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Abstract:
The problem of discrete-time multi-agent systems governed by general MIMO dynamics is addressed. By employing a PID-like distributed protocol, we aim to solve two relevant consensus problems, namely the leaderless consensus under disturbances and leader-follower under time-varying reference state ones. Sufficient conditions for stability as well as two LMI approaches to tune the controller gains are provided. The latter are either based on a $\mathcal{H}_\infty$ formulation of the problem or on fast response to a reference exogenous signal. Numerical simulations give some insight of which tuning should be considered according to the problem addressed.

1 INTRODUCTION

In recent years much research effort has been devoted to the area of multi-agent cooperative control because of its wide range of applications and potential benefits. Cooperation of a coordinated multi-agent network is sought via distributed algorithms as they present some interesting advantages over their centralized counterpart, e.g. avoiding single point of failure, reducing communication and computational burden, etc. The main problem in distributed coordination, known as *consensus* problem, is the one of achieving an agreement on some variables of interest of each agent via local interactions. These variables evolve according to a prescribed dynamics describing the physics of the problem, while interactions among agents are defined by a given communication graph. Finding a distributed protocol to solve the aforementioned problem has been extensively treated for single and double integrator dynamic agents, e.g. (Ren and Beard, 2008). However, in a more general framework, general dynamics need to be considered in order to describe the agents behavior.

The consensus problem for this latter case has been discussed for both continuous and discrete-time multi-agent systems. In addition, it can be further divided in two main classes of problems, namely leaderless and leader-follower ones. As far as the former is concerned, the most employed distributed protocol is given by a static state feedback law, also called P-like distributed control. One can cite, for instance, (Xi et al., 2010), (Li et al., 2013), (Yang-Zhou et al., 2014) for the continuous-time framework, and (Li et al., 2013), (You and Xie, 2011), (Su and Huang, 2012), (Ge et al., 2013) for the discrete one, where the consensus problem is led back to the one of simultaneously stabilizing multiple LTI systems. References (Li et al., 2013), and (Su and Huang, 2012) also solve a leader-follower problem where the leader has an autonomous time-invariant dynamics. Another interesting problem is the one of finding the optimal P-like protocol gain in order to improve consensus under system uncertainties, as in (Li et al., 2012), and disturbances as in (Oh et al., 2014), (Li et al., 2011), for continuous time systems, and (Wang and Gao, 2011) for discrete-time ones. The proposed approaches usually make use of some $\mathcal{H}_2$ or $\mathcal{H}_\infty$ constraints to be respected, and they are in general more involved than the one of simultaneously stabilizing multiple systems. For instance, (Li et al., 2011)

provide necessary and sufficient conditions, for the continuous-time case to solve the consensus problem while guaranteeing some properties on the aforementioned norms. On the other hand, for discrete-time systems only sufficient conditions are provided using results from robust control as in (Wang and Gao, 2011). Dynamic distributed controllers are also proposed for consensus achievement based on local output measurements, e.g. (Li et al., 2013). In the continuous-time framework, (Xi et al., 2012) provide a controller with limited energy, while a general full order one is presented in (Liu et al., 2009) to achieve some $\mathcal{H}_\infty$ performance. Other possible structures have been explored too. Indeed, given the common P-like controller, one can easily think of a more general PID-like structure. In continuous-time, for instance, (Carli et al., 2008) propose a PI-like distributed algorithm for single integrator dynamic agents, and (Ou et al., 2014) provide a PID-like controller for general high-order SISO systems. Similar control design is applied to solve a leader-follower consensus under time-varying reference state, as in (Ren, 2007), and in its sampled-data counterpart (Cao et al., 2009), where a PD-like protocol is given. Even though the presented literature review is nowhere near exhaustive, one can remark that poorer attention has been devoted to discrete-time dynamic protocols for general LTI MIMO systems, and this is where we wish to place our contribution.

In this paper we propose a PID-like distributed controller for the aforementioned systems, where the agents can communicate on a connected undirected graph, and we provide two possible ways of tuning the controller parameters, based on the solution of LMIs. To the best of our knowledge this distributed control structure has never been fully treated for the mentioned class of dynamic systems. The approach we propose is used to solve two different problems, namely the *leaderless consensus under the presence of disturbances*, and the *leader-follower consensus under a time-varying reference state*. Our main results are based on the work of (Wu et al., 2011), which we adapted for distributed coordination purposes. The fundamental feature of the aforesaid work is that MIMO PID parameter tuning can be performed via LMIs, avoiding in this way, the need for solving BMIs. Furthermore, in both the analyzed consensus problems the measurement matrix is kept general, allowing a more general problem formulation for the case in which the agents cannot directly measure the variables on which agreement is sought. Eventually, concerning the leaderless consensus, agreement can be focused on particular variables of interest via a proper selection of the controlled output matrix. As for classic control, the PID controller allows good performance despite being rather simple. Concerning the leaderless consensus problem, for instance, it enhances the disturbance rejection, and achieves results that a simple P-like protocol would not permit if the dynamics of the agents are general. Similar conclusions hold for the leader-follower consensus problem with a time-varying reference state, where a P-like control would undoubtedly reach lower performance.

The reminder of this paper is organized as follows. In Section 2 some preliminaries on graph theory are provided and the two main problems are stated. In Section 3 we provide sufficient conditions to solve a leaderless and a leader-follower consensus problem, and we give an LMI approach to tune the distributed PID controller gains. We carry out simulations to test the effectiveness of the proposed controller in Section 4. The paper ends with conclusions and future perspectives in Section 5.

2 PRELIMINARIES AND PROBLEM STATEMENT

2.1 Graph Theory

An undirected graph $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\dots,N\}$ is the set of nodes, and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of unordered pairs of nodes, named edges. Two nodes i, j are said to be adjacent if $(i,j)\in\mathcal{E}$. Under the assumption of *undirected* graph, the latter implies that $(j,i)\in\mathcal{E}$ too. An undirected graph is connected if there exists a path between every pair of distinct nodes, otherwise is disconnected. The adjacency matrix $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ associated with the undirected graph $\mathcal{G}$, considered in this paper, is defined by $a_{ii}=0$, i.e. self-loops are not allowed, and $a_{ij}=1$ if $(i,j)\in\mathcal{E}$. The Laplacian matrix $\mathcal{L}\in\mathbb{R}^{N\times N}$ is defined as $\mathcal{L}_{ii}=\sum_{j\neq i}a_{ij}$ and $\mathcal{L}_{ij}=-a_{ij}$, $i\neq j$. Considering an undirected graph we make use of the following

Lemma 1. *(Ren et al., 2005) The Laplacian matrix has the following properties: (i) $\mathcal{L}$ is symmetric and all its eigenvalues are either strictly positive or equal to 0, and $\mathbf{1}$ is the corresponding eigenvector to 0; (ii) 0 is a simple eigenvalue of $\mathcal{L}$ if and only if the graph is connected.*

We will also make use of another Laplacian matrix, according to the following

Lemma 2. *(Lin et al., 2008) Let* $\bar{\mathcal{L}} = [\bar{l}_{ij}] \in \mathbb{R}^{N\times N}$ *be a Laplacian matrix such that* $\bar{l}_{ij} = \frac{N-1}{N}$ *if* $i = j$, *and* $\bar{l}_{ij} = -\frac{1}{N}$ *otherwise, then the following hold: (i) the eigenvalues of* $\bar{\mathcal{L}}$ *are* 1 *with multiplicity* $N-1$, *and* 0 *with multiplicity* 1. $\mathbf{1}^\top$ *and* $\mathbf{1}$ *are respectively the left and right eigenvector associated to eigenvalue* 0; *(ii) there exists an orthogonal matrix* $U \in \mathbb{R}^{N\times N}$, *i.e.* $U : U^\top U = UU^\top = I$, *such that for any Laplacian matrix* $\mathcal{L}$ *associated to any undirected graph we have*

$$U^\top \bar{\mathcal{L}} U = \begin{bmatrix} I_{N-1} & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 0 \end{bmatrix} \triangleq \bar{\Lambda},$$
$$U^\top \mathcal{L} U = \begin{bmatrix} \mathcal{L}_1 & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 0 \end{bmatrix}$$

where $\mathcal{L}_1 \in \mathbb{R}^{(N-1)\times(N-1)}$ *is symmetric and positive definite if the graph is connected.*

In addition we employ the Kronecker product $\otimes$, for which we have

Lemma 3. *(Graham, 1981) Suppose that* $U \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{q\times q}$, $X \in \mathbb{R}^{p\times p}$, *and* $Y \in \mathbb{R}^{q\times q}$. *The following hold: (i)* $(U\otimes V)(X\otimes Y) = UX \otimes VY$; *(ii) suppose* U, *and* V *invertible, then* $(U\otimes V)^{-1} = U^{-1}\otimes V^{-1}$.

2.2 Problems Formulation

Problem 1. *We consider* N *identical agents governed by general discrete-time linear dynamics, according to*

$$\begin{cases} x_i^+ = Ax_i + B_2u_i + B_1\omega_i, \quad i = 1,\cdots,N \\ z_i = C_1x_i \\ y_i = C_2x_i \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_2 \in \mathbb{R}^{n\times l}$, $B_1 \in \mathbb{R}^{n\times h}$, $C_1 \in \mathbb{R}^{r\times n}$, $C_2 \in \mathbb{R}^{m\times n}$, $x_i \triangleq x_i(k) \in \mathbb{R}^n$ *and* $x_i^+ \triangleq x_i(k+1) \in \mathbb{R}^n$ *are respectively the agent state at the current step* k, *and at the next step* $k+1$, $u_i \triangleq u_i(k) \in \mathbb{R}^l$ *is the agent control,* $\omega_i \triangleq \omega_i(k) \in \mathbb{R}^h$ *its disturbance,* $z_i \triangleq z_i(k) \in \mathbb{R}^r$ *the variable on which agreement among the agents is sought, and* $y_i \triangleq y_i(k) \in \mathbb{R}^m$ *is the measured output. For the sake of leaderless consensus, a priori we do not require* A *to be Schur stable. Indeed, as shown by (Ge et al., 2013),* A *has a role in determining the consensus function to which the agents converge under proper control. Here it can be thought to be assigned by a previous control design step. The agents can communicate on an undirected connected graph whose Laplacian matrix* $\mathcal{L}$ *has positive minimum nonzero and maximum eigenvalues respectively equal to* $\underline{\lambda}_{\mathcal{L}}$, *and* $\bar{\lambda}_{\mathcal{L}}$. *At this point, we can state the problem in a general way as the one of finding a distributed control law for* u_i *such that* $\|z_i - z_j\|$ *is minimized for* $i,j = 1,\cdots,N$ *with respect to the disturbance* $\omega \triangleq [\omega_1^\top,\cdots,\omega_N^\top]^\top$. *In this work though, as previously stated, we focus on local controllers of the form*

$$\begin{cases} x_{c_i}^+ = A_cx_{c_i} + B_cs_i, \quad i = 1,\cdots,N \\ u_i = C_cx_{c_i} + D_cs_i \end{cases} \tag{2}$$

where $x_{c_i} \triangleq x_{c_i}(k) \in \mathbb{R}^{2l}$ *is the agent controller state, and*

$$A_c = \begin{bmatrix} I_l & I_l \\ 0_{l\times l} & 0_{l\times l} \end{bmatrix}_{2l\times 2l} \quad B_c = \begin{bmatrix} (K_i - K_d) \\ K_d \end{bmatrix}_{2l\times m}$$
$$C_c = \begin{bmatrix} I_l & 0_{l\times l} \end{bmatrix}_{l\times 2l} \quad D_c = [(K_p + K_i + K_d)]_{l\times m} \tag{3}$$

where $K_p, K_i, K_d \in \mathbb{R}^{l\times m}$ *are gain matrices to be tuned, and where* $s_i \triangleq s_i(k) \in \mathbb{R}^m$:

$$s_i \triangleq \sum_{j=1}^{N} a_{ij}(y_i - y_j) \tag{4}$$

Thus the closed-loop system for agent i *has dimension* $\bar{n} \triangleq n + 2l$. *As shown by (Wu et al., 2011), system (2) is a state representation of the discrete-time PID MIMO controller, whose z-transform is*

$$\frac{u_i(z)}{s_i(z)} = K_p + K_i\frac{z}{z-1} + K_d\frac{z-1}{z} \tag{5}$$

The problem can now be restated as the one of finding matrices B_c, *and* D_c *such that the effect of disturbance* ω *on the consensus is minimized.*

The second problem studied in this paper is the following

Problem 2. *Consider* $N+1$ *discrete-time linear agents, whose dynamics are described by*

$$\begin{cases} x_0^+ = Ax_0 + B_1u_0 \\ z_0 = C_1x_0 \\ y_0 = C_2x_0 \end{cases}$$
$$\begin{cases} x_i^+ = Ax_i + B_2u_i, \quad i = 1,\cdots,N \\ z_i = C_1x_i \\ y_i = C_2x_i \end{cases} \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times h}$, $B_2 \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{r\times n}$ $C_2 \in \mathbb{R}^{m\times n}$, $x_0 \triangleq x_0(k) \in \mathbb{R}^n$ *is the state of*

the $N+1$ agent, called leader, $y_0 \triangleq y_0(k) \in \mathbb{R}^m$ *is its measured output,* $u_0 \triangleq u_0(k) \in \mathbb{R}^h$ *is a time-varying unknown control acting on the leader dynamics, and* $z_0 \triangleq z_0(k) \in \mathbb{R}^r$ *is the variable on which we want the follower controlled outputs* z_i *to converge. Concerning the remaining* N *follower agents, system description similar to (1) holds. The followers are assumed to communicate on an undirected connected graph whose Laplacian matrix is* $\mathcal{L}$*. The leader can pass information to a subset of followers. If agent* i *receives information from the leader, then we set* a_{i0} *to* 1*, and* 0 *otherwise. Thus we define* $\mathcal{M} \triangleq \mathcal{L} - diag(a_{10}, \cdots, a_{N0})$*, which is symmetric and positive definite, and we name* $\underline{\lambda}_{\mathcal{M}}$*, and* $\bar{\lambda}_{\mathcal{M}}$ *respectively its minimum and maximum eigenvalue. Without loss of generality we consider* A *to be Schur stable. The aim of the present problem is indeed not the one of stabilizing each single agent, but rather to steer the follower agents state to the leader one despite the presence of* u_0*, which makes the leader dynamics time-varying. In order to accomplish such objective we aim to employ the controller of form (2), (3), where we consider a modified variable* s_i *to take into account the communication with the leader agent, according to*

$$s_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) + a_{i0}(y_i - y_0) \qquad (7)$$

Intuitively such a controller is not capable of solving the leader-follower tracking problem, i.e. $\lim_{k\to\infty} \|z_i - z_0\| \neq 0$ *for* $i = 1, \cdots, N$*, and for any vector signal* u_0*, because the latter acts as an unknown exogenous signal for the overall system including the* $N+1$ *agents. This is why we will focus on tuning the controller matrices* B_c*, and* D_c *such that* $\|z_i - z_0\|$ *is minimized for* $i = 1, \cdots, N$*.*

3 MAIN RESULT

3.1 $\mathcal{H}_\infty$ Output Consensus

In order to state our main result we introduce the following definition, similar to the one given in (Wang and Shao, 2015).

Definition 1. *System (1) is said to achieve an* $\mathcal{H}_\infty$ output consensus *with a performance index* $\gamma \in \mathbb{R}^+$ *if, for any initial condition,* $\lim_{k\to\infty} \|z_i - z_j\| = 0$ *for* $i, j = 1, \cdots, N$ *when* $\omega = 0$*, and the* $\mathcal{H}_\infty$ *norms of the transfer function matrices, for* $i = 1, \cdots, N$*, between* ω *and* $z_i - \frac{1}{N}\sum_{j=1}^{N} z_j$ *are inferior to* γ*.*

The following result is based on Theorem 3 in (Wu et al., 2011), reported in Theorem 4 in the Appendix.

Theorem 1. *Given* N *agents described by (1) on an undirected connected graph; consider the distributed protocol of equations (2),(3),(4); then the agents achieve* $\mathcal{H}_\infty$ *output consensus with performance index* γ *if there exist two symmetric positive definite matrices* $\underline{P}, \bar{P} \in \mathbb{R}^{\bar{n}\times\bar{n}}$ *such that the LMI conditions of Theorem 4 are simultaneously satisfied for two LTI systems whose matrices are respectively* $(A, B_2, \underline{\lambda}_{\mathcal{L}} C_2)$*, and* $(A, B_2, \bar{\lambda}_{\mathcal{L}} C_2)$*, and they both have controlled output matrix* C_1*, and disturbance input matrix* B_1*.*

Proof. The closed-loop dynamics for the generic agent i, by using (1),(2), and by defining the augmented state $\xi_i \triangleq \left[x_i^\top, x_{c_i}^\top\right]^\top \in \mathbb{R}^{\bar{n}}$, and matrices $\bar{C}_2 \triangleq [C_2 \quad 0_{m\times 2l}]$, $\bar{C}_1 \triangleq [C_1 \quad 0_{r\times 2l}]$, $\tilde{B} \triangleq \left[B_1^\top \quad 0_{h\times(2l)}\right]^\top$ is given by

$$\begin{cases} \xi_i^+ = \hat{A}\xi_i + \hat{B}\sum_{j=1}^{N} a_{ij}(\xi_i - \xi_j) + \tilde{B}\omega_i \\ z_i = \bar{C}_1 \xi_i \end{cases} \qquad (8)$$

where

$$\hat{A} = \begin{bmatrix} A & B_2 C_c \\ 0 & A_c \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_2 D_c \bar{C}_2 \\ B_c \bar{C}_2 \end{bmatrix}$$

Similar to ((Liu et al., 2009)), and ((Wang and Gao, 2011)), we define $\zeta_i \triangleq z_i - \frac{1}{N}\sum_{j=1}^{N} z_j$, and $\delta_i \triangleq \xi_i - \frac{1}{N}\sum_{j=1}^{N} \xi_j$, thus $\zeta_i = \bar{C}_1 \delta_i$. Note that if $\zeta_i = 0$ for $i = 1, \cdots, N$ then $z_i = z_j$, i.e. output consensus is achieved. If now we name $\xi \triangleq \left[\xi_1^\top, \cdots, \xi_N^\top\right]^\top$, $\delta \triangleq \left[\delta_1^\top, \cdots, \delta_N^\top\right]^\top$, and $\zeta \triangleq \left[\zeta_1^\top, \cdots, \zeta_N^\top\right]^\top$, we have that $\zeta = \left(I_N \otimes \bar{C}_1\right)\delta$, and $\delta = \xi - \mathbf{1} \otimes \frac{1}{N}\sum_{j=1}^{N} \xi_j = \left(\bar{\mathcal{L}} \otimes I_{\bar{n}}\right)\xi$, where $\bar{\mathcal{L}}$ satisfies the conditions of Lemma 2. Gathering together the equations of the closed-loop agents dynamics, we obtain

$$\begin{cases} \xi^+ = \left(I_N \otimes \hat{A} + \mathcal{L} \otimes \hat{B}\right)\xi + \left(I_N \otimes \tilde{B}\right)\omega \\ \zeta = \left(I_N \otimes \bar{C}_1\right)\left(\bar{\mathcal{L}} \otimes I_{\bar{n}}\right)\xi = \left(\bar{\mathcal{L}} \otimes \bar{C}_1\right)\xi \end{cases}$$

We now consider the following change of coordinates $\delta = \left(\bar{\mathcal{L}} \otimes I_{\bar{n}}\right)\xi$, which yields

$$\delta^+ = \left(\bar{\mathcal{L}} \otimes I_{\bar{n}}\right)\left(I_N \otimes \hat{A} + \mathcal{L} \otimes \hat{B}\right)\xi + \left(\bar{\mathcal{L}} \otimes I_{\bar{n}}\right)\left(I_N \otimes \tilde{B}\right)\omega =$$

$$
\begin{aligned}
&= \left(\bar{\mathcal{L}} \otimes \hat{A} + \bar{\mathcal{L}}\mathcal{L} \otimes \hat{B}\right) \left(\delta + \mathbf{1} \otimes \frac{1}{N} \sum_{j=1}^{N} \xi_j\right) + \\
&\quad \left(\bar{\mathcal{L}} \otimes \tilde{B}\right) \omega \\
&= \left(\bar{\mathcal{L}} \otimes \hat{A} + \bar{\mathcal{L}}\mathcal{L} \otimes \hat{B}\right) \delta + \left(\bar{\mathcal{L}} \otimes \tilde{B}\right) \omega
\end{aligned}
$$

where we used points (i) of Lemma 2, and 3. According to the (ii) point of the former, we employ the orthogonal matrix $U \in \mathbb{R}^{N \times N}$ to define the change of coordinates: $\hat{\delta} \triangleq \left(U^\top \otimes I_{\bar{n}}\right) \delta$, $\hat{\omega} \triangleq \left(U^\top \otimes I_h\right) \omega$, $\hat{\zeta} \triangleq \left(U^\top \otimes I_m\right) \zeta$, so that the system equations in the new coordinates are given by

$$
\begin{cases}
\hat{\delta}^+ &= \left(U^\top \otimes I_{\bar{n}}\right) \left(\bar{\mathcal{L}} \otimes \hat{A} + \bar{\mathcal{L}}\mathcal{L} \otimes \hat{B}\right) \left(U \otimes I_{\bar{n}}\right) \hat{\delta} \\
&\quad + \left(U^\top \otimes I_{\bar{n}}\right) \left(\bar{\mathcal{L}} \otimes \tilde{B}\right) \omega \\
&= \left(\bar{\Lambda} \otimes \hat{A} + \bar{\Lambda} U^\top \mathcal{L} U \otimes \hat{B}\right) \hat{\delta} + \left(\bar{\Lambda} \otimes \tilde{B}\right) \hat{\omega} \\
\hat{\zeta} &= \left(U^\top \otimes I_m\right) \left(I_N \otimes \bar{C}_1\right) \left(U \otimes I_{\bar{n}}\right) \hat{\delta} = \\
&\quad \left(I_N \otimes \bar{C}_1\right) \hat{\delta}
\end{cases}
\tag{9}
$$

As shown in Lemma 2, being the last rows of $\bar{\Lambda}$, and $U^\top \mathcal{L} U$ zeros, we can split the dynamics (9) in two parts by dividing the system variables as $\hat{\delta} = [\hat{\delta}_1^\top, \hat{\delta}_2^\top]^\top$, $\hat{\omega} = [\hat{\omega}_1^\top, \hat{\omega}_2^\top]^\top$, and $\hat{\zeta} = [\hat{\zeta}_1^\top, \hat{\zeta}_2^\top]^\top$. The dynamic equation of the second variable is then $\hat{\delta}_2^+ = 0$, and it does not influences $\hat{\delta}_1$. It follows that we can study the reduced order system

$$
\begin{cases}
\hat{\delta}_1^+ = \left(I_{N-1} \otimes \hat{A} + \mathcal{L}_1 \otimes \hat{B}\right) \hat{\delta}_1 + \left(I_{N-1} \otimes \tilde{B}\right) \hat{\omega}_1 \\
\hat{\zeta}_1 = \left(I_{N-1} \otimes \bar{C}_1\right) \hat{\delta}_1
\end{cases}
$$

From Lemma 2, it exists an orthogonal matrix $V \in \mathbb{R}^{(N-1)\times(N-1)} : V^\top \mathcal{L}_1 V \triangleq \Lambda = diag(\lambda_1, \cdots, \lambda_{N-1})$, where $0 < \underline{\lambda}_{\mathcal{L}} \leq \lambda_i \leq \bar{\lambda}_{\mathcal{L}}$ for $i = 1, \cdots, N-1$. Thus we can define a further change of coordinates, such that $\bar{\delta}_1 \triangleq \left(V^\top \otimes I_{\bar{n}}\right) \hat{\delta}_1$, $\bar{\omega}_1 \triangleq \left(V^\top \otimes I_h\right) \hat{\omega}_1$, and $\bar{\zeta}_1 \triangleq \left(V^\top \otimes I_m\right) \hat{\zeta}_1$. The latter yields

$$
\begin{cases}
\bar{\delta}_1^+ = \left(I_{N-1} \otimes \hat{A} + \Lambda \otimes \hat{B}\right) \bar{\delta}_1 + \left(I_{N-1} \otimes \tilde{B}\right) \bar{\omega}_1 \\
\bar{\zeta}_1 = \left(I_{N-1} \otimes \bar{C}_1\right) \bar{\delta}_1
\end{cases}
\tag{10}
$$

It is easy to see that the transfer function matrix of (10) satisfies

$$
\begin{aligned}
&\|T_{\bar{\zeta}_1 \bar{\omega}_1}(z)\|_\infty = \|T_{\hat{\zeta}_1 \hat{\omega}_1}(z)\|_\infty = \\
&\|T_{\hat{\zeta}\hat{\omega}}(z)\|_\infty = \|T_{\zeta\omega}(z)\|_\infty
\end{aligned}
\tag{11}
$$

It follows that we can impose an $\mathcal{H}_\infty$ constraint on transfer function matrix $T_{\zeta\omega}(z)$ by acting on $T_{\bar{\zeta}_1 \bar{\omega}_1}(z)$. We can now separate equation (10) in $N-1$ subsystems, each of them being governed by

$$
\begin{cases}
\bar{\delta}_{1_i}^+ = \begin{bmatrix} (A + B_2 D_c(\lambda_i C_2)) & B_2 C_c \\ B_c(\lambda_i C_2) & A_c \end{bmatrix} \bar{\delta}_{1_i} \\
\qquad\qquad + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \bar{\omega}_i \\
\bar{\zeta}_{1_i} = C_1 \bar{x}_{1_i}
\end{cases}
\tag{12}
$$

where $\bar{\delta}_{1_i} \triangleq [\bar{x}_{1_i}^\top \quad \bar{x}_{1,c_i}^\top]^\top$. System (12) can be equivalently seen as the closed-loop form of the two following systems

$$
\begin{cases}
\bar{x}_{1_i}^+ = A\bar{x}_{1_i} + B_2\bar{u}_i + B_1\bar{\omega}_i \\
\bar{y}_{1_i} \triangleq (\lambda_i C_2)\bar{x}_{1_i} \\
\bar{\zeta}_{1_i} = C_1\bar{x}_{1_i}
\end{cases}
\qquad
\begin{cases}
\bar{x}_{1,c_i}^+ = A_c\bar{x}_{1,c_i} + B_c\bar{y}_{1_i} \\
\bar{u}_i \triangleq C_c\bar{x}_{1,c_i} + D_c\bar{y}_{1_i}
\end{cases}
\tag{13}
$$

Thus, we can reformulate the problem as the one finding matrices B_c, and D_c such that for $i = 1, \cdots, N-1$ the closed-loop system of (13) is Schur stable when $\omega_i = 0$, and to guarantee that $\|T_{\bar{z}_i \bar{\omega}_i}(z)\|_\infty < \gamma$. A sufficient condition to prove the existence of such a solution and a relatively simple way to calculate the controller matrices are obtained by employing Theorem 4. In the latter it is proved that if it exists a symmetric positive definite matrix $P_i \in \mathbb{R}^{\bar{n}\times\bar{n}}$ such that if a given LMI condition is satisfied, then closed-loop system (12) using controller (2),(3),(4) is such that

$$
\begin{aligned}
&\bar{\delta}_{1_i}^\top(k+1) P_i \bar{\delta}_{1_i}(k+1) - \bar{\delta}_{1_i}^\top(k) P_i \bar{\delta}_{1_i}(k) \\
&\qquad < \gamma^2 \bar{\omega}_i^\top(k)\bar{\omega}_i(k) - \bar{z}_i^\top(k)\bar{z}_i(k)
\end{aligned}
$$

It is important to stress that such LMI condition is *affine* in the system matrices, variables and matrix P_i. We make use of this fact to provide sufficient conditions for which it exists a controller of the considered form such that the mentioned LMI is *simultaneously* verified for $i = 1, \cdots, N-1$. Since the generic eigenvalue of $\mathcal{L}_1 : \lambda_i$ is such that $\underline{\lambda}_{\mathcal{L}} \leq \lambda_i \leq \bar{\lambda}_{\mathcal{L}}$, then it always exists $\alpha_i \in \mathbb{R} : 0 \leq \alpha_i \leq 1$ so that $\lambda_i = \alpha_i \underline{\lambda}_{\mathcal{L}} + (1-\alpha_i)\bar{\lambda}_{\mathcal{L}}$. Notice that the systems to be stabilized, appearing in the first set of equation in (13), can be seen as *one* single system with an uncertain measurement matrix, whose parameter is λ_i. In other words, $C_{2_i} \triangleq \lambda_i C_2$, and $\exists \alpha_i : C_{2_i} = \alpha_i C_{2_{min}} + (1-\alpha_i) C_{2_{max}}$, where $C_{2_{min}} \triangleq \underline{\lambda}_{\mathcal{L}} C_2$, and $C_{2_{mix}} \triangleq \bar{\lambda}_{\mathcal{L}} C_2$, i.e. it can be written as a convex combination of the extreme matrices $C_{2_{min}}$, and $C_{2_{max}}$. Thus, as

in (Wang and Gao, 2011), the proof makes use of classic results of robust linear control, and in particular by introducing an affine parameter dependent Lyapunov matrix $P(\alpha_i) \triangleq \alpha_i \underline{P} + (1-\alpha_i)\bar{P}$, where $\underline{P}$, $\bar{P}$ are Lyapunov matrices solution of simultaneous LMI of Theorem 4 written for respectively $C_{2_{min}}$, and $C_{2_{max}}$. Eventually, it is easy to show that if $\underline{P}$, $\bar{P}$ exist, then the controller solves the problem $\forall \lambda \in \mathbb{R} : \underline{\lambda}_{\mathcal{L}} \leq \lambda \leq \bar{\lambda}_{\mathcal{L}}$, and in particular for $\lambda = \lambda_i$, for $i = 1, \cdots, N-1$. Such a controller is easily found from the solution of the aforementioned LMI condition. Indeed among the LMI variables there are matrices B_c, and D_c, from which it is easy to calculate the PID gains K_p, K_i, and K_d by employing relations in (3). □

Remark 1. *Note that the mentioned LMI conditions, if satisfied, guarantees that the consensus error is minimized with respect to the disturbance. However the latter still have a role in determining the consensus function to which the agents converge.*

3.2 Leader-Follower Consensus under time-varying reference

The result of Subsection 3.1 can be easily adapted for the sake of leader-follower consensus via an $\mathcal{H}_\infty$ formulation of the problem. Thus, we give the following

Definition 2. *System (6) is said to achieve an* $\mathcal{H}_\infty$ output leader-follower consensus *with a performance index* $\gamma \in \mathbb{R}^+$ *if, for any initial condition,* $\lim_{k\to\infty} \|z_i - z_0\| = 0$ *for* $i = 1, \cdots, N$ *when* $u_0(k) = \mathbf{0} \quad \forall k \in \mathbb{N}$, *and the* $\mathcal{H}_\infty$ *norms of the transfer function matrices, for* $i = 1, \cdots, N$, *between* u_0 *and* $z_i - z_0$ *are inferior to* γ.

Theorem 2. *Given the system described by (6), where N follower agents can communicate on an undirected connected graph, and one leader can communicate with a non-empty subset of followers; consider the distributed protocol of equations (2),(3),(7); then the systems achieve* $\mathcal{H}_\infty$ *output leader-follower consensus with performance index* γ *if there exist two symmetric positive definite matrices* $\underline{P}, \bar{P} \in \mathbb{R}^{\bar{n}\times\bar{n}}$ *such that the LMI conditions of Theorem 4 are simultaneously satisfied for two LTI systems whose matrices are respectively* $(A, B_2, \underline{\lambda}_{\mathcal{M}} C_2)$, *and* $(A, B_2, \bar{\lambda}_{\mathcal{M}} C_2)$, *and they both have controlled output matrix* C_1, *and disturbance input matrix* $-B_1$.

Proof. The proof is similar to proof of Subsection 3.1. By defining error $e_i \triangleq x_i - x_0$, $\xi \triangleq [e_i^\top, x_{c_i}^\top]^\top$, and $\zeta_i \triangleq C_1 e_i$ the closed-loop system for the generic follower agent i is given by

$$\begin{cases} \xi_i^+ = \hat{A}\xi_i + \hat{B}\left(\sum_{j=1}^N a_{ij}(\xi_i - \xi_j) + a_{i0}\xi_i\right) + \tilde{B}u_0 \\ \zeta_i = \bar{C}_1 \xi_i \end{cases}$$

where $\hat{A}$, $\hat{B}$, $\bar{C}_1$ are defined in (8), and $\tilde{B} \triangleq [-B_1^\top \quad 0_{h\times 2l}]^\top$. Defining $\boldsymbol{u_0} \triangleq \mathbf{1} \otimes u_0$, we then gather the N agent equations together

$$\begin{cases} \xi^+ = \left(I_N \otimes \hat{A} + \mathcal{M} \otimes \hat{B}\right)\xi + \left(I_N \otimes \tilde{B}\right)\boldsymbol{u_0} \\ \zeta = \left(I_N \otimes \bar{C}_1\right)\xi \end{cases} \tag{14}$$

From the definition of $\mathcal{M}$ in Section 2, there exists a orthogonal matrix $U : U^\top \mathcal{M} U \triangleq \Lambda = diag(\lambda_1, \cdots, \lambda_N)$, where $\lambda_i \in \mathbb{R} : \lambda_i > 0$ for $i = 1, \cdots, N$, so that we can define the change of coordinates $\xi \triangleq (U \otimes I_{\bar{n}})\hat{\xi}$, $\boldsymbol{u_0} \triangleq (U \otimes I_l)\hat{\boldsymbol{u}}_{\mathbf{0}}$, $\zeta \triangleq (U \otimes I_r)\hat{\zeta}$. By applying similar calculation as in the previous subsection, the global system in the new coordinates

$$\begin{cases} \hat{\xi}^+ = \left(I_N \otimes \hat{A} + \Lambda \otimes \hat{B}\right)\hat{\xi} + \left(I_N \otimes \tilde{B}\right)\hat{\boldsymbol{u}}_{\mathbf{0}} \\ \hat{\zeta} = \left(I_N \otimes \bar{C}_1\right)\hat{\xi} \end{cases} \tag{15}$$

As in (11), it results that $\|T_{\hat{\xi}\hat{\boldsymbol{u}}_{\mathbf{0}}}(z)\|_\infty = \|T_{\xi \boldsymbol{u_0}}(z)\|_\infty$, i.e. we can minimize the effect of $\boldsymbol{u_0}$ on the consensus error by acting on system (15). Similar to the passage from equations (10) to (12), splitting (15) in N subsystems yields the following equation for subsystem i

$$\begin{cases} \hat{\xi}_i^+ = \begin{bmatrix} (A + B_2 D_c(\lambda_i C_2)) & B_2 C_c \\ B_c(\lambda_i C_2) & A_c \end{bmatrix} \hat{\xi}_i \\ \qquad\qquad + \begin{bmatrix} -B_1 \\ 0 \end{bmatrix} \hat{u}_0 \\ \hat{\zeta}_i = C_1 \hat{e}_i \end{cases} \tag{16}$$

where $\hat{\xi}_i \triangleq [\hat{e}_i^\top \quad \hat{x}_{c_i}^\top]^\top$. Equivalently, it can be described as the connection of the two following systems

$$\begin{cases} \hat{e}_i^+ = A\hat{e}_i + B_2\hat{u}_i - B_1\hat{u}_0 \\ \hat{y}_i \triangleq (\lambda_i C_2)\hat{e}_i \\ \hat{\zeta}_i = C_1 \hat{e}_i \end{cases} \qquad \begin{cases} \hat{x}_{c_i}^+ = A_c\hat{x}_{c_i} + B_c\hat{y}_i \\ \hat{u}_i \triangleq C_c\hat{x}_{c_i} + D_c\hat{y}_i \end{cases} \tag{17}$$

The rest of the proof is equivalent to the last part of Subsection 3.1, and it is concluded by invoking Theorem 4, whose LMI conditions have to

be simultaneously satisfied for the two systems at the vertices of the polytope having matrices respectively $(A, B_2, \underline{\lambda}_{\mathcal{M}} C_2)$, and $(A, B_2, \bar{\lambda}_{\mathcal{M}} C_2)$, and same controlled output, and disturbance input matrices C_1, $-B_1$. From the solution of the aforementioned LMIs the controllers gains are easily found as in the proof of Theorem 1. If such a solution exists, then the system is stable. □

Having employed a PID structure for the distributed controller suggests that consensus should be reached for any $u_0(k) = \bar{u}_0$, where $\bar{u}_0$ is any constant vector. However this is not automatically guaranteed in the MIMO case by the mentioned LMI conditions, and in this framework it is verified a posteriori. Nonetheless, if such LMI has a solution then, according to the well-known Francis equation, a *necessary* conditions for the proposed controller to reject constant exogenous signals is that $l \geq r$.

In the leader-follower consensus framework a *different tuning* of the PID controller gains with respect to Theorem 2 could lead to better performance, as shown in section 4. Thus, by proposing the following definition we aim to focus on system *fast response* rather than imposing some $\mathcal{H}_\infty$ constraint. For this last development, we further consider $r = m$, thus we simply name $C \triangleq C_1 = C_2$.

Definition 3. *System (6) is said to achieve* fast leader-follower consensus *with performance index $\tau \in \mathbb{R}^+$ if for $u_0(k) = \mathbf{0}$, and any initial condition, $\lim_{k\to\infty} \|y_i - y_0\| = 0$ for $i = 1, \cdots, N$, and $(1 - e^{-1})\%$ of consensus is achieved in a maximum number of steps equal to $\lceil \tau \rceil$.*

Note that the same kind of definition can be considered for *sampled-data* systems, by saying that system (6) achieves fast leader-follower consensus with a time constant inferior to τT_s, where T_s is the system sampling time. The result we present in the following is based on Theorem 2 in (Wu et al., 2011), reported in Theorem 5 in the Appendix.

Theorem 3. *Given the system described by (6), where N follower agents can communicate on an undirected connected graph, and one leader can communicate with a non-empty subset of followers; consider the distributed protocol of equations (2),(3),(7); then the systems achieve fast leader-follower consensus with performance index $\tau = -\dfrac{1}{\log(R)}$, where $R \in \mathbb{R} : 0 \leq R < 1$, if there exist two symmetric positive definite matrices $\underline{P}, \bar{P} \in \mathbb{R}^{\bar{n}\times\bar{n}}$ such that the LMI conditions of Theorem 5 are simultaneously satisfied for two LTI systems whose matrices are respectively $(A, B_2, \underline{\lambda}_{\mathcal{M}} C)$, and $(A, B_2, \bar{\lambda}_{\mathcal{M}} C)$, and where the real constants (a, b) to be set in Theorem 5 are chosen to be $(a, b) = (0, R)$.*

Proof. The proof employs the same change of coordinates as in the previous one, so that we can restate the problem as the one of stabilizing the top system of equation (17), for $i = 1, \cdots, N$, with the bottom system in (17), i.e. a PID controller whose matrices are defined in (3). Unlike Theorem 2, as previously mentioned, we invoke Theorem 5, where it is stated that given two real constants (a, b), if there exists a symmetric positive definite matrix P_i such that a given LMI condition is satisfied, then system (16) is stable with all its eigenvalues λ laying in the complex plane region defined by $\mathcal{F}_D \triangleq \{(\Re[\lambda], \Im[\lambda]) : (\Re[\lambda] + a)^2 + \Im[\lambda]^2 < b^2\}$. As for the two previous proofs, we employ classic results of linear robust control to impose that this condition is simultaneously satisfied for two systems at the vertices of the polytope whose matrices are respectively $(A, B_2, \underline{\lambda}_{\mathcal{M}} C)$, and $(A, B_2, \bar{\lambda}_{\mathcal{M}} C)$. If such a solution exists then the eigenvalues of system (14) are guaranteed to lie in $\mathcal{F}_D$. In this framework we are interested in speeding up the system response to u_0. For this reason we set $a = 0$, and $b = R$, where $R : 0 \leq R < 1$. Thus, all system eigenvalues are guaranteed to have a module inferior to R. As a result, the system has the slowest time-constant inferior to $-\dfrac{T_s}{\log(R)}$. In terms of number of iterations it is easy to see that such performance is equal to a maximum value $\left\lceil -\dfrac{1}{\log(R)} \right\rceil$ of iterations. Eventually, from the LMI solution, the PID gains are found as in the two previous proofs. □

Remark 2. *In this latter problem too, having imposed a PID structure does not directly guarantee achievement of consensus for any constant $u_0(k)$ in the general MIMO case. According to Francis equations, if the mentioned LMI has a solution, a* necessary *condition though is given by $l \geq m$.*

4 Simulation Examples

First of all we carry out a numerical simulation to test the $\mathcal{H}_\infty$ output consensus control. We consider a network of 5 agents as show in Fig. 1.a.

Each of them is governed by (1), where

$$A = \begin{bmatrix} 0.8182 & 0.0452 & -0.0034 \\ 0 & 0.9888 & -0.1492 \\ 0 & 0.1492 & 0.9888 \end{bmatrix}, C_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^\top$$

$$B_2 = \begin{bmatrix} 1 & 0.4 \\ 0 & 1 \\ 0.5 & 0.5 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 \\ 0.05 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1.2 & 0.8 & 1.4 \\ 1.4 & -1.2 & 0.8 \\ -0.5 & 0.7 & 1.2 \end{bmatrix} \tag{18}$$

Note that (18) is not Schur stable because two of its eigenvalues lay on the unit circle. Each agent is perturbed by a disturbance of the form $\omega_i(k) = 0.8\nu_i(k) + c_i$, where ν_i is an aleatory variable with uniform distribution of probability in $[0,1]$, and c_i is some constant value. The PID gains found via LMIs allow an $\mathcal{H}_\infty$ performance index of $\gamma = 0.18$. Fig. 2 shows the 5 agents trajectories (colored dashed lines) as well as their average (blue continuous line). Then we compare the two proposed PID gain tuning for a leader-follower consensus problem. For this example we consider the graph of Fig. 1.b, where agent 0 is the leader. The system dynamics is governed by equation (6), where

$$A = \begin{bmatrix} 0.7711 & 0.4744 & 0.2475 \\ 0.1646 & 0.4487 & 0.1036 \\ -0.8959 & -0.8534 & -0.2198 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.5 & 0.3 & 0.4 \\ 0.7 & 0 & 1 \\ 0.4 & 0.9 & 0.3 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and C_2 as in (18). The controller tuned following Theorem 2 allows an $\mathcal{H}_\infty$ performance index of $\gamma = 2$, while the one tuned according to Theorem 3 guarantees a performance index of $\tau = 6.1531$. In Fig. 3 we simulate the system step response for a value of $u_0 = 3$. For ease of comparison, we plot here the only output associated to matrix C_1. As mentioned in section 3, fast consensus (green dashed lines) outperforms the $\mathcal{H}_\infty$ one (red dashed-dotted lines). Indeed, even if the latter respects Theorem 2, its consensus error goes slowly to zero with respect to the former one. Eventually, in Fig. 4 it is shown the system behavior for fast consensus tuning of PID gains when u_0 is a time-varying vector signal, and where we set $C = C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The

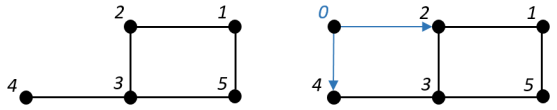

Figure 1: (a): leaderless communication graph (*left*); (b): leader-follower communication graph (*right*).

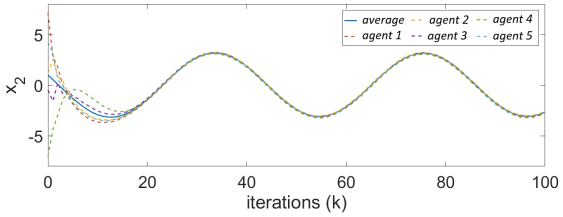

Figure 2: $\mathcal{H}_\infty$ output consensus.

blue and the dark green continuous signals are respectively leader states x_1, and x_2, while the followers states are represented respectively by the dashed green and red signals for x_1, and x_2.

5 Conclusion

We presented a PID-like distributed protocol for general LTI MIMO discrete-time agents communicating on an undirected connected graph. By employing LMIs we showed how the controller gains can be tuned to solve two different, yet similar, problems, namely a leaderless under system disturbances and a leader-follower under time-varying reference state consensus problem. Treating the system disturbances in the $\mathcal{H}_\infty$ framework revealed good performance, whereas a gain tuning based on fast response seems to be preferable when dealing with a leader-follower problem.
Our results are based on robust control to deal with the problem of simultaneous stabilization of a given number of systems. The given conditions are sufficient and therefore conservative. In near future work we are interested in studying less restrictive conditions when treating the discrete-time consensus problem in the $\mathcal{H}_\infty$ framework.

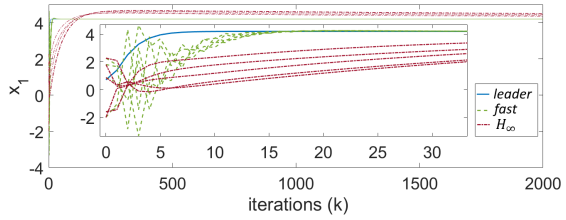

Figure 3: Step response comparison for $\mathcal{H}_\infty$ output and fast leader-follower consensus.

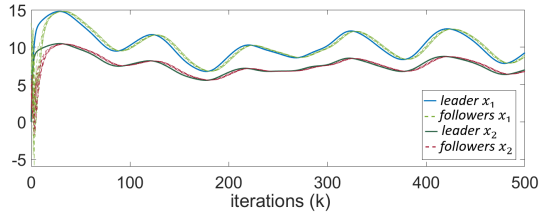

Figure 4: Leader-follower consensus under time-varying reference.

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APPENDIX

In the following we report the two cited theorems in (Wu et al., 2011). Consider the system of equations

$$\begin{cases} x^+ = Ax + B_1\omega + B_2 u \\ z = C_1 x, \quad y = C_2 x \end{cases} \tag{19}$$

where $A \in \mathbb{R}^{n\times n}$, $B_2 \in \mathbb{R}^{n\times l}$, $B_1 \in \mathbb{R}^{n\times h}$, $C_1 \in \mathbb{R}^{r\times n}$, $C_2 \in \mathbb{R}^{m\times n}$, $x \triangleq x(k) \in \mathbb{R}^n$ and $x^+ \triangleq x(k+1) \in \mathbb{R}^n$ are respectively the system state at the current step k, and at the next step $k+1$, $u \triangleq u(k) \in \mathbb{R}^l$ is the control input, $\omega \triangleq \omega(k) \in \mathbb{R}^h$ is an exogenous input signal, $z \triangleq z(k) \in \mathbb{R}^r$ the controlled output, and $y \triangleq y(k) \in \mathbb{R}^m$ is the measured one. Define the matrices $C_{cl} \triangleq \begin{bmatrix} C_1 & 0_{r\times(2l)} \end{bmatrix}$, $\tilde{B} \triangleq \begin{bmatrix} B_1^\top & 0_{h\times(2l)} \end{bmatrix}^\top$, $K \triangleq \begin{bmatrix} D_c^\top & B_c^\top \end{bmatrix}^\top$, and

$$\tilde{A} \triangleq \begin{bmatrix} A & B_2 C_c \\ 0_{2l\times n} & A_c \end{bmatrix}$$

where A_c, B_c, C_c, and D_c are defined in (3). Assuming B_2 to be of full column rank without loss of generality, there exists an invertible $T_b \in \mathbb{R}^{n\times n} : T_b B_2 = \begin{bmatrix} 0_{l\times(n-l)} & I_{l\times l} \end{bmatrix}^\top$. Finally define

$$T \triangleq \begin{bmatrix} T_b & 0_{n\times 2l} \\ 0_{2l\times n} & I_{2l\times 2l} \end{bmatrix}$$

Thus, we have the following theorems

Theorem 4. *Consider system (19). If there exists a positive definite matrix $P \in \mathbb{R}^{\bar{n}\times\bar{n}}$, where $\bar{n} \triangleq n+2l$, matrices*

$$F = \begin{bmatrix} F_{11} & 0_{(\bar{n}-q)\times 3l} \\ F_{21} & F_{22} \end{bmatrix}$$

$F_{22} \in \mathbb{R}^{q\times 3l}$, $1 \le q \le 3l$, $G_1 \triangleq [G_{11} \quad 0] \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $G_{11} \in \mathbb{R}^{\bar{n}\times(\bar{n}-3l)}$, $G_2 \triangleq [G_{21} \quad 0] \in \mathbb{R}^{h\times\bar{n}}$, $G_{21} \in \mathbb{R}^{h\times(\bar{n}-3l)}$, $G_3 \triangleq [G_{31} \quad 0] \in \mathbb{R}^{r\times\bar{n}}$, $G_{31} \in \mathbb{R}^{r\times(\bar{n}-3l)}$, $H_1 \in \mathbb{R}^{\bar{n}\times r}$, $H_2 \in \mathbb{R}^{\bar{n}\times r}$, $H_3 \in \mathbb{R}^{h\times r}$, $H_4 \in \mathbb{R}^{r\times r}$, $Y \in \mathbb{R}^{q\times m}$, and

$$N_1 = \begin{bmatrix} 0_{(\bar{n}-q)\times n} & 0_{(\bar{n}-q)\times 2l} \\ YC_2 & 0_{q\times 2l} \end{bmatrix}$$

and we further name $\Psi_{11} \triangleq P - FT - (FT)^\top$, $\Psi_{21} \triangleq N_1^\top + (FT\tilde{A})^\top - G_1 T + (H_1 C_{cl})^\top$, $\Psi_{22} \triangleq -P + G_1 T\tilde{A} + (G_1 T\tilde{A})^\top + H_2 C_{cl} + (H_2 C_{cl})^\top$, $\Psi_{31} \triangleq (FT\tilde{B})^\top - G_2 T$, $\Psi_{32} \triangleq G_2 T\tilde{A} + H_3 C_{cl} + (G_1 T\tilde{B})^\top$, $\Psi_{33} \triangleq -\gamma^2 I + G_2 T\tilde{B} + (G_2 T\tilde{B})^\top$, $\Psi_{41} \triangleq -G_3 T - H_1^\top$, $\Psi_{42} \triangleq G_3 T\tilde{A} + H_4 C_{cl} - H_2^\top$, $\Psi_{43} \triangleq G_3 T\tilde{B} - H_3^\top$, and $\Psi_{44} \triangleq I - H_4 - H_4^\top$, such that the following LMI has a solution

$$\begin{bmatrix} \Psi_{11} & * & * & * \\ \Psi_{21} & \Psi_{22} & * & * \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & * \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{bmatrix} < 0 \tag{20}$$

and if exists K such that $F_{22}K = Y$, then the $\mathcal{H}_\infty$ norm of the closed-loop system given by (19) and

$$\begin{cases} x_c^+ = A_c x_c + B_c y \\ u = C_c x_c + D_c y \end{cases} \tag{21}$$

satisfies $\|T_{z\omega}\|_\infty < \gamma$.

Theorem 5. *Consider system (19). If there exists a positive definite matrix $P \in \mathbb{R}^{\bar{n}\times\bar{n}}$, and a matrix*

$$J = \begin{bmatrix} J_{11} & 0_{(\bar{n}-q)\times 3l} \\ J_{21} & J_{22} \end{bmatrix}$$

$J_{22} \in \mathbb{R}^{3l\times 3l}$, and $X \in \mathbb{R}^{3l\times m}$, and we further name

$$\Omega \triangleq \begin{bmatrix} 0_{(\bar{n}-3l)\times n} & 0_{(\bar{n}-3l)\times 2l} \\ XC_2 & 0_{3l\times 2l} \end{bmatrix} \tag{22}$$

such that the following LMI has a solution

$$\begin{bmatrix} bP & * \\ \Omega + JT\tilde{A} + aJT & b(JT + (JT)^\top - P) \end{bmatrix} > 0 \tag{23}$$

and if J is nonsingular, then by choosing $K = J_{22}^{-1}X$, the eigenvalues of the following matrix

$$A_{cl} \triangleq \begin{bmatrix} (A + B_2 D_c C_2) & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix}$$

lie in the region $\mathcal{F}_D \triangleq \{(\Re[\lambda], \Im[\lambda]) : (\Re[\lambda] + a)^2 + \Im[\lambda]^2 < b^2\}$.