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# On the Coalescence of Spectral Values and its Effect on the Stability of Time-delay Systems: Application to Active Vibration Control

Islam Boussaada<sup>a,b,\*</sup>, Silviu-Iulian Niculescu<sup>a</sup>, Sami Tliba<sup>a</sup>, Tomáš Vyhřídál<sup>c</sup>

<sup>a</sup>L2S, CNRS, CentraleSupélec, Univ Paris-Sud, Université Paris-Saclay, 91190, Gif-sur-Yvette, France

<sup>b</sup>Institut Polytechnique des Sciences Avancées (IPSA), 94200 Ivry-sur-Seine, France

<sup>c</sup>Department of Instrumentation and Control Engineering, Czech Technical University in Prague, Czech Republic

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## Abstract

For linear delay-differential equations, a question of ongoing interest is to determine conditions on the equation parameters that guarantee exponential stability of solutions. Recent results have shown an unexpected link between the stable manifold and the variety characterizing multiple characteristic spectral values allowing to the right-most root assignment. In this paper, such an idea is presented and exploited in the control of active vibrations.

*Keywords:* Retarded functional differential equations, asymptotic stability, active vibration control, rightmost root, root assignment

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## 1. Introduction

To the best of the authors' knowledge, the theoretical use of time-delay in controllers design were first introduced in<sup>1</sup> where it is shown that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus can replace the proportional-derivative controller. The interest of considering such delayed control laws lies in the simplicity of the controller as well as in its easy practical implementation. In particular, it is proven in<sup>2</sup> that a chain of  $n$  integrators can be stabilized using  $n$  distinct delay blocks, where a delay block is described by two parameters: "gain" and "delay", see also<sup>3</sup>. In the context of mechanical engineering problems, the effect of time-delay was emphasized in<sup>4</sup> where concrete applications are studied, such as, for instance, the machine tool vibrations and robotic systems<sup>5</sup>, see also<sup>6</sup> where the stabilization of an inverted pendulum is considered.

In recent works, a new interesting property of time-delay systems was emphasized. As a matter of fact, the multiple spectral values for time-delay systems was characterized by using a Birkhoff/Vandermonde-based approach, (see for instance<sup>7,8,9,10</sup>). More precisely, in<sup>8</sup>, it is shown that the admissible multiplicity of the zero spectral value is bounded by the generic *Pólya and Szegő bound* denoted  $PS_B$ , which is nothing else than the *degree* of the corresponding quasipolynomial<sup>11</sup>. In<sup>7</sup>, it is shown that a given *Crossing Imaginary Root* (CIR) with non vanishing frequency never reaches  $PS_B$  and a sharper bound for its admissible multiplicities is established. Furthermore, in<sup>10</sup>, it is shown that the variety corresponding to a multiple root defines a stable variety for the steady state. Furthermore, it is emphasized that such a multiple spectral value corresponds to the spectral abscissa, see also<sup>12,13</sup>.

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\*Corresponding author. Tel.: +33-0169851760 ; fax: +33-0169851765.  
E-mail address: islam.boussaada@l2s.centralesupelec.fr

## 2. System description

The problem of active vibration damping of thin mechanical structures is a topic that has received a great attention by the control community since several years<sup>14</sup>, especially when actuators and sensors are based on piezoelectric materials. For mechanical structures that are deformable, piezoelectric materials are used as strain sensors or strain actuators. With an appropriate controller, they allow to achieve shape control<sup>15,16</sup> or the active damping of multi-modal vibrations thanks to their very large bandwidth. Moreover, their behavior is quite linear when they work in a specific range of use. This explains in part the great interest of using piezoelectric materials for the instrumentation of thin mechanical structures. In this area, the major challenge is the design of controllers able to damp the most vibrating modes in a specified low-frequency bandwidth while ensuring robustness against high-frequency modes, outside the bandwidth of interest, often unmodelled or weakly modelled. The inherent feature of this kind of systems is that they arise robustness issues when they are tackled with finite dimensional control tools. Many works have concerned the vibration control problem of the “*Euler-Bernoulli beam*” equipped with one rectangular piezoelectric actuator and sometime, another one, identical and collocated, but used as sensor. See for example<sup>17, 18</sup> where one edge of the beam is clamped whereas the other remains free. Other works dealt with the problem of vibration control for laminated rectangular plates<sup>19</sup> or complex plate like structures<sup>20</sup>.

In this work, we consider the flexible structure depicted in Fig. 1. It is an aluminium-based beam, embedded in a mobile support. The mobile support is subjected to an acceleration, denoted by  $w$  in the sequel, and it is moving along the  $z$  axis. This flexible beam is equipped with two piezoelectric patches made with *lead zirconate titanate* (also called *PZT*). One of them is used as an actuator and the other works as a sensor. These patches are supposed to be rigidly bounded on the beam, one on each side, located at the clamped edge. The whole device is called thereafter as a *piezo-actuated beam*. It can be deformed by the application of a voltage, denoted by  $u$ , across the actuator. The sensor delivers an electrical voltage which corresponds to a measure, denoted by  $y$ , of the local deformation under the piezoelectric patch. Very often, this equipped mechanical structure is partly described by the in-plane

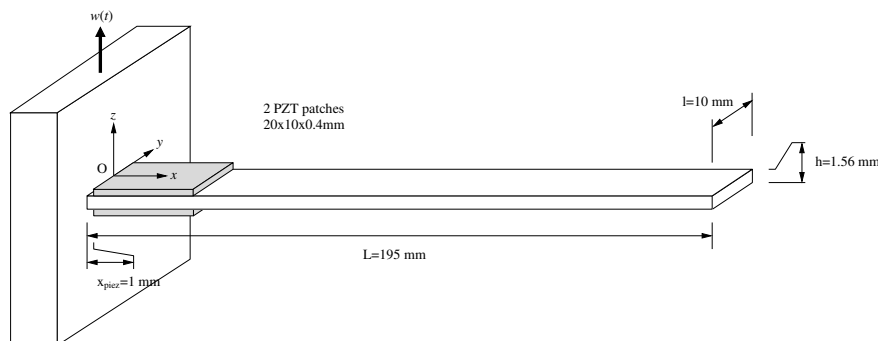


Figure 1: Sketch of the piezo-actuated flexible beam, clamped at one edge.

Euler-Bernoulli Partial Differential Equation (PDE) that suffers from the lack of precision in describing the electro-mechanical interactions between the passive structure and the piezoelectric components. Indeed, these latter are often withdrawn in the computation of the eigenfrequencies<sup>21</sup> of the whole structure. Nevertheless, such a structure obeys to fundamental equations of continuum mechanics in 3D space<sup>22</sup>, involving computations of gradient of displacement vector and divergence operator applied to strength tensor. When completed with Neumann and Dirichlet boundary conditions, the fundamental equations give several PDEs that are coupled, thus that are hardly or impossible to solve analytically. Then, for controller design purposes, one naturally turns toward numerical methods in order to get the inputs-to-outputs dynamical models<sup>23</sup>.

Finite Element Modelling (FEM) is a numerical method that approximates the displacement field with a displacement vector of finite dimension, denoted  $q \in \mathbb{R}^{n_{dof}}$ , containing as many components as needed to describe the displacement field in the 3D space, for both mechanical and electrical variables<sup>24</sup>. This method leads to the following

matricial Ordinary Differential Equations (ODE), time invariant but coupled and linear<sup>23</sup>:

$$\mathbb{M}_{qq}\ddot{q}(t) + \mathbb{D}_{qq}\dot{q}(t) + \mathbb{K}_{qq}q(t) = \mathbb{M}_{qw}w(t) - \mathbb{K}_{qu}u(t) \quad (1)$$

$$y(t) = \mathbb{K}_{qy}q(t) \quad (2)$$

$$z(t) = \mathbb{F}_{zw}w(t) - \mathbb{F}_{zu}u(t) - \mathbb{F}_{zq}q(t) - \mathbb{F}_{zv}\dot{q}(t) \quad (3)$$

where  $w(t) \in \mathbb{R}$  is the absolute acceleration ( $m/s^2$ ) of the movable support along axis  $z$ ,  $z(t) \in \mathbb{R}$  is the relative acceleration ( $m/s^2$ ) of the free end, derived from the equations of motion,  $u(t) \in \mathbb{R}$  is the piezoelectric voltage (V) across the actuator (control signal),  $y(t) \in \mathbb{R}$  is the piezoelectric sensor voltage (V) across the sensor (measured output signal  $y(t)$ ). Moreover, the terms  $\mathbb{M}_{qq}$ ,  $\mathbb{D}_{qq}$ ,  $\mathbb{M}_{qw}$ ,  $\mathbb{K}_{qq}$ ,  $\mathbb{K}_{qu}$ ,  $\mathbb{K}_{qy}$ ,  $\mathbb{F}_{zw}$ ,  $\mathbb{F}_{zu}$ ,  $\mathbb{F}_{zq}$  and  $\mathbb{F}_{zv}$  are all matrices derived from the assembly step of the FEM such that  $n_{dof} \# 1000$ , *i.e.* with several thousands degrees of freedom. The dynamical model given by equations (1)—(3) is not numerically tractable because of the very large number of degrees of freedom  $n_{dof} \# 10^3$  for our flexible system. After producing a FEM, a modal analysis is performed to the undamped motion equation (1). It consists in finding the eigenstructure of

$$\mathbb{M}_{qq}\ddot{q}(t) + \mathbb{K}_{qq}q(t) = 0,$$

and using the eigenvalues and eigenvectors as a new basis allowing the description of (1)—(3) into a new system of ODEs, still linear but decoupled, involving a new state vector, called vector of modes. The advantage of this form is that it allows to build a model in state-space form devoted to analysis (4), having its order sufficiently small to describe the dynamical behavior within a low-frequency bandwidth. Furthermore, a reduced order system, devoted to the controller synthesis step, is also available and differs from the analysis one by the presence of direct feedthrough terms between the outputs  $\{z, y\}$  and the inputs  $\{w, u\}$ . The analysis model is of order 12, containing two uncontrollable and unobservable modes, and describing the inputs-outputs behavior in the bandwidth  $[0 - 3500Hz]$ . The reduced order system is of order 2 and it only takes into account the first bending mode (see Fig. 3). This mode is controllable and observable. Let  $x_p \in \mathbb{R}^{n_p}$  be the state vector of the system, whatever its order. For the analysis model,  $n_p = 12$  and  $D_{p,zw}$ ,  $D_{p,zu}$ ,  $D_{p,yw}$  and  $D_{p,yu}$  are all equal to zero. For the reduced order model,  $n_p = 2$  and the direct feedthrough terms  $D_{p,zw}$ ,  $D_{p,zu}$ ,  $D_{p,yw}$  and  $D_{p,yu}$  are not equal to zero because of the model reduction technique that is used to keep the static responses of the analysis model. The frequency responses for the analysis and the reduced order models are shown in Fig. 2 and the shapes of the first three bending modes in Fig. 3.

$$\mathcal{P} \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_{p,w}w(t) + B_{p,u}u(t) \\ z(t) = C_{p,z}x_p(t) + D_{p,zw}w(t) + D_{p,zu}u(t) \\ y(t) = C_{p,y}x_p(t) + D_{p,yw}w(t) + D_{p,yu}u(t) \end{cases} \quad (4)$$

It is worth mentioning that the piezo-actuated beam is a SISO system, *i.e.* with only one actuator, driven by the controlled electrical voltage  $u$ , and only one measured output signal corresponding to the electrical voltage  $y$ . The perturbation input  $w$  is the total acceleration applied to the clamped edge of the structure. The vertical total acceleration of the free edge is our controlled output  $z$ . In the sequel, we shall use the transfer function based model, derived from (4) by applying to it Laplace transform, given by

$$\mathcal{P} \begin{cases} z(s) = \frac{N_{wz}(s)}{\psi(s)}w(s) + \frac{N_{uz}(s)}{\psi(s)}u(s) \\ y(s) = \frac{N_{wy}(s)}{\psi(s)}w(s) + \frac{N_{uy}(s)}{\psi(s)}u(s) \end{cases} \quad (5)$$

where  $s$  denotes the Laplace variable. For  $n_p = 2$ , we have the following numerical data for the reduced order model:

$$\begin{aligned} N_{wz}(s) &= -1.57233229405836 s^2 - 0.767039493121702 s + 0.114505932957013, \\ N_{uz}(s) &= 0.0407589609440159 s^2 + 0.019883667632349 s - 0.00264721568397969, \\ N_{wy}(s) &= -0.047734919434071 s^2 - 0.023286787751722 s - 24664.7202708044, \\ N_{uy}(s) &= 0.0824705565013658 s^2 + 0.0402320642368774 s + 5472.41008648971, \\ \psi(s) &= 1 s^2 + 0.487835488732404 s + 59495.8660165543. \end{aligned}$$

An interesting control objective would be to damp the peak of resonance of the first bending mode, by using an *output feedback controller*, without affecting the vibrating modes that were neglected in the reduced order model.

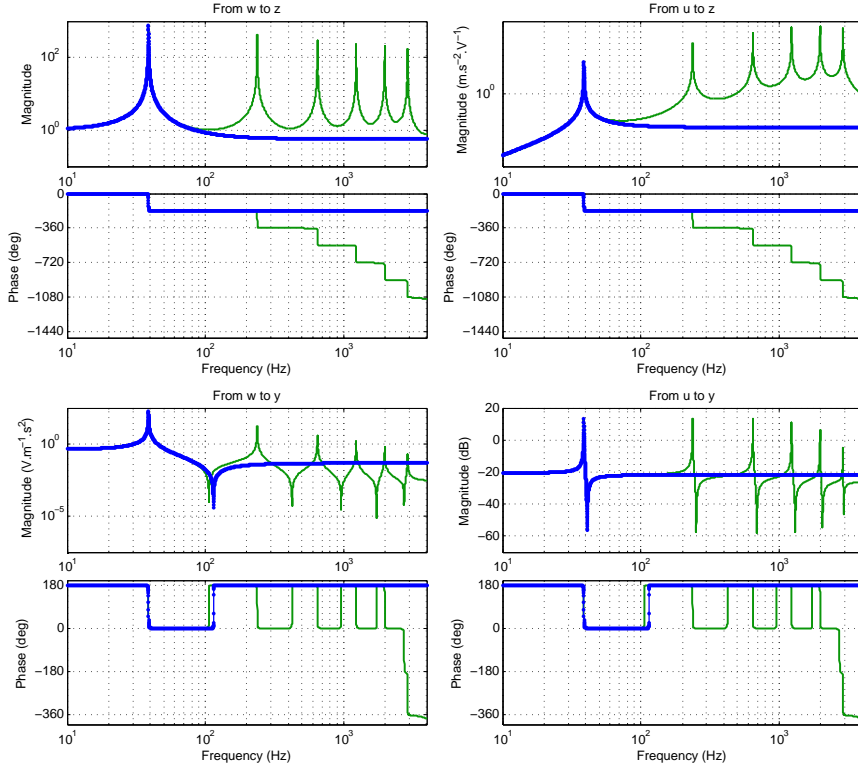


Figure 2: Frequency responses of the analysis (-) and reduced order (-) models.

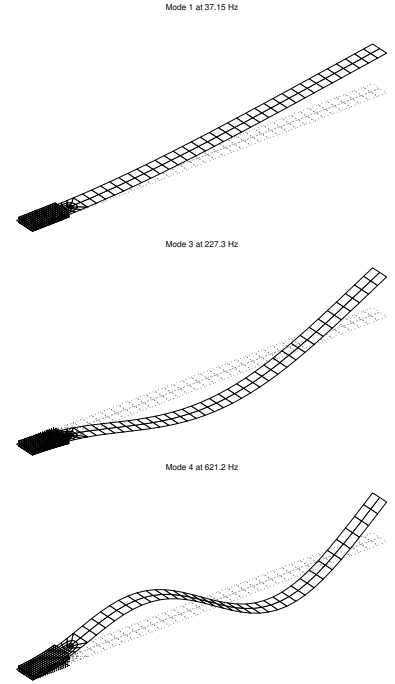


Figure 3: First three controllable and observable modes.

### 3. Quasipolynomial function's roots coalescence

Unlike ordinary differential equation, a scalar retarded differential equation admits an infinite number of spectral values. Furthermore, such spectral values may be complex and may be multiple as suggested by the following example. For the sake of simplicity, in the next section, we consider the linear scalar equation with two constant delays.

#### 3.1. The comprehensive scalar example

Consider the scalar linear time invariant equation with two delays:

$$\dot{x}(t) + b_0 x(t) + b_1 x(t - \tau_1) + b_2 x(t - \tau_2) = 0. \quad (6)$$

It is well known that the asymptotic behavior of the solutions of (6) is determined from the spectrum  $\aleph$  designating the set of the roots of the associated characteristic function (denoted in the sequel  $\Delta(s, \tau)$ ), often called *quasipolynomial*, that is a transcendental polynomial in the Laplace variable  $s$  in which appear exponential terms induced by delays, see for instance<sup>25,26</sup>.

The quasipolynomial function corresponding to (6) is given by:

$$\Delta(\lambda) = \lambda + b_0 + b_1 e^{-\lambda\tau_1} + b_2 e^{-\lambda\tau_2}. \quad (7)$$

where  $(\tau_1, \tau_2) = (\frac{\pi}{2}, \frac{\pi}{6})$ . Let us numerically explore if there exists a double spectral value of (6) such that  $\lambda = j\omega_0$  with  $\omega_0 \neq 0$ . A first computational step allows us concluding that a double crossing root may exist if and only if the parameters  $b_0$  and  $b_1$  satisfy:

$$b_0 = \frac{-6 - 2b_2\pi \cos\left(\frac{\omega\pi}{6}\right)}{3\pi} + j \frac{(2b_2 \sin\left(\frac{\omega\pi}{6}\right) - 3\omega)}{3}, \quad b_1 = \frac{6 \cos\left(\frac{\omega\pi}{2}\right) - b_2\pi \cos\left(\frac{\omega\pi}{3}\right)}{3\pi} + j \frac{(6 \sin\left(\frac{\omega\pi}{2}\right) - b_2\pi \sin\left(\frac{\omega\pi}{3}\right))}{3\pi}. \quad (8)$$

Since the system parameters are real, the imaginary parts of  $b_0$  and  $b_1$  have to vanish. This fact imposes two additional conditions: the vanishing of the imaginary part of  $b_0$  from (8), that is  $2b_2 \sin(1/6\omega) - 3\omega = 0$ , gives us the value of  $b_2$  parameterized by  $\omega$ , leading to  $b_2 = 3\omega/2 \sin(\frac{\omega\pi}{6})$ , then, the vanishing of the imaginary part of the expression of  $b_1$ , that is  $6 \sin(\frac{\omega\pi}{2}) - b_2\pi \sin(\frac{\omega\pi}{3}) = 0$ , in which one substitutes the obtained value for  $b_2$  given above allows us to an equation only in  $\omega$ . Indeed, one obtains  $F(\omega_0) = 0$ , where:

$$F(t) = 2 \cos\left(\frac{t\pi}{3}\right) - t\pi \sin\left(\frac{t\pi}{3}\right) - 2 \cos\left(\frac{2t\pi}{3}\right). \quad (9)$$

Thus, one concludes that  $\lambda = j\omega_0$  is a *double spectral value* of (6) (since  $\Delta''(j\omega_0) \neq 0$ ) if (9) admits a real solution. Numerical investigation shows that such a frequency  $\omega_0 \neq 0$  exists. As a matter of fact, since  $\omega_0$  is a real root of (9), then, under the parameter values defined by (10),  $j\omega_0$  is a double spectral value of (6).

$$b_0 = -\frac{2 \sin\left(\frac{\omega\pi}{6}\right) + \omega\pi \cos\left(\frac{\omega\pi}{6}\right)}{\sin\left(\frac{\omega\pi}{6}\right)\pi}, \quad b_1 = \frac{2 \sin\left(\frac{2\omega\pi}{3}\right) - \omega\pi \cos\left(\frac{\omega\pi}{3}\right) - 2 \sin\left(\frac{\omega\pi}{3}\right)}{2 \sin\left(\frac{\omega\pi}{6}\right)\pi}, \quad b_2 = \frac{3\omega}{2 \sin\left(\frac{\omega\pi}{6}\right)}. \quad (10)$$

Finally, one easily check such an assertion by substituting the set of conditions in the characteristic quasipolynomial function (6) as well as in its first derivative with respect to  $\lambda$  to obtain:

$$\begin{aligned} \Delta(j\omega) &= \mathcal{R}^0(\omega) + j\mathcal{I}^0(\omega) \quad \text{and} \quad \Delta'(j\omega) = \mathcal{R}^1(\omega) + j\mathcal{I}^1(\omega) \quad \text{where} \\ \mathcal{R}^0(\omega) &= 2 \sin\left(\frac{2\omega\pi}{3}\right) \cos\left(\frac{\omega\pi}{2}\right) - 4 \sin\left(\frac{\omega\pi}{6}\right) - 2 \sin\left(\frac{\omega\pi}{3}\right) \cos\left(\frac{\omega\pi}{2}\right) + \omega\pi \cos\left(\frac{\omega\pi}{6}\right) - \omega\pi \cos\left(\frac{\omega\pi}{3}\right) \cos\left(\frac{\omega\pi}{2}\right), \\ \mathcal{I}^0(\omega) &= \omega\pi \cos\left(\frac{\omega\pi}{3}\right) \sin\left(\frac{\omega\pi}{2}\right) - \omega \sin\left(\frac{\omega\pi}{6}\right)\pi + 2 \sin\left(\frac{\omega\pi}{3}\right) \sin\left(\frac{\omega\pi}{2}\right) - 2 \sin\left(\frac{2\omega\pi}{3}\right) \sin\left(\frac{\omega\pi}{2}\right), \\ \mathcal{R}^1(\omega) &= 2 \sin\left(\frac{\omega\pi}{3}\right) \cos\left(\frac{\omega\pi}{2}\right) - 2 \sin\left(\frac{2\omega\pi}{3}\right) \cos\left(\frac{\omega\pi}{2}\right) + \omega\pi \cos\left(\frac{\omega\pi}{3}\right) \cos\left(\frac{\omega\pi}{2}\right) + \sin\left(\frac{\omega\pi}{6}\right) - \omega\pi \cos\left(\frac{\omega\pi}{6}\right), \\ \mathcal{I}^1(\omega) &= \omega \sin\left(\frac{\omega\pi}{6}\right)\pi + 2 \sin\left(\frac{2\omega\pi}{3}\right) \sin\left(\frac{\omega\pi}{2}\right) - 2 \sin\left(\frac{\omega\pi}{3}\right) \sin\left(\frac{\omega\pi}{2}\right) - \omega\pi \cos\left(\frac{\omega\pi}{3}\right) \sin\left(\frac{\omega\pi}{2}\right). \end{aligned} \quad (11)$$

Recall that  $j\omega_0$  is a double crossing root for (6) if and only if  $\mathcal{R}^0(\omega_0) = \mathcal{I}^0(\omega_0) = \mathcal{R}^1(\omega_0) = \mathcal{I}^1(\omega_0) = F(\omega_0) = 0$ .

Numerical computations shows that the set of such a common roots is non empty and the problem admits at least a solution  $\omega_0$  that we are able to locate:  $\omega_0 \in ]\frac{8661\pi}{8192}, \frac{8662\pi}{8192}[$ . The following proposition, which is slightly different from the result of<sup>7</sup>, asserts the existence of spectral values with multiplicity three for the scalar equation with two delays. Its proof is in the same lines of the one from<sup>7</sup>.

**Proposition 1.** *For each pair  $(\tau_1, \tau_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$  ( $\tau_1 \neq \tau_2$ ), the equation (6) admits a triple spectral value at  $z_0 = -b_0 - \tau_2^{-1} - \tau_1^{-1}$  if, and only if,  $b_1 = -\tau_2 e^{-b_0\tau_1 - \frac{\tau_1}{\tau_2}} (\tau_1 - \tau_2)^{-1} \tau_1^{-1}$ ,  $b_2 = \tau_1 e^{-b_0\tau_2 - 1 - \frac{\tau_2}{\tau_1}} \tau_2^{-1} (\tau_1 - \tau_2)^{-1}$ . Moreover, if  $b_0 > \tau_2^{-1} - \tau_1^{-1}$ , the zero solution is asymptotically stable.*

### 3.2. Second order systems: Quasipolynomial function's rightmost characteristic root assignment

Roughly speaking, second-order linear systems capture the dynamic behavior of many natural phenomena, and have found wide applications in many fields, such as vibration and structural analysis. In this section, second-order linear systems of retarded type are explored.

$$\dot{x} = \sum_{k=0}^n A_k x(t - \tau_k) \quad (12)$$

with the state-vector  $x = (x_1, x_2) \in \mathbb{R}^2$ , under appropriate initial conditions belonging to the Banach space of continuous functions  $C([-\tau_N, 0], \mathbb{R}^2)$ . Here  $\tau_j$ ,  $j = 1 \dots n$  are strictly increasing positive constant delays,  $\tau_0 = 0$ ,  $\tau = (\tau_1, \dots, \tau_n)$  and the matrices  $A_j \in \mathcal{M}_2(\mathbb{R})$  for  $j = 0 \dots n$ .

The characteristic function corresponding to System (12) is a quasipolynomial  $\Delta : \mathbb{C} \times \mathbb{R}_+^n \rightarrow \mathbb{C}$  of the form:

$$\Delta(s, \tau) = \det\left(s I - A_0 - \sum_{k=1}^n A_k e^{-\tau_k s}\right). \quad (13)$$

More precisely, consider time-delay system where the corresponding characteristic function is written as:

$$\Delta(s, \tau) = s^2 + c_1 s + c_0 + \sum_{k=1}^n \beta_k e^{-\tau_k s}. \quad (14)$$

Using the change of variables  $s = \frac{c_1 \lambda}{2}$ , one obtains the corresponding normalized characteristic function:

$$\tilde{\Delta}(\lambda, \tilde{\tau}) = \lambda^2 + 2\lambda + a_0 + \sum_{k=1}^n \alpha_k e^{-\lambda \tilde{\tau}_k}, \quad \text{where } \alpha_k = \frac{4}{c_1^2} \beta_k, \quad \tilde{\tau}_k = \frac{c_1}{2} \tau_k, \quad a_0 = 4 \frac{c_0}{c_1^2}. \quad (15)$$

It is well known that if all  $\alpha_k = 0$  for  $k = 0, \dots, n$ , and the problem of spectral abscissa minimization is considered when  $a_0 = 1$  and the rightmost-root is  $\lambda_0 = -1$ , see for instance<sup>27</sup>. By exploiting the delay effect, increasing the solution's convergence rate follows from the decrease of the corresponding rightmost-root.

**Proposition 2.** *The quasi-polynomial function (15) admits a real spectral value at  $\lambda = \lambda_0$  with multiplicity  $n + 2$  if and only if*

$$\lambda_0 = -1 - \sum_{k=1}^n \frac{1}{\tilde{\tau}_k}, \quad a_0 = 1 + \sum_{k=1}^n \frac{1}{\tilde{\tau}_k^2}, \quad \alpha_k = -\frac{2 \prod_{l=1, l \neq k}^n \tilde{\tau}_l e^{-(1+\tilde{\tau}_k + \sum_{l=1, l \neq k}^n \tilde{\tau}_l)}}{\tilde{\tau}_k^2 \prod_{l=1, l \neq k}^n (\tilde{\tau}_k - \tilde{\tau}_l)}. \quad (16)$$

*Remark.* When (16) is satisfied and  $\lambda_0$  is the rightmost root associated with (15) then the corresponding steady state solution is asymptotically stable. Notice also that, increasing the number of delays in the loop induces the decrease of the rightmost root.

#### 4. Vibration damping

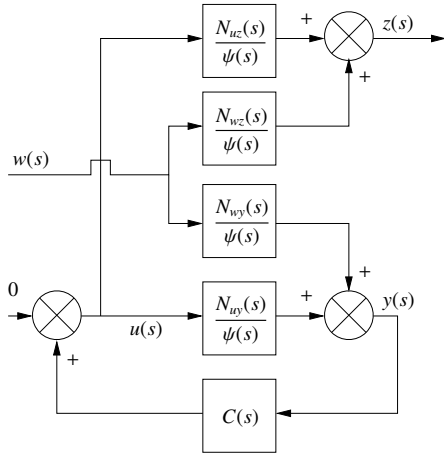


Figure 4: Feedback control structure.

Consider now the piezo-actuated system inserted in the typical output feedback control structure of Fig. 4, with a zero-reference signal and an input disturbance  $w$  corresponding to a rectangular impulse signal. The control problem consists in damping the vibrations due to the first mode when the mobile support is subjected to a shock like disturbance. We define the output feedback control law  $u(s) = C(s)y(s)$  involving the following Quasi Polynomial Based (QPB) controller given in Laplace domain by

$$C(s) := \frac{N(s, \tau)}{D(s, \tau)} \quad \text{where } N(s, \tau) := n_0 + n_{r_0} e^{-\tau s} \quad (17)$$

$$\text{and } D(s, \tau) := d_0 + d_{r_0} e^{-\tau s}.$$

By applying inverse Laplace transform, it can be easily shown that this control law is given in time domain by

$$u(t) = -\frac{d_{r_0}}{d_0} u(t - \tau) + \frac{n_0}{d_0} y(t) - \frac{n_{r_0}}{d_0} y(t - \tau) \quad (18)$$

which is an output feedback control law based on proportional actions plus delayed proportional actions. By omitting the variable  $s$  in the polynomials for sake of clarity, the closed loop relation between the disturbance  $w$  and the controlled output  $z$  is given by

$$z(s) = \frac{N_{wz} \psi D + (N_{uz} N_{wy} - N_{wz} N_{uy}) N}{\psi (\psi D - N_{uy} N)} w(s). \quad (19)$$

It can be proven that  $\psi$  divide the polynomial  $(N_{uz} N_{wy} - N_{wz} N_{uy})$  so that it can be removed from the numerator and the denominator. In that case, let  $Q(s)$  be a polynomial such that  $N_{uz}(s) N_{wy}(s) - N_{wz}(s) N_{uy}(s) := Q(s) \psi(s)$ . Then (19) becomes

$$z(s) = \frac{N_{wz} D + Q N}{\psi D - N_{uy} N} w(s). \quad (20)$$

Let consider the characteristic polynomial  $\Delta(s, \tau) := \psi(s)D(s, \tau) - N_{uy}(s)N(s, \tau)$ , which is of the form of the one tackled in subsection 3.2. By applying the results of Proposition 2 with  $\lambda_0 \simeq -244.1463568$ , we obtain the following numerical values for the parameters of the controller in (17)

$$n_0 \simeq 0.01309186753, \quad n_{r_0} \simeq 77.28704202, \quad d_0 \simeq 1.001079694, \quad d_{r_0} \simeq 6.373905366, \quad \tau \simeq 0.0041, \quad (21)$$

that assign  $\lambda_0$  as a rightmost root of multiplicity equal to 3. To show the efficiency of the proposed QPB-controller, we propose to compare, in Fig. 5, the time responses of both output signals in open-loop (blue) and in closed-loop (red) when the disturbance  $w$  is a rectangular impulse (black), say like a shock. We also put the time response of the control signal  $u$  that exhibits a peak of magnitude roughly equal to  $-90$  V which is reasonable for this application.

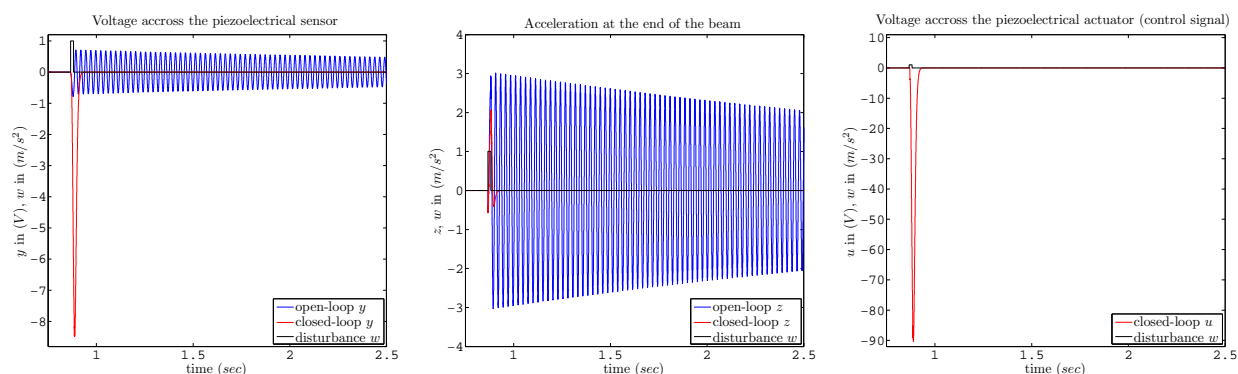


Figure 5: Time responses of the measured output  $y$  on the left, of the controlled output  $z$  on the middle and of the closed-loop control signal  $u$  on the right.

## 5. Conclusion

In this note, we illustrated the interest in using time-delay in the controller design as a control parameter. It also emphasized the applicability of the right-most root assignment in the problem of vibration's attenuation in the case of a single vibrating mode of a piezo-actuated beam. By lack of space, the whole properties of the QPB controller have not been shown, although interesting, especially the robustness of this controller with respect to the high frequency modes that have been neglected in the reduced order model. In future work, the proposed controller design will be applied to more vibrating modes and it will be compared with other optimal controllers of finite dimension.

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