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A family of piecewise affine control Lyapunov functions

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Abstract

This paper presents a novel method to construct a family of piecewise affine control Lyapunov functions. Unlike most of existing methods which require the contractivity of their domain of definition, the proposed control Lyapunov functions are defined over a so-called N -step controllable set, which is known not to be contractive. Accordingly, a robust control design procedure is presented which only requires solving a linear programming problem at each sampling time. The construction is finally illustrated via a numerical example.

Key words: Control Lyapunov function, Robust control, Convex lifting.

1 Introduction

As a fundamental concept in control theory Lyapunov (1907), Lyapunov stability has been applied in intensive studies related to stability analysis as well as control design. For the design purpose, control Lyapunov functions are usually employed to synthesize controllers guaranteeing closed-loop stability in the sense of Lyapunov, see among the others Zubov and Boron (1964); Khalil (2002). Such control Lyapunov functions are usually chosen a priori with special structural properties. More clearly, in the case of linear optimal control, suitable quadratic objective functions represent control Lyapunov candidates, see e.g. Anderson and Moore (2007); Chmielewski and Manousiouthakis (1996); Daafouz and Bernussou (2001). Moreover, model predictive control (MPC) usually employs finite/infinite horizon quadratic cost functions as control Lyapunov candidates, see for instance Kothare et al. (1996); Cuzzola et al. (2002); Mayne et al. (2000). Extensive studies about control Lyapunov functions for nonlinear systems have been found in the literature, see among the others Primbs et al. (1999). In case the underlying system is subject to constraints, such a control Lyapunov function should be determined such that the recursive feasibility is ensured. This problem is closely related to the determination of the domain of attraction.

Piecewise linear control Lyapunov functions date back to the studies in Gutman and Cwikel (1987); Nguyen et al. (2014) for the nominal case, and are subsequently extended for the robust case to cope with additive disturbances and/or polytopic uncertainty in Blanchini (1994); Rakovic and Baric

(2010); Nguyen et al. (2013), leading to simple design formulations as linear programming problems. However, these studies require that such control Lyapunov functions be defined over contractive sets to guarantee its strict decrease and the recursive feasibility.

This paper aims to present the construction of a more general family of control Lyapunov candidates in the context of constrained control, namely piecewise affine functions. These candidates are defined over a so-called N -step controllable set for a given positive integer N , which is obtained from an increasing sequence of N polytopes, and is known to be not necessarily (one step) contractive. Accordingly, we prove that the conditions of a Lyapunov function (the positivity and the strict decrease) are satisfied within this N -step controllable set with a suitable robust control algorithm. Note that the complexity of the proposed control Lyapunov functions increases as N becomes larger, leading to a more complex control algorithm over the ones using contractive set in Blanchini (1994); Nguyen et al. (2017c), since the number of constraints in the proposed algorithm is larger than the ones in two latter references. However, since this method only requires solving a linear program at each sampling instant, these results can be used for constrained control systems with fast dynamics, e.g. vibration attenuation system, c.f. Gulan et al. (2017b,a).

2 Generalities and basic notions

Throughout the paper, $\mathbb{R}, \mathbb{R}_+, \mathbb{N}, \mathbb{N}_{>0}$ denote the field of real numbers, the set of nonnegative real numbers, the set of nonnegative integers and the positive integer set, respectively. The following index set is also defined, for ease of presentation, with respect to a given $N \in \mathbb{N}_{>0}$:

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$\mathcal{I}_N = \{1, 2, \dots, N\}$. A polyhedron is defined as the intersection of finitely many closed halfspaces. A polytope is defined as a bounded polyhedron. Also, $\mathcal{V}(P)$ is understood as the set of vertices of polytope P . The distance from a point $x \in \mathbb{R}^d$ to a set $S \subset \mathbb{R}^d$ denoted by $\rho_S(x)$ is defined as $\rho_S(x) := \min_{y \in S} \sqrt{(x-y)^T(x-y)}$.

Given two sets $S_1, S_2 \subset \mathbb{R}^d$, we define the set $S_1 \setminus S_2 := \{x \in \mathbb{R}^d : x \in S_1, x \notin S_2\}$. Also, the Minkowski sum of these two sets, denoted by $S_1 \oplus S_2$, is defined as follows: $S_1 \oplus S_2 = \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\}$. We use ∂S to denote the boundary of a compact set S . Also, $\text{int}(S)$ represents the interior of a full-dimensional set S and $\text{conv}(S)$ denotes its convex hull. A function $\alpha(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$.

3 Problem settings

In this paper, we consider a linear time-varying system:

$$x_{k+1} = A(k)x_k + B(k)u_k + w_k, \quad (1)$$

where x_k, u_k, w_k denote the state, control variable and additive disturbance at time k . These variables satisfy

$$x_k \in \mathbb{X}, u_k \in \mathbb{U}, w_k \in \mathbb{W}, \quad (2)$$

where the constraint sets $\mathbb{X} \subset \mathbb{R}^{d_x}, \mathbb{U} \subset \mathbb{R}^{d_u}, \mathbb{W} \subset \mathbb{R}^{d_w}$ are assumed to be polytopes, containing the origin in their interior, with given $d_x, d_u \in \mathbb{N}_{>0}$. This assumption ensures that the origin as the equilibrium point satisfies the above constraints. In case the origin is not the equilibrium point, the system can be translated into the frame of the equilibrium point and the construction can be easily adapted. This assumption will be used later in (4) to build a control Lyapunov function which is only equal to 0 at the equilibrium point. Also, the state-space matrices $A(k), B(k)$ are assumed to belong to a given polytope, defined as below:

$$[A(k) \ B(k)] \in \Psi := \text{conv} \{[A_1 \ B_1], \dots, [A_L \ B_L]\}. \quad (3)$$

This paper aims to construct a new family of control Lyapunov functions, also referred to as convex liftings in the present framework. In particular, the control Lyapunov functions presented in this paper are more general than the piecewise linear family proposed in Blanchini (1994); Rakovic and Baric (2010), since besides their convex, piecewise affine properties, they are defined over the N -step controllable set, known to be non-necessarily contractive.

4 Construction of control Lyapunov functions

Before describing the main result, we need to recall some important ingredients which are instrumental for the proposed construction of control Lyapunov functions.

Positive invariance concept has been investigated in many studies Aubin and Cellina (2012); Bitsoris (1988b); Bitsoris and Vassilaki (1995); Bitsoris (1988a); Blanchini and Miani (2007) and used in different control strategy designs. In case the system is affected by disturbances, the robust positive invariance concept is of use instead.

Definition 4.1 *Given an admissible linear control law $u_k = Kx_k \in \mathbb{U}$, a set $\Omega \subseteq \mathbb{X}$ is called robust positively invariant with respect to system (1) subject to (2) and (3) iff*

$$(A(k) + B(k)K)\Omega \oplus \mathbb{W} \subseteq \Omega, \quad \forall [A(k) \ B(k)] \in \Psi.$$

In order to compute such a set Ω , one should determine a local controller $u = Kx \in \mathbb{U}$, which can cope with both the model uncertainty (3) and additive disturbances in $\mathbb{W}$. Such a gain K can be obtained by different methods, see among the others Kothare et al. (1996); Daafouz and Bernussou (2001). According to such a local controller, one can use existing algorithms to compute a robust positively invariant set, e.g., in Gilbert and Tan (1991); Kolmanovsky and Gilbert (1998); Nguyen et al. (2013).

The definition of another important ingredient N -step controllable set is recalled below.

Definition 4.2 *Consider system (1) subject to model uncertainty (3) and constraints (2). Given a robust positively invariant set Ω and $N \in \mathbb{N}_{>0}$, a set denoted by $\mathcal{K}_N(\Omega) \subseteq \mathbb{X}$ is called the N -step controllable set if any point belonging to this set can reach Ω in N steps in the presence of suitable admissible controller, while staying inside $\mathbb{X}$ despite any disturbances in $\mathbb{W}$ and model uncertainties in Ψ . It is mathematically defined below for all $i \in \mathcal{I}_N$:*

$$\begin{aligned} \mathcal{K}_0(\Omega) &= \Omega, \\ \mathcal{K}_i(\Omega) &= \{x_k \in \mathbb{X} : \exists u_k \in \mathbb{U} \text{ s.t. } \forall [A(k) \ B(k)] \in \Psi, \\ &\quad (A(k)x_k + B(k)u_k) \oplus \mathbb{W} \subseteq \mathcal{K}_{i-1}(\Omega)\}. \end{aligned}$$

To determine $\mathcal{K}_N(\Omega)$, the computation of the 1-step controllable set should be performed, i.e. $\mathcal{K}_i(\Omega)$ should be computed according to $\mathcal{K}_{i-1}(\Omega)$. Indeed, if one defines an intermediate set

$$\begin{aligned} S &:= \{[x^T \ u^T]^T \in \mathbb{R}^{d_x+d_u} : x \in \mathbb{X}, u \in \mathbb{U}, \\ &\quad (A_j x + B_j u) \oplus \mathbb{W} \subseteq \mathcal{K}_{i-1}(\Omega), \forall j \in \mathcal{I}_L\}, \end{aligned}$$

then $\mathcal{K}_i(\Omega)$ can be determined as the orthogonal projection of the set S defined above onto the space of x . Similar computation is repeated until $i = N$ to obtain $\mathcal{K}_N(\Omega)$. The interested reader is referred to Section 2.6 in Kerrigan (2001) for further detail. One can easily see that if Ω is empty, then so are $\mathcal{K}_i(\Omega)$ for $i \in \mathbb{N}$. Consequently, the proposed method cannot apply in this case. Also, if $\Omega \neq \emptyset$

We now present the construction of a control Lyapunov function defined over the N -step controllable set $\mathcal{K}_N(\Omega)$. For ease of presentation, let $\ell_N(x)$ denote such a control Lyapunov function. Function $\ell_N(x)$ is expected to be convex, continuous, piecewise affine. Also, the value of $\ell_N(x)$ at each vertex $v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega)$ should be strictly larger than the maximal value of $\ell_N(x)$ over $\mathcal{K}_{i-1}(\Omega)$. This requirement will be used later (more precisely (13)–(15) in Proposition 5.1) to prove the strict decrease of $\ell_N(x)$ along the trajectories outside Ω . For simplicity, one can choose the same height h_i for all $v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega)$ such that $\ell_N(v) = h_i$ for all $v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega)$. Further, function $\ell_N(x)$ should satisfy $\ell_N(x) = \ell_0(x)$ for all $x \in \Omega$. In order to find $\ell_N(x)$, we construct step by step intermediate functions $\ell_i(x)$ for $i \in \mathcal{I}_N$, defined over $\mathcal{K}_i(\Omega)$, such that these functions are convex, $\ell_i(v) = h_i$ for all $v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega)$, and $\ell_i(x) = \ell_{i-1}(x)$ for all $x \in \mathcal{K}_{i-1}(\Omega)$.

As proven in Lemma 4.2, $\ell_0(x)$ defined in (4), is a convex, continuous and piecewise affine function. For ease of presentation, let $\{\mathcal{X}_j^{(0)}\}_{j \in \mathcal{I}_{M^{(0)}}}$ denote the polytopic partition of Ω associated with $\ell_0(x)$ (the definition of a polytopic partition is referred to Nguyen et al. (2017b)) and we denote $\ell_0(x)$ as follows:

$$\ell_0(x) = (f_j^{(0)})^T x + g_j^{(0)} \text{ for } x \in \mathcal{X}_j^{(0)}, \quad (5)$$

where $f_j^{(0)} \in \mathbb{R}^{d_x}$, $g_j^{(0)} \in \mathbb{R}$, $j \in \mathcal{I}_{M^{(0)}}$ for a suitable $M^{(0)} \in \mathbb{N}_{>0}$. One can further prove that $M^{(0)}$ is equal to the number of facets of Ω . Moreover, $g_j^{(0)}$ can be proven to be 0 for all $j \in \mathcal{I}_{M^{(0)}}$.

Lemma 4.3 *Let function $\ell_0(x)$ be defined in (4) and denoted as in (5). Then, $\ell_0(x) = (f_j^{(0)})^T x$ for all $x \in \mathcal{X}_j^{(0)}$.*

The proof is referred to Subsection 8.2. This lemma shows that all the regions $\mathcal{X}_j^{(0)}$ for $j \in \mathcal{I}_{M^{(0)}}$ share a common vertex as the origin. For ease of presentation, define function:

$$\widehat{\ell}_0(x) := \max_{j \in \mathcal{I}_{M^{(0)}}} (f_j^{(0)})^T x + g_j^{(0)} \text{ for all } x \in \mathbb{R}^{d_x}. \quad (6)$$

In a general manner, the ingredients $\ell_i(x)$, $\widehat{\ell}_i(x)$, h_i , $\widehat{V}_i$, $\widehat{\Pi}_i$ for $i \in \mathcal{I}_N$ are defined step-by-step (recursively) in the sequel. Let function $\ell_{i-1}(x)$, known as a piecewise affine function over $\mathcal{K}_{i-1}(\Omega)$, be given in the following form: $\ell_{i-1}(x) = (f_j^{(i-1)})^T x + g_j^{(i-1)}$ for $x \in \mathcal{X}_j^{(i-1)}$, where $\{\mathcal{X}_j^{(i-1)}\}_{j \in \mathcal{I}_{M^{(i-1)}}}$ denotes the polytopic partition of $\mathcal{K}_{i-1}(\Omega)$ associated with $\ell_{i-1}(x)$ and $M^{(i-1)} \in \mathbb{N}_{>0}$. As $\ell_{i-1}(x)$ is restrictively defined over $\mathcal{K}_{i-1}(\Omega)$, we define the following function

$$\widehat{\ell}_{i-1}(x) := \max_{j \in \mathcal{I}_{M^{(i-1)}}} (f_j^{(i-1)})^T x + g_j^{(i-1)}, \forall x \in \mathbb{R}^{d_x}. \quad (7)$$

With respect to a given scalar constant $\epsilon > 0$, the function $\ell_i(x)$ over $\mathcal{K}_i(\Omega)$ is constructed as follows:

$$\begin{aligned} \ell_i(x) &:= \arg \min_z z \text{ s.t. } [x^T \ z]^T \in \widehat{\Pi}_i \\ \widehat{\Pi}_i &:= \text{conv}(\widehat{V}_i) \\ \widehat{V}_i &:= \{[v^T \ h_i]^T : v \in \mathcal{V}(\mathcal{K}_i(\Omega))\} \cup \widehat{V}_{i-1} \\ h_i &:= \min_h h \text{ s.t. } \widehat{\ell}_{i-1}(v) + \epsilon \leq h, \forall v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega). \end{aligned} \quad (8)$$

The constructions (7) and (8) are iteratively repeated until $i = N$. Recall that $\ell_i(x)$ at each vertex $v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega)$ is required to be strictly larger than the maximal value of $\ell_{i-1}(x)$ over $\mathcal{K}_{i-1}(\Omega)$, this requirement is enforced by the constraints

$$h_i \geq \widehat{\ell}_{i-1}(v) + \epsilon \text{ for all } v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega),$$

with a suitable $\epsilon > 0$. The proof for $h_i > \max_{x \in \mathcal{K}_{i-1}(\Omega)} \ell_{i-1}(x)$ will be shown later in Lemma 4.6. From the geometric viewpoint, the insertion of a constant $\epsilon > 0$ in (8) is to ensure that all the vertices $v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega)$ are lifted onto a level above all the hyperplanes composing $\widehat{\ell}_{i-1}(x)$.

To illustrate the construction (8), consider a simple system $x_{k+1} = x_k + u_k + w_k$, where $x_k \in [-10, 10]$, $u_k \in [-0.5, 0.5]$, $w_k \in [-0.1, 0.1]$. It can easily be observed that a robust positively invariant set is $\Omega = [-0.2, 0.2]$ with the local controller $u_k = -0.5x_k$. Accordingly, one determines $\mathcal{K}_1(\Omega) = [-0.6, 0.6]$, $\mathcal{K}_2(\Omega) = [-1, 1]$. As shown in Fig. 2, we choose $h_0 = 0.1$, $\epsilon = 0.1$; polytope $\widehat{\Pi}_0$ is thus computed by the convex hull of $[0 \ 0]^T$, $[-0.2 \ 0.1]^T$, $[0.2 \ 0.1]^T$ and $\ell_0(x)$ is represented by two blue segments above the interval $[-0.2, 0.2]$. Subsequently, the construction (8) results in $h_1 = 0.4$ and the vertices of $\widehat{\Pi}_1$ consist of $[0 \ 0]^T$, $[-0.2 \ 0.1]^T$, $[0.2 \ 0.1]^T$, $[-0.6 \ 0.4]^T$, $[0.6 \ 0.4]^T$. Similarly, (8) returns $h_2 = 0.8$, function $\ell_2(x)$ is thus composed of the colored segments above the x -axis, the blue ones correspond to $z = \pm 0.5x$, while the brown segments are represented by $z = \pm 0.75x - 0.05$ and the green segments are described by $z = \pm x - 0.2$.

Some essential properties of functions $\widehat{\ell}_i(x)$, $\ell_i(x)$ are presented in the sequel.

Lemma 4.4 *Functions $\ell_i(x)$, defined in (8), are convex, continuous, piecewise affine.*

PROOF. The proof follows the same arguments of claim 1) in Lemma 4.2. $\square$

Lemma 4.5 *Functions $\widehat{\ell}_i(x)$, defined in (7), satisfy $\widehat{\ell}_i(x) = \ell_i(x)$ for all $x \in \mathcal{K}_i(\Omega)$ and any $i \in \{0\} \cup \mathcal{I}_{N-1}$.*

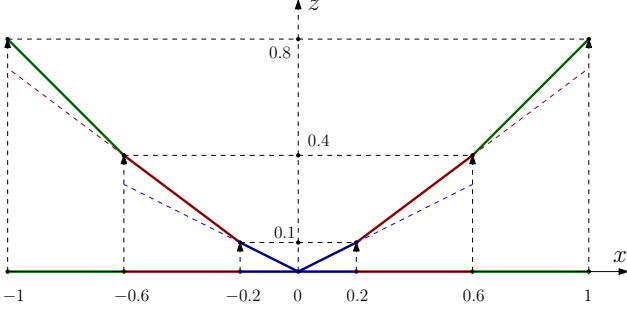


Fig. 2. Illustration for the construction of $\ell_N(x)$.

PROOF. As a convex, piecewise affine function over Ω , $\ell_0(x)$ can also be represented by:

$$\ell_0(x) = \max_{j \in \mathcal{I}_{M(0)}} (f_j^{(0)})^T x + g_j^{(0)} \quad \text{for all } x \in \Omega. \quad (9)$$

Therefore, the definition of $\widehat{\ell}_0(x)$ in (6) satisfies $\widehat{\ell}_0(x) = \ell_0(x)$ for any $x \in \Omega$. Following similar argument and using Lemma 4.4 lead to $\widehat{\ell}_i(x) = \ell_i(x)$ for all $x \in \mathcal{K}_i(\Omega)$. $\square$

We now prove that the functions $\ell_i(x)$, defined in (8), satisfy the aforementioned requirements. This is formally stated in the following result whose proof is presented in Subsection 8.3 for reading ease.

Lemma 4.6 Functions $\ell_i(x)$ for $i \in \mathcal{I}_N$ satisfy:

- (1) $0 \leq \ell_i(x) \leq h_i$ for all $x \in \mathcal{K}_i(\Omega)$;
- (2) $h_N > \dots > h_i > \dots > h_0 > 0$;
- (3) $\ell_i(x) = h_i$ for all $x \in \mathcal{V}(\mathcal{K}_i(\Omega)) \setminus \mathcal{K}_{i-1}(\Omega)$;
- (4) $\ell_i(x) = \ell_{i-1}(x)$ for all $x \in \mathcal{K}_{i-1}(\Omega)$;
- (5) $\ell_i(x) = h_j$ for $x \in \mathcal{V}(\mathcal{K}_j(\Omega)) \setminus \mathcal{K}_{j-1}(\Omega)$, $\forall j \in \mathcal{I}_i$;
- (6) $\ell_i(x) > 0$ for all $x \in \mathcal{K}_i(\Omega) \setminus \{0\}$;
- (7) $\ell_i(\beta x) \leq \beta \ell_i(x)$ for all $x \in \mathcal{K}_i(\Omega)$ and $0 \leq \beta \leq 1$.

It can also be proven that such a function $\ell_i(x)$ represents a convex lifting for the associated polytopic partition $\{\mathcal{X}_j^{(i)}\}_{j \in \mathcal{I}_{M(i)}}$ of $\mathcal{K}_i(\Omega)$. The interested reader is referred to Nguyen (2015); Nguyen et al. (2015a,b) for a precise definition of convex lifting and its existence conditions. For simplicity, we skip this analogy in this paper and stress that those existence conditions are fulfilled in the present context.

Lemma 4.7 Function $\ell_i(x) : \mathcal{K}_i(\Omega) \rightarrow \mathbb{R}$ represents a convex lifting of the polytopic partition $\{\mathcal{X}_j^{(i)}\}_{j \in \mathcal{I}_{M(i)}}$ of $\mathcal{K}_i(\Omega)$ for each $i \in \mathcal{I}_N$.

PROOF. The proof is referred to Lemma II.8 in Nguyen et al. (2017a) or Lemma 4.7 in Nguyen et al. (2017c). $\square$

Remark 4.2 From the mathematical point of view, the value of ϵ in (8) can be arbitrarily chosen as long as it is positive.

However, the value of h_N can be relatively large in some cases, since a subset of the vertices of $\mathcal{K}_N(\Omega)$ and $\mathcal{K}_{N-1}(\Omega)$ may be very close to each other, when N becomes larger. This end may present numerical sensitivity in determining $\ell_N(x)$, as the precision of convex hull operation is known to be limited, see Avis and Fukuda (1992). Therefore, ϵ in (8) should be chosen small enough to reduce the value of h_N and large enough to ensure the precision of the convex hull computation. A value in interval $[10^{-5}, 1]$ is basically a good choice.

5 Control design procedure

The construction and prominent properties of a family of control Lyapunov candidates have been presented in the preceding section. In this section, we need to prove that they satisfy the conditions of control Lyapunov functions. Before proceeding the proof, we recall a definition of a local input-to-state stability (ISS) Lyapunov function from Jiang and Wang (2001); Khalil (2002):

Definition 5.1 Consider system (1) subject to constraints (2), (3), the feasible region $\mathcal{X} \subseteq \mathbb{X}$ and a control law $u = \kappa(x) \in \mathbb{U}$. A continuous function $V(x) : \mathcal{X} \rightarrow \mathbb{R}_+$ is called a local ISS-Lyapunov function, if the following conditions hold

- $\chi_1(\|x\|) \leq V(x) \leq \chi_2(\|x\|)$ for all $x \in \mathcal{X}$, and
- $V(Ax + B\kappa(x) + w) - V(x) \leq -\chi_3(\|x\|) + \sigma(\|\kappa(x)\|)$ for all $x \in \mathcal{X}$ and time-varying $[A \ B] \in \Psi$, $w \in \mathbb{W}$,

where $\chi_1(\cdot), \chi_2(\cdot), \chi_3(\cdot), \sigma(\cdot)$ are class $\mathcal{K}$ -functions and $\|\cdot\|$ denotes any vector norm.

Note that as $\ell_N(x)$, constructed in the preceding section, is a convex, continuous, piecewise affine function defined over a bounded set $\mathcal{K}_N(\Omega)$, then there always exist two suitable class $\mathcal{K}$ -functions satisfying the first condition in Definition 5.1. This observation is formally stated in the following result.

Lemma 5.1 Given a function $\ell_N(x)$ constructed as in (8), then there exist two constant scalars $c_2 \geq c_1 > 0$ such that

$$c_1 \|x\|_\infty \leq \ell_N(x) \leq c_2 \|x\|_\infty \quad \text{for all } x \in \mathcal{K}_N(\Omega). \quad (10)$$

The proof is presented in Subsection 8.4. Lemma 5.1 shows that $\ell_N(x)$ satisfies the first condition in Definition 5.1. Accordingly, in order to prove $\ell_N(x)$ to be a control Lyapunov function, one needs to show that there exists a controller $u = \kappa(x) \in \mathbb{U}$ for all $x \in \mathcal{K}_N(\Omega)$ such that $\ell_N(x)$ satisfies the second condition in Definition 5.1. Such a controller can be determined by means of Algorithm 1. The main idea of (11) is to minimize the worst case of function $\ell_N(x_{k+1})$ despite any $[A(k) \ B(k)] \in \Psi$ and $w_k \in \mathbb{W}$, by means of minimizing an auxiliary variable $\gamma \geq 0$. Also, constraints

Algorithm 1 Control design procedure

Input: $\ell_N(x)$, $\mathcal{K}_N(\Omega)$.

Output: optimal control action $u^*(x_k)$ at each instant k .

- 1: Compute $\ell_N(x_k)$
- 2: Solve the following problem:

$$\begin{aligned} [\gamma^* (u_k^*)^T]^T &= \arg \min_{\gamma, u_k} \gamma \\ \text{s.t. } \ell_N(A_i x_k + B_i u_k + w) &\leq \gamma \ell_N(x_k), \\ A_i x_k + B_i u_k + w &\in \mathcal{K}_N(\Omega), \\ u_k &\in \mathbb{U}, \gamma \geq 0, \forall i \in \mathcal{I}_L, \forall w \in \mathcal{V}(\mathbb{W}). \end{aligned} \quad (11)$$

- 3: $u^*(x_k) = u_k^*$.
 - 4: $k \leftarrow k + 1$. Return to step 1.
-

$A_i x_k + B_i u_k + w \in \mathcal{K}_N(\Omega)$ for all $i \in \mathcal{I}_L$ and $w \in \mathcal{V}(\mathbb{W})$ are aimed at ensuring that function $\ell_N(x)$ is exclusively defined over $\mathcal{K}_N(\Omega)$, i.e. x_{k+1} stays inside $\mathcal{K}_N(\Omega)$ for all realizations of disturbances and uncertainties. These constraints can be removed if the set $\Xi := \{x \in \mathbb{R}^{d_x} : \ell_N(x) \leq h_N\}$ and $\mathcal{K}_N(\Omega)$ are identical.

Next, we concentrate on the proof that the convex lifting $\ell_N(x)$ and the controller designed in Algorithm 1 satisfy the second condition in Definition 5.1. More precisely, we prove that $\ell_N(x_k)$ is strictly decreasing along the trajectories outside Ω . This is formally stated in the following proposition.

Proposition 5.1 Consider function $\ell_N(x)$ defined in (8) and the controller designed in Algorithm 1, then for all $[A(k) \ B(k)] \in \Psi$ and $w_k \in \mathbb{W}$, any $x_k \in \mathcal{K}_N(\Omega) \setminus \Omega$ satisfies:

$$\ell_N(A(k)x_k + B(k)u^*(x_k) + w_k) < \ell_N(x_k).$$

PROOF. Consider any $x \in \mathcal{K}_N(\Omega)$, we define

$$i(x) := \arg \min_{j \in \{0\} \cup \mathcal{I}_N} j \quad \text{s.t. } x \in \mathcal{K}_j(\Omega). \quad (12)$$

It can be seen that the definition of the index $i(x)$, for each $x \in \mathcal{K}_N(\Omega)$, is unique. Consider now any point $v \in \bigcup_{j \in \mathcal{I}_N} \mathcal{V}(\mathcal{K}_j(\Omega)) \setminus \Omega$; according to definition (12) and claim 5) of Lemma 4.6, it yields

$$\ell_N(v) = h_{i(v)}. \quad (13)$$

As $v \in \mathcal{K}_{i(v)}(\Omega)$, there exists a controller, denoted by $u(v) \in \mathbb{U}$, such that $A(k)v + B(k)u(v) + w_k \in \mathcal{K}_{i(v)-1}(\Omega)$ for all $[A(k) \ B(k)] \in \Psi$ and all $w_k \in \mathbb{W}$ according to its definition. Consequently, claim 1) of Lemma 4.6 leads to:

$$\ell_N(A(k)v + B(k)u(v) + w_k) \leq h_{i(v)-1}. \quad (14)$$

Inclusions (13), (14) and claim 2) of Lemma 4.6 yield:

$$\ell_N(A(k)v + B(k)u(v) + w_k) < \ell_N(v), \quad (15)$$

for all $[A(k) \ B(k)] \in \Psi$ and all $w_k \in \mathbb{W}$. Otherwise, if $v \in \mathcal{V}(\Omega)$, there exists a control law $u(v) \in \mathbb{U}$ such that $A(k)v + B(k)u(v) + w_k \in \Omega$, since Ω is robust positively invariant. Accordingly, we obtain

$$\ell_N(A(k)v + B(k)u(v) + w_k) \leq h_0 = \ell_N(v). \quad (16)$$

Note that for any point $x_k \in \mathcal{K}_N(\Omega) \setminus \Omega$, there exists a region $\mathcal{X}_j^{(N)}$ in the polytopic partition $\{\mathcal{X}_j^{(N)}\}_{j \in \mathcal{I}_{M(N)}}$, associated with $\ell_N(x)$, such that $x_k \in \mathcal{X}_j^{(N)}$. Accordingly, x_k can be written in the following form: $x_k = \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v)v$, $\alpha(v) \geq 0$, $\sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) = 1$. As $x_k \in \mathcal{K}_N(\Omega) \setminus \Omega$, then there exists at least one vertex $v \in \mathcal{V}(\mathcal{X}_j^{(N)}) \setminus \Omega$ such that $\alpha(v) > 0$. Accordingly, it yields:

$$\ell_N(x_k) = \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) \ell_N(v) \quad (17a)$$

$$> \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) \ell_N(A(k)v + B(k)u(v) + w_k) \quad (17b)$$

$$\geq \ell_N(A(k)x_k + B(k) \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v)u(v) + w_k) \quad (17c)$$

$$\geq \ell_N(A(k)x_k + B(k)u^*(x_k) + w_k). \quad (17d)$$

Note that inclusion (17a) is obtained because function $\ell_N(x)$ is affine over $\mathcal{X}_j^{(N)}$. Meanwhile, inclusion (17b) is obtained due to inclusions (15) and (16). Also, inequality (17c) is induced from the convexity of $\ell_N(x)$, as proven in Lemma 4.4. Finally, inclusion (17d) is guaranteed since $\mathbb{U}$ is convex and $\sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v)u(v) \in \mathbb{U}$. The proof is complete. $\square$

Proposition 5.2 Consider function $\ell_N(x)$ defined in (8) and the controller designed in Algorithm 1, then for all $[A(k) \ B(k)] \in \Psi$ and $w_k \in \mathbb{W}$, any point $x_k \in \Omega$ satisfies

$$\ell_N(A(k)x_k + B(k)u^*(x_k) + w_k) \leq \ell_N(x_k) + \ell_N(w_k).$$

PROOF. As $x_k \in \Omega$, there exists a region $\mathcal{X}_j^{(0)}$ in the polytopic partition $\{\mathcal{X}_j^{(0)}\}_{j \in \mathcal{I}_{M(0)}}$ of Ω , associated with $\ell_0(x)$, such that $x_k \in \mathcal{X}_j^{(0)}$. Such a region $\mathcal{X}_j^{(0)}$ has its vertices as vertices of Ω and the origin. Therefore, x_k can be written in the following form: $x_k = \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(0)})} \alpha(v)v$, $\alpha(v) \geq 0$, $\sum_{v \in \mathcal{V}(\mathcal{X}_j^{(0)})} \alpha(v) = 1$. Also, since Ω represents a robust positively invariant set, then for any $v \in \mathcal{V}(\Omega)$, there exists a control law $u(v) \in \mathbb{U}$ such that $A(k)v + B(k)u(v) + w_k \in \Omega$ for all $[A(k) \ B(k)] \in \Psi$ and $w_k \in \mathbb{W}$. In other words, in-

clusion (16) holds at $v \in \mathcal{V}(\Omega)$. Accordingly, we obtain

$$\begin{aligned}\ell_N(x_k) &= \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(0)})} \alpha(v) \ell_N(v) \\ &\geq \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(0)}) \setminus \{0\}} \alpha(v) \ell_N(A(k)v + B(k)u(v) + w_k).\end{aligned}\quad (18)$$

If we choose $u(0) = 0$, then inclusion (18) yields

$$\begin{aligned}\ell_N(x_k) + \ell_N(w_k) &\geq \ell_N(x_k) + \alpha(0) \ell_N(w_k) \geq \\ &\sum_{v \in \mathcal{V}(\mathcal{X}_j^{(0)})} \alpha(v) \ell_N(A(k)v + B(k)u(v) + w_k).\end{aligned}\quad (19)$$

The convexity of $\ell_N(x)$ also leads to

$$\begin{aligned}&\sum_{v \in \mathcal{V}(\mathcal{X}_j^{(0)})} \alpha(v) \ell_N(A(k)v + B(k)u(v) + w_k) \\ &\geq \ell_N(A(k)x_k + B(k) \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(0)})} \alpha(v) u(v) + w_k)\end{aligned}\quad (20a)$$

$$\geq \ell_N(A(k)x_k + B(k)u^*(x_k) + w_k).\quad (20b)$$

Again, inclusion (20b) follows the convexity of $\mathbb{U}$ and $\sum_{v \in \mathcal{V}(\mathcal{X}_j^{(0)})} \alpha(v) u(v) \in \mathbb{U}$. Accordingly, incorporating inclusions (19) and (20b) yields

$$\ell_N(x_k) + \ell_N(w_k) \geq \ell_N(A(k)x_k + B(k)u^*(x_k) + w_k),$$

for all $[A(k) \ B(k)] \in \Psi$ and $w_k \in \mathbb{W}$. $\square$

Remark 5.1 Note that the choice of Ω is arbitrary as long as its polytopic topology and robust positively invariant properties are guaranteed. The minimal and maximal robust positively invariant sets, e.g. in Kolmanovsky and Gilbert (1998) are preferable.

The main result related to control design of the paper is formally stated in the following theorem:

Theorem 5.3 *Given system (1) subject to uncertainty (3) and constraint (2), the controller designed in Algorithm 1 guarantees the recursive feasibility and input-to-state stability.*

PROOF. The recursive feasibility is guaranteed by sufficiently large $\gamma > 0$ according to Propositions 5.1 and 5.2. Also, $\ell_N(x)$, constructed in (8), satisfies the conditions of an ISS Lyapunov function according to Definition 5.1. Therefore, input-to-state stability is proven. More intuitively, Proposition 5.1 shows that $\{\ell_N(x_k)\}_{k=0}^\infty$ outside Ω is strictly decreasing and bounded in $[h_0, h_N]$, therefore $\lim_{k \rightarrow \infty} \ell_N(x_k) = h_0$, consequently $\lim_{k \rightarrow \infty} \rho_\Omega(x_k) = 0$. The proof is complete. $\square$

Remark 5.2 It is worth stressing that neither finite cost function usually employed in model predictive control Mayne et al. (2000), nor the control Lyapunov function computed in Blanchini (1994) can guarantee their strict decrease outside a given robust positively invariant set Ω like function $\ell_N(x)$ as proven in Proposition 5.1.

Remark 5.3 Although the construction of $\mathcal{K}_N(\Omega)$ is much cheaper than the one of the maximal λ -contractive set, since the latter one is computed mainly relying on iterative procedures, see Blanchini (1994); Kerrigan (2001), the construction of $\ell_N(x)$ is however performed at the price of solving multiple parametric linear programming problems, see among the others Grancharova and Johansen (2012). Also, since the proposed construction relies on the vertex representation, it is thus limited to small-dimensional systems.

Remark 5.4 The number of regions in the polytopic partition associated with $\ell_N(x)$ may be relatively large in comparison to the methods in Blanchini (1994); Nguyen et al. (2017c), because its construction relies on combinations of the vertices of the sets $\mathcal{K}_i(\Omega)$ for $i \in \{0\} \cup \mathcal{I}_N$. This may thus slow down the online computation. However, since this problem is a linear program, it is not expensive.

6 Numerical example

This section aims to illustrate the above construction via a numerical example. To this end, we consider an angular antenna positioning system, presented in Kothare et al. (1996),

$$x_{k+1} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\beta_k \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u_k, \quad (21)$$

where the uncertain parameter β_k ranges in $[0.1 \ 10]$. Also, the state, control variables are subject to the following constraints:

$$\|x_k\|_\infty \leq 1, \ \|u_k\|_\infty \leq 2. \quad (22)$$

A stabilizing local controller $u = [-3.9922 \ -6.5135]x$ is chosen to compute Ω as the maximal output admissible set. Accordingly, Ω and the 10-step controllable set are presented in Fig. 3. In this example, we choose $h_0 = 0.1$ and $\epsilon = 10^{-4}$. The constructed control Lyapunov function is shown in Fig. 4, where the strict decrease of its value along the closed-loop dynamics outside the robust positively invariant set Ω is also proven with the controller designed by Algorithm 1. The closed-loop stability in the sense of Lyapunov is more clearly illustrated in Fig. 3. Also, the design problem (11) is composed of 887 constraints, as $\ell_N(x)$ associates with a partition of 402 regions. Note however that the method presented in Blanchini (1994) exploiting the maximal 0.999-contractive set for the same numerical example can formulate a control problem with 40 constraints, while the method in Nguyen et al. (2017c) with the same 0.999-contractive set requires solving a problem consisting of 61 constraints.

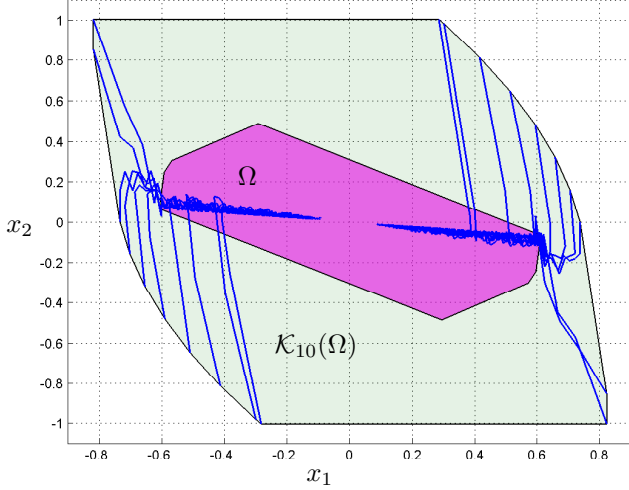


Fig. 3. Maximal output admissible set Ω , its 10-step controllable set $\mathcal{K}_{10}(\Omega)$ and the closed-loop dynamics.

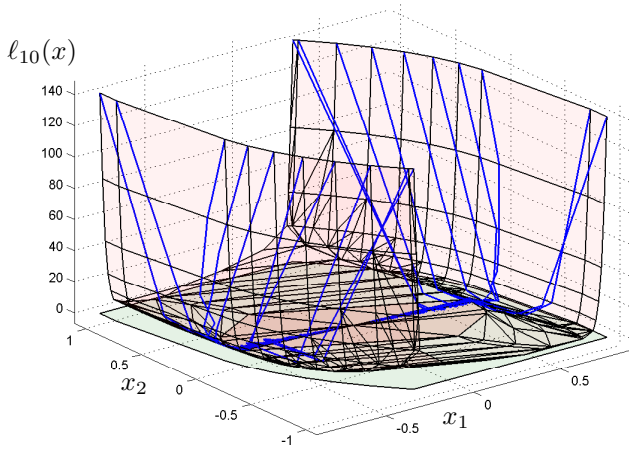


Fig. 4. The constructed control Lyapunov function with $h_0 = 0.1$, $\epsilon = 10^{-4}$ and its strict decrease along the closed-loop dynamics outside Ω .

We remark that Fig. 5 represents the 1-step controllable set $\mathcal{K}_1(\Omega)$, which is not contractive since two vertices of Ω , surrounded by two red circles, lie on the boundary of $\mathcal{K}_1(\Omega)$. This implies that for some points on the boundary of $\mathcal{K}_1(\Omega)$, there does not exist a controller satisfying the given constraints such that their next state stay in the interior of $\mathcal{K}_1(\Omega)$. Also, to illustrate more clearly control Lyapunov function $\ell_{10}(x)$, we show in Fig. 6 function $\ell_1(x)$ defined over set $\mathcal{K}_1(\Omega)$. Finally, the computation of $\ell_N(x)$ is performed in the environment of MPT 3.0 Herceg et al. (2013).

7 Conclusions

This paper presented a novel method to construct a family of piecewise affine control Lyapunov functions. The most remarkable property of this construction is that it does not require the contractivity of the domain over which these control Lyapunov functions are defined. Accordingly, this

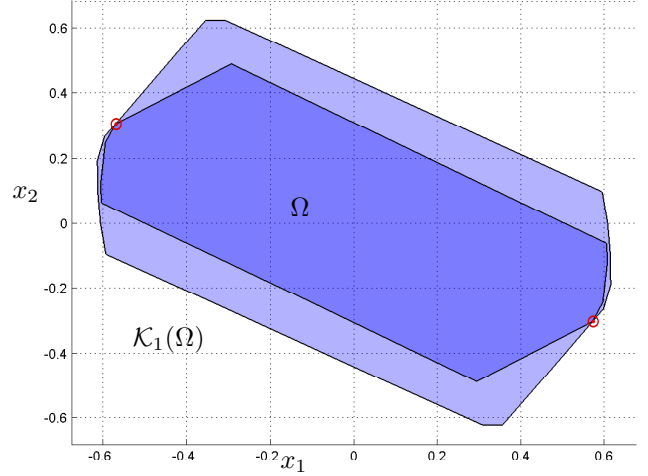


Fig. 5. $\mathcal{K}_1(\Omega)$ is not contractive.

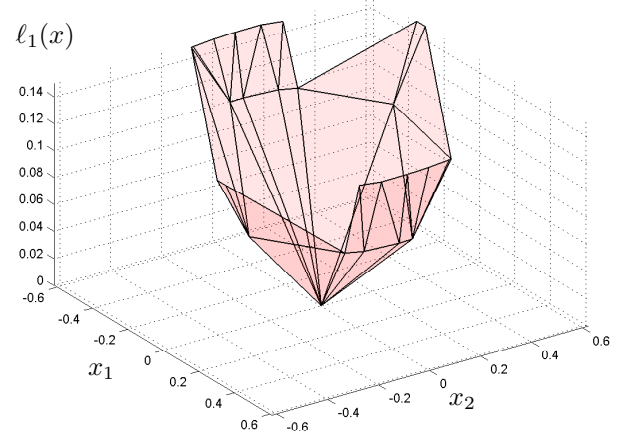


Fig. 6. Function $\ell_1(x)$ defined over $\mathcal{K}_1(\Omega)$.

construction was shown to lead to a simple robust control design procedure as a linear programming problem. The construction was finally illustrated via a numerical example.

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8 Appendix

8.1 Proof of Lemma 4.2

For claim 1), $\ell_0(x)$ is a piecewise affine function, since it is resulted from a pLP problem. Also, the convexity and continuity of $\ell_0(x)$ follow as direct consequences of Theorems 4.1 and 4.2. For claim 2), it is observed that any point $[x^T z]^T \in \hat{\Pi}_0$ can be expressed as a convex combination

of the points in $\widehat{V}_0$, defined in (4), i.e.,

$$\begin{aligned} [x^T \ z]^T &= \alpha(0)0 + \sum_{v \in \mathcal{V}(\Omega)} \alpha(v) [v^T \ h_0]^T \\ \alpha(0), \alpha(v) &\geq 0, \alpha(0) + \sum_{v \in \mathcal{V}(\Omega)} \alpha(v) = 1. \end{aligned} \quad (23)$$

According to inclusion (23), claim 2) is easily deduced by the argument: $0 \leq z = \sum_{v \in \mathcal{V}(\Omega)} \alpha(v)h_0 \leq h_0$. Also, $z = 0$ holds only if $\alpha(v) = 0$ for all $v \in \mathcal{V}(\Omega)$ and $\alpha(0) = 1$, leading to claim 3).

For claim 4), one can see that any $x \in \mathcal{V}(\Omega)$ can be expressed as in (23) only if $\alpha(v) = 0$ for $v \in \mathcal{V}(\Omega) \setminus \{x\}$ and $\alpha(v) = 1$ for $v = x$, since x represents a vertex of Ω . In other words, $\ell_0(x) = z = h_0$ for $x \in \mathcal{V}(\Omega)$. To prove claim 5), we note that any point $x \in \partial\Omega$ can be written in form (23) only if $\alpha(0) = 0$. This end leads to $\ell_0(x) = z = h_0$.

Finally, consider any point $x \in \text{int}(\Omega)$, then there exists a point $y(x) \in \partial\Omega$ and $0 \leq \beta < 1$ such that $x = \beta y(x)$. Accordingly, the convexity of $\ell_0(x)$ yields: $\ell_0(x) = \ell_0(\beta y(x)) \leq \beta \ell_0(y(x)) + (1 - \beta)\ell_0(0) = \beta h_0 < h_0$. This end completes the proof of claim 6). $\square$

8.2 Proof of Lemma 4.3

Consider any point $[x^T \ z]^T \in \widehat{\Pi}_0$ such that $x \in \Omega \setminus \{0\}$, this augmented point is represented by a convex combination of the points of $\widehat{V}_0$ as shown in (23). Accordingly, one obtains $z = (1 - \alpha(0))h_0$. It can be seen that the minimal value $\ell_0(x)$ of z holds at the maximal value of $\alpha(0)$, denoted by $\alpha^*(0)$. Since $x \in \Omega \setminus \{0\}$, then $\alpha^*(0) < 1$. Let $\alpha^*(v)$ denote the optimal value of $\alpha(v)$ such that $z = \ell_0(x)$, we will prove that $x/(1 - \alpha^*(0)) \in \partial\Omega$. Indeed, one can see that $\ell_0(x) = (1 - \alpha^*(0))h_0$. Note also that $\sum_{v \in \mathcal{V}(\Omega)} \alpha^*(v) = 1 - \alpha^*(0)$, leading to $x/(1 - \alpha^*(0)) = (1 - \alpha^*(0))^{-1} \sum_{v \in \mathcal{V}(\Omega)} \alpha^*(v)v \in \Omega$. Therefore, the convexity of $\ell_0(x)$ yields

$$h_0 = \ell_0(x)/(1 - \alpha^*(0)) \leq \ell_0(x/(1 - \alpha^*(0))) \leq h_0.$$

According to Lemma 4.2, $x/(1 - \alpha^*(0)) \in \partial\Omega$. Consequently, there exists a facet of Ω denoted by F such that $x/(1 - \alpha^*(0)) \in F$. Geometrically, the point $x/(1 - \alpha^*(0))$ can be obtained by the intersection between $\partial\Omega$ and the line going through $0, x$ such that x lies between 0 and $x/(1 - \alpha^*(0))$. Also, $[x^T \ \ell_0(x)]^T$ can be expressed by

$$\begin{aligned} \begin{bmatrix} x \\ \ell_0(x) \end{bmatrix} &= \alpha^*(0)0 + (1 - \alpha^*(0)) \sum_{v \in \mathcal{V}(F)} \beta(v) \begin{bmatrix} v \\ \ell_0(v) \end{bmatrix} \\ \beta(v) &\geq 0, \sum_{v \in \mathcal{V}(F)} \beta(v) = 1. \end{aligned} \quad (24)$$

Similarly, for any $x \in \text{conv}(\{0\} \cup \mathcal{V}(F))$, the point $[x^T \ \ell_0(x)]^T$ is described as in (24). In other words, $\text{conv}(\{0\} \cup \mathcal{V}(F))$ represents a region in the polytopic partition $\{\mathcal{X}_j^{(0)}\}_{j \in \mathcal{I}_{M^{(0)}}}$ of Ω associated with $\ell_0(x)$. Furthermore, the regions in this partition share a common point 0 . Also, $\ell_0(0) = 0$ according to claim 3) of Lemma 4.2, it yields $g_j^{(0)} = 0$ for all $j \in \mathcal{I}_{M^{(0)}}$. $\square$

8.3 Proof of Lemma 4.6

First, we will prove claims 1), 2) and 3) simultaneously. Obviously, $\ell_0(x) = h_0$ for $x \in \mathcal{V}(\Omega)$, we need to prove that $\ell_1(x) = h_1$ for all $x \in \mathcal{V}(\mathcal{K}_1(\Omega)) \setminus \Omega$, $0 \leq \ell_1(x) \leq h_1$ and $h_1 > h_0$. Indeed, any point $x \in \mathcal{K}_1(\Omega)$ is written as a convex combination of the vertices of $\mathcal{K}_1(\Omega)$, i.e.

$$x = \sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega))} \alpha(v)v, \alpha(v) \geq 0, \sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega))} \alpha(v) = 1.$$

According to the definition of h_1 in (8), we obtain

$$\begin{aligned} &\sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega)) \setminus \Omega} \alpha(v)h_1 + \sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega)) \cap \Omega} \alpha(v)h_0 \\ &\geq \sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega)) \setminus \Omega} \alpha(v)\epsilon + \sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega))} \alpha(v)\widehat{\ell}_0(v) \quad (25) \\ &\geq \sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega)) \setminus \Omega} \alpha(v)\epsilon + \widehat{\ell}_0(x). \end{aligned}$$

Note that the last inequality in inclusion (25) is obtained due to the convexity of function $\widehat{\ell}_0(x)$ over $\mathbb{R}^{d_x}$. Since this inclusion holds true for all $x \in \mathcal{K}_1(\Omega)$, thus if one chooses $x \in \mathcal{V}(\Omega)$ such that there exists at least one $v \in \mathcal{V}(\mathcal{K}_1(\Omega)) \setminus \Omega$, $\alpha(v) > 0$, then it yields $h_1 \geq h_0 + \epsilon > h_0$. This end leads to $0 \leq \ell_1(x) \leq h_1$ according to the definition of $\widehat{\Pi}_1$, $\ell_1(x)$ in (8) and similar argument as in the proof for claim 2) of Lemma 4.2.

To prove that $\ell_1(x) = h_1$ for all $x \in \mathcal{V}(\mathcal{K}_1(\Omega)) \setminus \Omega$, it can easily be seen that $\ell_1(x)$ is a convex function over $\mathcal{K}_1(\Omega)$ as proven in Lemma 4.4, then it attains its maximal value h_1 at its vertices. Note however that for the points $x \in \mathcal{V}(\mathcal{K}_1(\Omega)) \cap \Omega$, $\ell_1(x) \leq h_0 < h_1$. Therefore, $\ell_1(x) = h_1$ for $x \in \mathcal{V}(\mathcal{K}_1(\Omega)) \setminus \Omega$. For the other $i \in \mathcal{I}_N$, the proof follows similar arguments.

To prove claim 4), first we prove it holds with $i = 1$. Indeed, consider any point $[x^T \ z]^T \in \widehat{\Pi}_1$; this point can be expressed as a convex combination of the points in $\widehat{V}_1$ as:

$$\begin{aligned} [x^T \ z]^T &= \beta(0)0 + \sum_{j=0}^1 \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) [v^T \ h_j]^T \\ \beta(0), \alpha_j(v) &\geq 0, \beta(0) + \sum_{j=0}^1 \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) = 1. \end{aligned} \quad (26)$$

The above inclusion yields

$$\begin{aligned}
z &= \sum_{j=0}^1 \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) h_j \\
&\geq \sum_{v \in \mathcal{V}(\Omega)} \alpha_0(v) \widehat{\ell}_0(v) + \sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega))} \alpha_1(v) (\widehat{\ell}_0(v) + \epsilon) \\
&\geq \widehat{\ell}_0(x) + \sum_{v \in \mathcal{V}(\mathcal{K}_1(\Omega))} \alpha_1(v) \epsilon \geq \widehat{\ell}_0(x). \tag{27}
\end{aligned}$$

The equality in (27) holds only if $\alpha_1(v) = 0$ for all $v \in \mathcal{V}(\mathcal{K}_1(\Omega))$, leading to $x \in \Omega$. Suppose claim 4) holds true until $i-1$, we will prove it holds for i . In fact, claim 4) holding true until $i-1$ leads to $\ell_j(x) = \ell_{j-1}(x)$ for $x \in \mathcal{K}_{j-1}(\Omega)$, $j \in \mathcal{I}_{i-1}$. Accordingly, consider any point $[x^T \ z]^T \in \widehat{\Pi}_i$, then it can be written in the following form:

$$\begin{aligned}
[x^T \ z]^T &= \beta(0)0 + \sum_{j=0}^i \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) [v^T \ h_j]^T \\
\beta(0), \alpha_j(v) &\geq 0, \beta(0) + \sum_{j=0}^i \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) = 1. \tag{28}
\end{aligned}$$

Note that each $v \in \mathcal{V}(\mathcal{K}_j(\Omega))$ satisfies $h_j \geq \ell_j(v) = \ell_{i-1}(v)$ for $j \in \{0\} \cup \mathcal{I}_{i-1}$, therefore inclusion (28) leads to

$$\begin{aligned}
z &= \sum_{j=0}^i \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) h_j \\
&\geq \sum_{j=0}^{i-1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) \ell_{i-1}(v) + \sum_{v \in \mathcal{V}(\mathcal{K}_i(\Omega))} \alpha_i(v) h_i. \tag{29}
\end{aligned}$$

According to Lemma 4.5 and definition (8), we have:

$$\begin{aligned}
\widehat{\ell}_{i-1}(v) &= \ell_{i-1}(v) \text{ for all } v \in \mathcal{K}_{i-1}(\Omega), \\
\widehat{\ell}_{i-1}(v) + \epsilon &\leq h_i \text{ for all } v \in \mathcal{V}(\mathcal{K}_i(\Omega)). \tag{30}
\end{aligned}$$

Recall that proving $h_i \geq h_{i-1} + \epsilon$ follows similar arguments as showing $h_1 \geq h_0 + \epsilon$. According to claim 1), we obtain $h_i \geq \ell_{i-1}(v) + \epsilon = \widehat{\ell}_{i-1}(v) + \epsilon$ for $v \in \mathcal{V}(\mathcal{K}_i(\Omega)) \cap \mathcal{K}_{i-1}(\Omega)$. Therefore, inclusions (29) and (30) yield:

$$\begin{aligned}
z &\geq \sum_{j=0}^{i-1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) \widehat{\ell}_{i-1}(v) \\
&\quad + \sum_{v \in \mathcal{V}(\mathcal{K}_i(\Omega))} \alpha_i(v) (\widehat{\ell}_{i-1}(v) + \epsilon). \tag{31}
\end{aligned}$$

As $\widehat{\ell}_{i-1}(x)$ is a convex function, it satisfies:

$$\begin{aligned}
\beta(0) \widehat{\ell}_{i-1}(0) &+ \sum_{j=0}^{i-1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) \widehat{\ell}_{i-1}(v) \\
&+ \sum_{v \in \mathcal{V}(\mathcal{K}_i(\Omega))} \alpha_i(v) \widehat{\ell}_{i-1}(v) \geq \widehat{\ell}_{i-1}(x). \tag{32}
\end{aligned}$$

Inclusions (31) and (32) yield:

$$z \geq \widehat{\ell}_{i-1}(x) + \sum_{v \in \mathcal{V}(\mathcal{K}_i(\Omega))} \alpha_i(v) \epsilon \geq \widehat{\ell}_{i-1}(x). \tag{33}$$

The equality in (33) holds only if $\alpha_i(v) = 0$ for all $v \in \mathcal{V}(\mathcal{K}_i(\Omega))$. In other words, $z = \widehat{\ell}_{i-1}(x)$ only if $x = \beta(0)0 + \sum_{j=0}^{i-1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) v \in \mathcal{K}_{i-1}(\Omega)$. More precisely, $\ell_i(x) = \widehat{\ell}_{i-1}(x) = \ell_{i-1}(x)$ takes place only if $x \in \mathcal{K}_{i-1}(\Omega)$. Following similar argument, we conclude that claim 4) holds true for all $i \in \mathcal{I}_N$.

Claim 5) follows as a direct consequence of claims 3) and 4). Indeed, $\ell_i(x) = \ell_j(x)$ for all $x \in \mathcal{K}_j(\Omega)$ and $j \in \{0\} \cup \mathcal{I}_i$. On the other hand, claim 3) shows that $\ell_j(x) = h_j$ for $x \in \mathcal{V}(\mathcal{K}_j(\Omega)) \setminus \mathcal{K}_{j-1}(\Omega)$. Therefore, $\ell_i(x) = h_j$ for $x \in \mathcal{V}(\mathcal{K}_j(\Omega)) \setminus \mathcal{K}_{j-1}(\Omega)$ and $j \in \mathcal{I}_i$.

Claim 6) can easily be deduced from equation (29) that

$$z = \beta(0)0 + \sum_{j=0}^i \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega))} \alpha_j(v) h_j \geq 0, \tag{34}$$

as $h_j > 0$ for all $j \in \{0\} \cup \mathcal{I}_i$, thus the inequality (34) becomes equality only if $\alpha_j(v) = 0$ for all $v \in \bigcup_{j=0}^i \mathcal{V}(\mathcal{K}_j(\Omega))$ and $\beta(0) = 1$, leading to $x = 0$.

Claim 7) is obtained by the convexity of $\ell_i(x)$, i.e. $\ell_i(\beta x) \leq \beta \ell_i(x) + (1-\beta) \ell_i(0) = \beta \ell_i(x)$. The proof is complete. $\square$

8.4 Proof of Lemma 5.1

Consider the polytopical partition $\{\mathcal{X}_j^{(N)}\}_{j \in \mathcal{I}_{M(N)}}$ of $\mathcal{K}_N(\Omega)$ associated with $\ell_N(x)$. Denote also such a function $\ell_N(x)$ as follows: $\ell_N(x) = (f_j^{(N)})^T x + g_j^{(N)}$ for $x \in \mathcal{X}_j^{(N)}$. Consider a region $\mathcal{X}_j^{(N)}$ in this polytopical partition. Due to the boundedness and positivity of $\ell_N(x)$ except at 0, for each $v \in \mathcal{V}(\mathcal{X}_j^{(N)})$, there exists a $c_1^{(j)}(v) > 0$ such that

$$c_1^{(j)}(v) \|v\|_\infty \leq (f_j^{(N)})^T v + g_j^{(N)}. \tag{35}$$

Note that if region $\mathcal{X}_j^{(N)}$ contains the origin, then $g_j^{(N)} = 0$. Moreover, if $v = 0$, then inclusion (35) holds true for

all $c_1^{(j)}(0) \in \mathbb{R}$, since its two sides are equal to 0. As a consequence, if one chooses $c_1^{(j)} = \min_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} c_1^{(j)}(v) > 0$,

then one obtains:

$$c_1^{(j)} \|v\|_\infty \leq (f_j^{(N)})^T v + g_j^{(N)}, \quad \forall v \in \mathcal{V}(\mathcal{X}_j^{(N)}). \quad (36)$$

Note that any $x \in \mathcal{X}_j^{(N)}$ can be expressed as follows:

$$x = \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) v, \quad \alpha(v) \geq 0, \quad \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) = 1.$$

The convexity of function $\|\cdot\|_\infty$ yields

$$\begin{aligned} c_1^{(j)} \|x\|_\infty &\leq \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} c_1^{(j)} \alpha(v) \|v\|_\infty \\ &\leq \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) \left((f_j^{(N)})^T v + g_j^{(N)} \right) \\ &= (f_j^{(N)})^T x + g_j^{(N)}. \end{aligned} \quad (37)$$

Recall that the existence of $c_1^{(j)}$ satisfying (37) only holds true for all $x \in \mathcal{X}_j^{(N)}$. Therefore, if one chooses $c_1 = \min_{j \in \mathcal{I}_M(N)} c_1^{(j)} > 0$, then we can ensure that $c_1 \|x\|_\infty \leq \ell_N(x)$ for all $x \in \mathcal{K}_N(\Omega)$.

In order to prove the existence of c_2 , consider again region $\mathcal{X}_j^{(N)}$. First, we observe that $(f_j^{(N)})^T x \leq \|f_j^{(N)}\|_1 \|x\|_\infty$. Also, since this region is a polytope due to the boundedness of $\mathbb{X}$, then there exists a constant $b_j > 0$ such that

$$g_j^{(N)} \leq b_j \|x\|_\infty \quad \text{for all } x \in \mathcal{X}_j^{(N)}. \quad (38)$$

Note that if $\mathcal{X}_j^{(N)}$ contains the origin, then $g_j^{(N)} = 0$, therefore any $b_j > 0$ satisfies (38). Otherwise, one can choose b_j as follows: $b_j = \left(\min_{x \in \mathcal{X}_j^{(N)}} \|x\|_\infty \right)^{-1} |g_j^{(N)}|$, since $\mathcal{X}_j^{(N)}$ does not contain the origin, thus $\min_{x \in \mathcal{X}_j^{(N)}} \|x\|_\infty > 0$. Obviously, such a scalar b_j satisfies (38) and $b_j > 0$. We now define $c_2^{(j)} := \|f_j^{(N)}\|_1 + b_j > 0$, then one can ensure that

$$(f_j^{(N)})^T x + g_j^{(N)} \leq c_2^{(j)} \|x\|_\infty, \quad \forall x \in \mathcal{X}_j^{(N)}. \quad (39)$$

Therefore, if one chooses $c_2 = \max_{j \in \mathcal{I}_M(N)} c_2^{(j)} > 0$, then we ensure that $\ell_N(x) \leq c_2 \|x\|_\infty$ for all $x \in \mathcal{K}_N(\Omega)$. $\square$

References

Anderson, B. D., Moore, J. B., 2007. Optimal control: linear quadratic methods. Courier Corporation.

- Aubin, J.-P., Cellina, A., 2012. Differential inclusions: set-valued maps and viability theory. Vol. 264. Springer Science & Business Media.
- Avis, D., Fukuda, K., 1992. A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra. *Discrete & Computational Geometry* 8 (1), 295–313.
- Bitsoris, G., 1988a. On the positive invariance of polyhedral sets for discrete-time systems. *Systems & Control Letters* 11 (3), 243–248.
- Bitsoris, G., 1988b. Positively invariant polyhedral sets of discrete-time linear systems. *International Journal of Control* 47 (6), 1713–1726.
- Bitsoris, G., Vassilaki, M., 1995. Constrained regulation of linear systems. *Automatica* 31 (2), 223–227.
- Blanchini, F., 1994. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *Automatic Control, IEEE Transactions on* 39 (2), 428–433.
- Blanchini, F., Miani, S., 2007. Set-theoretic methods in control. Springer.
- Chmielewski, D., Manousiouthakis, V., 1996. On constrained infinite-time linear quadratic optimal control. *Systems & Control Letters* 29 (3), 121 – 129.
- Cuzzola, F. A., Geromel, J. C., Morari, M., 2002. An improved approach for constrained robust model predictive control. *Automatica* 38 (7), 1183–1189.
- Daafouz, J., Bernussou, J., 2001. Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties. *Systems & Control Letters* 43 (5), 355–359.
- Gal, T., 1995. Postoptimal analyses, parametric programming and related topics. Walter de Gruyter.
- Gilbert, E. G., Tan, K. T., 1991. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *Automatic Control, IEEE Transactions on* 36 (9), 1008–1020.
- Grancharova, A., Johansen, T. A., 2012. Explicit nonlinear model predictive control: theory and applications. Vol. 429. Springer Science & Business Media.
- Gulan, M., Takács, G., Nguyen, N. A., Olaru, S., Rodriguez-Ayerbe, P., Rohal'-Ilkiv, B., 2017a. Efficient embedded model predictive vibration control via convex lifting. *Control Systems Technology, IEEE Transactions on*.
- Gulan, M., Takács, G., Nguyen, N. A., Olaru, S., Rodriguez-Ayerbe, P., Rohal'-Ilkiv, B., July 9–14 2017b. Embedded linear model predictive control for 8-bit microcontrollers via convex lifting. In: the 20th IFAC World Congress. Toulouse, France.
- Gutman, P.-O., Cwikel, M., 1987. An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states. *Automatic Control, IEEE Transactions on* 32 (3), 251–254.
- Herceg, M., Kvasnica, M., Jones, C., Morari, M., July 2013. Multi-parametric toolbox 3.0. In: Control Conference (ECC), 2013 European. pp. 502–510.
- Jiang, Z. P., Wang, Y., 2001. Input-to-state stability for discrete-time nonlinear systems. *Automatica* 37 (6), 857–869.
- Kerrigan, E. C., 2001. Robust constraint satisfaction: Invari-

- ant sets and predictive control. Ph.D. thesis, University of Cambridge.
- Khalil, H. K., 2002. Nonlinear systems. Prentice Hall, Inc, 323.
- Kolmanovsky, I., Gilbert, E. G., 1998. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering* 4 (4), 317–367.
- Kothare, M. V., Balakrishnan, V., Morari, M., 1996. Robust constrained model predictive control using linear matrix inequalities. *Automatica* 32 (10), 1361–1379.
- Lyapunov, A. M., 1907. Problème général de la stabilité du mouvement. In: *Annales de la Faculté des Sciences de Toulouse*. Vol. 9. Université Paul Sabatier, pp. 203–474.
- Mayne, D. Q., Rawlings, J. B., Rao, C. V., Scokaert, P. O. M., 2000. Constrained model predictive control: Stability and optimality. *Automatica* 36 (6), 789–814.
- Nguyen, H.-N., Gutman, P.-O., Olaru, S., Hovd, M., 2013. Implicit improved vertex control for uncertain, time-varying linear discrete-time systems with state and control constraints. *Automatica* 49 (9), 2754–2759.
- Nguyen, N. A., 2015. Explicit robust constrained control for linear systems: analysis, implementation and design based on optimization. Ph.D. thesis, CentraleSupélec, Université Paris-Saclay, France.
- Nguyen, N. A., Gulan, M., Olaru, S., Rodríguez-Ayerbe, P., 2017a. Convex lifting: Theory and control applications. *Transactions on Automatic Control* (in press).
URL <https://hal-centralesupelec.archives-ouvertes.fr/hal-01326804/document>
- Nguyen, N. A., Olaru, S., Rodríguez-Ayerbe, P., 2015a. Any discontinuous PWA function is optimal solution to a parametric linear programming problem. In: *Decision and Control (CDC), IEEE 54th Annual Conference on*. IEEE, pp. 5926–5931.
- Nguyen, N. A., Olaru, S., Rodríguez-Ayerbe, P., 2015b. Inverse parametric linear/quadratic programming problem for continuous PWA functions defined on polyhedral partitions of polyhedra. In: *Decision and Control (CDC), IEEE 54th Annual Conference on*. IEEE, pp. 5920–5925.
- Nguyen, N. A., Olaru, S., Rodríguez-Ayerbe, P., Hovd, M., Necoara, I., 2017b. Constructive solution of inverse parametric linear/quadratic programming problems. *Journal of Optimization Theory and Applications* 172 (2), 623–648.
- Nguyen, N. A., Olaru, S., Rodríguez-Ayerbe, P., Kvasnica, M., 2017c. Convex liftings-based robust control design. *Automatica* 77, 206 – 213.
- Nguyen, T. T., Lazar, M., Spinu, V., 2014. Interpolation of polytopic control Lyapunov functions for discrete-time linear systems. *IFAC Proceedings Volumes* 47 (3), 2297–2302.
- Primbs, J. A., Nevistić, V., Doyle, J. C., 1999. Nonlinear optimal control: A control Lyapunov function and receding horizon perspective. *Asian Journal of Control* 1 (1), 14–24.
- Rakovic, S. V., Baric, M., 2010. Parameterized robust control invariant sets for linear systems: Theoretical advances and computational remarks. *Automatic Control, IEEE Transactions on* 55 (7), 1599–1614.
- Zubov, V. I., Boron, L. F., 1964. *Methods of AM Lyapunov and their application*. Noordhoff Groningen.