputs and iISS. In other words, there exist \( R > 0, \beta \in KL \) and \( \mu_1, \mu_2, \mu \in K_{\infty} \) such that, for all \( d \in U^p \), all \( x_0 \in \mathbb{R}^n \) and all \( t \geq 0 \), its solution satisfies the two properties:

\[
|x(t)| \leq \beta(|x_0|, t) + \mu_1 \int_0^t \mu_2(||d(s)||)ds
\]

\[

\|d\| < R \Rightarrow |x(t)| \leq \beta(|x_0|, t) + \mu(||d||). \tag{3}
\]

The constant \( R \) is then called an input threshold.

It can easily be seen from this definition that Strong iISS ensures GAS in the absence of disturbances, just like iISS and ISS. Moreover, the state remains bounded if the disturbance amplitude is below the input threshold \( R \). It was also shown in [4] that, if the disturbance \( d \) tends to zero, then the solution of a Strongly iISS system will also eventually tend to zero. Finally, like ISS, Strong iISS is well behaved under cascade interconnection [5].

2.4 From neutral stability to skew-symmetry

Reasoning as in [9, Section 3.2] the question of robustly stabilizing (1) when \( A \) is neutrally stable boils down to the case when \( A \) is skew-symmetric. This comes from the fact that any neutrally stable matrix is similar to \( \text{diag}(A_H, A_S) \) where \( A_H \) is a Hurwitz matrix and \( A_S \) is skew-symmetric. This observation lets us consider that \( A \) is skew-symmetric without loss of generality. This will be assumed in the remainder of the paper.

3 Main result

3.1 Strong iISS by saturated linear feedback

Beyond \( L_p \)-stability results for disturbances entering the function \( \sigma \), it was shown in [9] that the linear state feedback \( u = B^T x \) globally asymptotically stabilizes the saturated
Finally, let $\chi$ be the positive constant defined as

$$\chi = \frac{|2a - b|}{2a} |P_0 B|,$$  

and let $P := P_0 + \chi I$. Then (1) in closed loop with the control law $u = B^T x$ is Strongly iISS with input threshold

$$R = \frac{1}{2K |P B| + |P| (4K |P B|/3)^{1/3}}.$$  

Note that the existence of such constants $a, b, K$ and such a function $\tau$ is ensured by Fact 1. The existence of a matrix $P_0$ satisfying (7) is guaranteed by the controllability of the pair $(A, B)$. The proof of this result follows straightforwardly from that of Theorem 1. We provide its main steps in Section 5.2 for the sake of completeness.

For some particular saturations, the function $\tau$ of Fact 1 can be picked as a constant matrix. This includes for instance saturation functions of the form $\sigma(s) = (S(s), \ldots, S(s))^T$ where

$$S(s) = \text{sat}_s(s) := \text{sign}(s) \min\{\ell |s|; 1\}, \quad \forall s \in \mathbb{R},$$

where $\ell > 0$ denotes its linear slope. For such functions, the expression of the input threshold estimate is slightly simpler. We summarize this in the following corollary.

**Corollary 2** Let $S \in \mathcal{S}^m$ and let $A \in \mathbb{R}^{n \times n}$ be a skew-symmetric matrix such that the pair $(A, B)$ is controllable. Assume that there exist $\bar{r}, K > 0$ such that

$$|\sigma(u) - \bar{r} u| \leq K \sigma(u), \quad \forall u \in \mathbb{R}^m.$$  

Let $P_0 \in \mathbb{R}^{n \times n}$ be any symmetric positive definite matrix such that

$$(A - 2\bar{r} B B^T)^T P_0 + P_0 (A - 2\bar{r} B B^T) \leq -2I,$$

as ensured by the controllability of the pair $(A, B)$. Finally, let $P := P_0 + \frac{1}{2} |P_0 B| I$. Then the system (1) in closed loop with the control law $u = B^T x$ is Strongly iISS with the input threshold given in (9).

### 4 Example: the harmonic oscillator

We finally illustrate the above results with the following planar system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{sat}_\ell(u) + d,$$

where $\ell > 0$ and sat was introduced in (10). This system is in the form of (1) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma(s) = \text{sat}_s(s).$$

Note that the perturbation is not matching, as it affects the dynamics of $x_1$ although no control is available in that direction. The pair $(A, B)$ being controllable, we conclude with Theorem 1 that the state feedback $u = B^T x = x_2$ makes the system (13) Strong iISS.

In addition, we can estimate its input threshold by invoking Corollary 2. To that end, notice that $|\text{sat}_s(s) - \ell s| \leq \ell s \text{sat}_\ell(s)$ for all $s \in \mathbb{R}$, meaning that (11) is satisfied with $\bar{r} = K = \ell$. Furthermore, the Lyapunov equation (12) is satisfied with the following symmetric positive definite matrix:

$$P_0 = \frac{1}{2} \begin{pmatrix} 2\ell + 1/\ell & 1 \\ 1 & 1/\ell \end{pmatrix}.$$  

Noticing that $|P_0 B| = \frac{1}{2} \sqrt{1 + \ell^2}$, the input threshold of (13) in closed loop with $u = B^T x = x_2$ can be computed as a function of the slope $\ell$ of the saturation according to the estimate (9). We obtain the curve reported in Figure 1.
In order to study the derivative of the saturation function generated by Fact 1 and let $\tilde{V}_1 := |x| x^T P_x$, we will study the time derivative of $V_1$ and $V_2$ along the solutions of (1) separately. We start by the function $V_1$: recalling that $x^T A x = 0$ for all $x \in \mathbb{R}^n$ (as $A$ is skew-symmetric), direct computations show that

$$\dot{V}_1 = |x| x^T \dot{x} = - |x| x^T B \sigma(B^T x) + |x| x^T d. \quad (16)$$

In order to study the derivative of $V_2$, consider any initial state $x_0 \in \mathbb{R}^n$ and any input signal $d \in U$. Let $\tau$ be the function generated by Fact 1 and let $\tilde{A}(t) := A - B \tau(B^T x(t)) B^T$, $\forall t \geq 0$, where $x(\cdot)$ denotes the solution of (1) starting from $x_0$. Note that the saturation function $\sigma$ being locally Lipschitz and bounded, the existence of the solutions of (1) is ensured at all forward time, which makes the above matrix $\tilde{A}(t)$ well defined at all times $t \geq 0$. With this notation, the system (1) in closed loop with $u = B^T x$ can be rewritten as

$$\dot{x} = \tilde{A}(t)x + B \left[ (B^T x) B^T x - \sigma(B^T x) \right] + d. \quad (17)$$

Now, in view of Fact 1 and recalling that $\tau(u)$ is a diagonal matrix for each $u \in \mathbb{R}^n$, it holds that

$$\tau(B^T x(t)) + \tau(B^T x(t))^T \geq 2aI, \quad \forall t \geq 0,$$

for some constant $a > 0$ independent of $x_0$. It follows that all the conditions of Lemma 1 are fulfilled with $P = P^T$ such that

$$\tilde{A}(t)^T P + P \tilde{A}(t) \leq -I, \quad \forall t \geq 0.$$

The derivative of the function $V_2(x) = x^T P x$ with this particular matrix $P$ along the solutions of (17) reads

$$\dot{V}_2 = x^T \left( \tilde{A}(t)^T P + P \tilde{A}(t) \right) x$$

$$\quad + 2x^T PB \left[ \sigma(B^T x) - \sigma(B^T x) \right] + 2x^T P d \leq - |x|^2 + 2 |x| |P B| \left[ \sigma(B^T x) - \sigma(B^T x) \right] + 2x^T P d. \quad (18)$$

Recalling that, in view of Fact 1,

$$\left[ \sigma(B^T x) - \sigma(B^T x) \right] \leq K x^T B \sigma(B^T x),$$

it follows that

$$\dot{V}_2 \leq - |x|^2 + 2K |x| |P B| x^T B \sigma(B^T x) + 2x^T P d. \quad (19)$$

Combining (16) and (18) and picking $c = 2 |P B|$, we obtain that the derivative of $V(x) = c V_1(x) + V_2(x)$ along the solutions of (1) satisfies

$$\dot{V} \leq - |x|^2 + 2 |x| |d| + 2 |P| |x| |d|.$$

Thus the function $\tilde{V}$, defined in (14), satisfies

$$\tilde{V} \leq \frac{V(x)}{[1 + V(x)]^{2/3}}$$

$$\quad \leq - \frac{|x|^2}{[1 + V(x)]^{2/3}} + c \frac{|x|^2}{[1 + V(x)]^{2/3}} |d|$$

$$\quad + 2 |P| \frac{|x|}{[1 + V(x)]^{2/3}} |d|. \quad (19)$$

We now analyze the two nonnegative term of this upper bound separately. First, it holds that

$$\frac{|x|^2}{[1 + V(x)]^{2/3}} = \frac{|x|^2}{[1 + x^T P_x + \frac{c}{3} |x|^3]^{2/3}}$$

$$\quad \leq \frac{|x|^2}{[1 + \frac{c}{3} |x|^3]^{2/3}}$$

$$\quad \leq \left( \frac{3}{c} \right)^{2/3}.$$
In the same way, we have that

\[
\frac{|x|}{[1 + V(x)]^{2/3}} = \frac{|x|}{\left[1 + x^TPx + \frac{c}{3}|x|^3\right]^{2/3}} \\
\leq \frac{|x|}{\left[1 + \frac{c}{3}|x|^3\right]^{2/3}} \\
\leq \left(\frac{|x|^{3/2}}{1 + \frac{c}{3}|x|^3}\right)^{2/3} \\
\leq \left(\frac{3}{4c}\right)^{1/3}, \quad (21)
\]

where the last inequality follows from the fact that the function \(s \mapsto s/(1 + cs^2/3)\) reaches its maximum \(\sqrt{3/4c}\) at \(s = \sqrt{3/c}\). Plugging (20)-(21) into (19) leads, as claimed, to

\[
\dot{V}(x) \leq -W(x) + \gamma(|d|),
\]

where \(\gamma\) denotes the following \(K_\infty\) function:

\[
\gamma(s) := \begin{cases} (9c)^{1/3} + |P| \left(\frac{6}{c}\right)^{1/3} & \text{if } s > 0, \\
0 & \text{if } s = 0\end{cases},
\]

and \(W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\) is the function defined as

\[
W(x) := \frac{|x|^2}{\left[1 + x^TPx + \frac{c}{3}|x|^3\right]^{2/3}}, \quad \forall x \in \mathbb{R}^n,
\]

where we recall that \(c = 2K|PB|\). Notice that \(W\) is continuous and positive definite and satisfies

\[
\liminf_{|x| \to \infty} W(x) = \liminf_{|x| \to \infty} \frac{|x|^2}{(c/3)^{2/3}|x|^2} = \left(\frac{3}{c}\right)^{2/3}.
\]

Thus, we can apply [4, Theorem 1] to conclude that (1) is Strongly iISS with input threshold

\[
\gamma^{-1}\left((3/c)^{2/3}\right) = \frac{1}{c + |P|(2c/3)^{1/3}} = \frac{1}{2K|PB| + |P|(4K|PB|/3)^{1/3}}.
\]

5.2 Proof of Corollary 1

Since \(aI \leq \tau(u) \leq bI\), the constant \(\chi\) introduced in (8) satisfies (6). By Lemma 1, the proof of Theorem 1 can be repeated with \(P = P_0 + \chi I\). The rest of the proof is identical.

6 Conclusion

Focusing on the specific class of LTI systems whose internal dynamics is neutrally stable, we have shown that a simple linear static state-feedback ensures Strong iISS to both matching and non-matching additive disturbances acting outside the saturation. This result was illustrated on an academic example, that underlined the limits of the input threshold estimate and suggests that further work is needed to choose the feedback gains in order to obtain a tighter input threshold estimate. Finally, further research can be envisioned to check the robustness properties of saturated stabilization of chains of integrators; a first step in that direction will be presented in [7].

References