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# GLOBAL STABILIZATION OF LINEAR SYSTEMS WITH BOUNDS ON THE FEEDBACK AND ITS SUCCESSIVE DERIVATIVES\*

JONATHAN LAPORTE, ANTOINE CHAILLET AND YACINE CHITOUR †

**Abstract.** We address the global stabilization of linear time-invariant (LTI) systems when the magnitude of the control input and its successive time derivatives, up to an arbitrary order  $p \in \mathbb{N}$ , are bounded by prescribed values. We propose a static state feedback that solves this problem for any admissible LTI systems, namely for stabilizable systems whose internal dynamics has no eigenvalue with positive real part. This generalizes previous work done for single-input chains of integrators and rotating dynamics.

**Key words.** Bounded control, Bounded control rate, Linear systems, Stabilization of systems by feedback

**AMS subject classifications.** 93D05, 93D15

**1. Introduction.** The study of control systems subject to input constraints is motivated by the fact that signals delivered by physical actuators may be limited in amplitude, and may not evolve arbitrarily fast. An *a priori* bound on the amplitude of the control signal is usually referred to as *input saturation* whereas a bound on the variation of control signal is referred to as *rate saturation* (e.g [15]).

Stabilization of linear time-invariant systems (LTI for short) with input saturation has been widely studied in the literature. Such a system is given by

$$(S) \quad \dot{x} = Ax + Bu,$$

where  $x \in \mathbb{R}^n$ ,  $u$  belongs to a bounded subset of  $\mathbb{R}^m$ ,  $A$  is an  $n \times n$  matrix and  $B$  is an  $n \times m$  one. Global stabilization of  $(S)$  can be achieved if and only if the LTI system is asymptotically null controllable with bounded controls, i.e., it can be stabilized in the absence of input constraint and the eigenvalues of  $A$  have non positive real parts. Saturating a linear feedback law may fail at globally stabilizing  $(S)$  as it was observed first in [4] and then in [18] for the special case of integrator chains (i.e., when  $A$  is the  $n$ -th Jordan block and  $B = (0 \cdots 0 \ 1)^T$ ). As shown for instance in [12], optimal control can be used to define a globally stabilizing feedback for  $(S)$  but, when the dimension is greater than 3, deriving a closed form for this stabilizer becomes extremely difficult. The first globally stabilizing feedback with rather simple closed form (nested saturations) was provided in [20] for chains of integrators and then in [19] for the general case. In [9], a global feedback stabilizer for  $(S)$  was built by relying on control Lyapunov functions arising from a mere existence result. Other globally stabilizing feedback laws for  $(S)$  have been proposed with an additional property of robustness with respect to perturbations. In [14], using low-and-high gain techniques, a robust stabilizer was proposed to ensure semiglobal stability, meaning that the control gains can be tuned in such a way that the basin of attraction contains any prescribed compact subset of  $\mathbb{R}^n$ . This restriction has been removed in [13], where

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36 the authors provided a global feedback stabilizer for  $(S)$  which is robust with respect  
 37 to perturbations, based on an earlier idea due to Megretsky [11]. Nonetheless, the  
 38 feedback laws of [13] and [11] require to solve a nonlinear optimization problem at  
 39 every point  $x \in \mathbb{R}^n$ , which makes its practical implementation questionable. An  
 40 easily implementable global feedback stabilizer for  $(S)$  which is robust with respect to  
 41 perturbations was proposed in [2], but it only covers the multiple integrator case and  
 42 it is discontinuous since it is based on sliding mode techniques. Robust stabilization of  
 43  $(S)$  was also addressed in [1] by relying on the control Lyapunov function techniques  
 44 developed in [9]. Finally it is important to notice that, while global stabilizers own  
 45 the advantage to be valid for all initial conditions, they are typically low performance  
 46 and less suited for practical implementation because they do not allow the input to  
 47 exceed the saturation limits, which often results in input signals that stay well below  
 48 the maximum value.

49 In contrast to stabilization of LTI systems subject to input saturation, there are  
 50 much less results available in the literature regarding global stabilization under rate  
 51 saturation, i.e., when the first time derivative of the control signal is also *a priori*  
 52 bounded. When only rate saturation is considered (with no constraint on the input  
 53 magnitude), the objective can easily be reduced to the standard case of magnitude  
 54 saturation by considering the augmented system where the control is an extra variable  
 55 and its derivative becomes the new control. The control objective is more challenging  
 56 when both magnitude and rate constraints are considered. In [3], the authors rely  
 57 on a backstepping procedure to build a bounded globally stabilizing feedback with a  
 58 bounded rate, but the methodology does not allow to *a priori* impose a prescribed  
 59 rate. In [16], a dynamic feedback law inspired from [11] is constructed and can even be  
 60 generalized to take into account constraints on higher time derivatives of the control  
 61 signal. However, as mentioned previously, the numerical efficiency of such feedbacks is  
 62 definitely questionable. A rather involved global feedback stabilizer for  $(S)$  achieving  
 63 amplitude and rate saturations was also obtained in [17] for affine systems with a stable  
 64 free dynamics. This corresponds in our setting to requiring that the matrix  $A$  is stable,  
 65 i.e.,  $A^T + A \leq 0$  (up to similarity) and therefore fails at covering chains of integrators.  
 66 Finally, let us mention the references [8], [10] for semiglobal stabilization results and  
 67 [5] for local stabilization results using LMIs and anti-windup design. One should also  
 68 mention [21] where a nonlinear small gain theorem is given for the behaviour analysis  
 69 of control systems with saturation.

70 The results presented here encompass input and rate saturations as special cases.  
 71 More precisely, given any integer  $p$ , we construct a globally stabilizing feedback for  
 72  $(S)$  such that the control signal and its  $p$  first time derivatives are bounded by arbi-  
 73 trary prescribed positive values, along all trajectories of the closed-loop system. This  
 74 problem has already been solved by the authors in [7] for the multiple integrator and  
 75 skew-symmetric cases. The solution given in that paper for the multiple integrator  
 76 case consisted in considering appropriate nested saturation feedbacks. We also indi-  
 77 cated in [7] that these feedbacks fail at ensuring global stability in the skew-symmetric  
 78 case and we then provided an *ad hoc* feedback law for this specific case. Here, we solve  
 79 the general case with a unified strategy.

80 The paper should be seen as a first theoretical step towards the global stabiliza-  
 81 tion of an LTI system when the input signal is delivered by a dynamical actuator  
 82 that limits the control action in terms of magnitude and  $p$  first time derivatives. Fur-  
 83 ther developments are needed to explicitly take into account the dynamics of such an  
 84 actuator. Possible extensions of this work may also address the question of global sta-  
 85 bilization by smooth feedback laws (i.e.,  $C^\infty$  with respect to time) when *all* successive

86 derivatives need to be bounded by prescribed values.

87 The paper is organized as follows. In Section 2, we precisely state the problem we  
 88 want to tackle, the needed definitions as well as the main results we obtain, namely  
 89 Theorem 1 for the single input case and Theorem 2 for the multiple input case. Section  
 90 3 contains the proof of the main results. In Section 3.1.1 we show that the proof of  
 91 Theorem 1 is a consequence of two propositions. In the first one (Proposition 1), we  
 92 show that the feedback proposed in Theorem 1 is indeed a globally stabilizing feedback  
 93 for  $(S)$ . We actually prove a stronger result dealing with robustness properties of this  
 94 feedback, as it is required in [20] and [19]. The second proposition (Proposition 2)  
 95 specifically deals with bounding the  $p$  first derivatives of the control signal by relying  
 96 on delicate estimates. Section 3.2.1 contains the proof of Theorem 2 which is a  
 97 consequence of Proposition 1 and Proposition 3, the latter providing estimates on the  
 98 successive time derivatives of the control signal. In Section 4, we provide a numerical  
 99 validation of our main result based on a four-dimensional system made of a cascade of  
 100 two rotating dynamics. We close the paper by an Appendix, where we gather several  
 101 technical results used throughout the paper.

102 *Notations.* We use  $\mathbb{R}$  and  $\mathbb{N}$  to denote the sets of real numbers and the set of  
 103 non negative integers respectively. Given a set  $I \subset \mathbb{R}$  and a constant  $a \in \mathbb{R}$ , we let  
 104  $I_{\geq a} := \{x \in I : x \geq a\}$ . Given  $m, k \in \mathbb{N}$ , we define  $\llbracket m, k \rrbracket := \{l \in \mathbb{N} : l \in [m, k]\}$ . For  
 105 a given set  $M$ , the boundary of  $M$  is denoted by  $\partial M$ . The factorial of  $k$  is denoted  
 106 by  $k!$  and the binomial coefficient is denoted  $\binom{k}{m} := \frac{k!}{m!(k-m)!}$ .

107 Given  $k \in \mathbb{N}$  and  $n, p \in \mathbb{N}_{\geq 1}$ , we say that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$  is of class  
 108  $C^k(\mathbb{R}^n, \mathbb{R}^p)$  if its differentials up to order  $k$  exist and are continuous, and we use  $f^{(k)}$   
 109 to denote the  $k$ -th order differential of  $f$ . By convention,  $f^{(0)} := f$ .

110 Given  $n, m \in \mathbb{N}_{\geq 1}$ ,  $\mathbb{R}^{n,m}$  denotes the set of  $n \times m$  matrices with real coefficients.  
 111 The transpose of a matrix  $A$  is denoted by  $A^T$ . The identity matrix of dimension  $n$  is  
 112 denoted by  $\mathbb{I}_n$ . We say that an eigenvalue of  $A$  is *critical* if it has zero real part and  
 113 we set  $\mu(A) := s(A) + z(A)$  where  $s(A)$  is the number of conjugate pairs of nonzero  
 114 purely imaginary eigenvalues of  $A$  (counting multiplicity), and  $z(A)$  is the multiplicity  
 115 of the zero eigenvalue of  $A$ . We define  $A_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , and  $b_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

116 We use  $\|x\|$  to denote the Euclidean norm of an arbitrary vector  $x \in \mathbb{R}^n$ . Given  
 117  $\delta > 0$  and  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ , we say that  $f$  is eventually bounded by  $\delta$ , and we write  
 118  $\|f(\cdot)\| \leq_{ev} \delta$ , if there exists  $T > 0$  such that  $\|f(t)\| \leq \delta$  for all  $t \geq T$ .

119 **2. Problem statement and main results.** Given  $n \in \mathbb{N}_{\geq 1}$  and  $m \in \mathbb{N}_{\geq 1}$ ,  
 120 consider the LTI system defined by

$$121 \quad (1) \quad \dot{x} = Ax + Bu,$$

122 where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $A \in \mathbb{R}^{n,n}$ , and  $B \in \mathbb{R}^{n,m}$ . Assume that the pair  $(A, B)$   
 123 is stabilizable and that all the eigenvalues of  $A$  have non positive real parts. Recall  
 124 that these assumptions on  $(A, B)$  are necessary and sufficient for the existence of a  
 125 bounded continuous state feedback  $u = k(x)$  which globally asymptotically stabilizes  
 126 the origin of (1), see [19].

127 Given an integer  $p$  and a  $(p+1)$ -tuple of positive real numbers  $(R_j)_{0 \leq j \leq p}$ , we  
 128 want to derive a static state feedback law whose magnitude and  $p$ -first time derivatives  
 129 along all trajectories of the closed-loop system are bounded by  $R_j$ ,  $j \in \llbracket 0, p \rrbracket$ .

130 **DEFINITION 1** (*feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$* ). Given  $n \in \mathbb{N}_{\geq 1}$ ,  $m \in \mathbb{N}_{\geq 1}$   
 131 and  $p \in \mathbb{N}$ , let  $(R_j)_{0 \leq j \leq p}$  be a  $(p+1)$ -tuple of positive real numbers. We say that  
 132  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$  for system (1) if it is of class

133  $C^p(\mathbb{R}^n, \mathbb{R}^m)$  and, for every trajectory of the closed-loop system  $\dot{x} = Ax + B\nu(x)$ , the  
 134 control signal  $U : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ ,  $t \mapsto U(t) := \nu(x(t))$  satisfies  $\sup_{t \geq 0} \|U^{(j)}(t)\| \leq R_j$   
 135 for all  $j \in \llbracket 0, p \rrbracket$  and all  $t \geq 0$ . The function  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is said to be a feedback  
 136 law  $p$ -bounded for system (1), if there exists a  $(p+1)$ -tuple of positive real numbers  
 137  $(R_j)_{0 \leq j \leq p}$  such that  $\nu(\cdot)$  is a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$  for system (1).

138 We stress that the above definition includes only *static* state feedback laws, mean-  
 139 ing control laws that depend solely on the current state value. Based on this definition,  
 140 we can write our stabilization problem of Bounded Higher Derivatives as follows.

141 **PROBLEM (BHD).** Given  $p \in \mathbb{N}$  and a  $(p+1)$ -tuple of positive real numbers  
 142  $(R_j)_{0 \leq j \leq p}$ , design a state feedback law  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that the origin of the  
 143 closed-loop system  $\dot{x} = Ax + B\nu(x)$  is globally asymptotically stable (GAS for short)  
 144 and the feedback  $\nu$  is a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$  for system (1).

145 Our construction to solve Problem (BHD) will often use the property of *Small*  
 146 *Input Small State with linear gain* ( $SISS_L$  for short) developed in [19]. We recall  
 147 below its definition

148 **DEFINITION 2** ( $SISS_L$ , [19]). Given positive  $\Delta, N$ , the control system  $\dot{x} = f(x, u)$ ,  
 149 with  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , is said to be  $SISS_L(\Delta, N)$  if, for every  $\delta \in (0, \Delta]$  and every  
 150 bounded measurable signal  $e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$  eventually bounded by  $\delta$ , then any solution  
 151 of  $\dot{x} = f(x, e)$  is eventually bounded by  $N\delta$ . A control system  $\dot{x} = f(x, u)$  is said to  
 152 be  $SISS_L$  if it is  $SISS_L(\Delta, N)$  for some  $\Delta, N > 0$ . An input-free system  $\dot{x} = f(x)$  is  
 153 called  $SISS_L$ , if the control system  $\dot{x} = f(x) + u$  is  $SISS_L$ .

154 **REMARK 1.** Note that if  $\dot{x} = f(x)$  is  $SISS_L$ , then all its solutions converge to the  
 155 origin. To see this, pick a sequence  $(\delta_k)_{k \in \mathbb{N}}$  of positive numbers tending to zero so that  
 156  $\delta_0 \leq \Delta$  and apply the  $SISS_L$  property to every  $\delta_k$  with the zero input. Note, however,  
 157 that the  $SISS_L$  property does not necessarily ensure GAS in the absence of input, as  
 158 it does not imply stability of the origin.

159 When a state feedback law ensures both global asymptotic stability and  $SISS_L$ ,  
 160 we refer to it as an  $SISS_L$ -stabilizing feedback.

161 **DEFINITION 3** ( $SISS_L$ -stabilizing feedback). Given a control system  $\dot{x} = f(x, u)$   
 162 with  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , we say that a state feedback law  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is stabilizing if  
 163 the origin of the closed-loop system  $\dot{x} = f(x, \nu(x))$  is globally asymptotically stable. If,  
 164 in addition, this closed-loop system is  $SISS_L$ , then we say that  $\nu$  is  $SISS_L$ -stabilizing.

165 As mentioned before the state feedback law given in [7], which solves Problem  
 166 (BHD) for the special case of multiple integrators, simply made use of nested sat-  
 167 urations with carefully chosen saturation functions. We recall next why this state  
 168 feedback construction cannot work in general. For that purpose it is enough to con-  
 169 sider the 2D simple oscillator case which is the control system given by  $\dot{x} = \omega A_0 x + b_0 u$ ,  
 170 with  $x = (x_1, x_2)^T$ ,  $u \in \mathbb{R}$  and  $\omega > 0$ . This system is one of the two basic systems to  
 171 be stabilized by means of a bounded feedback, as explained in [19]. According to [7],  
 172 one must then consider a stabilizing feedback law  $u = -\sigma(k^T x)$ , where  $k = (k_1, k_2)^T$   
 173 is a fixed vector in  $\mathbb{R}^2$  and  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is a saturation function, i.e., a bounded, continu-  
 174 ously differentiable function satisfying  $s\sigma(s) > 0$  for  $s \neq 0$  and  $\sigma^{(1)}(0) > 0$ . Note that  
 175  $k$  is chosen so that the linearized system at  $(0, 0)$  is Hurwitz. In particular this implies  
 176 that  $k_2 \neq 0$ . Pick now the following sequence of initial conditions  $(l, -k_1 l / k_2)_{l \geq 1}$ . A  
 177 straightforward computation yields that the first time derivative of the control along  
 178 each trajectory satisfies  $\dot{u}(0) = -\sigma^{(1)}(0)\omega l(k_1^2/k_2 + k_2)$ , which grows unbounded as  $l$   
 179 tends to infinity. Therefore the feedback law proposed in [7] can not be a 1-bounded

180 feedback for this system.

181 In order to solve Problem (BHD) for the 2D oscillator, we showed in [7] that a  
 182 feedback law of the type  $u_{k,\alpha} := \frac{k^T x}{(1+\|x\|^2)^\alpha}$  with  $k \in \mathbb{R}^2$  and  $\alpha \geq 1/2$  does the job  
 183 and it also solves Problem (BHD) in case the matrix  $A$  in (1) is stable. However,  
 184 we are not able to show whether  $u_{k,\alpha}$  stabilizes or not the system in the case where  
 185  $A := \begin{pmatrix} A_0 & b_0^T b_0 \\ 0 & A_0 \end{pmatrix}$ . It turns out that the previous issue is as difficult as asking if a  
 186 saturated linear feedback stabilizes or not the above-mentioned 4D case, which is an  
 187 open problem. It is therefore not immediate how to address the general case. This  
 188 is why our main result is a non trivial extension of the solution of Problem (BHD)  
 189 provided for the two-dimensional oscillator.

190 **2.1. Single input case.** For the case of single input systems, the solution of  
 191 Problem (PHB) is given by the following statement.

192 **THEOREM 1 (Single input).** *Given  $n \in \mathbb{N}_{>0}$ , consider a single input system*  
 193  $\dot{x} = Ax + bu$  *where  $x \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n,n}$  and  $b \in \mathbb{R}^{n,1}$ . Assume that  $A$  has no eigenvalue*  
 194 *with positive real part and that the pair  $(A, b)$  is stabilizable. Then, given any  $p \in \mathbb{N}$*   
 195 *and any  $(p + 1)$ -tuple  $(R_j)_{0 \leq j \leq p}$  of positive real numbers, there exist vectors  $k_i \in \mathbb{R}^n$*   
 196 *and matrices  $T_i \in \mathbb{R}^{n,n}$ ,  $i \in \llbracket 1, \mu(A) \rrbracket$ , such that the feedback law  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}$  defined*  
 197 *as*

$$198 \quad (2) \quad \nu(x) = - \sum_{i=1}^{\mu(A)} \frac{k_i^T x}{(1 + \|T_i x\|^2)^{1/2}},$$

199 *is a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$  and  $SISS_L$ -stabilizing for system  $\dot{x} = Ax +$   
 200  $bu$ .*

201 In view of Definition 3, the feedback law (2) globally asymptotically stabilizes  
 202 the origin of (1), and thus solves Problem (BHD). We stress that, even though the  
 203 exact computation of the control gains  $k_i$  is quite involved (see proof in Section 3),  
 204 the structure of the proposed feedback law (2) is rather simple. It should also be  
 205 noted that, unlike the results developed in [7], this state feedback law applies to any  
 206 admissible single-input systems (including rotating dynamics and integrator chains)  
 207 in a unified manner.

208 **2.2. Multiple input case.** To give the main result for LTI system with multiple  
 209 inputs we need this following definition.

210 **DEFINITION 1 (Reduced controllability form).** *Given  $n \in \mathbb{N}$  and  $q \in \mathbb{N}$ , a LTI*  
 211 *system is said to be in reduced controllability form if it reads*

$$212 \quad (3) \quad \begin{aligned} \dot{x}_0 &= A_{00}x_0 + A_{01}x_1 + A_{02}x_2 + \dots + A_{0q}x_q + b_{01}u_1 + b_{02}u_2 + \dots + b_{0q}u_q, \\ \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + \dots + A_{1q}x_q + b_{11}u_1 + b_{22}u_2 + \dots + b_{1q}u_q, \\ \dot{x}_2 &= A_{22}x_2 + \dots + A_{2q}x_q + b_{22}u_2 + \dots + b_{2q}u_q, \\ &\vdots \\ \dot{x}_q &= A_{qq}x_q + b_{qq}u_q, \end{aligned}$$

213 *where, for some  $(q + 1)$ -tuple  $(n_i)_{0 \leq i \leq q+1}$  in  $\mathbb{N} \times (\mathbb{N}_{>0})^q$  with  $\sum_{i=0}^q n_i = n$ ,  $A_{00} \in$   
 214  $\mathbb{R}^{n_0, n_0}$  *is Hurwitz, for every  $i \in \llbracket 1, q \rrbracket$  all the eigenvalues of  $A_{ii} \in \mathbb{R}^{n_i, n_i}$  are critical,  
 215  $b_{ii} \in \mathbb{R}^{n_i, 1}$  *and the pairs  $(A_{ii}, b_{ii})$  are controllable.***

216 From Lemma 5.1 in [19], we can consider that system (1) is already given in the  
 217 reduced controllability form without loss of generality. We can now establish the  
 218 solution of Problem (BHD) for the multiple input case.

219 **THEOREM 2 (Multiple input).** *Let  $p \in \mathbb{N}$  and consider any  $(p+1)$ -tuple  $(R_j)_{0 \leq j \leq p}$   
 220 of positive real numbers. Given  $n \in \mathbb{N}$  and  $q \in \mathbb{N}$ , consider system (3). Then, there  
 221 exist  $\kappa_1, \dots, \kappa_q$  such that:*

- 222 *i) for every  $i \in \llbracket 1, q \rrbracket$ ,  $\kappa_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$  is a feedback law  $p$ -bounded and  $\text{SISS}_L$ -  
 223 stabilizing for  $\dot{x}_i = A_{ii}x_i + b_{ii}u_i$ ;*  
 224 *ii) the state feedback law  $\mu = [\mu_1, \dots, \mu_q]^T$  given by*

$$225 \quad (4) \quad \mu_i(x_i, \dots, x_q) := \frac{\kappa_i(x_i)}{(1 + \|x_{i+1}\|^2 + \dots + \|x_q\|^2)^{p+1}}, \quad \forall i \in \llbracket 1, q-1 \rrbracket,$$

$$226 \quad (5) \quad \mu_q(x_q) := \kappa_q(x_q),$$

227 *is a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$  and  $\text{SISS}_L$ -stabilizing for system  
 228 (3).*

229 This statement provides a unified control law solving Problem (BHD) for all  
 230 admissible LTI systems. It allows in particular multi-input systems, which was not  
 231 covered in [7].

### 232 3. Proof of the main results.

233 **3.1. Proof of Theorem 1.** In this section, we prove Theorem 1. For that  
 234 purpose, we first reduce the argument to establishing of Propositions 1 and 2 given  
 235 below. The first one states that the feedback given in Theorem 1 is  $\text{SISS}_L$  stabilizing  
 236 for (S) in the case of single input. The second proposition provides an estimate of  
 237 the successive time derivatives of the control signal.

238 **3.1.1. Reduction of the proof of Theorem 1 to the proofs of Proposi-**  
 239 **tions 1 and 2.** Let  $n \in \mathbb{N}_{\geq 1}$ ,  $p \in \mathbb{N}$  and  $(R_j)_{0 \leq j \leq p}$  be a  $(p+1)$ -tuple of positive  
 240 real numbers. Define  $\underline{R} := \min_{j \in \llbracket 0, p \rrbracket} R_j$ . Consider a single input linear system  
 241  $\dot{x} = Ax + bu$  where  $x \in \mathbb{R}^n$ ,  $A$  and  $b$  are  $n \times n$  and  $n \times 1$  matrices respectively. We  
 242 assume that the pair  $(A, b)$  is stabilizable and that all the eigenvalues of  $A$  have non  
 243 positive real parts. As observed in [19], it is sufficient to consider the case where the  
 244 pair  $(A, b)$  is controllable and all eigenvalues of  $A$  are critical. Indeed, since  $(A, b)$   
 245 is stabilizable there exists a linear change of coordinates transforming  $A$  and  $b$  into  
 246  $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$  and  $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ , where  $A_1$  is Hurwitz, the eigenvalues of  $A_2$  are critical and the  
 247 pair  $(A_2, b_2)$  is controllable. Then, it is immediate to see that we only have to treat  
 248 the case where  $A$  has only critical eigenvalues. From now on, we therefore assume that  
 249  $A$  has only eigenvalues with zero real parts, and that the pair  $(A, b)$  is controllable.

250 Our construction uses the following linear change of coordinates given by [19,  
 251 Lemma 5.2]. This decomposition puts the original system in a triangular form made  
 252 of one-dimensional integrators and two-dimensional oscillators.

253 **LEMMA 1 (Lemma 5.2 in [19]).** *Let  $\dot{x} = Ax + bu$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ , be a controllable  
 254 single input linear system. Assume that all the eigenvalues of  $A$  are critical. Let  
 255  $\pm i\omega_1, \dots, \pm i\omega_{s(A)}$  be the nonzero eigenvalues of  $A$ . Let  $(a_2, \dots, a_{\mu(A)})$  be a family of*

256 *positive numbers. Define*

$$\begin{aligned}
 257 \quad & \theta_{i,k} = 1, \quad \text{for } k = i + 1, \\
 258 \quad (6) \quad & \theta_{i,k} = \prod_{h=i}^{k-2} 1/a_{h+1}, \quad \text{for } i + 2 \leq k \leq \mu(A) + 1.
 \end{aligned}$$

259 *Then there exists a linear change of coordinates that puts  $\dot{x} = Ax + bu$  in the form*

$$\begin{aligned}
 260 \quad & \dot{y}_i = \omega_i A_0 y_i + b_0 \sum_{k=i+1}^{s(A)} \theta_{i,k} b_0^T y_k \\
 261 \quad & + b_0 \sum_{k=s(A)+1}^{\mu(A)} \theta_{i,k} y_k + \theta_{i,\mu(A)+1} b_0 u, \quad i = 1, \dots, s(A), \\
 262 \quad (7) \quad & \dot{y}_i = \sum_{k=i+1}^{\mu(A)} \theta_{i,k} y_k + \theta_{i,\mu(A)+1} u, \quad i = s(A) + 1, \dots, \mu(A) - 1,
 \end{aligned}$$

$$263 \quad \dot{y}_{\mu(A)} = u, \quad \text{if } \mu(A) > s(A),$$

265 *where  $y_i \in \mathbb{R}^2$  for  $i = 1, \dots, s(A)$ ,  $y_i \in \mathbb{R}$  for  $i = s(A) + 1, \dots, \mu(A) - 1$ , and  $y_{\mu(A)} \in \mathbb{R}$*   
 266 *if  $\mu(A) > s(A)$ .*

267 *With no loss of generality, we prove Theorem 1 for system (7). We rely on a*  
 268 *candidate feedback  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}$  under the form*

$$269 \quad (8) \quad \kappa(y) = - \sum_{i=1}^{s(A)} \frac{Q_{i,\mu(A)} b_0^T y_i}{\left(1 + \sum_{m=i}^{\mu(A)} \|y_m\|^2\right)^{1/2}} - \sum_{i=s(A)+1}^{\mu(A)} \frac{Q_{i,\mu(A)} y_i}{\left(1 + \sum_{m=i}^{\mu(A)} \|y_m\|^2\right)^{1/2}},$$

270 *with*

$$271 \quad (9) \quad Q_{i,\mu(A)} := \prod_{l=i}^{\mu(A)} a_l,$$

272 *where  $a_1, \dots, a_{\mu(A)}$  are positive constants that will be picked in such a way that the*  
 273 *feedback law (8) is a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$ , and  $SISS_L$ -stabilizing*  
 274 *for system (7). To that aim, we rely on the next two propositions, respectively proven*  
 275 *in Sections 3.1.2 and 3.1.3.*

276 **PROPOSITION 1.** *Let  $\dot{x} = Ax + bu$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ , be a controllable single input*  
 277 *linear system. Assume that all the eigenvalues of  $A$  are critical. Let  $\pm i\omega_1, \dots, \pm i\omega_{s(A)}$*   
 278 *be the nonzero eigenvalues of  $A$ . Then, there exist  $\mu(A) - 1$  functions  $\bar{a}_i : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ ,*  
 279  *$i \in \llbracket 1, \mu(A) - 1 \rrbracket$  such that for any constants  $a_1, \dots, a_{\mu(A)}$  satisfying*

$$280 \quad a_{\mu(A)} \in (0, 1], \quad a_i \in (0, \bar{a}_i(a_{i+1})], \quad \forall i \in \llbracket 1, \mu(A) - 1 \rrbracket,$$

282 *the feedback law (8) is  $SISS_L$ -stabilizing for system (7).*

283 **PROPOSITION 2.** *Let  $\dot{x} = Ax + bu$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ , be a controllable single input*  
 284 *linear system. Assume that all the eigenvalues of  $A$  are critical. Let  $\pm i\omega_1, \dots, \pm i\omega_{s(A)}$*   
 285 *be the nonzero eigenvalues of  $A$ . Let  $a_i$ ,  $i \in \llbracket 1, \mu(A) \rrbracket$ , be positive constants in  $(0, 1]$ .*  
 286 *Then, there exist a positive constant  $c_{\mu(A)}$ , and continuous functions  $c_i : \mathbb{R}_{>0}^{\mu(A)-i} \rightarrow$*



287  $\mathbb{R}_{>0}$ ,  $i \in \llbracket 1, \mu(A) - 1 \rrbracket$ , such that for any trajectory of the closed-loop system (7) with  
 288 the feedback law (8), the control signal  $U : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  defined by  $U(t) := \nu(y(t))$  for all  
 289  $t \geq 0$  satisfies, for all  $k \in \llbracket 0, p \rrbracket$ ,

$$290 \quad \left| U^{(k)}(t) \right| \leq a_\mu c_{\mu(A)} + \sum_{i=1}^{\mu(A)-1} a_i c_i(a_{\mu(A)}, \dots, a_{i+1}), \quad \forall t \geq 0.$$

291 Based on these two propositions, pick  $a_{\mu(A)} \in (0, 1]$  in such a way that

$$292 \quad a_{\mu(A)} \leq \frac{\underline{R}}{(p+1)c_{\mu(A)}}.$$

293 Choose recursively  $a_i \in (0, 1]$ ,  $i = \mu(A) - 1, \dots, 1$ , such that

$$294 \quad a_i \leq \bar{a}_i(a_{i+1}), \quad a_i \leq \frac{\underline{R}}{(p+1)c_i(a_{\mu(A)}, \dots, a_{i+1})},$$

295 where the functions  $c_i$  appearing above are defined in Proposition 2. By Proposition 1,  
 296 the feedback law (8) is  $SISS_L$ -stabilizing for system (7). Moreover, as a consequence  
 297 of Proposition 2, for any trajectory of the closed-loop system (7) with the feedback law  
 298 (8), the control signal  $U : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  defined by  $U(t) := \nu(y(t))$  for all  $t \geq 0$  satisfies  
 299  $\sup_{t \geq 0} |U^{(k)}(t)| \leq \underline{R}$  for all  $k \in \llbracket 0, p \rrbracket$ . Thus, the feedback law (8) is a feedback law  
 300  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$  for system (7). Since there is a linear change of coordinate  
 301 ( $y = Tx$ ) that puts (7) into the original form  $\dot{x} = Ax + bu$ , the feedback law defined  
 302 given in (2) can be picked as

$$303 \quad \nu(x) := \kappa(Tx)$$

304 and it is a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$ , and  $SISSSL$ -stabilizing for (1). To  
 305 sum up, the proof of Theorem 1 boils down to establishing Propositions 1 and 2.

306 **3.1.2. Proof of Proposition 1.** Proposition 1 is proved by induction on  $\mu(A)$ .  
 307 More precisely, we show that the following property holds true for every positive  
 308 integer  $\mu$ .

309  $(P_\mu)$ : Let  $s, z \in \mathbb{N}$  be such that  $s + z = \mu$  and  $\omega_1, \dots, \omega_s$  be positive constants.  
 310 Then there exist  $\mu - 1$  functions  $\bar{a}_i : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ ,  $i \in \llbracket 1, \mu - 1 \rrbracket$  such that for  
 311 any constants  $a_1, \dots, a_\mu$  satisfying

$$312 \quad a_\mu \in (0, 1], \quad a_i \in (0, \bar{a}_i(a_{i+1})], \quad \forall i \in \llbracket 1, \mu - 1 \rrbracket,$$

314 the feedback law (8) is  $SISS_L$ -stabilizing for system (7), with  $\mu(A) = \mu$ ,  
 315  $s(A) = s$ , and  $z(A) = z$ . Moreover the linearization of this closed-loop  
 316 system around the origin is asymptotically stable (AS).

317 In order to start the argument, we give intermediate results whose proofs are given  
 318 in Appendix and which will be used for the initialization step of the induction and  
 319 the inductive step. The first statement establishes  $SISS_L$  for the one-dimensional  
 320 integrator.

321 **LEMMA 2.** *Let  $\epsilon > 1$ . For every  $\beta > 0$ , the scalar system given by*

$$322 \quad (10) \quad \dot{x} = -\beta \frac{x}{(1+x^2)^{1/2}}$$

323 *is  $SISS_L(\frac{\beta}{2}, \frac{2\epsilon}{\beta})$ , its origin is GAS and its linearisation around zero is AS.*

324 The next lemma guarantees that the two-dimensional oscillator is  $SISS_L$ .

325 **LEMMA 3.** *For every  $\omega > 0$ , there exist  $\Gamma, N > 0$  such that for any  $\beta \in (0, 1]$  the*  
 326 *two-dimensional system given by*

$$327 \quad (11) \quad \dot{x} = \omega A_0 x - \beta b_0 \frac{b_0^T x}{(1 + \|x\|^2)^{1/2}}$$

328 *is  $SISS_L(\beta\Gamma, \frac{N}{\beta})$ , its origin is GAS and its linearisation around zero is AS.*

329 We now start the inductive proof of  $(P_\mu)$ . For  $\mu = 1$ , we have to consider two cases.  
 330 Either  $z = 1$  and  $s = 0$  corresponding to the simple integrator

$$331 \quad (12) \quad \dot{y}_1 = u, \quad \text{with} \quad u = \kappa(y_1) = -a_1 \frac{y_1}{(1 + y_1^2)^{1/2}},$$

332 or  $s = 1$  and  $z = 0$  corresponding to the simple oscillator

$$333 \quad (13) \quad \dot{y}_1 = \omega_1 A_0 y_1 + b_0 u, \quad \text{with} \quad u = \kappa(y_1) = -a_1 \frac{b_0^T y_1}{(1 + \|y_1\|^2)^{1/2}},$$

334 for some  $\omega_1 > 0$ . In both cases,  $(P_1)$  can be readily deduced by invoking Lemma 2  
 335 and 3 respectively. Given  $\mu \in \mathbb{N}_{>0}$ , assume that  $(P_\mu)$  holds. In order to establish  
 336  $(P_{\mu+1})$ , it is sufficient to consider the following two cases:

337 **case i)**  $z = \mu + 1$ , i.e, all the eigenvalues of  $A$  are zero (multiple integrator);

338 **case ii)**  $s \geq 1$ , i.e some eigenvalues of  $A$  have non zero imaginary part (multiple  
 339 integrator with rotating modes).

340 In both cases we reduce our problem to the choice of only one constant  $a_1$  using the  
 341 inductive hypothesis.

342 *Case i).* Let  $(a_1, \dots, a_{\mu+1})$  be a set of positive numbers to be chosen later. Con-  
 343 sider the multiple integrator given by

$$344 \quad \dot{y}_i = \sum_{k=i+1}^{\mu+1} \theta_{i,k} y_k + \theta_{i,\mu+2} u, \quad i = 1, \dots, \mu,$$

$$345 \quad \dot{y}_{\mu+1} = u,$$

347 where  $y_i \in \mathbb{R}$  for  $i = 1, \dots, \mu + 1$ . Let  $\tilde{y} = [y_2, \dots, y_{\mu+1}]^T$ . We then can rewrite this  
 348 system as

$$349 \quad \dot{y}_1 = \sum_{k=2}^{\mu+1} \theta_{1,k} y_k + \theta_{1,\mu+2} u,$$

$$350 \quad \dot{\tilde{y}} = \tilde{A} \tilde{y} + \tilde{b} u,$$

352 for some matrices  $\tilde{A}$  and  $\tilde{b}$  of appropriate dimensions. From the inductive hypothesis,  
 353 there exist  $\mu - 1$  functions  $\bar{a}_i : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$  for  $i \in \llbracket 2, \mu \rrbracket$  such that for any set  
 354 of positive constants  $a_2, \dots, a_{\mu+1}$  satisfying  $a_2, \dots, a_{\mu+1}$  satisfying  $a_{\mu+1} \in (0, 1]$  and  
 355  $0 < a_i \leq \bar{a}_i(a_{i+1})$ , for each  $i \in \llbracket 2, \mu \rrbracket$ , the feedback law  $\tilde{\kappa} : \mathbb{R}^\mu \rightarrow \mathbb{R}$  defined by

$$356 \quad \tilde{\kappa}(\tilde{y}) = - \sum_{i=2}^{\mu+1} \frac{Q_{i,\mu+1} y_i}{(1 + \sum_{m=i}^{\mu+1} \|y_m\|^2)^{1/2}}$$

357 is  $SISS_L$ -stabilizing for  $\dot{\tilde{y}} = \tilde{A}\tilde{y} + \tilde{b}u$ . Choose  $(a_2, \dots, a_{\mu+1})$  satisfying the above  
 358 conditions. The feedback law (8) is then given by

$$359 \quad \kappa(y) = -\tilde{\kappa}(\tilde{y}) - a_1 Q_{2,\mu+1} \frac{y_1}{(1 + \sum_{m=1}^{\mu+1} \|y_m\|^2)^{1/2}}.$$

360 Since  $\theta_{1,\mu+2} Q_{k,\mu+1} = \theta_{1,k}$  for all  $k \in \llbracket 2, \mu+1 \rrbracket$  (see (6) and (9)), the closed-loop system  
 361 can be rewritten as

$$362 \quad \dot{y}_1 = -a_1 \frac{y_1}{(1 + \|y_1\|^2)^{1/2}} + a_1 \rho_1(y) + g_1(\tilde{y}),$$

$$363 \quad (14) \quad \dot{\tilde{y}} = \tilde{A}\tilde{y} - \tilde{b}\tilde{\kappa}(\tilde{y}) - \tilde{b}a_1 f_1(y),$$

365 with

$$366 \quad (15) \quad \rho_1(y) = \frac{y_1}{(1 + \|y_1\|^2)^{1/2}} \left(1 - \frac{(1 + \|y_1\|^2)^{1/2}}{(1 + \sum_{m=1}^{\mu+1} \|y_m\|^2)^{1/2}}\right),$$

$$367 \quad (16) \quad g_1(\tilde{y}) = \sum_{k=2}^{\mu+1} \theta_{1,k} y_k \left(1 - \frac{1}{(1 + \sum_{m=k}^{\mu+1} \|y_m\|^2)^{1/2}}\right),$$

$$368 \quad (17) \quad f_1(y) = \frac{Q_{2,\mu+1} y_1}{(1 + \sum_{m=1}^{\mu+1} \|y_m\|^2)^{1/2}}.$$

369

370 We now move to the other case where the dynamics involves multiple integrators with  
 371 rotating modes.

372 *Case ii).* Let  $(a_1, \dots, a_{\mu+1})$  be a set of positive constants to be chosen later. Let  
 373  $s \in \mathbb{N}_{\geq 1}$ , and  $z \in \mathbb{N}$  be such that  $\mu = s + z$ . Let  $\omega_1, \dots, \omega_s$  be a set of non zero real  
 374 numbers. Consider the following linear control system

$$375 \quad \dot{y}_i = \omega_i A_0 y_i + b_0 \sum_{k=i+1}^s \theta_{i,k} b_0^T y_k + b_0 \sum_{k=s+1}^{\mu+1} \theta_{i,k} y_k + \theta_{i,\mu+2} b_0 u, \quad i = 1, \dots, s,$$

$$376 \quad \dot{y}_i = \sum_{k=i+1}^{\mu+1} \theta_{i,k} y_k + \theta_{i,\mu+2} u, \quad i = s+1, \dots, \mu,$$

$$377 \quad \dot{y}_{\mu+1} = u,$$

379 where  $y_i \in \mathbb{R}^2$  for  $i = 1, \dots, s$ , and  $y_i \in \mathbb{R}$  for  $i = s+1, \dots, \mu+1$ . Let  $\tilde{y} =$   
 380  $[y_2, \dots, y_{\mu+1}]^T$ . We then can rewrite this system as follows

$$381 \quad \dot{y}_1 = \omega_1 A_0 y_1 + b_0 \sum_{k=i+1}^s \theta_{i,k} b_0^T y_k + b_0 \sum_{k=s+1}^{\mu+1} \theta_{i,k} y_k + \theta_{i,\mu+2} b_0 u,$$

$$382 \quad \dot{\tilde{y}} = \tilde{A}\tilde{y} + \tilde{b}u.$$

384 From the inductive hypothesis, there exist  $\mu-1$  functions  $\bar{a}_i : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$  for  $i \in \llbracket 2, \mu \rrbracket$   
 385 such that for any set of positive constant  $a_2, \dots, a_{\mu+1}$  satisfying  $a_{\mu+1} \in (0, 1]$  and

386  $0 < a_i \leq \bar{a}_i(a_{i+1})$ , for each  $i \in \llbracket 2, \mu \rrbracket$ , the feedback law  $\tilde{\kappa} : \mathbb{R}^\mu \rightarrow \mathbb{R}$  defined by

$$387 \quad (18) \quad \tilde{\kappa}(\tilde{y}) = - \sum_{i=2}^s \frac{Q_{i,\mu+1} b_0^T y_i}{(1 + \sum_{m=i}^{\mu+1} \|y_m\|^2)^{1/2}} - \sum_{i=s+1}^{\mu+1} \frac{Q_{i,\mu+1} y_i}{(1 + \sum_{m=i}^{\mu+1} \|y_m\|^2)^{1/2}}$$

388 is  $SISS_L$ -stabilizing for  $\dot{\tilde{y}} = \tilde{A}\tilde{y} + \tilde{b}u$ . Choose  $a_2, \dots, a_{\mu+1}$  satisfying the above con-  
389 ditions. The feedback law (8) is then given by

$$390 \quad \kappa(y) = -\tilde{\kappa}(\tilde{y}) - a_1 Q_{2,\mu+1} \frac{b_0^T y_1}{(1 + \sum_{m=1}^{\mu+1} \|y_m\|^2)^{1/2}}.$$

391 By noticing that  $\theta_{1,\mu+2} Q_{k,\mu+1} = \theta_{1,k}$  for all  $k \in \llbracket 2, \mu + 1 \rrbracket$  (see (6) and (9)), the  
392 closed-loop system can be rewritten as

$$393 \quad \dot{y}_1 = \omega_1 A_0 y_1 - a_1 b_0 \frac{b_0^T y_1}{(1 + \|y_1\|^2)^{1/2}} + a_1 b_0 \rho_1(y) + b_0 g_1(\tilde{y}),$$

$$394 \quad (19) \quad \dot{\tilde{y}} = \tilde{A}\tilde{y} - \tilde{b}\tilde{\kappa}(\tilde{y}) - \tilde{b}a_1 f_1(y),$$

396 with

$$397 \quad (20) \quad \rho_1(y) = \frac{b_0^T y_1}{(1 + \|y_1\|^2)^{1/2}} \left(1 - \frac{(1 + \|y_1\|^2)^{1/2}}{(1 + \sum_{m=1}^{\mu+1} \|y_m\|^2)^{1/2}}\right),$$

$$398 \quad g_1(\tilde{y}) = \sum_{k=2}^s \theta_{1,k} b_0^T y_k \left(1 - \frac{1}{(1 + \sum_{m=k}^{\mu+1} \|y_m\|^2)^{1/2}}\right)$$

$$399 \quad (21) \quad + \sum_{k=s+1}^{\mu+1} \theta_{1,k} y_k \left(1 - \frac{1}{(1 + \sum_{m=k}^{\mu+1} \|y_m\|^2)^{1/2}}\right),$$

$$400 \quad (22) \quad f_1(y) = \frac{Q_{2,\mu+1} b_0^T y_1}{(1 + \sum_{m=1}^{\mu+1} \|y_m\|^2)^{1/2}}.$$

402 In both cases, it remains to show that there exists a function  $\bar{a}_1$  such that if  
403  $a_1 \in (0, \bar{a}_1]$  then the closed-loop systems (14) and (19) are  $SISS_L$  and globally asymp-  
404 totically stable with respect to the origin, and their respective linearization at zero is  
405 asymptotically stable as well. According to Remark 1, one only needs to prove that  
406 the closed-loop systems are  $SISS_L$  and their linearization at zero is asymptotically  
407 stable.

408 We start by showing the latter fact. For any  $a_1 > 0$ , the linearization at zero of  
409 the  $y_1$ -subsystem in (14) (respectively (19)) is asymptotically stable since it is given by  
410  $\dot{y}_1 = -a_1 y_1$  (respectively  $\dot{y}_1 = (\omega_1 A_0 - a_1 b_0 b_0^T) y_1$ ). Moreover, the linearization at zero  
411 of the  $\tilde{y}$ -subsystem in (14) (respectively (19)) is given by  $\dot{\tilde{y}} = (\tilde{A} - \tilde{b}\tilde{\kappa}^{(1)}(0))\tilde{y} - a_1 \tilde{b}y_1$   
412 (respectively  $\dot{\tilde{y}} = (\tilde{A} - \tilde{b}\tilde{\kappa}^{(1)}(0))\tilde{y} - a_1 \tilde{b}b_0^T y_1$ ). Due to the inductive hypothesis, the  
413 origin of  $\dot{\tilde{y}} = (\tilde{A} - \tilde{b}\tilde{\kappa}^{(1)}(0))\tilde{y}$  is asymptotically stable. Thus, local asymptotic stability  
414 of (14) and (19) follows easily.

415 It remains to prove that systems (14) and (19) are  $SISS_L$ . In both cases, using  
416 that  $1 - 1/(1+s)^{1/2} \leq s$  for all  $s \geq 0$ , it holds from (16) and (21) that

$$417 \quad (23) \quad \|g_1(\tilde{y})\| \leq \sum_{k=2}^{\mu+1} \theta_{1,k} \|y_k\| \left( \sum_{m=k}^{\mu+1} \|y_m\|^2 \right) \leq \|\tilde{y}\|^3 \sum_{k=2}^{\mu+1} \theta_{1,k},$$

418 and from (15) and (20) that

$$419 \quad (24) \quad |\rho_1(y)| \leq \|\tilde{y}\|^2.$$

420 Recall that, due to the inductive hypothesis,  $\dot{\tilde{y}} = \tilde{A}\tilde{y} - \tilde{b}\tilde{\kappa}(\tilde{y})$  is  $SISS_L(\tilde{\Delta}, \tilde{N})$  for  
421 some  $\tilde{\Delta} > 0$  and  $\tilde{N} > 0$ . We next prove the  $SISS_L$  property for *Case ii*), i.e., for  
422 system (19). Let

$$423 \quad (25) \quad C_1 := \tilde{N}(Q_{2,\mu+1} \|\tilde{b}\| + 1),$$

$$424 \quad (26) \quad C_2 := C_1^2 + C_1^3 \sum_{k=2}^{\mu+1} \theta_{i,k}.$$

425 From Lemma 3 (with  $\omega = \omega_1$ ), there exist  $\Gamma_1, N_1 > 0$  such that for any  $a_1 \in (0, 1]$   
426 the system  $\dot{y}_1 = \omega_1 A_0 y_1 - a_1 b_0 \frac{b_0^T y_1}{(1 + \|y_1\|^2)^{1/2}}$  is  $SISS_L(\Gamma_1 a_1, N_1/a_1)$ . Define

$$427 \quad (27) \quad \bar{a}_1 := \min \left\{ 1, \frac{\tilde{\Delta}\tilde{N}}{C_1}, \sqrt{\frac{\Gamma_1}{2C_2}}, \sqrt{\frac{C_1}{4Q_{2,\mu+1}\tilde{N}\|\tilde{b}\|N_1C_2}} \right\},$$

428 and choose  $a_1 \in (0, \bar{a}_1]$ . Let

$$429 \quad (28) \quad \Delta := \min \left\{ \frac{a_1\Gamma_1}{2}, a_1 \right\}.$$

430 Given  $\delta \leq \Delta$ , let  $e_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^2$  and  $e_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{2s+z-2}$  be two bounded measur-  
431 able functions, eventually bounded by  $\delta$ . Consider any trajectory  $(y_1(\cdot), \tilde{y}(\cdot))$  of the  
432 following system

$$433 \quad \dot{y}_1 = \omega_1 A_0 y_1 - a_1 b_0 \frac{b_0^T y_1}{(1 + \|y_1\|^2)^{1/2}} + a_1 b_0 \rho_1(y) + b_0 g_1(\tilde{y}) + e_1,$$

$$434 \quad (29) \quad \dot{\tilde{y}} = \tilde{A}\tilde{y} - \tilde{b}\tilde{\kappa}(\tilde{y}) - \tilde{b}a_1 f_1(y) + e_2,$$

436 In view of (19), (20), (21), (22) and (18) the above system is clearly forward complete.  
437 We next show that there exists a constant  $N > 0$  such that  $\|y_1(\cdot)\| \leq_{ev} N\delta$  and  
438  $\|\tilde{y}(\cdot)\| \leq_{ev} N\delta$ . From (22) and recalling that  $\|b_0\| = 1$ , a straightforward computation  
439 yields

$$440 \quad \left\| a_1 \tilde{b} f_1(y) \right\| \leq a_1 Q_{2,\mu+1} \|\tilde{b}\|.$$

441 Since  $\|e_2(\cdot)\| \leq_{ev} \delta$ , it follows that

$$442 \quad \left\| a_1 \tilde{b} f_1(y(\cdot)) + e_2(\cdot) \right\| \leq_{ev} a_1 Q_{2,\mu+1} \|\tilde{b}\| + \delta.$$

443 Moreover from (27), (28) and it follows that

$$444 \quad \left\| a_1 \tilde{b} f_1(y(\cdot)) + e_2(\cdot) \right\| \leq_{ev} a_1 (Q_{2,\mu+1} \left\| \tilde{b} \right\| + 1) \leq a_1 C_1 / \tilde{N} \leq \tilde{\Delta},$$

where  $C_1$  is defined in (25). Using the  $SISS_L(\tilde{\Delta}, \tilde{N})$  property of System  $\dot{\tilde{y}} = \tilde{A}\tilde{y} - \tilde{b}\tilde{\kappa}(\tilde{y})$ , it follows that the solution of (29) satisfies

$$\left\| \tilde{y}(\cdot) \right\| \leq_{ev} a_1 C_1.$$

445 Consequently, using (24) and (23), it follows that

$$446 \quad (30) \quad \left\| a_1 b_0 \rho_1(y(\cdot)) + b_0 g_1(\tilde{y}(\cdot)) \right\| \leq_{ev} a_1^3 C_2.$$

447 Using (27), we have  $a_1^3 C_2 \leq \frac{a_1 \Gamma_1}{2}$ . Moreover (28) ensures that  $\|e_1(\cdot)\| \leq_{ev} \frac{a_1 \Gamma_1}{2}$ . So it  
448 follows that

$$449 \quad \left\| a_1 b_0 \rho_1(y(\cdot)) + b_0 g_1(\tilde{y}(\cdot)) + e_1(\cdot) \right\| \leq_{ev} a_1 \Gamma_1.$$

450 The  $SISS_L(\Gamma_1 a_1, N_1/a_1)$  property of  $\dot{y}_1 = \omega_1 A_0 y_1 - a_1 b_0 \frac{b_0^T y_1}{(1+\|y_1\|^2)^{1/2}}$  ensures that

$$451 \quad (31) \quad \left\| y_1(\cdot) \right\| \leq_{ev} \frac{N_1}{a_1} (a_1^3 C_2 + \delta) \leq N_1 \Gamma_1.$$

452 Now let  $\theta > 0$  be defined as

$$453 \quad (32) \quad \theta := \limsup_{t \rightarrow +\infty} \left\| \tilde{y}(t) \right\|.$$

454 Then  $\left\| \tilde{y}(\cdot) \right\| \leq_{ev} 2\theta$ . There are two cases to consider, either  $2\theta \leq a_1 C_1$  or  $a_1 C_1 < 2\theta$ .  
455 In the case when  $2\theta \leq a_1 C_1$ , we have

$$456 \quad \left\| a_1 b_0 \rho_1(y(\cdot)) + b_0 g_1(\tilde{y}(\cdot)) + e_1(\cdot) \right\| \leq_{ev} 2\theta a_1^2 C_2 / C_1.$$

457 So invoking again the  $SISS_L(\bar{\rho}_1 \Gamma_1 a_1, N/a_1)$  property of  $\dot{y}_1 = \omega_1 A_0 y_1 - a_1 b_0 b_0^T y_1 / (1 + \|y_1\|^2)^{1/2}$ , one gets that the solution of (29) satisfies

$$459 \quad (33) \quad \left\| y_1(\cdot) \right\| \leq_{ev} \frac{N_1}{a_1} \left( \frac{2\theta a_1^2 C_2}{C_1} + \delta \right).$$

460 In the case when  $a_1 C_1 < 2\theta$ , the estimate (33) follows readily from (31). Exploiting  
461 again the  $SISS_L(\tilde{\Delta}, \tilde{N})$  property of System  $\dot{\tilde{y}} = \tilde{A}\tilde{y} - \tilde{b}\tilde{\kappa}(\tilde{y})$ , it follows that

$$462 \quad \left\| \tilde{y}(\cdot) \right\| \leq_{ev} \tilde{N} \left( \left\| \tilde{b} \right\| Q_{2,\mu+1} N_1 \left( \frac{2\theta a_1^2 C_2}{C_1} + \delta \right) + \delta \right)$$

$$463 \quad = \theta \frac{2Q_{2,\mu+1} \tilde{N} \left\| \tilde{b} \right\| N_1 a_1^2 C_2}{C_1} + \delta \tilde{N} (\left\| \tilde{b} \right\| Q_{2,\mu+1} N_1 + 1).$$

464 It then follows from (27) that

$$465 \quad \left\| \tilde{y}(\cdot) \right\| \leq_{ev} \frac{\theta}{2} + \delta \tilde{N} (\left\| \tilde{b} \right\| Q_{2,\mu+1} N_1 + 1).$$

466 Taking the limsup of the above estimate, we get from (32) that

$$467 \quad \theta \leq 2\delta\tilde{N}\left(\|\tilde{b}\| Q_{2,\mu+1}N_1 + 1\right).$$

468 Consequently, we obtain that

$$469 \quad \|\tilde{y}(\cdot)\| \leq_{ev} 2\tilde{N}\left(\|\tilde{b}\| Q_{2,\mu+1}N_1 + 1\right)\delta,$$

$$470 \quad \|y_1(\cdot)\| \leq_{ev} 2\frac{N_1}{a_1}\left(\frac{2a_1^2C_2}{C_1} + 1\right)\tilde{N}(N_1 + 1)\delta,$$

471 which finishes to establish  $(P_{\mu+1})$  for the case  $ii)$ . Proceeding as in case  $ii)$ , it can be  
472 shown that system (14) is  $\text{SISS}_L$ . This ends the inductive proof of  $(P_\mu)$ .

473

474 **3.1.3. Proof of Proposition 2.** Fix  $\mu \in \mathbb{N}_{\geq 1}$ . Let  $s$  and  $z$  be two integers such  
475 that  $s + z = \mu$ ,  $\omega_1, \dots, \omega_s$  be positive constant numbers, and  $a_1, \dots, a_\mu$  be positive  
476 numbers less than or equal to 1. Consider the system (7) with the feedback law (8),  
477 where  $\mu(A) = \mu$ ,  $s(A) = s$  and  $z(A) = z$ . We establish Proposition 2 by induction  
478 on the number of time derivatives, i.e.,  $p$ . More precisely we prove the following  
479 statement: for each  $p \in \mathbb{N}$ ,

480  $(H_p)$  : there exist a positive constant  $c_\mu$  and continuous functions  $c_i : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  
481  $i \in \llbracket 1, \mu - 1 \rrbracket$ , such that for every trajectory  $y(\cdot)$  of the closed-loop system  
482 (7) with the feedback law (8), the control signal  $U : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  defined by  
483  $U(t) := \kappa(y(t))$  for all  $t \geq 0$  satisfies, for all  $k \in \llbracket 0, p \rrbracket$ ,

$$484 \quad \left|U^{(k)}(t)\right| \leq a_\mu c_\mu + \sum_{i=1}^{\mu-1} a_i c_i(a_\mu, \dots, a_{i+1}), \quad \forall t \geq 0.$$

485 For  $p = 0$ , this statement  $(H_0)$  holds trivially. Indeed, it is easy to see that for  
486 any trajectory of the closed-loop system (7) with the feedback law (8) we have

$$487 \quad |U(t)| \leq a_\mu + \sum_{i=1}^{\mu-1} a_i Q_{i+1,\mu}, \quad \forall t \geq 0.$$

488 Now, assume that  $(H_p)$  holds true for some  $p \in \mathbb{N}$ . We next prove that  $(H_{p+1})$   
489 also holds true. To that aim, let  $y(\cdot)$  be any trajectory of the closed-loop system  
490 (7) with the feedback law (8), and the control signal  $U(t) := \kappa(y(t))$ ,  $\forall t \geq 0$ . By  
491 the induction hypothesis, there exist a positive constant  $\Upsilon_\mu$  and continuous functions  
492  $\Upsilon_i : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  $i \in \llbracket 1, \mu - 1 \rrbracket$ , such that for every  $k \in \llbracket 0, p \rrbracket$  it holds that

$$493 \quad (34) \quad \left|U^{(k)}(t)\right| \leq a_\mu \Upsilon_\mu + \sum_{i=1}^{\mu-1} a_i \Upsilon_i(a_\mu, \dots, a_{i+1}), \quad \forall t \geq 0.$$

494 It is sufficient to show that there exist a positive constant  $\tilde{\Upsilon}_\mu$  and continuous functions  
495  $\tilde{\Upsilon}_i : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  $i \in \llbracket 1, \mu - 1 \rrbracket$ , such that

$$496 \quad (35) \quad \left|U^{(p+1)}(t)\right| \leq a_\mu \tilde{\Upsilon}_\mu + \sum_{i=1}^{\mu-1} a_i \tilde{\Upsilon}_i(a_\mu, \dots, a_{i+1}), \quad \forall t \geq 0.$$

497 Indeed, the desired results will be obtained by setting  $c_\mu := \max\{\Upsilon_\mu, \tilde{\Upsilon}_\mu\}$ , and  $c_i(\cdot) :=$   
 498  $\max\{\Upsilon_i(\cdot), \tilde{\Upsilon}_i(\cdot)\}$  for  $i \in \llbracket 1, \mu - 1 \rrbracket$ . In order to establish (35), we start by defining  
 499 the following auxiliary functions:

$$500 \quad (36) \quad g(s) := s^{-1/2}, \quad \forall s > 0$$

502 and, for all  $t \geq 0$ ,

$$503 \quad (37) \quad f_i(t) := 1 + \sum_{l=i}^{\mu} \|y_l(t)\|^2, \quad i \in \llbracket 1, \mu \rrbracket.$$

504 Then, we can rewrite  $U(\cdot)$  as

$$505 \quad (38) \quad U(t) = - \sum_{i=1}^{\mu} U_i(t), \quad \forall t \geq 0,$$

506 where, for every  $i \in \llbracket 1, \mu \rrbracket$ ,

$$507 \quad (39) \quad U_i(t) := Q_{i,\mu} b_{0,i}^T y_i(t) g(f_i(t)), \quad \forall t \geq 0,$$

508 where  $b_{0,i} = b_0$  for all  $i \in \llbracket 1, s \rrbracket$  and  $b_{0,i} = 1$  otherwise, and  $Q_{i,\mu}$  is defined in (9).  
 509 The  $(p+1)$ -th time derivative of the control signal  $U(\cdot)$  is given, for all  $t \geq 0$ , by  
 510  $U^{(p+1)}(t) = - \sum_{i=1}^{\mu} U_i^{(p+1)}(t)$ . Therefore to prove  $(H_{p+1})$ , it is sufficient to show that,  
 511 for each  $i \in \llbracket 1, \mu \rrbracket$ , there exists continuous functions  $c_{i,l} : \mathbb{R}_{>0}^{\mu-l} \rightarrow \mathbb{R}_{>0}$ ,  $l \in \llbracket 1, i \rrbracket$ ,  
 512 such that, for all  $t \geq 0$ ,

$$513 \quad (40) \quad \left| U_i^{(p+1)}(t) \right| \leq \sum_{l=1}^i a_l c_{i,l}(a_\mu, \dots, a_{l+1}),$$

514  $c_{i,\mu}$  is actually a constant independent of  $a_\mu$ , we write it as  $c_{i,\mu}(a_\mu, a_{\mu+1})$  for the sake  
 515 of notation homogeneity.

516 For  $i \in \llbracket 1, \mu \rrbracket$ , we apply Leibniz's rule to (39) with respect to  $b_{0,i}^T y_i(t)$  and  $g(f_i(t))$   
 517 and obtain that the  $(p+1)$ -th time derivative of  $U_i(\cdot)$  is given, for all  $t \geq 0$ , by

$$518 \quad U_i^{(p+1)}(t) = a_i Q_{i+1,\mu} \left( \sum_{l_1=0}^{p+1} \binom{p+1}{l_1} b_{0,i}^T y_i^{(p+1-l_1)}(t) [g \circ f_i]^{(l_1)}(t) \right).$$

519 To obtain (40), it is sufficient to prove that for each  $i \in \llbracket 1, \mu \rrbracket$ , and  $l_1 \in \llbracket 0, p+1 \rrbracket$   
 520 there exist continuous functions  $\beta_{i,l,l_1} : \mathbb{R}_{>0}^{\mu-l} \rightarrow \mathbb{R}_{>0}$  for  $l \in \llbracket 1, i \rrbracket$  such that, for all  
 521  $t \geq 0$ ,

$$522 \quad (41) \quad \left| b_{0,i}^T y_i^{(p+1-l_1)}(t) [g \circ f_i]^{(l_1)}(t) \right| \leq \beta_{i,i,l_1}(a_\mu, \dots, a_{i+1}) + \sum_{l=1}^{i-1} a_l \beta_{i,l,l_1}(a_\mu, \dots, a_{l+1}).$$

523 In order to get (41) we next provide, for each  $i \in \llbracket 1, \mu \rrbracket$ , estimates of  $\|y_i^{(l_1)}(t)\|$ ,  
 524  $|f_i^{(l_1)}(t)|$  and  $[g \circ f_i]^{(l_1)}(t)$  for  $l_1 \in \llbracket 1, p+1 \rrbracket$ . One can observe that, for each  $i \in \llbracket 1, \mu \rrbracket$ ,  
 525  $\dot{y}_i$  depends on the constants  $a_{i+1}, \dots, a_\mu$ , the states  $y_i, \dots, y_\mu$  and  $u = \kappa(y)$ . By an  
 526 induction argument using differentiation of system (7), one can obtain the following  
 527 statement: for any  $k \in \llbracket 1, p+1 \rrbracket$ ,  $i \in \llbracket 1, \mu \rrbracket$ , there exist continuous functions

$$528 \quad \Psi_{k,i,l} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}, \quad l \in \llbracket i+1, \mu \rrbracket, \quad \Phi_{k,i,l} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}, \quad l \in \llbracket 0, p \rrbracket,$$



529 such that, for all positive times, it holds that

$$530 \quad \left\| y_i^{(k)}(t) \right\| \leq \sum_{l=i}^{\mu} \Psi_{k,i,l}(a_{\mu}, \dots, a_{i+1}) \|y_l(t)\| + \sum_{l=0}^{k-1} \Phi_{k,i,l}(a_{\mu}, \dots, a_{i+1}) \left| U^{(l)}(t) \right|,$$

531 where, by convention,  $\Psi_{k,i,\mu}$  are constant functions independent of  $a_{\mu}$  for  $k \in \llbracket 1, p+1 \rrbracket$   
 532 and  $i \in \llbracket 1, \mu \rrbracket$ . Using (34) in the above estimate, one gets that, for any  $k \in \llbracket 1, p+1 \rrbracket$   
 533 and  $i \in \llbracket 1, \mu-1 \rrbracket$ , there exist functions  $\tilde{v}_{l,k,i} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ , for  $l \in \llbracket i+1, \mu \rrbracket$ , and  
 534  $\tilde{\Phi}_{l,k,i} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$  such that, for all  $t \geq 0$ ,

$$535 \quad \left\| y_i^{(k)}(t) \right\| \leq \sum_{l=i}^{\mu} \Psi_{k,i,l}(a_{\mu}, \dots, a_{i+1}) \|y_l(t)\| + \tilde{\Phi}_{k,i}(a_{\mu}, \dots, a_{i+1}) + \sum_{l=1}^i a_l \tilde{v}_{l,k,i}(a_{\mu}, \dots, a_{l+1}).$$

536 Setting, for  $i \in \llbracket 1, \mu \rrbracket$ ,

$$537 \quad \bar{\Psi}_i(a_{\mu}, \dots, a_{i+1}) := \max\{\Psi_{k,i,i}(a_{\mu}, \dots, a_{i+1}) : k \in \llbracket 1, p+1 \rrbracket, l \in \llbracket i+1, \mu \rrbracket\},$$

$$538 \quad \bar{\Phi}_i(a_{\mu}, \dots, a_{i+1}) := \max\{\tilde{\Phi}_{k,i}(a_{\mu}, \dots, a_{i+1}) : k \in \llbracket 1, p+1 \rrbracket\},$$

$$539 \quad \tilde{v}_{l,i}(a_{\mu}, \dots, a_{l+1}) := \max\{\tilde{v}_{l,k,i}(a_{\mu}, \dots, a_{l+1}) : k \in \llbracket 1, p+1 \rrbracket\}, \quad l \in \llbracket 1, i \rrbracket,$$

540 one can obtain that, for all  $k \in \llbracket 1, p+1 \rrbracket$ , all  $i \in \llbracket 1, \mu \rrbracket$ , and all  $t \geq 0$ ,  
 (42)

$$541 \quad \left\| y_i^{(k)}(t) \right\| \leq \bar{\Psi}_i(a_{\mu}, \dots, a_{i+1}) \sum_{l=i}^{\mu} \|y_l(t)\| + \bar{\Phi}_i(a_{\mu}, \dots, a_{i+1}) + \sum_{l=1}^i a_l \tilde{v}_{l,i}(a_{\mu}, \dots, a_{l+1}).$$

542 It follows that (41) for  $l_1 = 0$  holds true. For any  $i \in \llbracket 1, \mu \rrbracket$  and  $k \in \llbracket 1, p+1 \rrbracket$ , the  
 543  $k$ -th time derivative of  $f_i(\cdot)$ , defined in (37), is given, for all  $t \geq 0$ , by

$$544 \quad f_i^{(k)}(t) = \sum_{l_1=0}^k \binom{k}{l_1} \sum_{l_2=i}^{\mu} (y_{l_2}^{(l_1)}(t))^T y_{l_2}^{(k-l_1)}(t).$$

545 Thus, one can get that

$$546 \quad \left| f_i^{(k)}(t) \right| \leq 2 \sum_{l_2=i}^{\mu} \|y_{l_2}(t)\| \left\| y_{l_2}^{(k)}(t) \right\| + \sum_{l_1=1}^{k-1} \binom{k}{l_1} \sum_{l_2=i}^{\mu} \left\| y_{l_2}^{(l_1)}(t) \right\| \left\| y_{l_2}^{(k-l_1)}(t) \right\|,$$

$$547 \quad \leq \sum_{l_2=i}^{\mu} \left( \|y_{l_2}(t)\|^2 + \left\| y_{l_2}^{(k)}(t) \right\|^2 \right) + \sum_{l_1=1}^{k-1} \binom{k}{l_1} \sum_{l_2=i}^{\mu} \left( \left\| y_{l_2}^{(l_1)}(t) \right\|^2 + \left\| y_{l_2}^{(k-l_1)}(t) \right\|^2 \right).$$

548 From (42), and using the fact that  $\left( \sum_{i_1=1}^m |x_{i_1}| \right)^2 \leq m \sum_{i_1=1}^m x_{i_1}^2$ , one can obtain that for  
 549 each  $l_2 \in \llbracket 1, \mu \rrbracket$  and  $l_1 \in \llbracket 1, p+1 \rrbracket$  it holds that, for all  $t \geq 0$ ,

$$550 \quad \left\| y_{l_2}^{(l_1)}(t) \right\|^2 \leq (\mu+2) \left( \bar{\Psi}_{l_2}(a_{\mu}, \dots, a_{l_2+1})^2 \sum_{l=l_2}^{\mu} \|y_l(t)\|^2 + \bar{\Phi}_{l_2}(a_{\mu}, \dots, a_{l_2+1})^2 \right)$$

$$551 \quad (43) \quad + \sum_{l=1}^{l_2} (a_l \tilde{v}_{l,l_2}(a_{\mu}, \dots, a_{l+1}))^2).$$

552 Since the right-hand side of (43) is independent of  $l_1$ , and  $a_l \leq 1$  for all  $l \in \llbracket 1, \mu \rrbracket$ , one  
 553 can get that there exist continuous functions

$$\begin{aligned} 554 \quad & \tilde{\Psi}_l : \mathbb{R}_{>0}^{\mu-l} \rightarrow \mathbb{R}_{>0}, \quad l \in \llbracket 1, \mu \rrbracket, \\ 555 \quad & \tilde{\Phi}_l : \mathbb{R}_{>0}^{\mu-l} \rightarrow \mathbb{R}_{>0}, \quad l \in \llbracket 1, \mu \rrbracket, \\ 556 \quad & \tilde{v}_{l,l_1} : \mathbb{R}_{>0}^{\mu-l} \rightarrow \mathbb{R}_{>0}, \quad l_1 \in \llbracket 1, \mu \rrbracket, l \in \llbracket 1, l_1 \rrbracket, \end{aligned}$$

557 such that, for any  $k \in \llbracket 1, p \rrbracket$  and all  $t \geq 0$ , it holds

$$558 \quad \left| f_i^{(k)}(t) \right| \leq \tilde{\Psi}_{l_2}(a_\mu, \dots, a_{i+1}) \sum_{l=i}^{\mu} \|y_l(t)\|^2 + \tilde{\Phi}_{l_2}(a_\mu, \dots, a_{i+1}) + \sum_{l=1}^i a_l \tilde{v}_{l,i}(a_\mu, \dots, a_{l+1}).$$

559 A trivial estimate for any  $k \in \llbracket 1, p+1 \rrbracket$ , any  $i \in \llbracket 1, \mu \rrbracket$ , and all  $t \geq 0$  is given by

$$560 \quad (44) \quad \left| f_i^{(k)}(t) \right| \leq \tilde{\Psi}_i(a_\mu, \dots, a_{i+1}) f_i(t) + \tilde{\Phi}_i(a_\mu, \dots, a_{i+1}) + \sum_{l=1}^i a_l \tilde{v}_{l,l_2}(a_\mu, \dots, a_{l+1}).$$

561 By the Faà di Bruno's formula (given in Lemma 5 in Appendix), for each  $i \in \llbracket 1, \mu \rrbracket$ ,  
 562 and  $l_1 \in \llbracket 1, p+1 \rrbracket$ , the  $l_1$ -th time derivative of  $g \circ f_i(\cdot)$  is given, for all  $t \geq 0$ , by

$$563 \quad [g \circ f_i]^{(l_1)}(t) = \sum_{l_2=1}^{l_1} g^{(l_2)}(f_i(t)) \sum_{\delta \in \mathcal{P}_{l_1, l_2}} c_\delta \prod_{l=1}^{l_1-l_2+1} (f_i^{(l)}(t))^{\delta_l},$$

564 where  $\mathcal{P}_{l_1, l_2}$  denotes the set of  $(l_1 - l_2 + 1)$ -tuples  $\delta := (\delta_1, \delta_2, \dots, \delta_{l_1-l_2+1})$  of positive  
 565 integers satisfying  $\delta_1 + \delta_2 + \dots + \delta_{l_1-l_2+1} = l_2$  and  $\delta_1 + 2\delta_2 + \dots + (l_1-l_2+1)\delta_{l_1-l_2+1} = l_1$ .  
 566 Observe that the  $k$ -th derivative of the function  $g$  defined in (36) reads

$$567 \quad (45) \quad g^{(k)}(s) = d_k s^{-1/2-k}, \quad \forall s > 0,$$

568 with  $d_k = (-1)^k \prod_{l=0}^{k-1} (1/2 + l)$ . Using (45), and taking the absolute value, one can get,  
 569 for all  $t \geq 0$ ,

$$570 \quad \left| [g \circ f_i]^{(l_1)}(t) \right| \leq \sum_{l_2=1}^{l_1} d_{l_2} \frac{1}{(f_i(t))^{l_2+1/2}} \sum_{\delta \in \mathcal{P}_{l_1, l_2}} c_\delta \prod_{l=1}^{l_1-l_2+1} \left| f_i^{(l)}(t) \right|^{\delta_l}.$$

571 Using (44), one can obtain that, for any  $l_1 \in \llbracket 1, p+1 \rrbracket$ , any  $l_2 \in \llbracket 1, l_1 \rrbracket$  and for all  
 572  $t \geq 0$ ,

$$\begin{aligned} 573 \quad & \sum_{\delta \in \mathcal{P}_{l_1, l_2}} c_\delta \prod_{l=1}^{l_1-l_2+1} \left| f_i^{(l)}(t) \right|^{\delta_l} \leq \left( \tilde{\Psi}_i(a_\mu, \dots, a_{i+1}) f_i(t) + \tilde{\Phi}_i(a_\mu, \dots, a_{i+1}) \right. \\ 574 \quad & \left. + \sum_{l_3=1}^i a_{l_3} \tilde{v}_{l_3, i}(a_\mu, \dots, a_{i+1}) \right)^{l_2} \sum_{\delta \in \mathcal{P}_{l_1, l_2}} c_\delta. \end{aligned}$$

575 It follows that, for all  $l_1 \in \llbracket 1, p+1 \rrbracket$ ,  $t \geq 0$ ,

$$\begin{aligned}
576 \quad \left| [g \circ f_i]^{(l_1)}(t) \right| &\leq \sum_{l_2=1}^{l_1} d_{l_2} \frac{\sum_{\delta \in \mathcal{P}_{l_1, l_2}} c_\delta}{(f_i(t))^{1/2}} \left( \right. \\
577 \quad &\quad \left. \frac{\tilde{\Psi}_i(a_\mu, \dots, a_{i+1}) f_i(t) + \tilde{\Phi}_i(a_\mu, \dots, a_{i+1}) + \sum_{l_3=1}^i a_{l_3} \tilde{v}_{l_3, i}(a_\mu, \dots, a_{i+1})}{f_i(t)} \right)^{l_2}, \\
578 \quad &\leq \sum_{l_2=1}^{l_1} d_{l_2} \frac{\sum_{\delta \in \mathcal{P}_{l_1, l_2}} c_\delta}{(f_i(t))^{1/2}} \left( \right. \\
579 \quad &\quad \left. \tilde{\Psi}_i(a_\mu, \dots, a_{i+1}) + \tilde{\Phi}_i(a_\mu, \dots, a_{i+1}) + \sum_{l_3=1}^i a_{l_3} \tilde{v}_{l_3, i}(a_\mu, \dots, a_{i+1}) \right)^{l_2},
\end{aligned}$$

580 Thus, it can be seen that, for every  $i \in \llbracket 1, \mu \rrbracket$  and  $l_1 \in \llbracket 1, p+1 \rrbracket$ , there exist  
581 continuous functions  $\Gamma_{i, l_1} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$  and  $\Gamma_{i, l_1, l} : \mathbb{R}_{>0}^{\mu-l} \rightarrow \mathbb{R}_{>0}$ ,  $l \in \llbracket 1, i+1 \rrbracket$ , such  
582 that, for all  $t \geq 0$ ,

$$583 \quad (46) \quad \left| [g \circ f_i]^{(l_1)}(t) \right| \leq \frac{1}{(f_i(t))^{1/2}} \left( \Gamma_{i, l_1}(a_\mu, \dots, a_{i+1}) + \sum_{l=1}^i a_l \Gamma_{i, l_1, l}(a_\mu, \dots, a_{i+1}) \right).$$

584 Then, from (46) and (42) it follows that (41) holds true for any  $l_1 \in \llbracket 1, p+1 \rrbracket$ . This  
585 ends the inductive proof of  $(H_p)$ .

## 586 3.2. Proof of Theorem 2.

587 **3.2.1. Reduction of the proof of Theorem 2 to the proof of Propositions**  
588 **1 and 3.** We prove Theorem 2 by induction on the number of inputs  $q$ . We show  
589 that the inductive step reduces to Proposition 1 and Proposition 3 which is proven in  
590 Section 3.2.2.

591 For  $q = 1$ , the conclusion follows from Theorem 1. For a given  $q \in \mathbb{N}_{\geq 1}$  assume  
592 that Theorem 2 holds. We show that Theorem 2 then holds for LTI systems given  
593 in the reduced controllability form with  $q+1$  inputs. Let  $p \in \mathbb{N}$  and  $(R_j)_{0 \leq j \leq p}$  be  
594 a  $(p+1)$ -tuple of positive real numbers. Define  $\underline{R} := \min_{j \in \llbracket 0, p \rrbracket} R_j$ . Given  $n \in \mathbb{N}_{\geq 2}$   
595 consider a LTI system given in the reduced controllability form with  $\tilde{q} := q+1$  inputs  
596 by

$$\begin{aligned}
&\dot{x}_0 = A_{00}x_0 + A_{01}x_1 + A_{02}x_2 + \dots + A_{0\tilde{q}}x_{\tilde{q}} + b_{01}u_1 + b_{02}u_2 + \dots + b_{0q}u_{\tilde{q}}, \\
&\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \dots + A_{1\tilde{q}}x_{\tilde{q}} + b_{11}u_1 + b_{22}u_2 + \dots + b_{1q}u_{\tilde{q}}, \\
597 \quad \dot{x}_2 = A_{22}x_2 + \dots + A_{2\tilde{q}}x_{\tilde{q}} + b_{22}u_2 + \dots + b_{2q}u_{\tilde{q}}, \\
&\quad \vdots \\
&\dot{x}_{\tilde{q}} = A_{\tilde{q}\tilde{q}}x_{\tilde{q}} + b_{\tilde{q}\tilde{q}}u_{\tilde{q}},
\end{aligned}$$

598 where  $x_i \in \mathbb{R}^{n_i}$  and  $u_i \in \mathbb{R}$  for each  $i \in \llbracket 0, q+1 \rrbracket$ ,  $A_{00}$  is Hurwitz, for every  $i \in \llbracket 1, q+1 \rrbracket$   
599 all the eigenvalues of  $A_{ii}$  are critical, and the pairs  $(A_{ii}, b_{ii})$  are controllable.

600 Since  $A_{00}$  is Hurwitz, if we find a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$ , and  
601  $SISS_L$ -stabilizing for  $(x_1, \dots, x_{\tilde{q}})$ -subsystem then, clearly, this feedback does the job  
602 for the complete system. From now on, we only consider the  $(x_1, \dots, x_{\tilde{q}})$ -subsystem

603 and we rewrite it compactly as

$$604 \quad (47a) \quad \dot{x}_1 = A_{11}x_1 + b_{11}u_1 + \tilde{A}z + \tilde{B}\bar{u},$$

$$605 \quad (47b) \quad \dot{z} = \bar{A}z + \bar{B}\bar{u},$$

607 where  $z := [x_2, \dots, x_{\bar{q}}]^T$ ,  $u := [u_2, \dots, u_{\bar{q}}]^T$ .

608 We next provide a key technical lemma.

LEMMA 4. *Let  $\dot{x} = Ax + bu$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ , be a controllable single input linear system. Assume that all the eigenvalues of  $A$  are critical. Let  $\pm i\omega_1, \dots, \pm i\omega_{s(A)}$  be the nonzero eigenvalues of  $A$ ,  $(a_2, \dots, a_{\mu(A)})$  be a sequence of positive numbers and  $T \in \mathbb{R}^{n,n}$  be such that the linear change of coordinate  $y = Tx$  transforms  $\dot{x} = Ax + bu$  into system (7) compactly written as  $\dot{y} = Jy + bu$ . Rewrite  $T$  as*

$$T = [T_1, \dots, T_{s(A)}, T_{s(A)+1}, \dots, T_{\mu(A)}]^T,$$

609 where  $T_i \in \mathbb{R}^{2,n}$  if  $i \in \llbracket 1, s(A) \rrbracket$  otherwise  $T_i \in \mathbb{R}^{1,n}$ . Then  $T$  has the following  
610 property

611  $(I) : T_{\mu(A)}$  is independent of  $(a_2, \dots, a_{\mu(A)})$ , and each  $T_i$  depend only on  $(a_{i+1},$   
612  $\dots, a_{\mu(A)})$ .

613 Moreover, given  $r, k \in \mathbb{N}$ , let  $M \in \mathbb{R}^{n,r}$  be independent of the constants  $a_i$ , then the  
614 matrices  $TM$  and  $J^k T$  satisfy property (I).

615 The proof of Lemma 4 follows from a careful examination of the proofs of Lemmas  
616 3.1 and 5.1 in [19].

617 . Let  $(a_2, \dots, a_{\mu(A_{11})})$  be a sequence of positive numbers (to be chosen later). Let  
618  $T$  be the linear change of coordinate that transforms  $\dot{x} = A_{11}x + b_{11}u_1$  into the form  
619 of system (7) compactly written as  $\dot{y} = Jy + bu$ . We now make the following changes  
620 of coordinates  $y = Tx$ , and system (47) is then given by

$$621 \quad (48a) \quad \dot{y} = Jy + bu_1 + T\tilde{A}z + T\tilde{B}\bar{u},$$

$$622 \quad (48b) \quad \dot{z} = \bar{A}z + \bar{B}\bar{u}.$$

624 Let  $\kappa$  be a feedback law  $p$ -bounded by  $(R_j/2)_{0 \leq j \leq p}$ , and  $SISS_L(N_2, \Delta_2)$ -stabilizing  
625 for subsystem (48b), for some  $N_2, \Delta_2 > 0$  (thanks to the inductive hypothesis, we  
626 know that this feedback law exists). Let  $a_1 > 0$ , to be chosen later. We seek the  
627 following state feedback law:

$$628 \quad (49a) \quad u_1(y, z) := \frac{\mu(y)}{(1 + \|z\|^2)^p},$$

$$629 \quad (49b) \quad \bar{u}(z) := \kappa(z),$$

631 where  $\mu(y)$  is defined in (8). We now show that there exist positive constants  
632  $(a_1, a_2, \dots, a_{\mu(A_{11})})$  such that the feedback law (49) is a feedback law  $p$ -bounded and  
633  $SISS_L$ -stabilizing for system (48). This choice is based on Proposition 1 and the  
634 following statement which is proven in Section 3.2.2.

635 PROPOSITION 3 ( $p$ -bounded feedback). *Let  $a_i$ , for  $i \in \llbracket 1, \mu(A_{11}) \rrbracket$ , be positive  
636 constants in  $(0, 1]$ . Consider system (48) with the feedback law (49). Assume that  $\kappa$   
637 is a feedback law  $p$ -bounded by  $(R_j/2)_{0 \leq j \leq p}$ , and  $SISS_L(N_2, \Delta_2)$ -stabilizing for sub-  
638 system (48b). Then, there exist a positive constant  $c_{\mu(A_{11})}$ , and continuous functions*

639  $c_i : \mathbb{R}_{>0}^{\mu(A_{11})-i} \rightarrow \mathbb{R}_{>0}$ ,  $i \in \llbracket 1, \mu(A_{11}) - 1 \rrbracket$ , such that for any trajectory of the closed-  
 640 loop system (48) with the feedback law (49), the control signal  $U_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  defined  
 641 by  $U_1(t) := u_1(y(t), z(t))$  for all  $t \geq 0$  satisfies, for all  $k \in \llbracket 0, p \rrbracket$ ,

$$642 \quad \left| U_1^{(k)}(t) \right| \leq a_\mu c_{\mu(A_{11})} + \sum_{i=1}^{\mu(A_{11})-1} a_i c_i(a_{\mu(A_{11})}, \dots, a_{i+1}), \quad \forall t \geq 0.$$

Pick  $a_{\mu(A_{11})} \in (0, 1]$  in such a way that

$$a_{\mu(A_{11})} \leq \frac{\underline{R}}{2(p+1)c_{\mu(A_{11})}}.$$

643 Choose recursively  $a_i \in (0, 1]$ ,  $i = \mu(A_{11}) - 1, \dots, 1$ , such that

$$644 \quad a_i \leq \bar{a}_i(a_{i+1}), \quad a_i \leq \frac{\underline{R}}{2(p+1)c_i(a_{\mu(A)}, \dots, a_{i+1})},$$

645 where the functions  $c_i$  appearing above are defined in Proposition 3 and the functions  
 646  $\bar{a}_i$  are defined in Proposition 1. By Proposition 1, the feedback law  $\mu(y)$  is  $SISS_L$ -  
 647 stabilizing for system  $\dot{x} = Jx + bu$ . We now prove that the closed-loop system (48)  
 648 with the feedback (49) is  $SISS_L$  (now, all the coefficients have been chosen). To that  
 649 aim, first notice that there exist  $\alpha_1, \alpha_2 > 0$  such that, for all  $\|z\| \leq 1$ ,

$$650 \quad \left\| T\tilde{A}z + T\tilde{B}\kappa(z) \right\| \leq \alpha_1 \|z\|,$$

$$651 \quad \left\| b\mu(y) \left( 1 - \frac{1}{(1 + \|z\|^2)^p} \right) \right\| \leq \alpha_2 \|z\|.$$

Let

$$\Delta := \min \left\{ 1, \Delta_2, \frac{1}{N_2}, \frac{\Delta_1}{(\alpha_2 + \alpha_1)N_2 + 1} \right\}.$$

652 Given  $\delta \leq \Delta$ , let  $e_1, e_2$  be two bounded measurable functions of the appropriate di-  
 653 mension, eventually bounded by  $\delta$ . Consider any trajectory  $(y(\cdot), z(\cdot))$  of the following  
 654 system

$$655 \quad (50) \quad \dot{y} = Jy + b\mu(y) - b\mu(y) \left( 1 - \frac{1}{(1 + \|z\|^2)^p} \right) + T\tilde{A}z + T\tilde{B}\kappa(z) + e_1,$$

$$656 \quad (51) \quad \dot{z} = \bar{A}z + \bar{B}\kappa(z) + e_2,$$

658 From the  $SISS_L(\Delta_2, N_2)$  property of  $z$ -subsystem it follows that  $\|z(\cdot)\| \leq_{ev} N_2\delta \leq 1$ .  
 659 Thus, using the above estimate, it is immediate to see that

$$660 \quad \left\| b\mu(y(\cdot)) \left( 1 - \frac{1}{(1 + \|z(\cdot)\|^2)^p} \right) + T\tilde{A}z(\cdot) + T\tilde{B}\kappa(z(\cdot)) + e_1(\cdot) \right\| \leq_{ev} \delta((\alpha_1 + \alpha_2)N_2 + 1)$$

$$661 \quad \leq \Delta_1.$$

Therefore, invoking the  $SISS_L(\Delta_1, N_1)$  property of  $\dot{x} = Jx + b\mu(y)$ , it follows that  
 $\|y(\cdot)\| \leq_{ev} \delta((\alpha_1 + \alpha_2)N_2 + 1)N_1$ . So, the closed-loop system (48) with the feedback  
 (49) is  $SISS_L$ . Moreover, as a consequence of Proposition 3 and of the inductive  
 hypothesis, for any trajectory of the closed-loop system (7) with the feedback law

(49), the control signal  $U : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ , defined by  $U(\cdot) := [U_1(\cdot), U_2(\cdot)]^T$  with  $U_1(t) := u_1(y(t), z(t))$  and  $U_2(t) := \kappa(z(t))$  for all  $t \geq 0$ , satisfies

$$\sup_{t \geq 0} \|U^{(k)}(t)\| \leq R_k$$

for all  $k \in \llbracket 0, p \rrbracket$ . Thus, the feedback law (49) is a feedback law  $p$ -bounded by  $(R_j)_{0 \leq j \leq p}$  for system (48).

**3.2.2. Proof of Proposition 3.** For the sake of notation compactness let  $\mu = \mu(A_{11})$ . To prove Proposition 3, we establish by induction on  $k$  that the following property holds, for all  $k \in \llbracket 0, p \rrbracket$ :

$(\overline{H}_k)$  : There exist a positive constant  $c_\mu$ , and continuous functions  $c_i : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  $i \in \llbracket 1, \mu - 1 \rrbracket$ , such that for any trajectory of the closed-loop system (48) with the feedback law (49), the control signal  $U_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  defined by  $U_1(t) := u_1(y(t), z(t))$  for all  $t \geq 0$  satisfies, for all  $j \in \llbracket 0, k \rrbracket$ ,

$$|U_1^{(j)}(t)| \leq a_\mu c_\mu + \sum_{i=1}^{\mu-1} a_i c_i(a_\mu, \dots, a_{i+1}), \quad \forall t \geq 0.$$

For  $k = 0$ , the statement  $(\overline{H}_0)$  holds trivially. Now, assume that  $(\overline{H}_k)$  holds true for some  $k \in \llbracket 0, p - 1 \rrbracket$ . We next prove that  $(\overline{H}_{k+1})$  also holds true. Let  $(y(\cdot), z(\cdot))$  be any trajectory of the closed-loop system (48) with the feedback law (49), and the control signal  $U_1(t) := u_1(y(t), z(t))$  and  $U_2(t) := \kappa(z(t))$ ,  $\forall t \geq 0$ . As in the proof of Proposition 2, it is sufficient to prove that there exist a positive constant  $\tilde{\Upsilon}_\mu$  and continuous functions  $\tilde{\Upsilon}_i : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  $i \in \llbracket 1, \mu - 1 \rrbracket$ , such that

$$|U_1^{(k+1)}(t)| \leq a_\mu \tilde{\Upsilon}_\mu + \sum_{i=1}^{\mu-1} a_i \tilde{\Upsilon}_i(a_\mu, \dots, a_{i+1}), \quad \forall t \geq 0.$$

Let  $\tilde{q}(s) := s^{-(p+1)}$ , for all  $s > 0$ . Define  $h(t) := 1 + \|z(t)\|^2$ , for all  $t \geq 0$ . With the same notation given in the proof of Proposition 2, one can write  $U_1(\cdot)$  as

$$U_1(t) = - \sum_{i=1}^{\mu} U_{1i}(t), \quad \forall t \geq 0,$$

where, for every  $i \in \llbracket 1, \mu \rrbracket$ ,

$$U_{1i}(t) := Q_{i,\mu} b_{0,i}^T y_i(t) [g \circ f_i](t) [\tilde{q} \circ h](t), \quad \forall t \geq 0.$$

As in the proof of Proposition 2, we next show that for each  $i \in \llbracket 1, \mu \rrbracket$ , there exist continuous functions  $c_{i,l} : \mathbb{R}_{>0}^{\mu-l} \rightarrow \mathbb{R}_{>0}$ ,  $l \in \llbracket 1, i \rrbracket$ , such that, for all  $t \geq 0$ ,

$$|U_{1i}^{(k+1)}(t)| \leq \sum_{l=1}^i a_l c_{i,l}(a_\mu, \dots, a_{l+1}),$$

$c_{i,\mu}$  is actually a constant independent of  $a_\mu$ , we write it as  $c_{i,\mu}(a_\mu, a_{\mu+1})$  for the sake of notation homogeneity. For  $i \in \llbracket 1, \mu \rrbracket$ , we apply Leibniz's rule to (54) and obtain that the  $(k+1)$ -th time derivative of  $U_{1i}(\cdot)$  is given, for all  $t \geq 0$ , by

$$U_{1i}^{(k+1)}(t) = a_i Q_{i+1,\mu} \left( \sum_{l_1=0}^{k+1} \sum_{l_2=0}^{l_1} \binom{k+1}{l_1} \binom{l_1}{l_2} [\tilde{q} \circ h]^{(k+1-l_1)}(t) [g \circ f_i]^{(l_2)}(t) b_{0,i}^T y_i^{(l_1-l_2)}(t) \right).$$

Then, to get (55), it is sufficient to show that :

a) there exists  $C > 0$  such that, for any  $\tilde{l} \in \llbracket 0, k+1 \rrbracket$  and for all  $t \geq 0$ ,

$$\left| [\tilde{q} \circ h]^{(\tilde{l})}(t) \right| \leq C[\tilde{q} \circ h](t).$$

692 b) for each  $i \in \llbracket 1, \mu \rrbracket$ , there exist  $\Psi_i, \Theta_i, \Phi_i : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ , and  $v_{i,j} : \mathbb{R}_{>0}^{\mu-j} \rightarrow$   
693  $\mathbb{R}_{>0}$  for  $j \in \llbracket 1, i \rrbracket$  such that, for any  $\tilde{l} \in \llbracket 0, k+1 \rrbracket$  and for all  $t \geq 0$ ,

$$\begin{aligned} 694 \quad \left\| y_i^{(\tilde{l})}(t) \right\| &\leq \bar{\Psi}_i(a_\mu, \dots, a_{i+1}) \sum_{l=i}^{\mu} \|y_l(t)\| + \Theta_i(a_\mu, \dots, a_{i+1}) \|z(t)\| \\ 695 \quad &+ \bar{\Phi}_i(a_\mu, \dots, a_{i+1}) + \sum_{l=1}^i a_l \tilde{v}_{l,i}(a_\mu, \dots, a_{l+1}). \end{aligned}$$

696 c) for each  $i \in \llbracket 1, \mu \rrbracket$ , there exist  $\Gamma_i, \theta_i : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ , and  $\Gamma_{i,j} : \mathbb{R}_{>0}^{\mu-j} \rightarrow \mathbb{R}_{>0}$   
697 for  $j \in \llbracket 1, i \rrbracket$  such that, for any  $\tilde{l} \in \llbracket 0, k+1 \rrbracket$  and for all  $t \geq 0$ ,

$$\begin{aligned} 698 \quad \left| [g \circ f_i]^{(\tilde{l})}(t) \right| &\leq [g \circ f_i](t) \left( \Gamma_i(a_\mu, \dots, a_{i+1}) + \sum_{l=1}^i a_l \tilde{v}_{l,i}(a_\mu, \dots, a_{l+1}) \right. \\ 699 \quad &\left. + \theta_i(a_\mu, \dots, a_{i+1}) \|z(t)\|^{2\tilde{l}} \right). \end{aligned}$$

We now establish *a)*. By an inductive argument using differentiation of the  $z$ -subsystem (48b) coupled with the fact that the feedback law  $\kappa$  is  $p$ -bounded, one easily shows that there exist  $C_0, C_1 > 0$  such that for any  $\tilde{l} \in \llbracket 1, k+1 \rrbracket$  and for any  $t \geq 0$ ,

$$\left\| z^{(\tilde{l})}(t) \right\| \leq C_0 + C_1 \|z(t)\|.$$

Using the Leibniz rule, it can be establish that there exist  $\tilde{C}_0, \tilde{C}_1 > 0$  such that, for any  $l \in \llbracket 1, k+1 \rrbracket$ ,

$$\left| h^{(\tilde{l})}(t) \right| \leq \tilde{C}_0 + \tilde{C}_1 \|z(t)\|^2,$$

700 for all  $t \geq 0$ . Thanks to Faà Di Bruno Formula (Lemma 5) applied to  $[q \circ h]$ , item *a)*  
701 follows.

702 We now deal with item *b)*. From Lemma 4 and an induction argument using  
703 differentiation of system (48a), one can obtain the following statement: for any  $l_1 \in$   
704  $\llbracket 1, k+1 \rrbracket$ ,  $i \in \llbracket 1, \mu \rrbracket$ , there exist continuous functions  $\bar{\Psi}_{l_1, i, l} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  $l \in$   
705  $\llbracket i+1, \mu \rrbracket$ ,  $\bar{\Phi}_{l_1, i, l} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  $l \in \llbracket 0, p \rrbracket$ ,  $\bar{\Theta}_{l_1, i, l} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  $l \in \llbracket 0, p \rrbracket$ , and  
706  $\bar{\Xi}_{l_1, i, l} : \mathbb{R}_{>0}^{\mu-i} \rightarrow \mathbb{R}_{>0}$ ,  $l \in \llbracket 0, p \rrbracket$ , such that, for all  $t \geq 0$ ,

$$\begin{aligned} 707 \quad \left\| y_i^{(l_1)}(t) \right\| &\leq \sum_{l=i}^{\mu} \bar{\Psi}_{l_1, i, l}(a_\mu, \dots, a_{i+1}) \|y_l(t)\| + \bar{\Theta}_{l_1, i, l}(a_\mu, \dots, a_{i+1}) \|z(t)\| \\ 708 \quad &+ \sum_{l=0}^{l_1-1} \bar{\Phi}_{l_1, i, l}(a_\mu, \dots, a_{i+1}) \left| U_1^{(l)}(t) \right| + \bar{\Xi}_{l_1, i, l}(a_\mu, \dots, a_{i+1}) \left\| U_2^{(l_1)}(t) \right\|. \end{aligned}$$

709 So, using the inductive hypothesis and the fact that  $\kappa$  is a  $p$ -bounded feedback law,  
710 one can obtain item *b)*.

711 Proceeding as in Proposition 2, one can get item *c)*. This ends the proof of  
712 Proposition 3.

713 **4. Numerical validation.**

714 In order to test the validity of our main result, we consider the single-input system  
 715 discussed after Definition 3, namely  $\dot{x} = Ax + bu$  with

$$716 \quad A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

717

718 It can easily be checked that  $A$  has no eigenvalue with positive real part (it actually  
 719 has two pairs of purely imaginary eigenvalues:  $s(A) = \mu(A) = 2$ ) and that the pair  
 720  $(A, b)$  is controllable, thus making the assumptions of Theorem 1 fulfilled. The system  
 721 is readily in the form (7) and the proposed control law (8) reads

$$722 \quad \kappa(y) = -\frac{Q_{1,2}x_2}{(1 + \|x\|^2)^{1/2}} - \frac{Q_{2,2}x_4}{(1 + x_3^2 + x_4^2)^{1/2}},$$

723

724 where  $Q_{1,2} = a_1a_2$  and  $Q_{2,2} = a_2$  for some  $a_1, a_2 > 0$ . In Figure 1, we run 30  
 725 simulations, each of them starting from initial conditions randomly picked in the  
 726 interval  $[-5; 5]$ , with  $a_1 = 0.5$  and  $a_2 = 1$ . A particular solution is reported in bold  
 727 black.

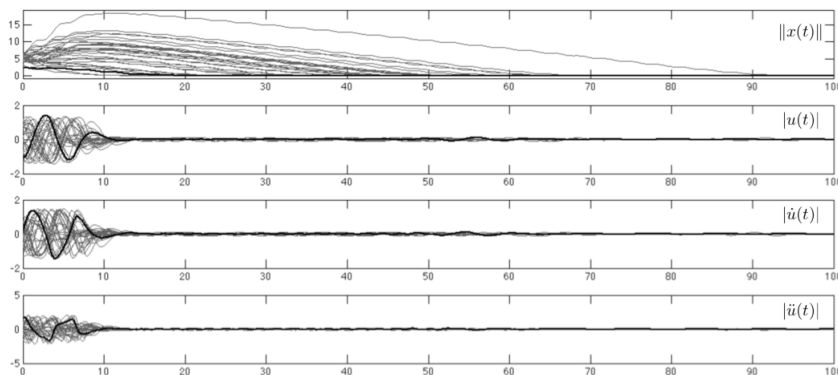


FIG. 1. Time evolution of the state norm, the control signal, and its first two derivatives for 30 random initial states within  $[-5; 5]$ .

728 These simulations indicate that, despite poor closed-loop performance, stabiliza-  
 729 tion is achieved for all considered initial states, while  $u(t)$ ,  $\dot{u}(t)$ , and  $\ddot{u}(t)$  all remain  
 730 within  $[-2; 2]$ .

731 Varying the parameter  $a_1$  between 0.25 and 5, and running again 30 simulations  
 732 for each of these values, Figure 2 reports the largest transient state norm and conver-  
 733 gence time (top), and the values reached by  $|u(t)|$ ,  $|\dot{u}(t)|$  and  $|\ddot{u}(t)|$  respectively  
 734 (bottom).

735 It can be seen that the control signal and its first two derivatives can be con-  
 736 strained to smaller values, while still achieving stabilization. Not surprisingly, the  
 737 price to pay is a larger overshoot and a slower convergence towards the origin as  $a_1$   
 738 decreases.

739 **5. Appendix.**



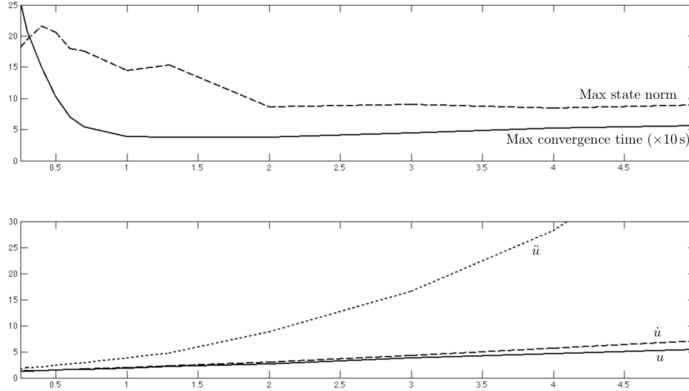


FIG. 2. *Top: Maximum value of the transient state norm and maximum convergence time over 30 random simulations, as a function of the feedback gain  $a_1$ . Bottom: Maximum norm of the control signal and of its first two derivatives over 30 random simulations, as a function of the feedback gain  $a_1$ .*

740 **5.1. Proof of Lemma 2 .** Let  $\epsilon > 1$  and  $\beta > 0$ . We first prove forward  
741 completeness of

$$742 \quad (56) \quad \dot{x} = -\beta \frac{x}{(1+x^2)^{1/2}} + d_1$$

743 in response to any locally bounded function  $d_1(\cdot)$ . For this, let  $V(x) := x^2/2$ . Its  
744 derivative along trajectories of (56) satisfies

$$745 \quad (57) \quad \dot{V}(x) = -\beta \frac{x^2}{(1+x^2)^{1/2}} + x^T d_1(t).$$

746 Then, a straightforward computation leads to  $\dot{V}(x) \leq V(x) + d_1(t)^2$  and forward  
747 completeness follows using classical comparison results. Moreover when  $d_1 = 0$ , (57)  
748 ensures that the origin of (56) is G.A.S.

749 We then prove the  $SISS_L(\beta/2, \frac{2\epsilon}{\beta})$  property of the system (56) with respect to  
750  $d_1(\cdot)$ . Given  $\delta \leq \beta/2$ , let  $d_1$  be a bounded measurable function on  $\mathbb{R}_{\geq 0}$  eventually  
751 bounded by  $\delta$ . Since the system is forward complete, we can consider without loss of  
752 generality that  $d_1(t) \leq \delta$  for all  $t \geq 0$ . From (57) and the fact that  $(1+x^2)^{1/2} \leq 1+|x|$ ,  
753 one can obtain that

$$754 \quad \dot{V}(x) = -\beta \frac{x^2}{(1+x^2)^{1/2}} + \frac{1}{(1+x^2)^{1/2}} (|d_1(t)||x| + |d_1(t)|x^2).$$

755 Observing that

$$756 \quad (58) \quad \frac{|d_1(t)|x^2}{(1+x^2)^{1/2}} \leq \frac{\beta x^2}{2(1+x^2)^{1/2}},$$

757 it follows that

$$758 \quad (59) \quad \dot{V}(x) \leq -\beta \frac{|x|}{(1+x^2)^{1/2}} \left( |x| - \frac{2}{\beta} \delta \right).$$

759 Consequently,  $\dot{V} < 0$  whenever  $|x| > \frac{2\delta}{\beta}$ . It follows that every trajectory of (10)  
760 eventually enters and remains in the set  $S = \{x \in \mathbb{R} : x^2 \leq \epsilon^2 (\frac{2\delta}{\beta})^2\}$  (indeed,  $\dot{V} < 0$   
761 for all  $x \notin S$  and  $x \in \partial S$ ). Thus Lemma 2 can be easily established.

762 **5.2. Proof of Lemma 3.** Let  $\omega > 0$ . Given any  $0 < \beta < 1$ , let  $A_\beta :=$   
 763  $\omega A_0 - \beta b_0 b_0^T$ , which is Hurwitz since  $A_0$  is skew-symmetric and  $(A_0, b_0)$  is controllable.  
 764 Therefore there exists a symmetric positive definite matrix  $P_\beta$  satisfying the following  
 765 Lyapunov equation

$$766 \quad (60) \quad P_\beta A_\beta + A_\beta^T P_\beta = -\mathbb{I}_2.$$

767 A simple computation gives

$$768 \quad P_\beta = \begin{pmatrix} \frac{\beta}{2\omega^2} + \frac{1}{\beta} & \frac{1}{2\omega} \\ \frac{1}{2\omega} & \frac{1}{\beta} \end{pmatrix}.$$

769 The smallest and largest eigenvalues of  $P_\beta$  denoted by  $\underline{\sigma}_\beta$  and  $\bar{\sigma}_\beta$  respectively are  
 770 given by

$$771 \quad \underline{\sigma}_\beta := \beta \|P_\beta b_0\|^2 - \frac{\beta}{2\omega} \|P_\beta b_0\|,$$

$$772 \quad \bar{\sigma}_\beta := \beta \|P_\beta b_0\|^2 + \frac{\beta}{2\omega} \|P_\beta b_0\|,$$

773 with

$$774 \quad \|P_\beta b_0\| = \sqrt{\frac{1}{4\omega^2} + \frac{1}{\beta^2}}.$$

775 Define  $V: \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$  as

$$776 \quad (61) \quad V(x) := x^T P_\beta x + \frac{(\bar{\sigma}_\beta + \underline{\sigma}_\beta)}{3} \left( (1 + \|x\|^2)^{3/2} - 1 \right), \quad \forall x \in \mathbb{R}^2.$$

777 Given  $C > 1$ , let  $\alpha_1$  and  $\alpha_2$  be class  $\mathcal{K}_\infty$  functions given by

$$778 \quad \alpha_1(r) := \frac{(\bar{\sigma}_\beta + \underline{\sigma}_\beta)}{C} \max\{r^2, r^3\},$$

$$779 \quad \alpha_2(r) := C(\bar{\sigma}_\beta + \underline{\sigma}_\beta) \max\{r^2, r^3\}.$$

780 There exists  $C > 1$  such that

$$781 \quad \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^2.$$

782 Moreover, there exists a constant  $M > 0$ , independent of  $\beta$ , such that

$$783 \quad (62) \quad \alpha_1^{-1} \circ \alpha_2(r) \leq Mr, \quad \forall r \geq 0.$$

784 Proceeding as in the proof of Lemma 2, forward completeness of

$$785 \quad (63) \quad \dot{x} = \omega A_0 x - \beta b_0 \frac{b_0^T x}{(1 + \|x\|^2)^{1/2}} + d_1$$

786 can easily be derived in response to any locally measurable bounded function  $d_1$ . We  
 787 next show that the system (63) is  $SISS_L(\beta\Gamma, N/\beta)$  with respect to  $d_1$ , for some  $N > 0$   
 788 and with

$$789 \quad (64) \quad \Gamma := \frac{1}{8\left(\frac{1}{4\omega^2} + 1\right)}.$$

790 Since (63) is forward complete, we can assume without loss of generality that  $d_1$   
 791 satisfies  $\|d_1(t)\| \leq \delta$ ,  $\forall t \geq 0$ , for some  $\delta \leq \beta\Gamma$ . Consider the Lyapunov function  
 792  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined in (61). By noticing that (63) can be rewritten as

$$793 \quad \dot{x} = A_\beta x + \beta b_0 b_0^T x \left( 1 - \frac{1}{(1 + \|x\|^2)^{1/2}} \right) + d_1,$$

794 one gets that the time derivative of  $V$  along trajectories of (63) satisfies

$$795 \quad \begin{aligned} \dot{V} = & x^T P_\beta \left( A_\beta x + \beta b_0 b_0^T x \left( 1 - \frac{1}{(1 + \|x\|^2)^{1/2}} \right) + d_1 \right) \\ 796 & + \left( x^T A_\beta^T + \beta b_0^T b_0^T x \left( 1 - \frac{1}{(1 + \|x\|^2)^{1/2}} \right) + d_1^T \right) P_\beta x \\ 797 & + (\bar{\sigma}_\beta + \underline{\sigma}_\beta) (1 + \|x\|^2)^{1/2} \left( -\beta \frac{(b_0^T x)^2}{(1 + \|x\|^2)^{1/2}} + x^T d_1 \right). \end{aligned}$$

799 Since  $P_\beta$  is a symmetric matrix satisfying the Lyapunov equation (60), it follows that

$$800 \quad \begin{aligned} \dot{V} = & -\|x\|^2 + 2\beta x^T P_\beta b_0 b_0^T x \left( 1 - \frac{1}{(1 + \|x\|^2)^{1/2}} \right) + 2x^T P_\beta d_1 - \beta(\bar{\sigma}_\beta + \underline{\sigma}_\beta) (b_0^T x)^2 \\ 801 & + (\bar{\sigma}_\beta + \underline{\sigma}_\beta) (1 + \|x\|^2)^{1/2} x^T d_1. \end{aligned}$$

803 By completing the squares it holds that, for all  $t \geq 0$ ,

$$804 \quad \left| 2\beta x^T P_\beta b_0 b_0^T x \left( 1 - \frac{1}{(1 + \|x\|^2)^{1/2}} \right) \right| \leq \frac{\|x\|^2}{2} + 2\beta^2 \|P_\beta b_0\|^2 (b_0^T x)^2.$$

805 Therefore, one can get that

$$806 \quad \dot{V} \leq -\frac{1}{2} \|x\|^2 + 2x^T P d_1 + 2\beta \|P b_0\|^2 (1 + \|x\|^2)^{1/2} x^T d_1.$$

807 Using the fact that  $(1 + \|x\|^2)^{1/2} \leq 1 + \|x\|$  for all  $x \in \mathbb{R}^2$ , and exploiting (64), it  
 808 follows that

$$809 \quad \dot{V} \leq -\frac{1}{4} \|x\|^2 + 2\|x\| \delta \left( 2\beta \|P_\beta b_0\|^2 + \frac{\beta}{2\omega} \|P_\beta b_0\| \right).$$

810 Consequently, it holds that  $\dot{V} < 0$  whenever  $\|x\| > 8\delta(2\beta \|P_\beta b_0\|^2 + \frac{\beta}{2\omega} \|P_\beta b_0\|)$ . Let  
 811  $\mu > 1$  and set  $r := 8\mu(2\beta \|P_\beta b_0\|^2 + \frac{\beta}{2\omega} \|P_\beta b_0\|)$ . Define  $S := \{x \in \mathbb{R}^2 : V(x) \leq$   
 812  $\alpha_2(r\delta)\}$ . If  $x \notin S$  then  $\|x\| > r\delta$ . Consequently, any trajectory eventually enters and  
 813 stay in  $S$ . Moreover, we have that  $\alpha_1(\|x(\cdot)\|) \leq_{ev} V(x(t)) \leq \alpha_2(r\delta)$ . From (62),  
 814 it follows that  $\|x(\cdot)\| \leq_{ev} rM\delta$ . Moreover, one can see that there exists a constant  
 815  $D > 0$  such that for any  $\beta \leq 1$  we have  $r \leq \frac{D}{\beta}$ . So we obtain

$$816 \quad \|x(\cdot)\| \leq_{ev} \frac{N\delta}{\beta},$$

817 for some  $N > 0$ , which concludes the proof.

818 **5.3. Faà Di Bruno's Formula.**

819 LEMMA 5 (Faà Di Bruno's formula, [6], p. 96). For  $k \in \mathbb{N}$ , let  $\phi \in C^k(\mathbb{R}_{\geq 0}, \mathbb{R})$   
 820 and  $\rho \in C^k(\mathbb{R}, \mathbb{R})$ . Then the  $k$ -th order derivative of the composite function  $\rho \circ \phi$  is  
 821 given by

$$822 \quad [\rho \circ \phi]^{(k)}(t) = \sum_{a=1}^k \rho^{(a)}(\phi(t)) B_{k,a} \left( \phi^{(1)}(t), \dots, \phi^{(k-a+1)}(t) \right),$$

823 where  $B_{k,a}$  is the Bell polynomial given by

$$824 \quad B_{k,a} \left( \phi^{(1)}(t), \dots, \phi^{(k-a+1)}(t) \right) := \sum_{\delta \in \mathcal{P}_{k,a}} c_{\delta} \prod_{l=1}^{k-a+1} \left( \phi^{(l)}(t) \right)^{\delta_l},$$

825  
 826 where  $\mathcal{P}_{k,a}$  denotes the set of  $(k-a+1)$ -tuples  $\delta := (\delta_1, \delta_2, \dots, \delta_{k-a+1})$  of positive  
 827 integers satisfying

$$828 \quad \delta_1 + \delta_2 + \dots + \delta_{k-a+1} = a,$$

$$829 \quad \delta_1 + 2\delta_2 + \dots + (k-a+1)\delta_{k-a+1} = k,$$

$$830 \quad c_{\delta} := \frac{k!}{(\delta_1! \cdots \delta_{k-a+1}! (1!)^{\delta_1} \cdots ((k-a+1)!)^{\delta_{k-a+1}})}.$$

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