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► **To cite this version:**

Jonathan Laporte, Antoine Chaillet, Yacine Chitour. Global stabilization of classes of linear control systems with bounds on the feedback and its successive derivatives. *Systems and Control Letters*, 2017, 99, pp.17-24. 10.1016/j.sysconle.2016.11.002 . hal-01633370

HAL Id: hal-01633370

<https://centralesupelec.hal.science/hal-01633370>

Submitted on 3 Apr 2020

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Global stabilization of classes of linear control systems with bounds on the feedback and its successive derivatives

Jonathan Laporte, Antoine Chaillet and Yacine Chitour ^{*}

Abstract

In this paper, we address the problem of globally stabilizing a linear time-invariant (LTI) system by means of a static feedback law whose amplitude and successive time derivatives, up to a prescribed order p , are bounded by arbitrary prescribed values. We solve this problem for two classes of LTI systems, namely integrator chains and skew-symmetric systems with single input. For the integrator chains, the solution we propose is based on the nested saturations introduced by A.R. Teel. We show that this construction fails for skew-symmetric systems and propose an alternative feedback law. We illustrate these findings by the stabilization of the third order integrator with prescribed bounds on the feedback and its first two derivatives, and similarly for the harmonic oscillator with prescribed bounds on the feedback and its first derivative.

1 Introduction

Actuator constraints constitute an important practical issue in control applications since they are a possible source of instability or performance degradation. Strong research efforts have been devoted to the stabilization of linear time-invariant (LTI) plants. LTI systems are known to be global stabilizable despite actuator saturations (i.e., by bounded inputs) if and only if they are stabilizable in the absence of input constraints and their internal dynamics has no eigenvalues with

^{*}This research was partially supported by a public grant overseen by the French ANR as part of the Investissements d'Avenir program, through the iCODE institute, research project funded by the IDEX Paris-Saclay, ANR-11-IDEX-0003-02.

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positive real part [?]. For systems that do not fulfill these constraints, several approaches provide stabilization from an arbitrarily large compact set of initial conditions (semiglobal stability); these include for instance [?, ?, ?, ?]. Some of these semiglobal approaches can be extended to robust stabilization in presence of exogenous disturbances [?, ?].

The objective of globally stabilizing plants by a bounded feedback remains of practical relevance, since the resulting control gains do not depend on the magnitude of initial states. Procedures ensuring global stability of nonlinear plants by bounded feedback have been proposed in [?, ?] and robustness to exogenous inputs have been addressed in [?, ?]. Among the LTI systems that can be globally stabilized by bounded feedback, chains of integrators have received specific attention. The simple saturation of a linear feedback fails at ensuring global stability as soon as the integrator chain is of dimension greater than or equal to three [?, ?]. In [?] a globally stabilizing feedback was constructed using *nested saturations* for a chain of integrators of arbitrary length. This construction has been extended to all LTI plants that can be stabilized by bounded feedback in [?], in which a family of stabilizing feedback laws was proposed as a linear combination of saturation functions. Global approaches ensuring robustness to exogenous disturbances have also been investigated. The first general solution to the L_p finite-gain stabilization problem was provided in [?], based on a gain scheduled feedback initially proposed in [?]. An alternative easily implementable solution to that problem was recently proposed in [?] for chains of integrators. As for neutrally stable systems, such a solution was first given in [?].

While actuation magnitude is often the main concern in practical applications, limited actuation reactivity can also be an issue. Indeed, technological constraints may affect not only the amplitude of the delivered control signal, but also the amplitude of its time derivative. This latter problem is known as rate saturation and has been addressed for instance in [?, ?, ?, ?, ?]. Semiglobal stabilization has been achieved via a gain scheduling technique [?], or through low-gain feedback or low-and-high-gain feedback [?]. In [?, ?], regional stability was ensured through LMI-based conditions. In [?], this problem has been addressed for nonlinear plants using backstepping procedure ensuring global stability.

In this paper, we deepen the investigations on global stabilization of integrator chains subject to bounded actuation with rate constraints. We consider rate constraints that affect not only the first time derivative of the control signal, but also its successive p first derivatives, where p denotes an arbitrary positive integer. We specifically study two classes of systems that can be globally stabilized by bounded state feedback, namely chains of integrator and skew-symmetric dynamics with single input. No restriction is imposed on the dimension of these systems. We show that solving the problem for these two cases actually cover wider classes of systems, namely all systems with either only zero eigenvalues or

only simple eigenvalues with zero real part. For both these classes of LTI systems, we propose a bounded static feedback law that ensures global asymptotic stability of the closed-loop system, and whose magnitude and p first time derivatives are bounded by arbitrary prescribed values. For the chains of integrators, the proposed control law is based on the nested saturations procedure introduced in [?]. We rely on specific saturation functions, which are linear in a neighborhood of the origin and constant for large values of their argument. Unfortunately, we show that this nested saturations feedback fails solve the problem for skew-symmetric dynamics. For the latter class of systems, we propose an alternative construction.

This paper is organized as follows. In Section 2, we provide definitions and state our main results for both considered classes of LTI systems. The proofs of the main results are provided in Section 3 based on several technical lemmas. In Section ??, we test the efficiency of the proposed control laws via numerical simulations on the third order integrator and the harmonic oscillator, where we bound with prescribed values the feedback, as well as its first two time derivatives for the third order integrator and its first time derivative for the harmonic oscillator respectively. We provide some conclusions and possible future extensions in Section ??.

Notations. The function $\text{sign} : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is defined as $\text{sign}(r) := r/|r|$. Given a set $I \subset \mathbb{R}$ and a constant $a \in \mathbb{R}$, we let $I_{\geq a} := \{x \in I : x \geq a\}$. Given $k \in \mathbb{N}$ and $n, p \in \mathbb{N}_{\geq 1}$, we say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is of class $C^k(\mathbb{R}^n, \mathbb{R}^p)$ if its differentials up to order k exist and are continuous, and we use $f^{(k)}$ to denote the k -th order differential of f . By convention, $f^{(0)} := f$. The factorial of k is denoted by $k!$ and the binomial coefficient is denoted $\binom{k}{m} := \frac{k!}{m!(k-m)!}$. We define $\llbracket m, k \rrbracket := \{l \in \mathbb{N} : l \in [m, k]\}$. We use $\mathbb{R}^{n,n}$ to denote the set of $n \times n$ matrices with real coefficients. The matrices I_n and $J_n \in \mathbb{R}^{n,n}$ denote the identity matrix of dimension n and the n -th Jordan block respectively, i.e., the $n \times n$ matrix given by $(J_n)_{i,j} = 1$ if $i = j - 1$ and zero otherwise. For each $i \in \llbracket 1, n \rrbracket$, $e_i \in \mathbb{R}^n$ refers to the column vector with coordinates equal to zero except the i -th one equal to one. We use $\|x\|$ to denote the Euclidean norm of an arbitrary vector $x \in \mathbb{R}^n$. For two sets A and B , the relative complement of A in B is denoted by $B \setminus A$.

2 Statement of the main results

2.1 Problem statement

We start by introducing in more details the general problem we address. Given $n \in \mathbb{N}_{\geq 1}$, consider LTI systems with single input:

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$, A and B are $n \times n$ and $n \times 1$ matrices respectively. Assume that the pair (A, B) is stabilizable and that all the eigenvalues of A have non positive real part. Recall that these assumptions on (A, B) are necessary and sufficient for the existence of a bounded continuous state feedback $u = k(x)$ which globally asymptotically stabilizes the closed-loop system [?]. We say that an eigenvalue of A is *critical* if it has zero real part.

Given a family of prescribed bounds $(R_j)_{0 \leq j \leq p}$ on the control signal and its successive p -first derivative, we start by introducing the notion of p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ for system (1). This terminology will be used all along the document.

Definition 1. Given $n \in \mathbb{N}_{\geq 1}$ and $p \in \mathbb{N}$, let $(R_j)_{0 \leq j \leq p}$ denote a family of positive constants. We say that $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is a p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ for system (1) if it is of class $C^p(\mathbb{R}^n, \mathbb{R})$ and, for every trajectory of the closed-loop system $\dot{x} = Ax + Bv(x)$, the control signal $U : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ defined by $U(t) := v(x(t))$ for all $t \geq 0$ satisfies, for all $j \in \llbracket 1, p \rrbracket$,

$$\sup_{t \geq 0} \left\{ \left| U^{(j)}(t) \right| \right\} \leq R_j.$$

Based on this definition, we can restate our stabilization problem as follows.

Problem. Given $p \in \mathbb{N}$ and a family of positive real numbers $(R_j)_{0 \leq j \leq p}$, design a feedback law v such that

- the origin of the closed-loop system $\dot{x} = Ax + Bv(x)$ is globally asymptotically stable;
- the feedback $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is a p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ for System (1).

The case $p = 0$ corresponds to global stabilization with bounded state feedback and has been addressed in e.g. [?, ?, ?, ?]. The case $p = 1$ corresponds to global stabilization with bounded state feedback and limited rate, in the line of e.g. [?, ?, ?, ?, ?].

In this paper, we present a general solution to the problem at stake for two classes of LTI systems:

Case 1: All the critical eigenvalues of A are zero;

Case 2: All the critical eigenvalues of A are simple and have zero real parts.

Since the pair (A, B) is stabilizable, there exists a linear change of coordinates transforming the matrices A and B into $\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}$ and $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, where A_1 is Hurwitz,

the eigenvalues of A_2 have zero real parts and the pair (A_2, B_2) is controllable. Then, it is immediate to see that we only have to treat the case where A has only critical eigenvalues. From now on, we therefore assume that A has only eigenvalues with zero real part, and that the pair (A, B) is controllable.

2.2 Multiple integrators

In Case 1, up to a linear change of coordinates, A can be put in a block-diagonal form with Jordan blocks J_r on the diagonal. It is then clear that, up to an additional linear change of coordinates, addressing Case 1 amounts to dealing with the sole case of a multiple integrator of arbitrary length n , i.e. the LTI control system given by

$$\begin{cases} \dot{x}_1 = x_2, \\ \vdots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = u. \end{cases} \quad (2)$$

Letting $x := (x_1, \dots, x_n)^T$, system (2) can be compactly written as

$$\dot{x} = J_n x + e_n u.$$

Inspired by [?], our design of a p -bounded feedback for this system is based on a nested saturations feedback. We focus on the specific class of saturations that are linear around zero, and constant for large values of their argument.

Definition 2. Given $p \in \mathbb{N}$, $\mathcal{S}(p)$ is defined as the set of all odd functions σ of class $C^p(\mathbb{R}, \mathbb{R})$ such that there exists positive constants α , L , σ^{max} and S satisfying, for all $r \in \mathbb{R}$,

(i) $r\sigma(r) > 0$, when $r \neq 0$,

(ii) $\sigma(r) = \alpha r$, when $|r| \leq L$,

(iii) $|\sigma(r)| = \sigma^{max}$, when $|r| \geq S$.

In the sequel, we associate with every $\sigma \in \mathcal{S}(p)$ the 4-tuple $(\sigma^{max}, L, S, \alpha)$.

The constants σ^{max} , L , α , and S will be extensively used throughout the paper and are referred to as the saturation level, the linearity threshold, the gain of the saturation in the linear part and the saturation threshold respectively. See Figure 1 to fix ideas. Notice that it necessarily holds that $S \geq L$ and the equality may only hold when $p = 0$. We also stress that the successive derivatives up to order p of an element of $\mathcal{S}(p)$ are bounded. An example of such function is given in Section ?? for $p = 2$.

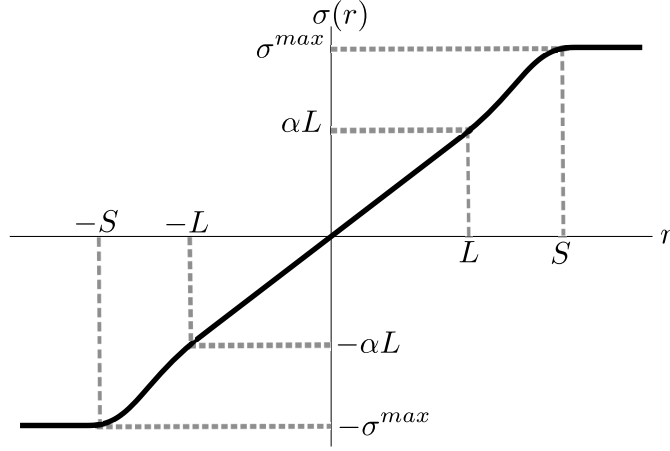


Figure 1: A typical example of a $\mathcal{S}(p)$ saturation function with constants $(\sigma^{max}, L, S, \alpha)$.

The first result of this paper establishes that global stabilization on any chain of integrators by bounded feedback with constrained p first derivatives can always be achieved by a particular choice of nested saturations. In other words, it solves the Problem in Case 1.

Theorem 1. *Given $n \in \mathbb{N}_{\geq 1}$ and $p \in \mathbb{N}$, let $(R_j)_{0 \leq j \leq p}$ be a family of positive constants. For every set of saturation functions $\sigma_1, \dots, \sigma_n \in \mathcal{S}(p)$, there exists vectors k_1, \dots, k_n in \mathbb{R}^n , and positive constants a_1, \dots, a_n such that the feedback law v defined, for each $x \in \mathbb{R}^n$, as*

$$v(x) = -a_n \sigma_n \left(k_n^T x + a_{n-1} \sigma_{n-1} \left(k_{n-1}^T x + \dots + a_1 \sigma_1 (k_1^T x) \right) \dots \right) \quad (3)$$

is a p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ for system (2), and the origin of the closed-loop system $\dot{x} = J_n x + e_n v(x)$ is globally asymptotically stable.

The proof of this result is given in Section 3.1 and the argument also provides an explicit choice of the gain vectors k_1, \dots, k_n and constants a_1, \dots, a_n .

Remark 1. *It can be seen along the proof of Theorem 1 that the proposed construction allows to chose the magnitude of control signal independently of the magnitude of its p first derivatives. More precisely, a_0 can be chosen to ensure that $\max_{x \in \mathbb{R}^n} |v(x)| = R_0$ and the gain vectors k_1, \dots, k_n and constants a_1, \dots, a_{n-1} can be taken in such a way that the p first derivatives of the feedback are bounded by $(R_j)_{1 \leq j \leq p}$.*

Remark 2. In [?], a stabilizing feedback law was constructed using linear combinations of saturated functions. That feedback with saturation functions in $\mathcal{S}(p)$ cannot be a p -bounded feedback for System (2) for $p \geq 1$. To see this, consider the double integrator, given by $\dot{x}_1 = x_2$, $\dot{x}_2 = u$. Any stabilizing feedback using a linear combination of saturation functions in $\mathcal{S}(p)$ is given by $v(x_1, x_2) = -a\sigma_1(bx_2) - c\sigma_2(d(x_2 + x_1))$, where the constants a , b , c , and d are chosen to insure stability of the closed-loop system according to [?]. Let $u(t) = v(x_1(t), x_2(t))$ for all $t \geq 0$. A straightforward computation yields, for $t \geq 0$,

$$\dot{u}(t) = -ab\sigma_1^{(1)}(ax_2(t))u(t) - cd\sigma_2^{(1)}(d(x_2(t) + x_1(t)))(x_2(t) + u(t)).$$

Consider now consider a solution with initial condition $x_2(0) = x_{20}$, and $x_1(0) = -x_{20}$ such that $\sigma_1^{(1)}(ax_{20}) = 0$. We then have $\dot{u}(0) = -cd\sigma_2^{(1)}(0)(x_{20} + u(0))$, whose norm is greater than $c_1(|x_{20}| - c_2)$ for some positive constants c_1, c_2 . Thus $|\dot{u}(0)|$ grows unbounded as $|x_{20}|$ tends to infinity, which contradicts the definition of a 1-bounded feedback.

2.3 Harmonic oscillators

In Case 2, up to a linear change of coordinates, A can be put in a block-diagonal form with skew-symmetric matrices on the diagonal. Addressing the stabilization problem under concern therefore amounts to only considering the following control system:

$$\dot{x} = Ax + bu, \tag{4}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n,n}$ is skew-symmetric, $b \in \mathbb{R}^n$ and the pair (A, b) is controllable.

Unfortunately, the nested saturations feedback law given in (3) is a generic solution to the Problem for this class of systems only in the scalar case ($n = 1$) or when when no rate constraint is imposed ($p = 0$). To see why it may fail for $n \geq 2$, consider for instance the harmonic oscillator given by $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + u$ (which we address in more details in Section ??) with a bounded stabilizing law given by $u = -\sigma(x_2)$ with $\sigma \in \mathcal{S}(p)$ for some integer p . The time derivative of u then satisfies

$$|\dot{u}(t)| \geq \left| \sigma^{(1)}(x_2(t)) \right| (|x_1(t)| - |u(t)|),$$

which grows unbounded as x_1 goes unbounded and $|x_2|$ remains small (i.e. in the linear zone of σ). This prevents the feedback $-\sigma(x_2)$ to be a 1-bounded feedback, hence a p -bounded feedback for all $p \geq 1$.

Our second result provides an alternative p -bounded feedback for the harmonic oscillator, thus solving the Problem in Case 2.

Theorem 2. Given $n \in \mathbb{N}_{\geq 1}$ and $p \in \mathbb{N}$, let $(R_j)_{0 \leq j \leq p}$ be a family of positive constants, let $A \in \mathbb{R}^{n,n}$ be a skew-symmetric matrix, and let $b \in \mathbb{R}^n$ be such that the pair (A, b) is controllable. Then, for any $\alpha \geq 1/2$, there exists a positive constant β such that the feedback law $v : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$v(x) := -\beta \frac{b^T x}{(1 + \|x\|^2)^\alpha} \quad (5)$$

is a p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ for system (4), and the origin of the closed-loop system $\dot{x} = Ax + bv(x)$ is globally asymptotically stable.

The proof of this theorem is given in Section 3.2.

Remark 3. Unlike for chains of integrators (see Remark 1), the magnitude of the proposed feedback cannot be chosen independently of the amplitude of its p first derivatives.

3 Proof of main results

3.1 Multiple integrators

3.1.1 Technical lemma

We start by estimating upper bounds on composed saturation functions of the class $\mathcal{S}(p)$. This estimate, presented in Lemma 2, relies on Faà di Bruno's formula recalled below.

Lemma 1 (Faà Di Bruno's formula, [?], p. 96). Given $k \in \mathbb{N}$, let $\phi \in C^k(\mathbb{R}_{\geq 0}, \mathbb{R})$ and $\rho \in C^k(\mathbb{R}, \mathbb{R})$. Then the k -th order derivative of the composite function $\rho \circ \phi$ is given by

$$\frac{d^k}{dt^k} \rho(\phi(t)) = \sum_{a=1}^k \rho^{(a)}(\phi(t)) B_{k,a}(\phi^{(1)}(t), \dots, \phi^{(k-a+1)}(t)), \quad (6)$$

where $B_{k,a}$ is the Bell polynomial given by

$$B_{k,a}(\phi^{(1)}(t), \dots, \phi^{(k-a+1)}(t)) := \sum_{\delta \in \mathcal{P}_{k,a}} c_\delta \prod_{l=1}^{k-a+1} (\phi^{(l)}(t))^{\delta_l} \quad (7)$$

where $\mathcal{P}_{k,a}$ denotes the set of $(k-a+1)$ -tuples $\delta := (\delta_1, \delta_2, \dots, \delta_{k-a+1})$ of positive integers satisfying

$$\begin{aligned} \delta_1 + \delta_2 + \dots + \delta_{k-a+1} &= a, \\ \delta_1 + 2\delta_2 + \dots + (k-a+1)\delta_{k-a+1} &= k, \end{aligned}$$

and $c_\delta := k! / (\delta_1! \cdots \delta_{k-a+1}! (1!)^{\delta_1} \cdots ((k-a+1)!)^{\delta_{k-a+1}})$.

Remark 4. We stress that the Bell polynomial $B_{k,a}\left(\phi^{(1)}(t), \dots, \phi^{(k-a+1)}(t)\right)$ is of (homogeneous) degree a with respect to the $(k-a+1)$ -dimensional vector representing the argument of $B_{k,a}$.

The proof of Theorem 1 extensively relies on the following upper bound on composition of functions of $\mathcal{S}(p)$, which exploits their constant value in their saturation region.

Lemma 2. Given $k \in \mathbb{N}$, let f and g be functions of class $C^k(\mathbb{R}_{\geq 0}, \mathbb{R})$, σ be a saturation function in $\mathcal{S}(k)$ with constants $(\sigma^{\max}, L, S, \alpha)$, and E and F be subsets of $\mathbb{R}_{\geq 0}$ such that $E \subseteq F$. Assume that

$$|f(t)| > S, \quad \forall t \in F \setminus E, \quad (8)$$

and that there exist positive constants M, Q_1, \dots, Q_k such that

$$|f^{(a)}(t)| \leq Q_a, \quad \forall t \in E, \forall a \in \llbracket 1, k \rrbracket, \quad (9)$$

$$|g^{(k)}(t)| \leq M, \quad \forall t \in F. \quad (10)$$

Then the k th-order derivative of the function $h: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ defined by $h(\cdot) := g(\cdot) + \sigma(f(\cdot))$, satisfies

$$|h^{(k)}(t)| \leq M + \sum_{a=1}^k \bar{\sigma}_a B_{k,a}(Q_1, \dots, Q_{k-a+1}), \quad \forall t \in F, \quad (11)$$

where $B_{k,a}(Q_1, \dots, Q_{k-a+1})$ is a polynomial function of Q_1, \dots, Q_{k-a+1} of degree a , and $\bar{\sigma}_a := \max_{s \in \mathbb{R}} |\sigma^{(a)}(s)|$ for each $a \in \llbracket 1, k \rrbracket$.

Proof of Lemma 2. Using Lemma 1, a straightforward computation yields

$$h^{(k)}(t) = g^{(k)}(t) + \sum_{a=1}^k \sigma^{(a)}(f(t)) B_{k,a}\left(f^{(1)}(t), \dots, f^{(k-a+1)}(t)\right), \quad \forall t \geq 0,$$

where the polynomials $B_{k,a}$ are defined in (7). Since $\sigma \in \mathcal{S}(k)$, (8) ensures that the set $F \setminus E$ is contained in the saturation zone of σ . It follows that

$$\frac{d^k}{dt^k} \sigma(f(t)) = 0, \quad \forall t \in F \setminus E. \quad (12)$$

Furthermore, from (9) and (7) it holds that, for all $t \in E$,

$$\begin{aligned} \left| B_{k,a}\left(f^{(1)}(t), \dots, f^{(k-a+1)}(t)\right) \right| &\leq \sum_{\delta \in \mathcal{P}_{k,a}} c_\delta \prod_{l=1}^{k-a+1} Q_l^{\delta_l}, \\ &= B_{k,a}(Q_1, \dots, Q_{k-a+1}). \end{aligned}$$

From the definition of $\bar{\sigma}_a$ and (6), we get that

$$\left| \frac{d^k}{dt^k} \sigma(f(t)) \right| \leq \sum_{a=1}^k \bar{\sigma}_a B_{k,a}(Q_1, \dots, Q_{k-a+1}), \quad \forall t \in E. \quad (13)$$

In view of (12), the estimate (13) is valid on the whole set F . Thanks to (10), a straightforward computation leads to the estimate (11). \square

3.1.2 Intermediate results

The proof of Theorem 1 relies on the two propositions presented in this section. The first one provides bounds on the successive derivatives of the arguments of the nested saturations and of the resulting control signal. The second proposition provides conditions on the parameters involved to guarantee global asymptotic stability of the closed-loop system. Both these propositions rely on notations and change of variables that we introduce first.

Given $n \in \mathbb{N}_{\geq 1}$ and $p \in \mathbb{N}$, let μ_1, \dots, μ_n be saturations in $\mathcal{S}(p)$ with respective constants $(\mu_i^{max}, L_{\mu_i}, S_{\mu_i}, \alpha_{\mu_i})$, $i \in \llbracket 1, n \rrbracket$. We define, for each $i \in \llbracket 1, n \rrbracket$,

$$\bar{\mu}_{i,j} := \max \left\{ \left| \mu_i^{(j)}(r) \right| : r \in \mathbb{R} \right\}, \quad \forall j \in \llbracket 1, p \rrbracket, \quad (14)$$

$$b_{\mu_i} := \max \left\{ |r - \mu_i(r)| : |r| \leq S_{\mu_i} + 2\mu_{i-1}^{max} \right\}. \quad (15)$$

We also let

$$\bar{b}_{\mu_n} := \max \left\{ \frac{\mu_n(r)}{r} : 0 < |r| \leq S_{\mu_n} \right\}, \quad (16)$$

$$\underline{b}_{\mu_n} := \min \left\{ \frac{\mu_n(r)}{r} : 0 < |r| \leq S_{\mu_n} \right\}. \quad (17)$$

Note that these quantities are well defined since the functions μ_i are all in $\mathcal{S}(p)$.

We also make a linear change of coordinates $y = Hx$, with $H \in \mathbb{R}^{n,n}$, that puts system (2) into the form

$$\dot{y}_i = \alpha_{\mu_n} \sum_{l=i+1}^n y_l + u, \quad \forall i \in \llbracket 1, n \rrbracket. \quad (18)$$

The matrix H can be determined from the following relation:

$$y_{n-i} = \sum_{k=0}^i \binom{i}{k} (\alpha_{\mu_n})^k x_{n-k}, \quad \forall i \in \llbracket 0, n-1 \rrbracket, \quad (19)$$

and since H is triangular with non zero elements on the main diagonal, it is invertible. For this system, we define a nested saturations feedback law $\Upsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\Upsilon(y) = -\mu_n(y_n + \mu_{n-1}(y_{n-1} + \dots + \mu_1(y_1)) \dots). \quad (20)$$

Note that, in the original x -coordinates, this feedback law reads $\Upsilon(y) = \Upsilon(Hx)$ therefore the bounds of the successive time derivatives of $\Upsilon(y)$ coincide with those of $\Upsilon(Hx)$. The global stabilization (18) with a p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ is thus equivalent to that of the original system (2). So, from now on, we will rely on this expression. Let $y(\cdot)$ be a trajectory of the system

$$\dot{y}_i = \alpha_{\mu_n} \sum_{l=i+1}^n y_l + \Upsilon(y), \quad \forall i \in \llbracket 1, n \rrbracket, \quad (21)$$

which is the closed-loop system (18) with the feedback defined in (20). For each $i \in \llbracket 1, n \rrbracket$, let $z_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be the time function defined recursively as

$$z_i(\cdot) := y_i(\cdot) + \mu_{i-1}(z_{i-1}(\cdot)),$$

with $\mu_0(\cdot) = 0$. With the above functions, the closed-loop system (21) can be rewritten as

$$\begin{cases} \dot{y}_i = \alpha_{\mu_n} z_n - \mu_n(z_n) + \alpha_{\mu_n} \sum_{l=i+1}^{n-1} (z_l - \mu_l(z_l)) - \alpha_{\mu_n} \mu_i(z_i), & \forall i \in \llbracket 1, n-1 \rrbracket, \\ \dot{y}_n = -\mu_n(z_n). \end{cases} \quad (22)$$

For each $i \in \llbracket 1, n \rrbracket$, we also let

$$E_i := \{y \in \mathbb{R}^n : |y_v| \leq S_{\mu_v} + \mu_{v-1}^{max}, \forall v \in \llbracket i, n \rrbracket\}, \quad (23)$$

with $\mu_0^{max} = 0$, and

$$I_i := \{t \in \mathbb{R}_{\geq 0} : y(t) \in E_i\}. \quad (24)$$

Note that from the definitions of I_i and E_i , we have $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n$, and a straightforward computation yields

$$|z_i(t)| > S_{\mu_i}, \quad \forall t \in I_{i+1} \setminus I_i, \forall i \in \llbracket 1, n-1 \rrbracket, \quad (25)$$

$$|z_n(t)| > S_{\mu_n}, \quad \forall t \in \mathbb{R}_{\geq 0} \setminus I_n, \quad (26)$$

which allows us to determine when saturation occurs. Moreover from the definitions of saturation functions of class $\mathcal{S}(p)$, E_i , I_i , (16) and (17), the following estimates can easily be derived:

$$|z_i(t) - \mu_i(z_i(t))| \leq b_{\mu_i}, \quad \forall t \in I_i, \quad (27)$$

$$|\alpha_{\mu_n} z_n(t) - \mu_n(z_n(t))| \leq (\bar{b}_{\mu_n} - \underline{B}_{\mu_n})(S_{\mu_n} + 2\mu_{n-1}^{max}), \quad \forall t \in I_n, \quad (28)$$

with $\underline{B}_{\mu_n} := \min \left\{ \underline{b}_{\mu_n}, \frac{\mu_n^{\max}}{S_{\mu_n} + 2\mu_{n-1}^{\max}} \right\}$.

The following statement provides explicit bounds on the successive derivatives of each functions $y_i(t)$, $z_i(t)$ for each $i \in \llbracket 1, n \rrbracket$ and the control input given by $U(\cdot) := \Upsilon(y(\cdot))$.

Proposition 1. *Given $n \in \mathbb{N}_{\geq 1}$ and $p \in \mathbb{N}$, let μ_1, \dots, μ_n be saturation functions in $\mathcal{S}(p)$ with respective constants $(\mu_i^{\max}, L_{\mu_i}, S_{\mu_i}, \alpha_{\mu_i})$ for each $i \in \llbracket 1, n \rrbracket$. With the notation introduced in this section and the Bell polynomials introduced in (7), every trajectory of the closed-loop system (21) satisfies, for each $i \in \llbracket 1, n \rrbracket$ and each $j \in \llbracket 1, p \rrbracket$,*

$$(P_1(i, j)) : \quad \left| y_i^{(j)}(t) \right| \leq Y_{i,j}, \quad \forall t \in I_i; \quad (29)$$

$$(P_2(i, j)) : \quad \left| z_i^{(j)}(t) \right| \leq Z_{i,j}, \quad \forall t \in I_i; \quad (30)$$

$$(P_3(j)) : \quad \sup_{t \geq 0} \left\{ \left| U^{(j)}(t) \right| \right\} \leq \sum_{q=1}^j G_{q,j} \bar{\mu}_{n,q}; \quad (31)$$

where $Y_{i,j}$, $Z_{i,j}$, and $G_{q,j}$ are independent of initial conditions and are obtained recursively as follows: for $j = 1$,

$$Y_{n,1} := \mu_n^{\max},$$

$$Y_{i,1} := (\bar{b}_{\mu_n} - \underline{B}_{\mu_n})(S_{\mu_n} + 2\mu_{n-1}^{\max}) + \alpha_{\mu_n} \sum_{l=i+1}^{n-1} b_{\mu_l} + \alpha_{\mu_n} \mu_i^{\max}, \quad \forall i \in \llbracket 1, n-1 \rrbracket,$$

$$Z_{1,1} := Y_{1,1},$$

$$Z_{i,1} := Y_{i,1} + \bar{\mu}_{i-1,j} Z_{i-1,1}, \quad \forall i \in \llbracket 2, n \rrbracket,$$

$$G_{1,1} := Z_{n,1}$$

and, for each $j \in \llbracket 2, p \rrbracket$,

$$Y_{i,j} := \alpha_{\mu_n} \sum_{b=i+1}^n Y_{b,j-1} + \sum_{q=1}^{j-1} G_{q,j-1} \bar{\mu}_{n,q}, \quad \forall i \in \llbracket 1, n-1 \rrbracket,$$

$$Z_{1,j} := Y_{1,j},$$

$$Z_{i,j} := Y_{i,j} + \sum_{a=1}^j \bar{\mu}_{i-1,a} B_{j,a}(Z_{i-1,1}, \dots, Z_{i-1,j-1+a}), \quad \forall i \in \llbracket 2, n \rrbracket,$$

$$G_{q,j} := B_{j,q}(Z_{n,1}, \dots, Z_{n,j-q+1}), \quad \forall q \in \llbracket 1, j \rrbracket.$$

Proof of Proposition 1. Let $y(\cdot)$ be a trajectory of the closed-loop system (21). The right-hand side of (21) being of class $C^p(\mathbb{R}^n, \mathbb{R}^n)$ and globally Lipschitz, system (21) is forward complete and its trajectories are of class $C^{p+1}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. In

particular, the successive time derivatives of $y_i(\cdot)$, $z_i(\cdot)$, and $U(\cdot)$ are well defined for all positive time.

We establish the result by induction on j . We start by $j = 1$. We begin to prove that $P_1(i, 1)$ holds for all $i \in \llbracket 1, n \rrbracket$. Let $i \in \llbracket 1, n-1 \rrbracket$. From (22), (27), and (28) a straightforward computation leads to

$$|\dot{y}_i(t)| \leq (\bar{b}_{\mu_n} - \underline{B}_{\mu_n})(S_{\mu_n} + 2\mu_{n-1}^{max}) + \alpha_{\mu_n} \sum_{l=i+1}^{n-1} b_{\mu_l} + \alpha_{\mu_n} \mu_i^{max},$$

for all $t \in I_{i+1}$. Since $I_i \subseteq I_{i+1}$, the above estimate is still true on I_i . Moreover, from (22) it holds that $|\dot{y}_n(t)| \leq \mu_n^{max}$ at all positive time. Thus, $P_1(i, 1)$ has been proven for each $i \in \llbracket 1, n \rrbracket$.

We now prove by induction on i the statement $P_2(i, 1)$. Since $z_1(\cdot) = y_1(\cdot)$, the case $i = 1$ is done. Assume that, for $i \in \llbracket 1, n-1 \rrbracket$, $P_2(i_2, 1)$ holds true for every $1 \leq i_2 \leq i$. From Lemma 2 (with $k = 1$, $f(\cdot) = z_i(\cdot)$, $g(\cdot) = y_{i+1}(\cdot)$, $h(\cdot) = z_{i+1}(\cdot)$, $\sigma(\cdot) = \mu_i(\cdot)$, $Q_1 = Z_{i,1}$, $M = Y_{i+1,1}$, $\bar{\sigma}_1 = \bar{\mu}_{i,1}$, $E = I_i$, $F = I_{i+1}$, and (25)), we can establish that $P_2(i+1, 1)$ holds. Thus $P_2(i, 1)$ holds for all $i \in \llbracket 1, n \rrbracket$.

Notice that the applied control input reads $U(\cdot) = -\mu_n(z_n(\cdot))$. We then can establish $P_3(1)$ from Lemma 2 (with $k = 1$, $f = z_n$, $g \equiv 0$, $h = u$, $\sigma = \mu_n$, $Q_1 = Z_{1,i}$, $M = 0$, $\bar{\sigma}_1 = \bar{\mu}_{n,1}$, $E = I_n$, $F = \mathbb{R}_{\geq 0}$ and (26)). This ends the case $j = 1$.

Now, assume that for a given $j \in \llbracket 1, p-1 \rrbracket$, statements $P_1(i, j_2)$, $P_2(i, j_2)$ and $P_3(j_2)$ hold for all $j_2 \leq j$ and all $i \in \llbracket 1, n \rrbracket$. Let $i \in \llbracket 1, n \rrbracket$. From (18), a straightforward computation yields

$$\left| y_i^{(j+1)}(t) \right| \leq \alpha_{\mu_n} \sum_{l=i+1}^n \left| y_l^{(j)}(t) \right| + \left| u^{(j)}(t) \right|, \quad \forall t \geq 0.$$

From $P_3(j)$, $P_1(i+1, j), \dots, P_1(n, j)$, we obtain that

$$\left| y_i^{(j+1)}(t) \right| \leq \alpha_{\mu_n} \sum_{l=i+1}^n Y_{l,j} + \sum_{q=1}^j G_{q,j} \bar{\mu}_{n,q}, \quad \forall t \geq I_i.$$

Thus the statement $P_1(j+1, i)$ is proven for all $i \in \llbracket 1, n \rrbracket$.

We now prove by induction on i the statement $P_2(i, j+1)$. As before, since $z_1 = y_1$, the case for $i = 1$ is done. Assume that for a given $i \in \llbracket 1, n-1 \rrbracket$, the statement $P_2(i_1, j+1)$ holds for all $i_1 \leq i$. From Lemma 2 (with $k = j+1$, $f = z_i$, $g = y_{i+1}$, $h = z_{i+1}$, $\sigma = \mu_i$, $Q_{k_1} = Z_{i,k_1}$, $M = Y_{i+1,j+1}$, $\bar{\sigma}_a = \bar{\mu}_{i,a}$, $E = I_i$, $F = I_{i+1}$, and (25)), we can establish that $P_2(i+1, j+1)$ holds. $P_2(i, j+1)$ is thus satisfied for all $i \in \llbracket 1, n \rrbracket$.

Finally, we can establish $P_3(j+1)$ from Lemma 2 (with $k = j+1$, $f = z_n$, $g \equiv 0$, $h = u$, $\sigma = \mu_n$, $Q_{k_1} = Z_{n,k_1}$, $M = 0$, $\bar{\sigma}_a = \bar{\mu}_{n,a}$, $E = I_n$, $F = \mathbb{R}_{\geq 0}$ and (26)). This ends the proof of Proposition 1. \square

We next provide sufficient conditions on the parameters of the saturation functions in $\mathcal{S}(p)$ guaranteeing global asymptotic stability of the closed-loop system (21).

Proposition 2. *Given $n \in \mathbb{N}$ and $p \in \mathbb{N}$, let μ_1, \dots, μ_n be saturation functions in $\mathcal{S}(p)$ with respective constants $(\mu_i^{\max}, L_{\mu_i}, S_{\mu_i}, \alpha_{\mu_i})$ for each $i \in \llbracket 1, n \rrbracket$ and assume that, for all $i \in \llbracket 1, n-1 \rrbracket$,*

$$\alpha_{\mu_i} = 1, \quad (32a)$$

$$\mu_i^{\max} < L_{\mu_{i+1}}/2. \quad (32b)$$

Then the origin of the closed-loop system (21) is globally asymptotically stable.

Actually the above proposition is almost the same as the one given in [?], except that we allow the first level of saturation μ_n to have a slope different from 1. This feature is instrumental for the proof of the main result, as reducing this slope will allow to decrease the amplitudes of the successive time derivatives on the control signal.

Proof of Proposition 2. We prove that after a finite time any trajectory of the closed-loop system (21) enters a region in which the feedback (20) becomes simply linear.

To that end, we consider the Lyapunov function candidate $V_n(y_n) := \frac{1}{2}y_n^2$. Its derivative along the trajectories of (21) reads

$$\dot{V}_n = -y_n \mu_n(y_n + \mu_{n-1}(z_{n-1})).$$

From (32b), we obtain the following implication:

$$|y_n| \geq L_{\mu_n}/2 \quad \Rightarrow \quad \dot{V}_n \leq -\theta L_{\mu_n}/2, \quad (33)$$

where $\theta := \inf_{r \in [L_{\mu_n}/2 - \mu_{n-1}^{\max}, S_{\mu_n}]} \{\mu_n(r)\}$.

We next show that there exists a time $T_1 \geq 0$ such that $|y_n(t)| \leq L_{\mu_n}/2$, for all $t \geq T_1$. To prove that, we consider the following alternative: either for every $t \geq 0$, $|y_n(t)| \leq L_{\mu_n}/2$ and we are done, or there exist $T_0 \geq 0$ such that $|y_n(T_0)| > L_{\mu_n}/2$. In that case there exists $\tilde{T}_0 \geq T_0$ such that $y_n(\tilde{T}_0) = L_{\mu_n}/2$ (otherwise thanks to (33), $V_n(t) \rightarrow -\infty$ as $t \rightarrow \infty$ which is impossible). Due to (33), we have $|y_n(t)| < L_{\mu_n}/2$ in a right open neighbourhood of \tilde{T}_0 . Suppose that there exists a positive time $\tilde{T}_1 > \tilde{T}_0$ such that $|y_n(\tilde{T}_1)| \geq L_{\mu_n}/2$. Then by continuity, there must exist $\tilde{T}_2 \in (\tilde{T}_0, \tilde{T}_1)$ such that $|y_n(\tilde{T}_2)| = L_{\mu_n}/2$, and $|y_n(t)| < L_{\mu_n}/2$ for all $t \in (\tilde{T}_0, \tilde{T}_2)$. However, it then follows from (33) that for a left open neighbourhood of \tilde{T}_2 we have $|y_n(t)| > |y_n(\tilde{T}_2)| = L_{\mu_n}/2$. This is a contradiction with the fact that on a right

open neighbourhood of \tilde{T}_0 we have $|y_n(t)| < L_{\mu_n}/2$. Therefore, for every $\tilde{T}_1 > \tilde{T}_0$, one has $|y_n(\tilde{T}_1)| < L_{\mu_n}/2$ and the claim is proved.

It follows from (32b) that

$$|y_n(t) + \mu_{n-1}(z_{n-1}(t))| \leq L_{\mu_n}, \quad \forall t \geq T_1.$$

Therefore μ_n operates in its linear region after time T_1 . Similarly, we now consider $V_{n-1}(y_{n-1}) := \frac{1}{2}y_{n-1}^2$, whose derivative along the trajectories of (21) satisfies

$$\dot{V}_n = -\alpha_{\mu_n} y_{n-1} \mu_{n-1} (y_{n-1} + \mu_{n-2}(y_{n-2} + \dots)), \quad \forall t \geq T_1.$$

Reasoning as before and invoking (32b), there exists a time $T_2 > 0$ such that $|y_{n-1}(t)| \leq L_{\mu_{n-1}}/2$ and μ_{n-1} operates in its linear region for all $t \geq T_2$.

By repeating this procedure, we construct a time T_n such that for every $t \geq T_n$ the whole feedback law becomes linear, that is

$$\Upsilon(y(t)) = -\alpha_{\mu_n} (y_n(t) + \dots + y_1(t)), \quad \forall t \geq T_n.$$

Thus, after time T_n , the system (21) becomes linear and its local exponential stability follows readily. Thus the origin of the closed-loop system (21) is globally asymptotically stable, which concludes the proof of Proposition 2. \square

3.1.3 Proof of Theorem 1

Based on Propositions 2 and 1, we now proceed to the final stage of the proof of Theorem 1. We explicitly construct the vectors k_1, \dots, k_n and the constants a_1, \dots, a_n guaranteeing global asymptotic stability with a bounded feedback law whose successive derivatives remain below prescribed bounds at all times. This proof can thus be used as an algorithm to compute the nested saturations feedback proposed in Theorem 1.

Given $p \in \mathbb{N}$ and $n \in \mathbb{N}_{\geq 1}$, let σ_i be saturation functions in $\mathcal{S}(p)$ with constants $(\sigma_i^{max}, L_{\sigma_i}, S_{\sigma_i}, \alpha_{\sigma_i})$ for each $i \in \llbracket 1, n \rrbracket$, and let $(R_j)_{0 \leq j \leq p}$ be the family of prescribed positive bounds on the amplitude and the successive time derivatives

of the control signal. We let

$$\begin{aligned}
\underline{R} &:= \min\{R_j : j \in \llbracket 1, p \rrbracket\}, \\
\bar{\sigma}_{n,j} &:= \max_{r \in \mathbb{R}} \left\{ \left| \sigma_n^{(j)}(r) \right| \right\}, \quad \forall j \in \llbracket 1, p \rrbracket, \\
\alpha_{\bar{\mu}} &:= R_0 L_{\sigma_n} \alpha_{\sigma_n} / \sigma_n^{\max}, \\
\tilde{\mu}_{n,j} &:= \frac{R_0 \bar{\sigma}_{n,j} (L_{\sigma_n})^j}{\sigma_n^{\max}}, \quad \forall j \in \llbracket 1, p \rrbracket, \\
\bar{b}_{\sigma_n} &:= \max \left\{ \frac{\sigma_n(r)}{r} : 0 < |r| \leq S_{\sigma_n} \right\}, \\
\underline{b}_{\sigma_n} &:= \min \left\{ \frac{\sigma_n(r)}{r} : 0 < |r| \leq S_{\sigma_n} \right\}.
\end{aligned}$$

Note that all these quantities are well defined since $\sigma_n \in \mathcal{S}(p)$. We first construct saturations μ_1, \dots, μ_n in order to use the results of Section 3.1.2. Let $(\mu_i^{\max})_{1 \leq i \leq n-1}$ and $(L_{\mu_i})_{1 \leq i \leq n-1}$ be two families of positive constants such that

$$\mu_{n-1}^{\max} < \frac{1}{2}, \quad L_{\mu_{n-1}} = \frac{\mu_{n-1}^{\max} L_{\sigma_{n-1}} \alpha_{\sigma_{n-1}}}{\sigma_{n-1}^{\max}}, \quad (34)$$

and, for each $i \in \llbracket 1, n-2 \rrbracket$,

$$\mu_i^{\max} < \frac{1}{2} L_{\mu_{i+1}}, \quad L_{\mu_i} = \frac{\mu_i^{\max} L_{\sigma_i} \alpha_{\sigma_i}}{\sigma_i^{\max}}. \quad (35)$$

For each $i \in \llbracket 1, n-1 \rrbracket$, the saturation function $\mu_i \in \mathcal{S}(p)$ with constants $(\mu_i^{\max}, L_{\mu_i}, S_{\mu_i}, 1)$, where $S_{\mu_i} = S_{\sigma_i} L_{\mu_i} / L_{\sigma_i}$, is then given by

$$\mu_i(s) := \frac{\mu_i^{\max}}{\sigma_i^{\max}} \sigma_i \left(s \frac{L_{\sigma_i}}{L_{\mu_i}} \right), \quad \forall s \in \mathbb{R}.$$

For $\lambda \geq 1$, to be chosen later, we define the saturation function μ_n as

$$\mu_n(s) := \frac{R_0}{\sigma_n^{\max}} \sigma_n \left(\frac{L_{\sigma_n}}{\lambda} s \right), \quad \forall s \in \mathbb{R},$$

with $\mu_n^{\max} = R_0$, $L_{\mu_n} = \lambda$, $S_{\mu_n} = S_{\sigma_n} \lambda / L_{\sigma_n}$, and $\alpha_{\mu_n} = \alpha_{\bar{\mu}} / \lambda$. Then μ_n belongs to the class of saturation functions $\mathcal{S}(p)$. From (34) and (35) we can establish that the functions μ_1, \dots, μ_n satisfy conditions (32). It follows from Proposition 2 that the nested saturations feedback $\Upsilon(y)$ defined in (20) globally asymptotically stabilizes the origin of (18).

We next choose λ in such a way that $\Upsilon(y)$ is a p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ for System (18). To that end, first notice that

$$\bar{b}_{\mu_n} = \frac{\alpha_{\bar{\mu}} \bar{b}_{\sigma_n}}{\lambda \alpha_{\sigma_n}}, \quad (36)$$

$$\underline{b}_{\mu_n} = \frac{\alpha_{\bar{\mu}} \underline{b}_{\sigma_n}}{\lambda \alpha_{\sigma_n}}, \quad (37)$$

$$\underline{B}_{\mu_n} = \frac{1}{\lambda} \min \left\{ \frac{\alpha_{\bar{\mu}} \underline{b}_{\sigma_n}}{\alpha_{\sigma_n}}, \frac{R_0}{S_{\sigma_n}/L_{\sigma_n} + 2\sigma_{n-1}^{max}/\lambda} \right\}, \quad (38)$$

$$\bar{\mu}_{n,q} = \tilde{\mu}_{n,q}/\lambda^q, \quad (39)$$

where \bar{b}_{μ_n} , \underline{b}_{μ_n} , and $\bar{\mu}_{n,q}$ are defined in (16), (17), and (14) respectively. Using Proposition 1 it follows that, along every trajectory of the closed-loop system (21), the control input $U(\cdot) = \Upsilon(y(\cdot))$ satisfies, for each $j \in \llbracket 1, p \rrbracket$,

$$\sup_{t \geq 0} \left\{ \left| U^{(j)}(t) \right| \right\} \leq \sum_{q=1}^j G_{q,j} \frac{\tilde{\mu}_{n,q}}{\lambda^q}. \quad (40)$$

By substituting (36), (37), (38), and (39) into the recurrence of Proposition 1, it can be seen that, for each $j \in \llbracket 1, p \rrbracket$, $\sum_{q=1}^j \frac{G_{q,j} \tilde{\mu}_{n,q}}{\lambda^q} = \frac{1}{\lambda} P(\frac{1}{\lambda})$ where P is a polynomial with positive coefficients. This sum is thus decreasing in λ . Hence, we can pick $\lambda \geq 1$ in such a way that

$$\sum_{q=1}^j \frac{G_{q,j} \tilde{\mu}_{n,q}}{\lambda^q} \leq \underline{R}, \quad \forall j \in \llbracket 1, p \rrbracket.$$

It follows that, for each $j \in \llbracket 1, p \rrbracket$,

$$\sup_{t \geq 0} \left\{ \left| u^{(j)}(t) \right| \right\} \leq \underline{R} \leq R_j.$$

Recalling that the feedback Υ is bounded by R_0 , we conclude that it is p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ for System (18).

With the linear change $y = Hx$ proposed in (21), the closed-loop system (21) can be put into the form of the original system (2) in closed loop with $u = \Upsilon(Hx)$. Thus, the sought feedback law v of Theorem 1 is obtained by $v(x) = \Upsilon(Hx)$. This

leads to the following choices of parameters:

$$\begin{aligned}
a_n &= R_0 / \sigma_n^{\max}, \\
a_i &= \frac{L_{\sigma_{i+1}} \mu_i^{\max}}{L_{\mu_{i+1}} \sigma_i^{\max}}, \quad \forall i \in \llbracket 1, n-1 \rrbracket, \\
k_n^T x &= \frac{L_{\sigma_n}}{L_{\mu_n}} x_n, \\
k_{n-i}^T x &= \frac{L_{\sigma_{n-i}}}{L_{\mu_{n-i}}} \sum_{k=0}^i \frac{i!}{k!(i-k)!} \left(\frac{\alpha_{\bar{\mu}}}{L_{\mu_n}} \right)^k x_{n-k}, \quad \forall i \in \llbracket 1, n-1 \rrbracket,
\end{aligned}$$

and $L_{\mu_n} = \lambda$.

3.2 Skew-symmetric systems with scalar input

The proof of Theorem 2 is divided into two steps:

Step 1: We will prove that for any $\beta > 0$ and any $\alpha \geq 1/2$, the origin of the closed-loop system

$$\dot{x} = Ax - \beta b \frac{b^T x}{(1 + \|x\|^2)^\alpha} \quad (41)$$

is globally asymptotically stable;

Step 2: We will show that for any $\alpha \geq 1/2$ there exists a positive constant β such the feedback law (5) is a p -bounded feedback law by $(R_j)_{0 \leq j \leq p}$ for system (4).

3.2.1 Step 1

Let β be a positive constant and $\alpha \geq 1/2$. We define $A_\beta := A - \beta b b^T$. The matrix A_β is then Hurwitz. To see this, observe that the Lyapunov equation $A_\beta^T P + A_\beta P = -2\beta b b^T$ holds and that the pair (A_β, b) is controllable. Thus there exists a real symmetric positive definite matrix $P \in \mathbb{R}^{n,n}$ such that

$$PA_\beta + A_\beta^T P = -I_n. \quad (42)$$

Furthermore, the closed-loop system (??) can be rewritten as

$$\dot{x} = A_\beta x + \beta b b^T x \left(1 - \frac{1}{(1 + \|x\|^2)^\alpha} \right).$$

Consider the candidate Lyapunov function given by

$$V(x) := x^T P x + K \left((1 + \|x\|^2)^{\alpha+1} - 1 \right),$$

with $K := \frac{4(\|b^T P\| \beta)^2}{\alpha+1}$. The derivative of V along trajectories of (??) is then given by

$$\begin{aligned} \dot{V}(x) = & x^T (P A_\beta + A_\beta^T P) x + 2\beta \left(1 - \frac{1}{(1 + \|x\|^2)^\alpha} \right) x^T P b b^T x \\ & - \frac{K(\alpha+1)}{2} (b^T x)^2. \end{aligned}$$

Using that $|x^T P b| \leq \|x\| \|P b\|$ and

$$\left(1 - \frac{1}{(1 + \|x\|^2)^\alpha} \right) |x^T P b b^T x| \leq \frac{\|x\|^2}{2} + \beta \|P b\|^2 (b^T x)^2,$$

we get from (??) that

$$\dot{V}(x) \leq -\frac{\|x\|^2}{2}.$$

Therefore the origin of the closed-loop system (??) is globally asymptotically stable.

3.2.2 Step 2

We first give and prove two lemmas which give explicit formula of the successive derivatives of the trajectories and of the control signal.

Lemma 3. *Given any $\beta > 0$ and $\alpha \geq 1/2$, each trajectory $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ of the closed-loop system (??) is C^∞ and satisfies, for any $k \in \mathbb{N}_{\geq 1}$,*

$$x^{(k)}(t) = A^k x(t) + \sum_{j=0}^{k-1} A^j b U^{(k-j+1)}(t), \quad \forall t \geq 0, \quad (43)$$

where $U(\cdot) := v(x(\cdot))$.

Proof of Assertion ??. The right hand side of (??) is C^∞ and globally Lipschitz. Therefore system (??) is forward complete and its trajectories are of class $C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. In particular, the successive derivatives of $U(\cdot)$ and $x(\cdot)$ are well defined. Equation (??) then follows by a trivial induction argument using differentiation of (??) at any order. \square

Lemma 4. Given $\beta > 0$ and $\alpha \geq 1/2$, let $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ be any trajectory of the closed-loop system (??) and let $G(\cdot) := 1 + \|x(\cdot)\|^2$. Then, for any $k \in \mathbb{N}_{\geq 1}$ and all $t \in \mathbb{R}_{\geq 0}$, it holds that

$$G^{(k)}(t) = \sum_{m=0}^k \binom{k}{m} \sum_{i=1}^n x_i^{(m)}(t) x_i^{(k-m)}(t), \quad (44)$$

and

$$U^{(k)}(t) = -\beta \left(\frac{b^T x^{(k)}(t)}{(1 + \|x(t)\|^2)^\alpha} + \sum_{l=1}^k \binom{k}{l} \sum_{a=1}^l d_a \frac{B_{l,a}(G^{(1)}(t), \dots, G^{(l-a+1)}(t)) b^T x^{(k-l)}(t)}{(1 + \|x(t)\|^2)^{\alpha+a}} \right), \quad (45)$$

where $d_a := (-1)^a \prod_{i=0}^a (\alpha + i)$ and $B_{l,a}$ are the Bell polynomials introduced in (7).

Proof of Assertion ??. Expression (??) is readily obtained from the general Leibniz rule. In order to establish (??), let $f : \mathbb{R}_{> 0} \rightarrow \mathbb{R}_{> 0}$ be defined as $f(z) := \frac{1}{z^\alpha}$. The feedback law can then be rewritten as

$$U(t) = -\beta b^T x(t) f(G(t)).$$

Using the general Leibniz rule we get that, for any $k \in \mathbb{N}_{\geq 1}$ and any $t \in \mathbb{R}_{\geq 0}$,

$$U^{(k)}(t) = -\beta b^T x^{(k)}(t) f(G(t)) + \sum_{l=1}^k \binom{k}{l} \frac{d^l}{dt^l} [f(G(t))] b^T x^{(k-l)}(t). \quad (46)$$

Thanks to Faà Di Bruno's formula (Lemma 1), we obtain that, for each $l \in \llbracket 1, k \rrbracket$ and each $t \in \mathbb{R}_{\geq 0}$,

$$\frac{d^l}{dt^l} f(G(t)) = \sum_{a=1}^l f^{(a)}(G(t)) B_{l,a}(G^{(1)}(t), \dots, G^{(l-a+1)}(t)). \quad (47)$$

Since $f^{(a)}(z) = (-1)^a \prod_{i=0}^a (\alpha + i) z^{-(a+\alpha)}$ for all $a \in \llbracket 1, k \rrbracket$, Assertion ?? is proven. \square

We now proceed with Step 2. Let $p \in \mathbb{N}$, $\alpha \geq 1/2$ and $(R_j)_{0 \leq j \leq p}$ be a family of positive constants. Let $\underline{R} := \min \{R_0, \dots, R_p\}$. We now prove by induction on $j \in \llbracket 0, p \rrbracket$ that there exist $\beta_j > 0$ such that, for any $\beta \leq \beta_j$, each trajectory of the closed-loop system (??) satisfies

$$\sup_{t \geq 0} \left\{ \left| U^{(j_1)}(t) \right| \right\} \leq \underline{R}, \quad \forall j_1 \in \llbracket 1, j \rrbracket.$$

Note that this in turn ensures that the feedback law (5) is a j -bounded feedback law by $(R_{j_1})_{0 \leq j_1 \leq j}$.

We start by $j = 0$. Since $\sup_{x \in \mathbb{R}^n} \{|v(x)|\} \leq \beta$, the base case follows by choosing $\beta_0 = \underline{R}$.

Now assume that, for a given $j \in \llbracket 0, p-1 \rrbracket$, there exists $\beta_j > 0$ such that for any $\beta \leq \beta_j$ the feedback law (5) is a j -bounded feedback law by $(R_{j_1})_{0 \leq j_1 \leq j}$. Using Lemma ??, we get that for each $\beta \leq \beta_j$ there exists for each $i \in \llbracket 1, n \rrbracket$ and $k \in \llbracket 1, j+1 \rrbracket$ a multivariate polynomial $P_{i,k} : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree 1 (which not depend on β) such that

$$\left| x_i^{(k)}(t) \right| \leq P_{i,k}(|x_1(t)|, \dots, |x_2(t)|), \quad \forall t \geq 0. \quad (48)$$

From (??) and (??) it follows that, for each $k \in \llbracket 1, j+1 \rrbracket$, there exists a multivariate polynomial $\underline{P}_k : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree 2, which not depend on β , such that

$$\left| G^{(k)}(t) \right| \leq \underline{P}_k(|x_1(t)|, \dots, |x_2(t)|), \quad \forall t \geq 0. \quad (49)$$

In view of Remark 4, (??), and (??), we conclude that, for each $l \in \llbracket 1, j+1 \rrbracket$ and each $a \in \llbracket 1, l \rrbracket$, there exists a multivariate polynomial $\bar{P}_{l,a} : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree $2a+1$, which does not depend on β , such that

$$\left| B_{l,a}(G^{(1)}(t), \dots, G^{(l-a+1)}(t)) b^T x^{(k-l)}(t) \right| \leq \bar{P}_{l,a}(|x_1(t)|, \dots, |x_2(t)|), \quad \forall t \geq 0. \quad (50)$$

Since $\bar{P}_{l,a}$ and $P_{i,j+1}$ are respectively of degree $2a+1$ and 1 and recalling that $\alpha \geq 1/2$, we conclude that, for each $l \in \llbracket 1, j+1 \rrbracket$ and each $a \in \llbracket 1, l \rrbracket$, there exists a positive constant $M_{l,a,j}$ such that

$$\sup_{x \in \mathbb{R}^n} \left\{ \left| \frac{\bar{P}_{l,a}(x)}{(1 + \|x\|^2)^{\alpha+a}} \right| \right\} \leq M_{l,a,j}, \quad (51)$$

and, for each $i \in \llbracket 1, n \rrbracket$, there exists a positive constant $Q_{i,j}$ such that

$$\sup_{x \in \mathbb{R}^n} \left\{ \left| \frac{P_{i,j+1}(x)}{(1 + \|x\|^2)^\alpha} \right| \right\} \leq Q_{i,j}. \quad (52)$$

Let β_j and $x(\cdot)$ be a trajectory of the closed-loop system (??). Thanks to Assertion ?? (with $k = j+1$), we get that, for all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \left| U^{(j+1)}(t) \right| &\leq \beta \left(\frac{\left| b^T x^{(j+1)}(t) \right|}{(1 + \|x(t)\|^2)^\alpha} \right. \\ &\quad \left. + \sum_{l=1}^{j+1} \binom{j+1}{l} \sum_{a=1}^l d_a \frac{\left| B_{l,a}(G^{(1)}(t), \dots, G^{(l-a+1)}(t)) b^T x^{(j+1-l)}(t) \right|}{(1 + \|x(t)\|^2)^{\alpha+a}} \right). \end{aligned}$$

Therefore a straightforward computation using (??), (??), (??), (??), and (??) leads to the existence of a positive constant M_j such that

$$\left|U^{(j+1)}(t)\right| \leq \beta M_j, \quad \forall t \geq 0. \quad (53)$$

Thus for all $\beta \leq \min_{0 \leq j \leq p} \{\beta_j, \underline{R}/M_j\}$, it follows that $\sup_{t \geq 0} \left\{ \left|U^{(j+1)}(t)\right| \right\} \leq \underline{R}$. This ends the induction on j and concludes the proof of Theorem 2.

4 Numerical examples

4.1 The triple integrator

In this subsection, we illustrate the applicability and the performance of the feedback law proposed for Case 1 on a particular example. We use the procedure described in Section 3.1.3 in order to compute a 2-bounded feedback law by (2, 20, 18) for the multiple integrator of length three. Our set of saturation functions is $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, where σ is an $\mathcal{S}(2)$ saturation function with constants (2, 1, 2, 1) given by

$$\sigma(r) := \begin{cases} r & \text{if } |r| \leq 1, \\ h_1(r) & \text{if } 1 \leq |r| \leq 1.5, \\ h_2(r) & \text{if } 1.5 \leq |r| \leq 2, \\ 2\text{sign}(r) & \text{otherwise,} \end{cases}$$

where h_1 and h_2 were picked in order to ensure sufficient smoothness for σ :

$$\begin{aligned} h_1(r) &:= \text{sign}(r)(-4 + 15|r| - 18r^2 + 10|r|^3 - 2r^4), \\ h_2(r) &:= 2\text{sign}(r)(25 - 60|r| + 54r^2 - 21|r|^3 + 3r^4). \end{aligned}$$

In accordance with (34) and (35), we choose $\mu_2^{\max} = 2/5$, $L_{\mu_2} = 1/5$, $\mu_1^{\max} = 1/12$, and $L_{\mu_1} = 1/24$. Following the procedure, we obtain that the two first time deriva-

tives of the control signal $U(\cdot) = \Upsilon(y(\cdot)) = \Upsilon(Hx(\cdot))$, with $H = \begin{pmatrix} 1/\lambda^2 & 2/\lambda & 1 \\ 0 & 1/\lambda & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

satisfies

$$\begin{aligned} \sup_{t \geq 0} \left\{ \left|u^{(1)}(t)\right| \right\} &\leq (7.91 + 4.35\lambda)/\lambda^2, \\ \sup_{t \geq 0} \left\{ \left|u^{(2)}(t)\right| \right\} &\leq \frac{26.2\lambda^3 + 396\lambda^2 + 1147.2\lambda + 125.2}{\lambda^4}. \end{aligned}$$

Choosing $\lambda = 6.5$, we obtain that $\sup_{t \geq 0} \left\{ \left| U^{(1)}(t) \right| \right\} \leq 0.9$, and $\sup_{t \geq 0} \left\{ \left| U^{(2)}(t) \right| \right\} \leq 18$. Observing that the amplitude of U is below 2 by construction, this confirms the fact that this is a 2-bounded feedback by $(2, 20, 18)$. The desired feedback is then given by

$$v(x) = -\sigma\left(\frac{1}{6.5}\left(x_3 + \frac{1}{5}\sigma\left(5(x_2/6.5 + x_3 + \frac{1}{24}\sigma\left(24(x_3 + 2x_2/6.5 + x_1/6.5^2)\right)\right)\right)\right)\right).$$

This feedback law is tested in simulations and the results are presented in Figure ???. Trajectories of triple integrator with the above feedback are plotted in grey for several initial conditions. The corresponding values of the control law and its time derivatives up to order 2 are shown in Figure ??. These grey curves validate the fact that asymptotic stability is reached and that the control feedback magnitude, and two first derivatives, never overpass the prescribed values $(2, 20, 18)$. In order to illustrate the behavior of one particular trajectory, the specific simulation obtained for initial condition $x_{10} = 446.7937$, $x_{20} = -69.875$ and $x_{30} = 11.05$ is highlighted in bold black.

It can be seen from Figure ?? that our procedure shows some conservativeness as the amplitude of the second derivative of the feedback never exceeds the value 2, although maximum value of 18 was tolerated.

4.2 The harmonic oscillator

We finally test the performance of the control law proposed for Case 2 through example of a 1-bounded feedback law by $(1, 1)$ for an harmonic oscillator. We consider the following system

$$\begin{cases} \dot{x}_1 &= 10x_2, \\ \dot{x}_2 &= -10x_1 + u. \end{cases}$$

In accordance with Theorem 2, we take $u(x_1, x_2) = -\beta x_2 / \sqrt{1 + x_1^2 + x_2^2}$ with $\beta = (-10 + \sqrt{108})/4$. The behavior of the resulting closed-loop system is presented in Figure ??? for initial conditions $x_{10} = 1$ and $x_{20} = -1$. The corresponding values of the feedback and its first time derivative are shown in Figure ???, where it can be seen that these values stay below 1.

5 Conclusion

In this paper, we have shown that any LTI system admitting either only zero eigenvalues or simple eigenvalues with zero real part can be globally asymptotically sta-

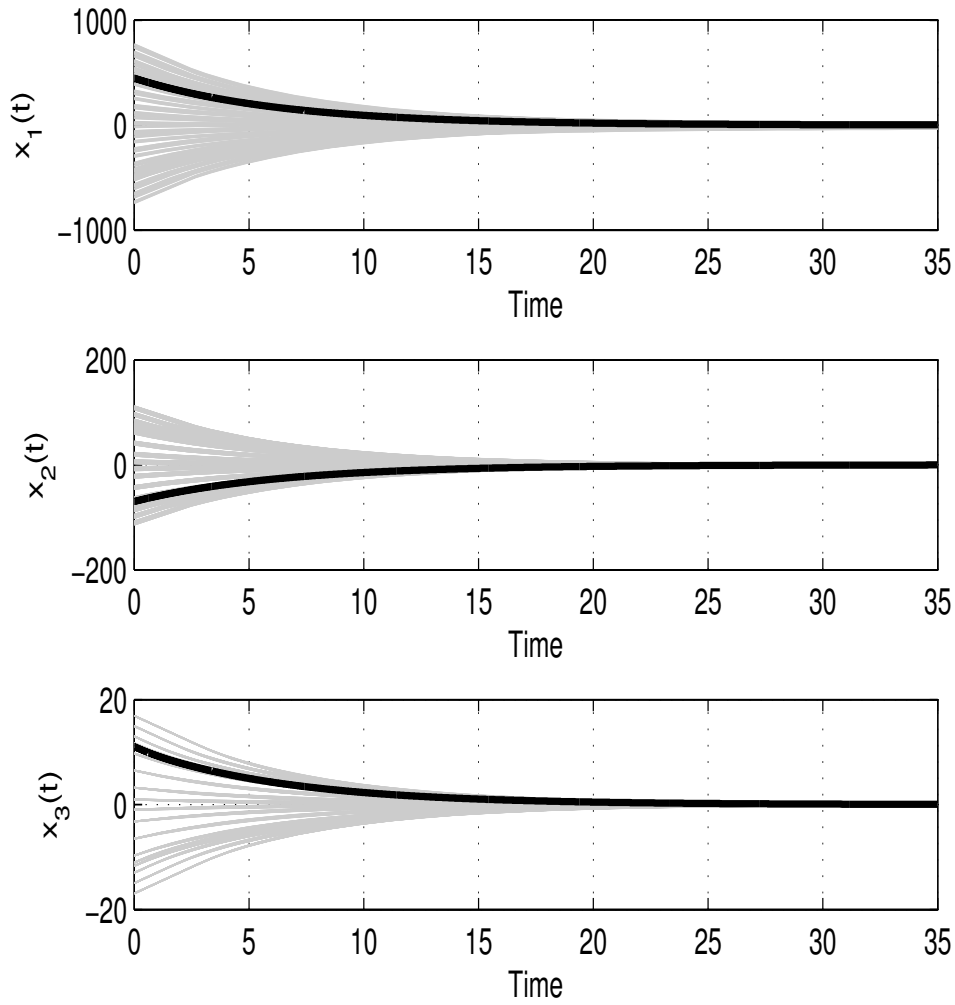


Figure 2: Evolution of the states for a set of initial conditions.

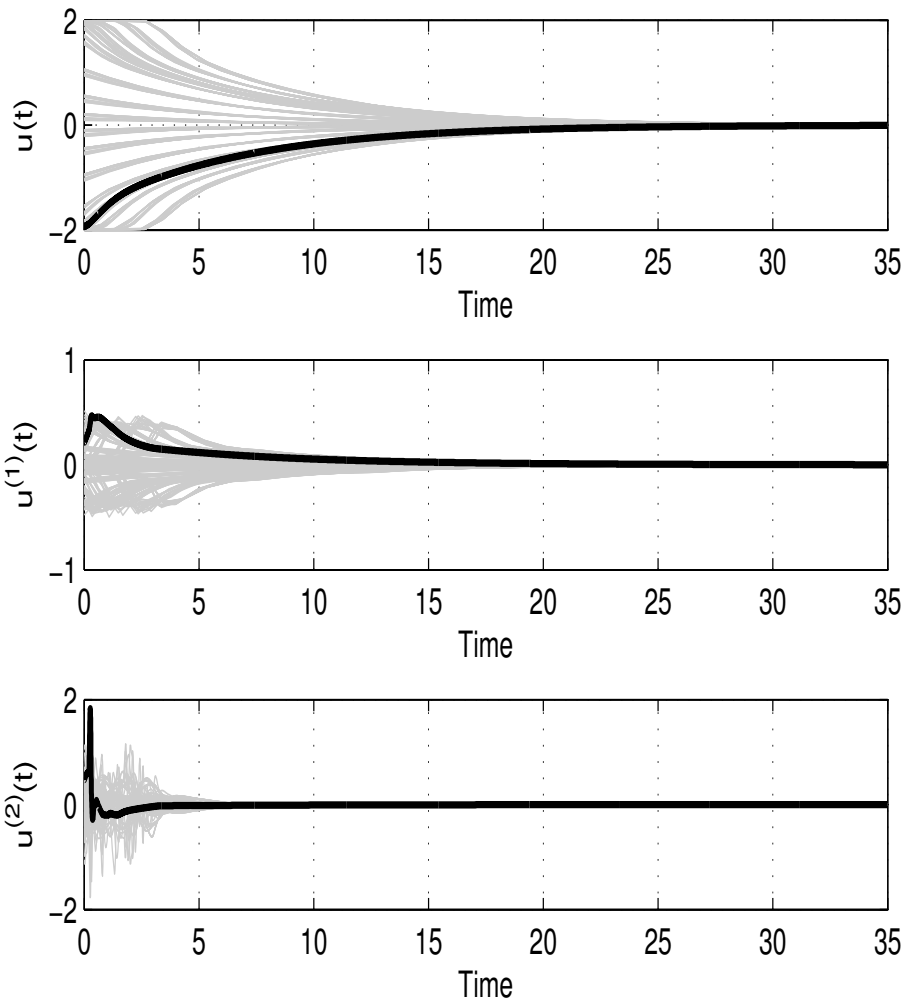


Figure 3: Evolution of the control and its derivative up to order 2 for the same set of initial conditions.

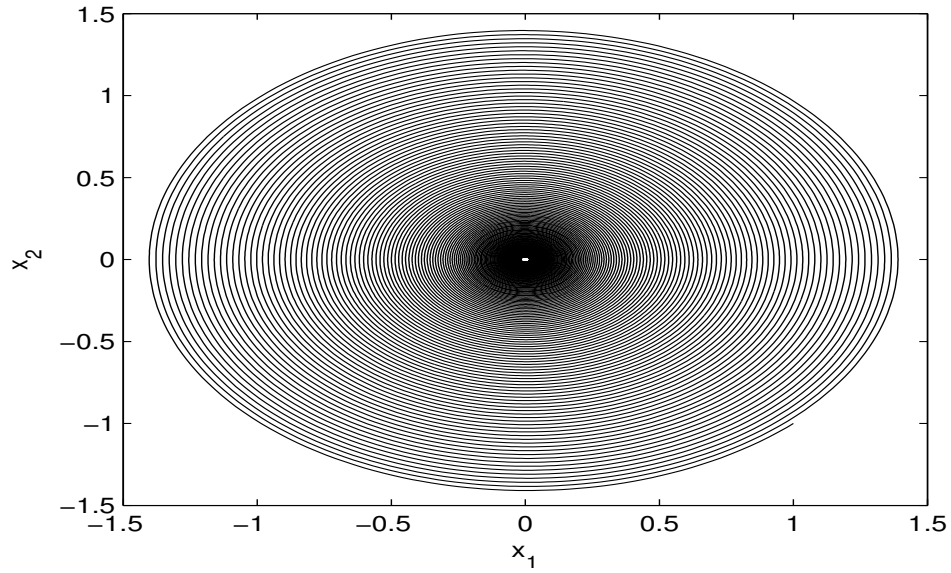


Figure 4: Trajectory of the harmonic oscillator with a feedback law of the form (5) for initial conditions $x_{10} = 1$ and $x_{20} = -1$.



Figure 5: Evolution of the control and its first derivative for the same initial conditions.

bilized by a static feedback whose amplitude and p first time derivatives along any trajectory of the closed-loop system can be bounded arbitrary prescribed values. We have shown that the former case boils down to studying chains of integrators, while the latter one sums up to LTI systems with skew-symmetric internal dynamics and single input. For integrator chains, the design of this feedback relies on the technique of nested saturations first introduced in [?]. We also showed that this strategy does not work in general for the case of skew-symmetric systems. For this second case, we have proposed an alternative control law. The applicability of the design procedure and the performance of the resulting closed-loop system was tested on two particular examples: the triple integrator and the harmonic oscillator.

The first natural issue to address next is designing p -bounded feedback laws without restrictions on the spectrum of A , namely to just assume that all the eigenvalues of A have non positive real parts. The second problem that one may consider is to relax the requirement of the order p of considered successive derivatives, that is, to design (if possible) a smooth bounded feedback law such that the feedback itself and *all* its successive derivatives along any trajectory of the closed-loop system are bounded arbitrary prescribed values.

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