Combinatorial Approach towards Multi-Parametric Quadratic Programming based on Characterizing Adjacent Critical Regions

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Parisa Ahmadi-Moshkenani[1] , Tor Arne Johansen [2], and Sorin Olaru [3]

***Abstract*—Several optimization-based control design techniques can be cast in the form of parametric optimization problems. The multi-parametric quadratic programming (mpQP) represents a popular class often related to the control of constrained linear systems. The complete solution to mpQP takes the form of explicit feedback functions with a piecewise affine structure, valid in polyhedral partitions of the feasible parameter space known as critical regions. The recently proposed combinatorial approach for solving mpQP has shown better efficiency than geometric approaches in finding the complete solution to problems with high dimensions of the parameter vectors. The drawback of this method, on the other hand, is that it tends to become very slow as the number of constraints increases in the problem. This paper presents an alternative method for enumerating all optimal active sets in a mpQP based on theoretical properties of adjacent critical regions and their corresponding optimal active sets. Consequently, it results in excluding a noticeable number of feasible but not optimal candidate active sets from investigation. Therefore, the number of linear programs that should be solved decreases noticeably and the algorithm becomes faster. Simulation results confirm the reliability of the suggested method in finding the complete solution to the mpQPs while decreasing the computational time compared favourably with the best alternative approaches.**

I. INTRODUCTION

EXPLOITING multi-parametric quadratic programming (mpQP) for solving model predictive control (MPC) problems enables the main online computational burden of the problem to be moved offline [1], [2] and [3]. Consequently, application of MPC can be extended to systems with relatively fast dynamics. In a mpQP problem, the Karush-Kuhn-Tucker (KKT) optimality conditions can be used to characterize the affine local parametric optimal solution for every fixed combination of optimal active constraints as well as the representation of the polyhedral critical region (CR) which is the domain of validity of affine optimal solution for that optimal active set. There are basically two approaches towards solving a mpQP problem. i) Geometric approaches that iteratively build a partition of parameter space using geometric (polyhedral) computations [4]–[9] ii) Combinatorial approaches which are based on implicitly enumerating all possible combinations of active constraints in a combinatorial search tree [10]–[12]. The advantage of geometric approaches is that mostly optimal combinations of active sets are considered, avoiding unnecessary computations due to the combinatorial number of possible active sets. However, for problems of high dimension of the parameter space, geometric computations become numerically sensitive and these algorithms, therefore, tend to become slow and unreliable. This is due to the fact that high-dimensional geometric problems (such as computing the centers of lower-dimensional facets) cannot be solved reliably even with state-of-the-art solvers [13]. Combinatorial approaches, on the other hand, avoid geometric computations and hence deal quite effectively and efficiently with mpQP problems having a higher number of parameters where the geometric methods perform poorly and may fail finding the complete solution [13]. Furthermore, the enumerative feature of these methods makes them suitable for region-free explicit model predictive controls suggested by [14] and [15] where creating the critical regions, which is computationally demanding in high dimensional parameter spaces, is not required. Another enumeration-based method for solving linear and semi-definite quadratic multi-parametric programs is recently proposed in [13] based on reformulating these problems into parametric linear complementarity problems (PLCP). This method has shown to be, in the best case reported in [13], twice as fast as method of [10]. The pruning criterion in all these enumerative methods is to simultaneously cut off branches with infeasible active sets which is crucial for achieving optimal efficiency in enumeration. A drawback of these methods, however, is that the number of possible combinations of active constraints increases exponentially with the number of constraints. Therefore, their applications are limited to problems with few constraints [12]. Very recently [16] has introduced a connected-graph approach towards solving mpQPs which bridges the division between geometrical and combinatorial approaches. Similarly to the method suggested by [5], identifying the type of each facet of a full-dimensional CR, i.e. investigating which constraint becomes active or inactive on that facet, is required during the offline procedure in order to find adjacent CRs which can result in the same drawbacks as geometric approaches when dealing with mpQPs with large number of parameters. Moreover, when two or more lower-dimensional critical regions overlap along a facet of a full-dimensional critical

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region due to violation of strict complementarity slackness condition, characterising that facet as one of the intended types is not possible. This paper suggests an alternative combinatorial approach towards solving mpQPs which avoids geometric computations completely, resulting in faster and more reliable computation of solution for high number of parameters compared to other approaches. The objective of this method is to exclude a noticeable number of feasible active sets that are not optimal from the combinatorial tree in order to accelerate the enumeration of all optimal sets. To this aim, [17] has suggested a downward and upward exploration of combinatorial tree which exploits the underlying relationship between two full-dimensional adjacent critical regions when degeneracy does not occur on their common facet. This method is guaranteed to find all critical regions in non-degenerate cases while reducing the number of LPs that should be solved. Hence the required computational time decreases significantly. A modification to the method in [17] is presented in [18] to handle degeneracies based on theoretical properties of full-dimensional adjacent critical regions for which degeneracy occurs on their common facet and the relation between their corresponding optimal sets. This method guarantees enumeration of all optimal active sets in a general case which can be subject to degeneracies as well. This paper completes this trend of development by presenting the complete theoretical framework exploited in combinatorial approach and offers additional discussions, numerical studies, comparisons and examples.

The first part of the this paper presents the combinatorial approach towards mpQP in conjunction with the suggested downward and upward exploration of the combinatorial tree. The algorithm for exploring the combinatorial tree is presented in section III along with a series of theorems describing the theoretical foundation. Simulation results are presented in section IV. Moreover, the comparison between different methods for solving mpQP problems, implemented in Multi-Parametric Toolbox [9], is presented which confirms the superiority of the suggested method w.r.t. other approaches for problems with a large number of constraints, and finally the paper is concluded in section V.

II. COMBINATORIAL APPROACH TOWARDS MULTI-PARAMETRIC QUADRATIC PROGRAMMING

Consider the following multi-parametric quadratic program:

$$V_N^*(x) = \min_z \ \frac{1}{2} z^T H z \tag{1a}$$
$$\text{s.t.} \quad Gz \leq Sx + W \tag{1b}$$

which is an equivalent to the standard multi-parametric quadratic program including quadratic, linear and constant terms in the cost function and is derived by applying appropriate transformation. See for example [1]. Here $z \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ denote the vectors of optimization variables and parameters, respectively. Assume that the problem is strictly convex, i.e. $H > 0$. As shown by [1], the Karush-Kuhn-Tucker (KKT) optimality conditions can be used to characterize the analytic solutions to the mpQP problem:

$$Hz + G^T\lambda = 0, \qquad \lambda \in \mathbb{R}^q, \tag{2a}$$
$$\lambda^i(G^i z - W^i - S^i x) = 0, \qquad i = 1, \ldots, q, \tag{2b}$$
$$\lambda \geq 0, \quad Gz \leq Sx + W \tag{2c}$$

Defining $\mathcal{Q} = \{1, \ldots, q\}$ as the index set of all constraints in (1b), we recall that a constraint among q constraints in (1b) is said to be *active* if it holds with equality for a given z and x, and *inactive* if it holds with strict inequality. Thus the active set $\mathcal{A}(z, x)$ can be described as $\mathcal{A}(z, x) := \{i \in \mathcal{Q} \mid G^i z - S^i x - W^i = 0\}$ while the corresponding inactive set $\mathcal{J}(z, x)$ is given by the set difference of $\mathcal{Q}$ and $\mathcal{A}$ i.e. $\mathcal{J}(z, x) := \mathcal{Q} \backslash \mathcal{A}(z, x)$. Denoting $\mathcal{A}$ and $\mathcal{J}$ as the active and inactive sets, one can rewrite the KKT conditions as follows:

$$Hz + {G^{\mathcal{A}}}^T \lambda^{\mathcal{A}} = 0, \tag{3a}$$
$$G^{\mathcal{A}} z - W^{\mathcal{A}} - S^{\mathcal{A}} x = 0, \tag{3b}$$
$$G^{\mathcal{J}} z - W^{\mathcal{J}} - S^{\mathcal{J}} x \leq 0, \tag{3c}$$
$$\lambda^{\mathcal{A}} \geq 0, \quad \lambda^{\mathcal{J}} = 0 \tag{3d}$$

Before going further, we recall some definitions and theorems.

Definition 1. Redundant constraints: Let a polyhedron $\boldsymbol{\Theta}$ be represented by $A\theta \leq b$. We say that $A^i\theta \leq b^i$ is *redundant* if $A^j\theta \leq b^j, \forall j \neq i \Rightarrow A^i\theta \leq b^i$ (i.e., it can be removed from the description of the polyhedron).

Definition 2. Minimal representation: A representation of a polyhedron is minimal if there are no redundant constraints.

Assumption 1. The constraints in (1) are assumed, without loss of generality, to form a minimal representation of the polyhedral feasible set.

Definition 3. Linear Independence Constraints Qualification (LICQ), (Nocedal and Wright, 1999): Given $z^*(x)$ as the optimal solution of (1) at which KKT conditions are satisfied and the corresponding active set $\mathcal{A}$, we say that LICQ holds if the set of active constraint gradients $\{G^i \mid i \in \mathcal{A}(z^*(x), x)\}$ is linearly independent, i.e., $G^{\mathcal{A}}$ has full row rank.

Definition 4. Strict Complementarity Slackness (SCS), (Nocedal and Wright, 1999): Given the pair $(z^*(x), \lambda^*(x))$ satisfying the KKT conditions, SCS holds if exactly one of $\lambda^{i*}(x)$ and $G^i z^*(x) - S^i x - W^i$ is zero for each $i \in \mathcal{Q}$, i.e., $\lambda^{i*}(x) > 0$ for each $i \in \mathcal{A}(z^*(x), x)$ and $s^i > 0$ for each $i \in \mathcal{J}(z^*(x), x)$ where s^i is the slack variable of inactive constraint $i \in \mathcal{J}$ such that $G^i z^*(x) + s^i = S^i x + W^i$.

For a constraint that is assumed to be active, if (3) is feasible with the associated Lagrange multiplier λ^{i*} equal to zero, we define that constraint as *weakly active constraint*. On the other hand, if (3c) holds with strict equality for a constraint that is assumed to be inactive, we call that constraint as *weakly inactive constraint*. Furthermore, an optimization problem for which both the LICQ condition and the SCS condition hold is known to be *non-degenerate* according to the definition of degeneracy in [5].

Definition 5. Full-dimensional polyhedron: Let X be a polyhedron in $\mathbb{R}^n$. If the dimension of the affine hull of X, defined as the set of affine combinations of points in X, is equal to n, then X is full-dimensional.

Theorem 1:
Consider the problem in (1) *with* $H > 0$. *Let* $X \subseteq \mathbb{R}^n$ *be the problem's polyhedral feasible set and let* $x \in X$. *Then the solution* $z^*(x)$ *and the Lagrange multipliers* $\lambda^*(x)$ *of a mpQP are piecewise affine functions of the parameter* x *and* $z^*(x)$ *is continuous. Moreover, if LICQ holds for all* $x \in X$, $\lambda^*(x)$ *is also continuous.*
Proof: See [1]

Assuming that we know an optimal active set $\mathcal{A}$ and that LICQ holds, we can use (2a) and (2b) to derive the parameter-dependent optimizer [1]:

$$z_{\mathcal{A}}(x) = H^{-1}(G^{\mathcal{A}})^T H_{G^{\mathcal{A}}}^{-1}(W^{\mathcal{A}} + S^{\mathcal{A}}x) \tag{4}$$

where the existence of $H_{G^{\mathcal{A}}}^{-1} := (G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})^T)^{-1}$ is guaranteed due to the LICQ and positive definiteness of H. The set of inequalities in (2c) characterize the so-called critical region (CR) for the considered optimal active set $\mathcal{A}$. The CR is in the form of a polyhedron in the parameter space defined by the following inequalities:

$$H_{G^{\mathcal{A}}}^{-1}(W^{\mathcal{A}} + S^{\mathcal{A}}x) \leq 0 \tag{5a}$$

$$GH^{-1}(G^{\mathcal{A}})^T H_{G^{\mathcal{A}}}^{-1}(W^{\mathcal{A}} + S^{\mathcal{A}}x) \leq W + Sx \tag{5b}$$

This polyhedron is the largest set of parameters $x \in X$ for which the combination of active constraints $\mathcal{A}$ at the optimizer remains unchanged and hence, the optimizer is given by (4).

To enumerate all optimal active sets, [10] suggests to choose the candidate active sets from the power set of $\mathcal{Q}$ in the order of increasing cardinality. It should be noted that for a QP with m decision variables and q constraints, only a maximum of $\tilde{m} = min\{m, q\}$ linearly independent constraints can be strongly active at the optimal solution [19]. For each candidate active set, $\mathcal{A}_i$, the following LP should be solved to check whether it can be optimal or not:

$$\max_{z,x,\lambda^{\mathcal{A}_i},s^{\mathcal{J}_i}} t \tag{6a}$$

$$\text{s.t.} \quad te_1 \leq \lambda^{\mathcal{A}_i}, te_2 \leq s^{\mathcal{J}_i} \tag{6b}$$

$$t \geq 0, \lambda^{\mathcal{A}_i} \geq 0, s^{\mathcal{J}_i} \geq 0 \tag{6c}$$

$$Hz + (G^{\mathcal{A}_i})^T \lambda^{\mathcal{A}_i} = 0 \tag{6d}$$

$$G^{\mathcal{A}_i}z - S^{\mathcal{A}_i}x - W^{\mathcal{J}_i} = 0 \tag{6e}$$

$$G^{\mathcal{J}_i}z - S^{\mathcal{J}_i}x - W^{\mathcal{A}_i} + s^{\mathcal{J}_i} = 0 \tag{6f}$$

Here t is a scalar optimization variable and $e_1 = [1, \dots, 1]^T$ and $e_2 = [1, \dots, 1]^T$ are vectors of appropriate sizes corresponding to the vector of Lagrangian multipliers $\lambda^{\mathcal{A}_i}$ and the vector of slack variables $s^{\mathcal{J}_i}$, respectively. Inequalities (6b) form an upper bound on the optimization variable t as the minimal value contained in $\lambda^{\mathcal{A}_i}$ and $s^{\mathcal{J}_i}$. This formulation allows the immediate identification of failure of the SCS condition whenever $t = 0$. Note that, according to the formulation in (6), we adopt the freedom to split the set of constraints in

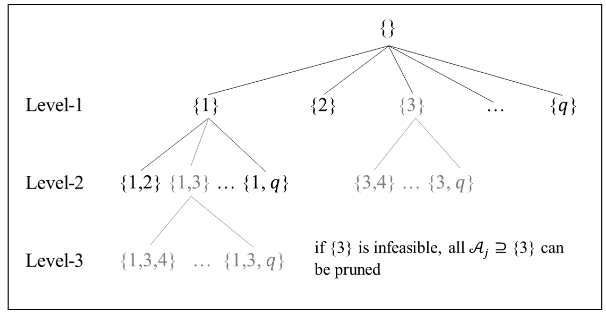


Fig. 1: Combinatorial enumeration strategy used in [10]

Active and Inactive while both are capable of violating the SCS condition through a zero Lagrange multiplier or a zero slack variable, respectively. However, since the objective in (6) is optimized over the parameter space x as well, (6) does not yield a zero Lagrange multiplier or a zero slack variable unless it is zero over the entire critical region corresponding to $\mathcal{A}_i$ whether it is full-dimensional or lower-dimensional. Hence, the situations where both $\lambda_i = 0$ and $G^{\mathcal{A}_i}z - S^{\mathcal{A}_i}x - W^{\mathcal{J}_i} = 0$ hold for constraint i on the boundaries of a full-dimensional critical are not considered as violation of SCS condition. If the candidate active set is found not to be optimal, i.e., if the optimization problem in (6) is not feasible, another optimization problem should be solved by removing all constraints arising from the optimality condition (namely all constraints including $\lambda^{\mathcal{A}_i}$ in (6)), to check for the feasibility of the candidate active set. If this optimization problem is not feasible, we can exclude $\mathcal{A}_i$ and all its supersets from the combinatorial tree. This is the only pruning criterion in this method which is based on the infeasibility of a combination of active constraints. A graphical illustration of the combinatorial enumeration strategy and the involved pruning process is given in the form of a combinatorial tree diagram in Fig. 1. As it can be seen from Fig. 1, all feasible combinations of active constraints remain in the combinatorial tree for exploring the levels below while for many cases, none of their supersets become optimal in future.

In order to exclude a noticeable number of feasible candidate active sets which are not optimal from the combinatorial tree, a joint downward and upward method for exploration of the combinatorial tree is suggested in [17] based on finding all the adjacent critical regions of any critical region while avoiding the geometric computations. As it is explained in [4], critical regions can be considered as nodes of a finite, fully connected graph. There are no isolated regions that could not be reached by starting from any region and going from one neighbour to another neighbour. Thus we can explore the entire feasible space starting from anywhere, while all critical regions are guaranteed to be found.
The downward and upward exploration method is based on the following theorem from [5].

Theorem 2 (mpQP without Degeneracy):
Consider an optimal active set $\{i_1, i_2, \dots, i_k\}$ *and its corresponding minimal representation of the critical region* CR_0. *Let* CR_i *be a full-dimensional neighbouring critical region to* CR_0 *and assume LICQ holds on their common*

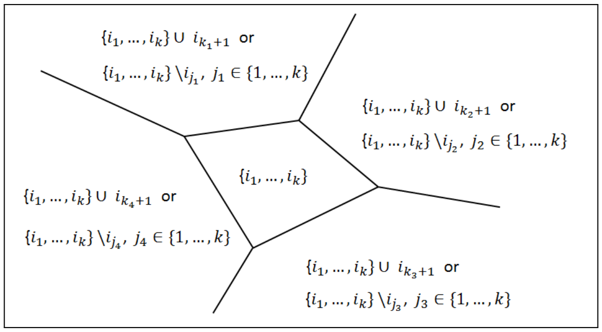


Fig. 2: Combinations of optimal active constraints in adjacent critical regions in a non-degenerate system

facet $\mathscr{F} = CR_0 \cap \mathscr{H}$ where $\mathscr{H}$ is the separating hyperplane between CR_0 and CR_i. Moreover, assume that there are no constraints which are weakly active at the optimizer $z^*(x)$ for all $x \in CR_0$. Then:

Type I: If $\mathscr{H}$ is given by $G^{i_{k+1}} z_0^*(x) = W^{i_{k+1}} + S^{i_{k+1}}x$, then the optimal active set in CR_i is $\{i_1, i_2, \ldots, i_k, i_{k+1}\}$.

Type II: If $\mathscr{H}$ is given by $\lambda_0^{i_k}(x) = 0$, then the optimal active set in CR_i is $\{i_1, i_2, \ldots, i_{k-1}\}$.

According to Theorem 2, the combinations of optimal active sets in two adjacent CRs differ only in one constraint in non-degenerate mpQPs. Therefore, one can only keep the track of optimal active sets and for every optimal active set which is found with a full-dimensional CR, find all optimal active sets corresponding to its adjacent CRs by adding one feasible constraint to or removing one existing constraint from the current optimal active set (See Fig. 2 for illustration). Repeating this for all optimal active sets which are found, guarantees finding the complete solution in non-degenerate cases. Therefore, this method for finding optimal active sets requires joint downward and upward exploration of the combinatorial tree. To this aim, one can explore the combinatorial tree as before, in the order of increasing cardinality. The difference is that in this method, we only use the optimal active sets for building the levels below (downward exploration). Hence if a combination of active constraints is not optimal, the feasibility check of LP (6) is not required any more. For every optimal active set found during downward exploration, we should explore the combinatorial tree upward to check for the optimality of all its subsets with one element less if they are not enumerated yet (upward exploration). Then for every newly found optimal set during upward exploration, we should explore the combinatorial tree downward and upward again, until no new non-enumerated combination is found. For each eliminated feasible but not optimal combination of active constraints, the number of LPs in the form of (6) that should be solved decreases by two (one for checking the optimality and the other for checking the feasibility of the candidate active set). However, when the non-degeneracy assumption is not fulfilled for some combinations of optimal active constraints, some CRs may remain unexplored using this procedure. One way to handle this limitation is to do a post-processing, using geometric approaches, to find the regions that could be missed as it is suggested in [17]. In accordance with our work in [18], we suggest an alternative approach for handling degenerate cases rather than post processing in the next section. This approach is not based on geometric operations and hence is faster and more reliable when the number of parameter variables and the number of constraints increases.

III. MPQP ALGORITHM WITH DEGENERACY HANDLING

Theorem 2 implies that when the optimal active sets in two adjacent full-dimensional CRs differ in more than one constraint, at least one of the LICQ condition or SCS condition is violated. In order to explain different degenerate cases that might happen in the problem and propose proper methods for handling each of them, let us split different combinations of optimal active constraints in two adjacent critical regions which do not fulfill the conditions of Theorem 2 into two categories.

Categ. I: Let CR_i and CR_j be two adjacent critical regions with the corresponding optimal sets $\mathcal{A}_i$ and $\mathcal{A}_j$, respectively. If $\mid (\mathcal{A}_i \setminus \mathcal{A}_j) \mid = \mid (\mathcal{A}_j \setminus \mathcal{A}_i) \mid = 1$ where $\mid \cdot \mid$ denotes the cardinality of a set, then CR_i and CR_j lie in Categ. I.

Categ. II: Let CR_i and CR_j be two adjacent critical regions with the corresponding optimal sets $\mathcal{A}_i$ and $\mathcal{A}_j$, respectively. If $max\{\mid (\mathcal{A}_i \setminus \mathcal{A}_j) \mid, \ \mid (\mathcal{A}_j \setminus \mathcal{A}_i) \mid\} \geq 2$, then CR_i and CR_j lie in Categ. II.

For all adjacent CRs classified in Categ.I, the following theorem states the two possible circumstances which can be characterised on their common facet.

Theorem 3 (Categ.I degeneracy)
Let two full-dimensional neighbouring CRs with the minimal representation be classified as Categ.I, i.e., the optimal active sets in these two regions can be defined by $\mathcal{A}_i = [i_1, \ldots i_k, i_{k+1}]$ and $\mathcal{A}_j = [i_1, \ldots i_k, i_{k+2}]$. Then one of these conditions holds:
a) LICQ is violated for the combination of optimal active constraints on their common facet.
b) LICQ holds on the common facet and SCS is violated for the optimal sets of those two CRs.

Proof: Since the combinations of the optimal active constraints in two adjacent CRs differ in more than one constraint, the possibility of violation of LICQ condition on the common facet follows directly from Theorem 2. Now, assume that LICQ holds on the common facet $\mathcal{F}$. If none of the constraints in $\mathcal{A}_i$ are weakly active, then we have that $\lambda^{i_p} > 0,\ \forall p \in \{1, \ldots k+1\}$. Furthermore, inactiveness of i_{k+1} in CR_j leads in $\lambda^{i_{k+1}} = 0$ for $x \in CR_j$ and since $\lambda^{i_{k+1}}$ is continuous due to Theorem 1 and the fact that LICQ holds on the common facet, $\lambda^{i_{k+1}}$ should be equal to zero on $\mathcal{F}$ as well. This means than the common facet for CR_i can be expressed by $\lambda^{i_{k+1}} \geq 0$. On the other hand, if there is no constraint being weakly inactive in $\mathcal{A}_i$, we have $G^{k+2}z^* < S^{k+2}x + W^{k+2},\ \forall x \in CR_i$ except from on the common facet where $G^{k+2}z^* = S^{k+2}x + W^{k+2}$ (since i_{k+2} is active in $\mathcal{A}_j$, and due to continuity of the optimizer) Hence, $G^{k+2}z^* \leq S^{k+2}x + W^{k+2}$ is also defining the common facet for CR_i. This contradicts with the minimal representation of

CR_i as for the minimal representation to hold, each facet should be represented by *only* one of the inequalities (5a) or (5b). Hence either $\lambda^{i_{k+1}}$ must be zero on the entire CR_i meaning that i_{k+1} is weakly active in CR_i or if this cannot hold, due to strict convexity of the problem, $s^{i_{k+2}}$ must be zero on CR_i which means i_{k+2} is weakly inactive over the entire CR_i. ■

Theorem 4 describes the characteristic of combinations of active constraints on the common facet between two CRs that are classified as Categ.II.

Theorem 4 (Categ.II degeneracy)
Let two full-dimensional neighbouring CRs be classified as Categ. II, i.e., the optimal active set in one of the regions have at least two constraints which do not appear in the optimal set of the adjacent CR. Then SCS condition is violated on $\mathcal{F}$, i.e., the common facet between these two critical regions.

Proof: Let us denote the critical region containing at least two constraints which do not appear in the optimal active set of the neighbouring critical region as CR_i, those two constraints as i_{k+1} and i_{k+2}, and $\mathcal{A}_j$ as the optimal active set in the adjacent critical region CR_j. It can be proved that $\mathcal{A}_{\mathcal{F}_1} \triangleq \mathcal{A}_j \cup i_{k+1}$ is an optimal active set on the common facet with the associated critical region $CR_{\mathcal{F}_1}$ due to feasibility of the LP in (6) with $\mathcal{A}_j$ for all $x \in \mathcal{F}$ and the trivial value for $\lambda^{i_{k+1}}$ equal to zero (Note that $\lambda^{i_{k+1}} = 0$ gives a feasible point for LP in (6) with $\mathcal{A}_{\mathcal{F}_1} \triangleq \mathcal{A}_j \cup i_{k+1}$ which guarantees the optimality of $\mathcal{A}_{\mathcal{F}_1}$ there. However, this does not declare that the obtained optimal value for $\lambda^{i_{k+1}}$ should be necessarily zero). Similarly it can be proved that $\mathcal{A}_{\mathcal{F}_2} \triangleq \mathcal{A}_j \cup i_{k+1} \cup i_{k+2}$ is an optimal active set on $\mathcal{F}$ with the trivial values $\lambda^{i_{k+1}} = \lambda^{i_{k+2}} = 0$ in (6) and the corresponding critical region $CR_{\mathcal{F}_2}$. Since the optimizer $z^*(x)$ is unique due to positive definiteness of H, for all $x \in \mathcal{F}$ we have that $G^{k+2}z^*(x) + s^{k+2} = S^{k+2}x + W^{k+2}$ with some $s^{k+2} \geq 0$ as $x \in CR_{\mathcal{F}_1}$ and simultaneously we have $G^{k+2}z^*(x) = S^{k+2}x + W^{k+2}$ as $x \in CR_{\mathcal{F}_2}$. This means that $s^{k+2} = 0$ for all $x \in CR_{\mathcal{F}_1}$, which completes the proof that i_{k+2} is weakly inactive on $\mathcal{F}$. ■

Remark 1. Whenever the facet-to-facet property [20] does not hold for two adjacent critical regions, the same results as in Theorem 3 and Theorem 4 still hold by substituting $\mathcal{F}$ with the part of the facet that is common between CR_i and CR_j in the proofs.

Exploiting the results in Theorem 3 and Theorem 4, we can now modify the downward-upward algorithm in [17] such that the degenerate cases are explicitly considered. As a result, all critical regions are found during exploration of the combinatorial tree while on average, the number of LPs needed to be solved reduces. To this aim, in the downward-upward exploration we consider combinations of active constraints for which either LICQ condition or SCS condition is violated as well. If in the exploration of the entire tree, no combination of active constraints with failure of SCS condition is found, then due to Theorem 4, no adjacent CRs which can be classified as Categ.II exists in the whole partitioned feasible parameter domain. Thus, the only possibility for the combinations of optimal active sets in two adjacent CRs, except for the cases for which degeneracy does not occur on their common facet, is due to Theorem 3-a. Hence one can explore the combinatorial tree up to level-$(\tilde{m}+1)$ where $\tilde{m} = min\{m, q\}$, simultaneously considering combinations of optimal active constraints for which LICQ is violated, and for all optimal sets with LICQ violation explore their subsets which have one constraint less and are not explored yet. This procedure guarantees the enumeration of all optimal active sets in such cases.

Remark 2. Note that the exploration of combinatorial tree up to level-$(\tilde{m}+1)$, which is one level deeper than what is considered in the exploration method suggested by [10], is crucial for assuring that all optimal sets are enumerated. This is due to the fact that optimal sets which lie in the first category may appear in the last level of the combinatorial tree, i.e. level-$(\tilde{m})$, and the violation of LICQ condition takes place for the optimal active set in level-$(\tilde{m}+1)$ forming the common facet between two adjacent critical regions. However, this does not impose significant computational burden to the problem as we built the lower levels using only the optimal sets (not all the feasible sets) in the level above.

On the other hand, if the SCS condition fails for some combinations of active constraints in a full-dimensional CR or in a lower-dimensional CR which corresponds to the common facet between full-dimensional CRs, identifying the combination of optimal active constraints in the adjacent CR is not straightforward. This is, in particular, due to the possibility of many overlapping lower-dimensional CRs which leads to a significantly different combination of active constraints in the full-dimensional adjacent CR. To make it more clear, consider the following example.

Example 1 (Lower-dimensional critical regions with SCS violation): Fig. 3 shows the partition of the feasible parameter domain for the first example in [20]. As it can be seen, two full-dimensional CRs with the optimal sets $\mathcal{A}_i = [1, 3, 6]$ and $\mathcal{A}_j = [2, 4, 5]$ are adjacent, which shows 6 different constraints in the neighbouring critical regions. In other words, the combination of optimal active sets in these two regions are completely different. This is due to violation of the SCS condition in the overlapping lower-dimensional CRs which form the common facet (or part of it) between them. More detailed, a possible transition of combinations of optimal active constraints from $\mathcal{A}_i$ to $\mathcal{A}_j$ takes place via $\mathcal{A}_i = [1, 3, 6] \longrightarrow [1, 3, 5, 6] \longrightarrow [1, 5, 6] \longrightarrow [1, 2, 5, 6] \longrightarrow [1, 2, 5] \longrightarrow [1, 2, 4, 5] \longrightarrow \mathcal{A}_j = [2, 4, 5]$ where the SCS condition fails for all the intermediate optimal active sets and their corresponding CRs partially overlap. Fig. 4 depicts this overlapping lower-dimensional CRs.

Since any of the weakly inactive constraints may appear in the full-dimensional adjacent CR, depending on which lower-dimensional critical regions with violation of SCS overlap on the common facet, one way to deal with such situations is to

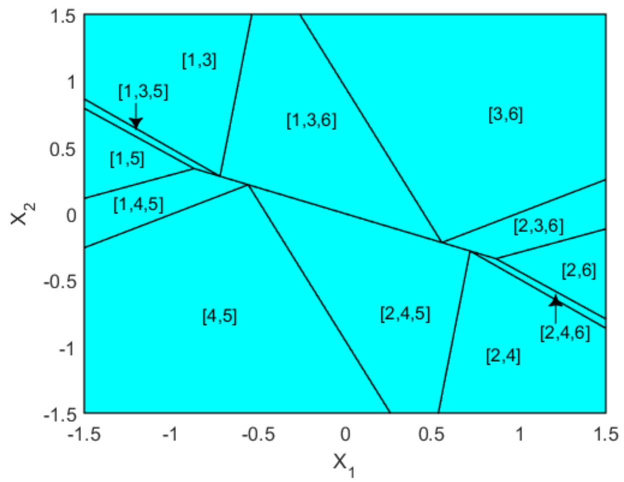


Fig. 3: Example with violation of the SCS condition in lower-dimensional CRs from [20].

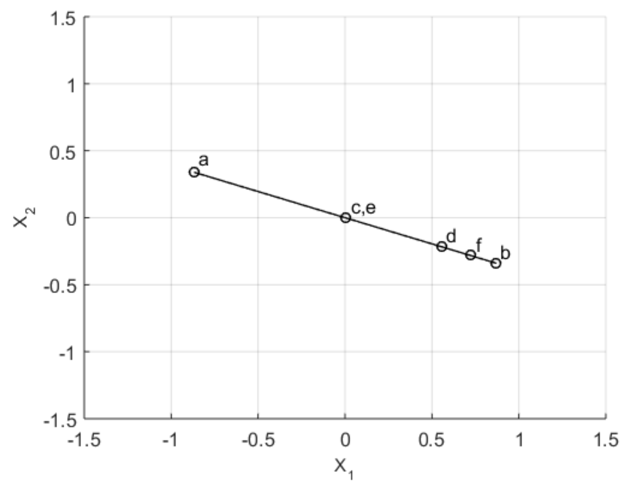


Fig. 4: A closer consideration of Fig. 3 with respect to the lower-dimensional critical regions, where line “ab”, “ac”, “ad”, “ae” and “af” show critical regions corresponding to optimal sets $[1, 3, 5, 6]$, $[1, 5, 6]$, $[1, 2, 5, 6]$, $[1, 2, 5]$ and $[1, 2, 4, 5]$ respectively

determine the set including all constraints that are active or weakly inactive for each optimal active set with violation of SCS condition and then explore all its unexplored full row rank subsets which have at most $\tilde{m}$ elements.

For illustration, in the presented example constraints 2, 3 and 4 are weakly inactive for $\mathcal{A} = [1, 5, 6]$. Hence we can build the superset $Sup = [1, 2, 3, 4, 5, 6]$ and then explore all its full row rank subsets which have at most $\tilde{m} = 3$ elements if they have not been already explored. This is the same exploration method of the combinatorial tree as suggested by [10]. However, as we have observed a priori, the combination of all these constraints is feasible. Therefore, there is no need to check the feasibility of non-optimal subsets as we are sure that all of them are feasible. By doing so, $\mathcal{A}_j = [2, 4, 5]$ will be found even if it is not found as an adjacent critical region of its other neighbouring critical regions. Note that the indices of all weakly inactive constraints can be simply obtained by identifying all slack variables equal to zero.

Theoretically, this can be done for every optimal set with SCS failure separately. But as the constructed supersets can share many constraints in common or even they can be exactly identical (e.g., the supersets for all intermediate optimal active sets in the above example are identical and equal to $Sup = [1, 2, 3, 4, 5, 6]$), we have observed in the numerical examples that it would be beneficial if we first determine the union of all found supersets for optimal sets with violation of SCS, and then explore all its unexplored full row rank subsets as mentioned before. Consequently we avoid constructing too many repetitive subsets. On the other hand, if the cardinality of the obtained superset is considerably large with respect to each of such sets, meaning that the sets with SCS violation do not share many constraints, it can happen that considering the sets with SCS violation individually results in less computational complexity. The approximate number of LPs that should be solved in each case can be computed first in order to help choosing the best strategy. Note that using the superset, the maximum number of $C(r, 1) + C(r, 2) + \cdots + C(r, n_z)$ LPs should be solved while the number of required LPs considering each set separately is approximately $n_s \times [C(r_a, 1) + C(r_a, 2) + \cdots + C(r_a, n_z)]$, where $C(r, k)$ denotes the combination of k elements out of r and r, n_z, n_s and r_a are the cardinality of the superset, number of control variables, number of sets with SCS violation and the average cardinality of all sets with SCS violation, respectively.

The following theorem shows that it is not required to consider the optimal active sets for which LICQ is violated in the downward exploration of the combinatorial tree.

Theorem 5

If a superset $\mathcal{A}_l$ of an optimal active set $\mathcal{A}_j$ for which LICQ is violated, is also optimal, then the SCS condition is violated for the optimal active set $\mathcal{A}_j$.

Theorem 5 guarantees the enumeration of all sets which have similar characteristics to $\mathcal{A}_l$ while dealing with their subsets which are optimal with violation in SCS condition. Hence it preserves us from solving the optimization problem (6) for candidate active sets which can arise from exploring the supersets of optimal sets with LICQ violation if it is not needed. Before proceeding further, we state the following lemma which gives us required tools for proving Theorem 5.

Lemma 1.

If the LICQ condition fails for the optimal active set $\mathcal{A}_i$ in a full-dimensional critical region, then all its subsets $\mathcal{A}_j \subset \mathcal{A}_i$ with $G^{\mathcal{A}_j}$ having full row rank, are optimal active sets with violation of the SCS condition.

Proof: Assume the full-dimensional critical region CR_i with corresponding optimal set $\mathcal{A}_i = [i_1, \ldots, i_{k-1}, i_k]$ and the Lagrange multipliers $\{\lambda^1, \ldots, \lambda^k\}$ for which LICQ is violated. Moreover, assume that $\mathcal{A}_j = [i_1, \ldots, i_{k-1}]$ is one of its full row rank subsets. This means that the k^{th} row of matrix $G^{\mathcal{A}_i}$

can be written as $G^{\mathcal{A}_i,k} = c_1 G^{\mathcal{A}_i,1} + \ldots + c_{k-1} G^{\mathcal{A}_i,k-1}$ where $G^{\mathcal{A}_i,j}$ represents the j^{th} row of matrix $G^{\mathcal{A}_i}$. Let x_0 be a point in the interior of CR_i. Since $\mathcal{A}_i$ is optimal at x_0, we have the optimality condition as $Hz^* + (G^{\mathcal{A}_i,1})^T \lambda^1 + \ldots + (G^{\mathcal{A}_i,k-1})^T \lambda^{k-1} + (G^{\mathcal{A}_i,k})^T \lambda^k$. Using the equality $G^{\mathcal{A}_i,k} = c_1 G^{\mathcal{A}_i,1} + \ldots + c_{k-1} G^{\mathcal{A}_i,k-1}$ we can rewrite the optimality condition as $Hz^* + (G^{\mathcal{A}_i,1})^T \lambda^1 + \ldots + (G^{\mathcal{A}_i,k-1})^T \lambda^{k-1} + (c_1 G^{\mathcal{A}_i,1} + \ldots + c_{k-1} G^{\mathcal{A}_i,k-1})^T \lambda^k = Hz^* + (G^{\mathcal{A}_i,1})^T (\lambda^1 + c_1 \lambda^k) + \ldots + (G^{\mathcal{A}_i,k-1})^T (\lambda^{k-1} + c_{k-1} \lambda^k)$. This means that $\mathcal{A}_j = [i_1, \ldots, i_{k-1}]$ is also optimal active set at x_0 with $\overline{\lambda}^l = \lambda^l + c_l \lambda^k, \forall l \in \{1, \ldots, k-1\}$ and the slack variable corresponding to the k^{th} constraint is equal to zero ($s^k = 0$). Hence $\mathcal{A}_j = [i_1, \ldots, i_{k-1}]$ is an optimal set for which SCS does not hold. ■

Regarding Lemma 1 it is worth noting that an optimal active set for which LICQ is violated can have a full-dimensional critical region as pointed out in [1]. Such CRs can be obtained by a projection algorithm [See Appendix for more details]. The following example gives an illustration for these cases.

Example 2 (LICQ violation in a full-dimensional CR): Consider the following mpQP,

$$V_N^*(x) = \min_{z} \ \frac{1}{2} z^T z \tag{7a}$$

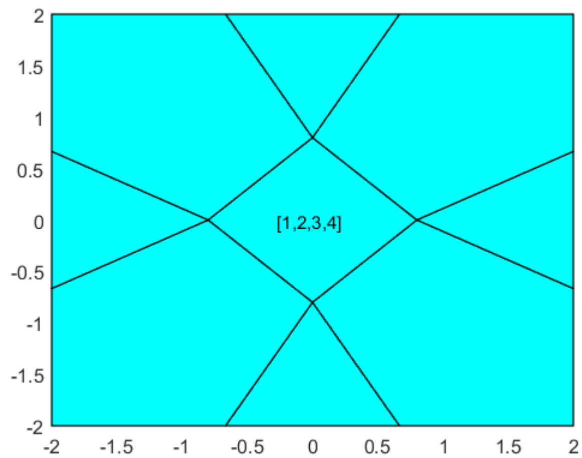
$$\text{s.t.} \quad G = \begin{bmatrix} 1 & 0 & -1 & 0.5 \\ -1 & 0 & -1 & 0.5 \\ 0 & 1 & -1 & 0.5 \\ 0 & -1 & -1 & 0.5 \end{bmatrix} z \leq \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \tag{7b}$$

and $-2 \leq x_i \leq 2; i = 1, 2, 3, 4$

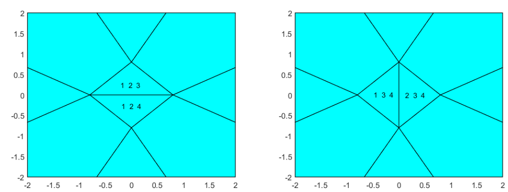
where $n_z = 4$ indicates that up to 4 different constraints can appear in the optimal active sets with full-dimensional CRs. Fig. 5a shows the critical region corresponding to the optimal set $\mathcal{A} = \{1, 2, 3, 4\}$ which is obtained by projection. Based on Lemma 1, we expect that any arbitrarily chosen subset of $\mathcal{A}$ with 3 constraints should be degenerate in the sense of SCS violation. The simulation results meet this expectation and confirm that for all optimal sets $\mathcal{A}_1 = [1, 2, 3]$, $\mathcal{A}_2 = [1, 2, 4]$, $\mathcal{A}_3 = [1, 3, 4]$ and $\mathcal{A}_4 = [2, 3, 4]$ SCS condition is violated while their corresponding CRs are full-dimensional as shown in Fig. 5b.

Lemma 1 proves that an optimal set with violation of LICQ condition ($\mathcal{A}_j$), built by adding one feasible constraint to an optimal set for which both LICQ and SCS hold, should be lower-dimensional. The low-dimensionality of $\mathcal{A}_j$ is then used in the proof of Theorem 5 as follows:

Proof (Theorem 5): Assume that $\mathcal{A}_i = [i_1, \ldots, i_k]$ is an optimal active set with a full-dimensional critical region CR_i where both LICQ and SCS conditions hold. Further assume that its superset $\mathcal{A}_j = [i_1, \ldots, i_k, i_{k+1}]$ is an optimal active set with violation of the LICQ condition. By Lemma 1 it is clear that the corresponding critical region CR_j cannot be full-dimensional since otherwise, SCS condition should fail for $\mathcal{A}_i$. Then if $\mathcal{A}_l = [i_1, \ldots, i_{k+1}, i_{k+2}]$ which is built by adding the feasible constraint i_{k+2} to $\mathcal{A}_j$ is also optimal with CR_l, two different situations may happen. i) CR_l is low-dimensional: This means that two low-dimensional critical region CR_j and CR_l are neighbouring. Therefore they must overlap. Hence i_{k+2} is weakly inactive for $\mathcal{A}_j$. ii) CR_l is full-dimensional: This means that CR_i and CR_l are two full-dimensional CRs which are adjacent. Therefore they lie in the Categ.II and the SCS condition fails on their common facet (with $\mathcal{A}_j$) as a result of Theorem 4. ■



(a) a full-dimensional critical region corresponding to an optimal set with LICQ violation



(b) Optimal sets with violation of SCS condtion

Fig. 5: Optimal sets and their corresponding CRs for Example 2

Based on the above theories, the downward-upward algorithm can be summarized as in Algorithm 1.

IV. SIMULATION RESULTS

In this section, the simulation results of the combinatorial approach using the suggested method in Algorithm 1 for three different cases are shown and compared with other methods implemented in MPT3 .

Case 1: As the first case, we consider the fuel cell breathing control system with 8 state variables and 1 input and discretize it with $T_d = 1$ sec. This case does not have optimal sets in which SCS fails. However, the condition in Theorem 3-a occurs in the fuel cell system with $N = 6$ in which $\tilde{m} = 3$ and $\mathcal{A}_i = [3, 11, 13]$ and $\mathcal{A}_j = [11, 13, 16]$ are the optimal sets in two full-dimensional adjacent CRs and $\mathcal{A}_{\mathcal{F}} = [3, 11, 13, 16]$ is the optimal set on their common facet with the violation of the LICQ condition as $|\mathcal{A}_{\mathcal{F}}| > \tilde{m}$. Algorithm 1 is implemented in MATLAB using GLPK, intended for solving large-scale linear programmings, as the LP solver. The simulation results using this routine and the algorithm in [10], implemented in

Algorithm 1 Downward-upward exploration strategy of the combinatorial tree

Phase I (Initialization):

1) $i = 1$, Explore the entire level-1, use (6) to check the optimality of each constraint. For each optimal constraint with violation of the SCS condition, create its superset including the active and all weakly inactive constraints and store it in "*SCS Set*". If the constraint is not optimal, use (6) without optimality conditions to check the feasibility of that constraint. Store all optimal constraints for which the SCS condition holds in "*Optimal Set*" and all feasible constraints, whether they are optimal or not, in "*Feasible Set*";
   - ∟ if no constraint is found to be optimal without violation of SCS condition in 1), then:
     - ∟ while *Optimal Set* is empty, explore the entire level-$(i + 1)$, check *only* for optimality of the generated combinations. For each found optimal set with violation in SCS condition, create its superset including all active and weakly inactive constraints and store them in *SCS Set*;
     - ∟ $i := i + 1$;

Phase II (Recursive Exploration):

2) *(Downward Exploration)* Construct level-$(i + 1)$ by adding one feasible constraint from level-1 to all sets in *Optimal Set* which are found in level-i and check *only* for the optimality of new combinations whether LICQ holds for them or not. For each optimal active set which is found during this step:
   - ∟ if both LICQ and SCS hold (Theorem 2) compute control law and critical region, and add the combination to *Optimal Set*;
   - ∟ elseif SCS fails (Theorem 4) compute the superset including all active and weakly inactive constraints, and add it to *SCS Set*;
   - ∟ elseif LICQ fails (Theorem 3-a and Theorem 5) add it to *LICQ Set* to check its subsets with one element less to find possibly missed CRs as in Theorem 3-a;

   $i := i + 1$;
   - ∟ if $i < \tilde{m} = min\{m, q\}$ then go to 2), else go 3);

3) *(Upward Exploration)* For all optimal active sets which are added to *Optimal Set* or *LICQ Set*, check the optimality of all its subsets with one element less that have not been enumerated yet. Store all newly found optimal sets in "*New Set*";
4) For each optimal set $\mathcal{A}_i \in$ *New Set*:
   New Set := *New Set* $\backslash \mathcal{A}_i$
   - ∟ if both LICQ and SCS hold (Theorem 2) add $\mathcal{A}_i$ to *Optimal Set* and compute the corresponding critical region and control law, check the optimality of all its subsets with one element less and supersets with one element more that have not been enumerated yet (joint upward and downward exploration of the tree for a newly found non-degenerate optimal set). Add all found optimal sets to *New Set*;
   - ∟ elseif SCS fails (Theorem 4) compute the superset including all active and weakly inactive constraints and add it to *SCS Set*;
   - ∟ esleif LICQ fails (Theorem 3-a and Theorem 5) add it to *LICQ Set* and explore only its subsets with one element less and check for the optimality, add all found optimal sets to *New Set*;
   - ∟ if *New Set* is empty then go to 5), else go to 4);

Phase III (Handling Cases with SCS Violation):

5) Compute the union of all sets in *SCS Set*. Explore all its full row rank subsets with cardinality less than or equal to $\tilde{m}$ if it is not enumerated yet;
   - ∟ add all newly found optimal active set for which SCS holds to *Optimal Set* and go to 4);
   - ∟ if no new set for which SCS holds is found, stop

MPT3, on a 3.2 GHz core i5 CPU running MATLAB 2014a are shown in Table I, where N, n_{CR}, n_{LP} and SF represent the prediction (and control) horizon, number of found CRs, number of solved LPs and the speedup factor defined as the ratio of the computational time using algorithm in [10] to the computational time using the suggested algorithm here. It can be seen that as the prediction horizon increases, the speedup factor increases dramatically which indicates the superiority of the suggested algorithm for systems with a large number of constraints.

Case 2: As an example for cases with violation of SCS condition, we augmented example 1 from [20] by adding random matrices to G, S and W such that the number of inputs and the number of constraints are increased in the problem. Table II shows the comparison for four different randomly augmented examples for which SCS condition fails for some of the combinations of active constraints. Here n_z, q, n_{CR}, n_{LP}, t_{comp} and SF represent the number of control variables, number of constraints, number of found CRs, number of solved LPs, computational time required by different algorithms and speedup factor, respectively. It can be seen that the suggested algorithm has a significant reduction

TABLE I: Comparison between different algorithms for fuel cell breathing system

Method	N	n_{CR}	n_{LP}	$t_{comp}[s]$	SF
Alg. 1	3	71	574	2.7608	
Alg. 2		71	287	2.0150	0.7298
Alg. 1	4	133	1551	5.9922	
Alg. 2		133	1701	5.0730	0.8466
Alg. 1	5	191	2653	9.4219	
Alg. 2		191	6001	11.4970	1.2202
Alg. 1	6	241	3888	13.2536	
Alg. 2		241	18561	29.5160	2.2270
Alg. 1	7	279	4622	16.3629	
Alg. 2		279	47017	69.8420	4.2683
Alg. 1	8	307	5840	19.6120	
Alg. 2		307	149319	230.5860	11.7574

Alg.1: Algorithm suggested here
Alg.2: Algorithm by Gupta et al.

TABLE II: Comparison between different algorithms for the system with violation in the SCS condition

Method	n_z	q	n_{CR}	n_{LP}	$t_{comp}[s]$	SF
Alg. 1	4		34	263	2.0614	
Alg. 2		10	34	452	0.8500	0.4123
Alg. 1	6		54	925	3.7631	
Alg. 2		20	54	14376	16.6200	4.4165
Alg. 1	8		70	1763	7.3106	
Alg. 2		30	70	223211	423.8800	57.9815
Alg. 1	10		79	2835	9.6660	
Alg. 2		40	-	6439332*	5h*	-

* Matlab ran out of memory in the ninth-level, after approximately 5 hours of execution and solving 6439332 LPs.
Alg.1: Algorithm suggested here
Alg.2: Algorithm by Gupta et al.

TABLE III: Comparison between different algorithms for the system in (8)

Method	N	n_x	n_{CR}	$t_{enum}[s]$	$t_{tot}[s]$
mpt _ solvemp	4	4	19		0.6460
mpt _ enum _ plcp			19		1.7340
mpt _ enumpqp			19	0.6026	1.0170
suggested method			19	0.3523	0.7404
mpt _ solvemp	4	10	23		5.0120
mpt _ enum _ plcp			22		3.1640
mpt _ enumpqp			27	0.8254	1.6530
suggested method			27	0.5346	1.5370
mpt _ solvemp	10	4	21		0.7600
mpt _ enum _ plcp			12		201.1730
mpt _ enumpqp			21	788.1386	788.5990
suggested method			21	0.8911	1.3498
mpt _ solvemp	10	10	499		67.8480
mpt _ enum _ plcp			45		3235.0
mpt _ enumpqp			536	8h*	-
suggested method			537	39.6771	72.8494

* Matlab ran out of memory in the ninth-level, after approximately 8 hours of execution.

of computational time in comparison with the algorithm in [10] for the combinatorial approach and as the the number of control variables and the number of constraints increase, the superiority of the suggested algorithm becomes significantly noticeable.

Case 3: In the following, we show how Algorithm 1 compares to other methods for solving mpQP implemented in MPT3, i.e. the geometric approach using the function *mpt-solve*, the enumeration based method of [10] using function *mpt-enumpqp* and the enumeration based partial complementarity problem using function *mpt-enum-plcp*. The simulations are performed by considering the example in [13], i.e. a mpQP constructed from the typical MPC setup of the form

$$\min_{u_0,\cdots,u_{N-1}} \quad x_N^T P x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \tag{8a}$$

$$\text{s.t.} \quad x_{k+1} = Ax_k + Bu_k \tag{8b}$$

$$x \in \mathcal{X}, \ u \in \mathcal{U} \tag{8c}$$

with $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}$, $P = Q = I_{n_x}$, $R = 1$, $\mathcal{X} = \{x| -10 \leq x_i \leq 10, \ i = 1, \cdots, n_x\}$, $\mathcal{U} = \{-1 \leq u \leq 1\}$. The prediction model (8b) is obtained by discretizing the model $1/(s+1)^{n_x}$ with sampling time of 1 second and then converting the discretized model to a state-space form. The number of optimization variables depends on the control horizon N. Different values for n_x and N are considered in simulations in order to assess the performance of various parametric solvers for varying dimension of the parameter space and optimization variables. Simulation results are summarized in Table III, where t_{enum} and t_{tot} indicate the required time for enumerating all optimal active sets and the total time needed for enumerating all optimal active sets and creating their corresponding CRs. It can be seen that for low dimensional parameter spaces, the geometric approach succeeds to find all critical regions with a small computational time, which indicates its priority for such cases. The computational time of the suggested method in these cases, however, is not far from the computational time for geometric approaches, specially when building the critical regions is not of interest, e.g. in region-free explicit model predictive control. It can be seen that for cases having a higher number of parameters, the enumeration based methods show better performance in finding all CRs. While the method by [10] does not scale well with increasing control horizon in terms of the computational time, the suggested method is able to find the complete solution in a considerably shorter time.

V. CONCLUSION

In this paper, a new enumeration-based method for solving the mpQP problems was suggested based on exploiting the properties of full-dimensional adjacent critical regions. By excluding a large number of feasible but not optimal combinations of active constraints from the combinatorial tree, the computational time decreases dramatically while all critical regions in both non-degenerate and degenerate cases are guaranteed to be found. Furthermore, its enumerative nature makes it a suitable method for region-free explicit model predictive control purposes. Simulation results confirm the efficiency and priority of the suggested method for problems with a large number of parameters and constraints.

APPENDIX

CRITICAL REGION OF AN OPTIMAL ACTIVE SET WITH VIOLATION OF LINEAR INDEPENDENT CONSTRAINTS QUALIFICATION

Consider the multi-parametric quadratic program in (1) and the optimal active set $\mathcal{A}$ such that the rows of $G^{\mathcal{A}}$ are linearly dependent. Since $G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})^T$ is not invertible due to rank deficiency, the KKT conditions in (3) do not lead directly to (5a) and (5b), but only to a polyhedron expressed in the (λ, x)-space which can be lower-dimensional or full-dimensional. In the sequel, the conditions under which the critical region is forced to be lower-dimensional is investigated.
The optimality condition in (3a) yields $z = -H^{-1}(G^{\mathcal{A}})^T\lambda$. Inserting this to (3b), we will have the following equality which must hold for the optimal set $\mathcal{A}$:

$$-G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})^T\lambda - S^{\mathcal{A}}x - W^{\mathcal{A}} = 0 \tag{9}$$

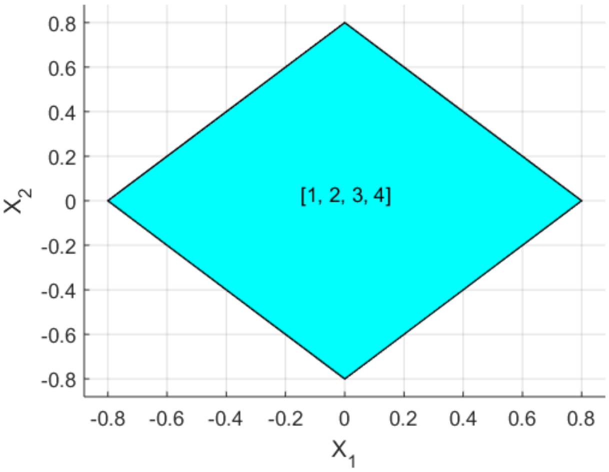


Fig. 6: Full-dimensional critical region of an optimal active set with LICQ violation

Denoting $-G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})^T\lambda = U\Sigma V^T$, using singular value decomposition, where U and V are unitary matrices and Σ is a rectangular diagonal matrix with non-negative real numbers on the diagonal, we can rewrite (9) as:

$$U\Sigma V^T\lambda = S^{\mathcal{A}}x + W^{\mathcal{A}} \tag{10}$$

Since U is unitary matrix and hence invertible, (10) reads:

$$\Sigma V^T\lambda = U^{-1}S^{\mathcal{A}}x + U^{-1}W^{\mathcal{A}} \tag{11}$$

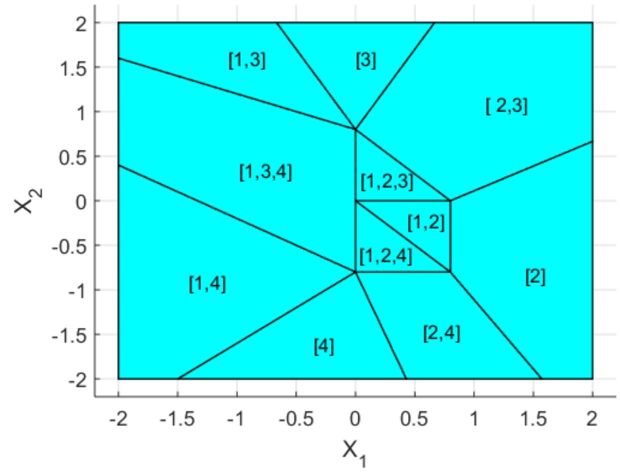
For a rank deficient matrix, Σ has p zero rows where p is difference between number of rows in $-G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})^T$ and its row rank. For simplicity, assume that $p = 1$. This means Σ has one zero row, and the same holds for ΣV^T, i.e., $\Sigma V^T = \begin{bmatrix}\tilde{\Sigma}\\0\end{bmatrix}$. Using this, we can rewrite the equality constraints of (11) as follows:

$$\tilde{\Sigma}\lambda = S_1x + W_1 \tag{12a}$$
$$0 = S_2x + W_2 \tag{12b}$$

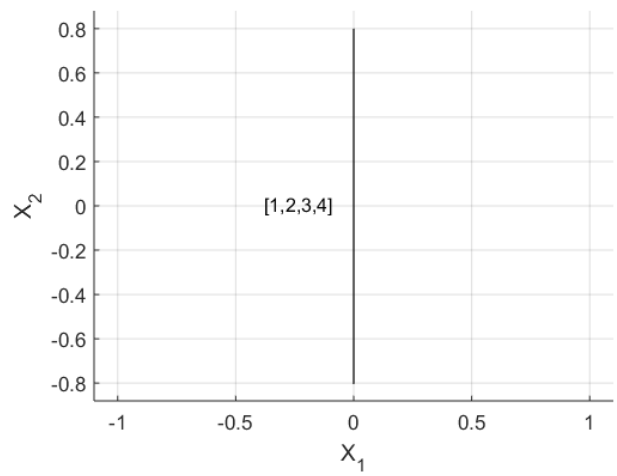
Where $\begin{bmatrix}S_1\\S_2\end{bmatrix} = U^{-1}S^{\mathcal{A}}$ and $\begin{bmatrix}W_1\\W_2\end{bmatrix} = U^{-1}W^{\mathcal{A}}$.

Regarding (12b), different cases may happen:

- If $S_2 \neq 0$, the critical region of optimal set $\mathcal{A}$ will be lower-dimensional since (12b) imposes a restriction on the values of state variables.
- If $S_2 = 0$ and $W_2 = 0$, (12b) evidently holds. Therefore, the critical region can be full-dimensional as there is no restriction on the values of state variables. This full-dimensional critical region can be obtained by a projection algorithm [21] which projects the polyhedron expressed in the (λ, x)-space, resulted from KKT conditions, onto the state space.
- The case $S_2 = 0$ and $W_2 \neq 0$ leads to infeasibility and thus to a contradiction since $\mathcal{A}$ is assumed to be a feasible active set.



(a) State space partitions



(b) Lower-dimensional critical region for optimal set $\mathcal{A} = [1, 2, 3, 4]$

Fig. 7: Optimal sets and their corresponding CRs for Example 2 with altered S matrix

Consider for illustration Example 2 and the optimal active set $\mathcal{A} = [1, 2, 3, 4]$ for which $G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})^T$ is rank deficient. For this case, we have $S_2 = 0$. Hence (12b) yields $0 = \begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + 0$ which does not impose any restriction on state variables. Therefore, the corresponding critical region can be full-dimensional. Fig. 6 shows this CR, obtained by firstly computing its representation in (λ, x)-space and then projecting it on x-space using the command *projection* of MPT3.

Let us now change the last row of matrix S in Example 2 to $\begin{bmatrix}1 & 1\end{bmatrix}$. From (12b) we have $0 = \begin{bmatrix}0.5 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + 0$. This enforces x_1 to be zero over the entire critical region of optimal set $\mathcal{A}$. Therefore, this CR is lower-dimensional (as x_1 is constant). Fig. 7a depicts the critical regions of the this new problem, and Fig. 7b shows the CR for $\mathcal{A} = [1, 2, 3, 4]$ which is obtained by projection.

Besides, the latter case can be considered as another example for the critical regions in Categ II and the associated

Theorem 4. Based on the definition of Categ II, the adjacent critical regions with optimal sets $\mathcal{A} = [1, 3, 4]$ and $\mathcal{A} = [1, 2, 3]$ lie in Categ II. The same holds for the critical regions with optimal sets $\mathcal{A} = [1, 3, 4]$ and $\mathcal{A} = [1, 2, 4]$. Hence, we expect to have SCS violation on the common facets of these CRs due to Theorem 4. Results from solving LP in (6) for $\mathcal{A} = [2, 3, 4]$ is consistent with this expectation as it yield $s^1 = 0$. This implies that constraint $\{1\}$ is weakly inactive on the entire critical region of $\mathcal{A} = [2, 3, 4]$ which exactly overlaps the critical region of $\mathcal{A} = [1, 2, 3, 4]$ shown in Fig. 7b.

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