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Abstract

The instants of time emission of photoelectrons generated by a detector immersed in an optical field constitute a point compound Poisson process. A complete definition of such a process is introduced to calculate some average values of the distribution. The shot noise due to this point process is also considered and we study the difference .between the deterministic and the random shot noises. They are completely defined by the set of their characteristic functions. We consider also the asymptotic properties of the shot noise and we show that for large mean density of the point process the fluctuations are not described by a Gaussian, but by a Gaussian compound random function. Thus the central limit theorem is not strictly valid. An experimental setup to obtain these fluctuations is described and some statistical properties of the asymptotic shot noise are presented.

I. INTRODUCTION

Several papers have been recently devoted to the study of statistical properties of optical fields. Particularly, the time instants t_i at which a photon is absorbed by a detector in an electromagnetic

field or a photoelectron is emitted constitute a point stochastic process, whose many properties are now clarified. This process can be defined by using either the classical representation of fields [1 - 3] or the quantum one.[4, 5]. The relations between the classical and quantum descriptions are now well established [6 - 8]. We also have many experimental results which confirm the theoretical calculations, particularly on photon counting [9 - 13] and photon coincidences.[14 - 17] One of the main results of those studies is that the time instants t_i define not a simple Poisson process, but a compound Poisson process, for which there are some correlation properties which appear, for example, in bunching effects.

In order to describe a point process, it is convenient to introduce the random function $N(t)$, number of random points t_i between an arbitrary origin of time and the time instant t. For Poisson processes, $N(t)$ has independent increments $dN(\theta)$, and is almost surely a discontinuous function [18]. The statistical properties of the random increments $dN(\theta)$, or number of points t_i between 0 and $0 + d\theta$, can be used to specify the probability distribution of the process $N(t)$. We will study this point in the case of a compound Poisson process. Particularly, we will compute expectation values, such as $E[dN(\theta_1)dN(\theta_1)\ldots dN(\theta_n)]$ and their Fourier transforms, which are used in many applications, and particularly for the photoelectron shot noise.

Usually shot noise appears when a point process passes through a linear time invariant system with impulse function $R(t)$. Therefore, the shot noise is described by the random function

$$X(t) = \int_{-\infty}^{+\infty} R(t - \theta)dN(\theta) \tag{1}$$

Nevertheless, for optical detectors this expression is not always convenient. In fact, the output to one photoelectron at the time instant θ is generally not a deterministic function $R(t - \theta)$, but a random impulsion $R(t - \theta, \omega)$. This is particularly true in the case of the photomultiplier (PM): for very low light intensity it is well known that at random times we observe pulses which are also random in shape. This fact is due to the amplification by secondary emission which is essentially a random process of amplification. Thus it is necessary to introduce the concepts of "deterministic" shot noise, described by Eq. (1), and "random" shot noise in which R is a function of a point ω in some probability space. In the two cases, we will define completely the statistical properties of these noises by the set of the characteristic functions of the random variable $X(t_1 \ldots X(t_n))$ for arbitrary t_i and n.

With the complete definition of the shot noise X(t), we can now explore its asymptotic properties. This problem is well known and very important in classical theory of shot noise, i.e., in the case where the $dN(\theta)$ are the increments of a stationary pure Poisson process with density ρ, and $R(t)$ a deterministic function. The result is that, for large ρ, $X(t)$ becomes a Gaussian random function, which is a particular case of the central limit theorem. In practice this result is very significant because ρ is often large, and

therefore the shot noise Gaussian. Therefore it is important to explore the same problem for photoelectrons, i.e., for a Poisson compound process.

At first we will show that, as for a pure Poisson process, the fluctuations of the deterministic or random shot noises become very small compared to the mean value. This means that for high light intensity the instantaneous output of the detector is a good estimation of this intensity. Nevertheless, if we study these fluctuations we obtain that, for large mean density of the point process, they are no longer described by a Gaussian stochastic process, but by a new process defined by its characteristic function and called compound Gaussian stochastic process. Therefore, the central limit theorem is no longer valid for photoelectrons. But the question arises if it is possible to separate the fluctuations from the mean value of the shot noise, because for large ρ the fluctuations are very small compared to the mean value. For this purpose, we describe an experimental method which allows this observation by using two optical detectors in the same optical field, as in the Hanbury Brown and Twiss experiment, and an adapted signal processing of the outputs of the detectors.

To explain this experiment, some properties of Gaussian compound random functions are reviewed. Particularly, we calculate some probability distributions of the asymptotic shot noise fluctuations for optical fields, obtained by superposition of thermal light and ideal laser light.

II. Definitions and some Properties of Poisson Compound Poisson Processes

Let us call t_i the time instants at which photoelectrons are emitted by a detector in an optical field. The instants t_i constitute a random point process [19]. The first derivation of some properties of such process by using a quantum theory of electromagnetic measurement was given by Kelley and Kleiner [5]. We will present more general results, but without discussing the microscopy physical origin of the process.

This point process is represented by the random function $N(t) = \int_0^t dN(\theta)$, where $dN(\theta)$ are the random increments of $N(t)$. The probability distribution of $N(t)$ can be defined if, for every n and every set of $\theta_1, \theta_2, \ldots \theta_n$ we know the probability distribution of the n-dimensional random variable $dN(\theta_1), dN(\theta_2), \ldots dN(\theta_2)$.

We characterize a Poisson compound point process by the following properties:

(i) $\Pr[dN(\theta) > 1] = o(d\theta)$, which means that the random variable $dN(\theta)$ takes only two values, 0 or 1. As a consequence we have $dN^k(\theta) = dN(\theta)$.

(ii) For distinct θ_1

$$\Pr\{[dN(\theta_1) = 1][dN(\theta_2) = 1] \ldots [dN(\theta_n) = 1]\} =$$

$$E[\rho(\theta_1)\rho(\theta_2) \ldots \rho(\theta_n)]d\theta_1 d\theta_2 \ldots d\theta_n, \tag{2}$$

where $\rho(\theta)$ is a stationary, nonnegative random function. Evidently, if $\rho(\theta)$ is a nonrandom constant, Eq. (2) defines a stationary pure Poisson process; if now $\rho(\theta)$ is a nonrandom function of time, we have a nonstationary pure Poisson process [20]. We obtain a compound (or a posteriori) process, because, for a given trial of the random function $\rho(\theta)$, the a posteriori distribution of $dN(\theta)$ is given by

$$\Pr\{[dN(\theta_1) = 1] \dots [dN(\theta_n) = 1] \mid \rho\} = \rho(\theta_1) \dots \rho(\theta_n) d\theta_1 \dots d\theta_n, \tag{3}$$

which is the definition of a Poisson nonstationary process. Obviously a priori distribution given by Eq. (2) is obtained by averaging a posteriori distribution given by Eq. (3).

In the case of optical fields, it is well established that $\rho(\theta)$ is proportional to the instantaneous light intensity [1,5]. For the following discussion, we will write

$$\rho(\theta) = \alpha F(\theta), \tag{4}$$

where α is a nonrandom parameter and $F(\theta)$ a nonnegative random function. By varying α, we describe the variations of the mean light intensity which can be obtained by various means and for the asymptotic problem we will study the case where $\alpha \to \infty$, which appears for fields with very large intensities.

For many problems, as for example shot noise, it is important to have expression of the moments of $dN(\theta)$, and we will now establish some properties of such moments.

A. First-Order Moment

The random variable $dN(\theta)$ takes only two values, 0 and 1. Therefore,

$$E[dN(\theta)] = \Pr[dN(\theta) = 1] = \alpha E[F(\theta)]d\theta. \tag{5}$$

If $F(t)$ is stationary, we can introduce the mean density ρ of the process and write

$$E[dN(\theta)] = \rho d\theta, \tag{6}$$

where ρ is evidently $E[\rho(\theta)] = \alpha E[F]$.

B. Second-Order Moment

The notation of the second-order moment $E[dN(\theta_1)dN(\theta_2)]$ has the meaning of a distribution on the space $\mathbf{R}^2(\theta_1 \times \theta_2)$. This distribution can be decomposed in two parts. First, if $\theta_1 \neq \theta_2$, we obtain, by using (2) and the same method as for the first-order moment,

$$E[dN(\theta_1)dN(\theta_2)]_1 = \alpha^2 E[F(\theta_1)F(\theta_2)]d\theta_1 d\theta_2 \tag{7}$$

Moreover, if $\theta_1 = \theta_2$ we have

$$E[dN^2(\theta_1)]_2 = E[dN(\theta_1)] = \alpha E[F(\theta_1)]d\theta_1, \tag{8}$$

which is proportional to $d\theta_1$ and therefore is a distribution on the curve $\theta_1 = \theta_2$ of the space $\mathbf{R}^2$. Therefore, we can write the complete expression of the moment

$$E[dN(\theta_1)dN(\theta_2)] = \{\alpha^2 E[F(\theta_1)F(\theta_2)] + \alpha E[F(\theta_1)]\delta(\theta_1 - \theta_2)\}d\theta_1 d\theta_2, \tag{9}$$

where δ is the Dirac distribution.

With this notation, $\int \int_I E[dN(\theta_1)dN(\theta_2)]$, where $I \in \mathbf{R}^2$, remains finite.

Now, if $F(t)$ is a wide sense stationary random function, this moment depends only on $\theta_1 - \theta_2$ and is

$$E[dN(\theta_1)dN(\theta_2)] = \{\alpha^2 \Gamma_F(\theta_1 - \theta_2) + \rho\delta(\theta_1 - \theta_2)\}d\theta_1 d\theta_2, \tag{10}$$

where $\Gamma(\tau)$ is the correlation function of $F(t)$. In this equation, the first term describes the Hanbury Brown, and Twiss effect, or bunching effect of photoelectrons, and the second is the contribution of a stationary Poisson process with density ρ.

For the following, it is convenient to introduce the random function $\tilde{F}(\theta)$ defined by

$$\tilde{F}(\theta) = F(\theta) - E[F], \tag{11}$$

which is clearly 0, if $F(\theta)$ is nonrandom (Poisson process). Thus, by using $\alpha E[F] = \rho$, we can introduce the second-order density $M(\theta_1, \theta_2)$, defined by

$$E[dN(\theta_1)dN(\theta_2)] = M(\theta_1, \theta_2)d\theta_1 d\theta_2, \tag{12}$$

and from Eqs. (10) and (11) we obtain

$$M(\theta_1, \theta_2) = \rho^2 + \alpha^2 \tilde{\Gamma}_F(\theta_1 - \theta_2) + \rho\delta(\theta_1 - \theta_2) \tag{13}$$

where $\Gamma_F(\tau)$ is the covariance function of $F(t)$.

The Fourier transform of $M(\theta_1, \theta_2)$ plays an important role in the following discussion. As we suppose that $F(t)$ is stationary, this Fourier transform can be written as

$$\gamma(\nu_1, \nu_2) = \rho^2 \delta(\nu_1)\delta(\nu_2) + \delta(\nu_1 + \nu_2)[\alpha^2 \tilde{\gamma}_F(\nu_1) + \rho], \tag{14}$$

where $\tilde{\gamma}_F(\nu)$ is the power spectrum of $\tilde{F}(t)$, or the Fourier transform of $\tilde{\Gamma}_F(\tau)$. Obviously, the term $\delta(\nu_1)\delta(\nu_2)$ means that $dN(t)$ has a nonzero mean value, and $\delta(\nu_1 + \nu_2)$ that it is stationary.

C. Third-Order Moment

By a simple extension of the preceding calculations, we obtain

$$E\left[\prod_{i=1}^{3} dN(\theta_i)\right] = \{\alpha^3 E\left[\prod_{i=1}^{3} F(\theta_i)\right] + \alpha^2 \sum_{[i,j]} \delta(\theta_i - \theta_j)E[F(\theta_i)F(\theta_k)] + \rho\delta(\theta_1 - \theta_2)\delta(\theta_1 - \theta_3)\}d\theta_1 d\theta_2 d\theta_3, \tag{15}$$

In the sum $\sum_{[i,j]}$ we have three terms corresponding to the three combinations $(\theta_1 - \theta_2)$, $(\theta_1 - \theta_3)$, and $(\theta_2 - \theta_3)$.

D. Fourth-Order Moment

For a general discussion, it is necessary to write explicitly this moment which has the structure of higher-order moments, but is considerably simpler to write down. By extension of previous calculations, this moment can be written as

$$E\left[\prod_{i=1}^{4} dN(\theta_i)\right] = M(\theta_1, \ldots, \theta_4) \prod_{i=1}^{4} d\theta_i, \tag{16}$$

where

$$M(\theta_1, \ldots, \theta_4) = \alpha^4 E[F(\theta_1) \ldots F(\theta_4)] + \sum_{1}^{4} T_i, \tag{17}$$

with

$$T_1 = \alpha^3 \sum_{P_1} \delta(\theta_i - \theta_j) E[F(\theta_i) F(\theta_k) F(\theta_l)],$$

where P_1 is the list of permutations of two distinct times θ_i taken in the set of four instants θ_k, $1 \leq k \leq 4$. They are$\binom{4}{2}$ such combinations. Similarly we have

$$T_2 = \alpha^2 \sum_{P_2} \delta(\theta_i - \theta_j) \delta(\theta_i - \theta_k) E[F(\theta_i) F(\theta_k)],$$

where P_2 are the combinations of three distinct times θ_i in the same set as just above? Their number is now $\binom{4}{3}$. Similarly we have

$$T_3 = \alpha^2 \sum_{P_G} \delta(\theta_i - \theta_j) \delta(\theta_k - \theta_l) E[F(\theta_i) F(\theta_k)],$$

where P_G means the three so called Gaussian combinations [21] (θ_1, θ_2) (θ_3, θ_4), (θ_1, θ_3) (θ_2, θ_4), and (θ_1, θ_4) (θ_2, θ_3). Finally the last term is

$$T_4 = \rho \delta(\theta_1 - \theta_2) \delta(\theta_1 - \theta_3) \delta(\theta_1 - \theta_4).$$

As for the second-order moment, it is necessary to write Eq. (17) with the function $\tilde{F}(\theta)$. After some algebra we obtain

$$M(\theta_1, \ldots, \theta_4) = \alpha^4 E[F(\theta_1) \ldots F(\theta_4)] + \sum_{1}^{4} T_i', \tag{18}$$

with

$$T_1' = \rho^4 + \rho^2 \alpha^2 \sum_{P_1} E[\tilde{F}(\theta_i) \tilde{F}(\theta_j)] + \rho \alpha^3 \sum E[\tilde{F}(\theta_i) \tilde{F}(\theta_j) \tilde{F}(\theta_k)]$$

$$T_2' = \alpha^4 E[\tilde{F}(\theta_1) \tilde{F}(\theta_2) \tilde{F}(\theta_3) \tilde{F}(\theta_4)] + \sum \delta(\theta_i - \theta_j) \left\{ \rho^3 + \rho \alpha^2 \sum E[\tilde{F}(\theta_i) \tilde{F}(\theta_k)] + \alpha^3 [\tilde{F}(\theta_i) \tilde{F}(\theta_k) \tilde{F}(\theta_l)] \right\}$$

$$T_3' = \sum \delta(\theta_i - \theta_j) \delta(\theta_i - \theta_k) \{\rho^2 + \alpha^2 E[\tilde{F}(\theta_i) \tilde{F}(\theta_l)]\}$$

$$T_4' = \sum \delta(\theta_i - \theta_j) \delta(\theta_k - \theta_l) \{\rho^2 + \alpha^2 E[\tilde{F}(\theta_i) \tilde{F}(\theta_k)]\} + \rho \delta(\theta_1 - \theta_2) \delta(\theta_1 - \theta_3) \delta(\theta_1 - \theta_4).$$

We will not write explicitly the Fourier transform of the function $M(\theta_1, \ldots, \theta_4)$, which is obviously not very simple. For our discussion it is only necessary to extract from this Fourier transform the part which is distributed on the "Gaussian manifolds" of the space $\mathbf{R}^4$, $\nu_1, \ldots, \nu_4$ [22]. These manifolds are defined by the equations

$$\nu_i + \nu_j = 0 \quad ; \quad \nu_k + \nu_l = 0 \, , \tag{19}$$

and in $\mathbf{R}^4$ there are evidently 3 different Gaussian manifolds corresponding to the 3 Gaussian permutations of (i, j, k, I).

Let us, for instance, consider the Gaussian manifold defined by

$$\nu_1 + \nu_2 = 0 \quad ; \quad \nu_3 + \nu_4 = 0 \, , \tag{20}$$

The distribution $g(\nu_1, \nu_3)$ on this manifold is the coefficient of $\delta(\nu_1 + \nu_2)\delta(\nu_3 + \nu_4)$ in the Fourier transform of $M(\theta_1, \ldots, \theta_4)$. By inspection of all the terms of Eq. (18), we find

$$g(\nu_1, \nu_3) = \alpha^4 h(\nu_1, \nu_3) + \rho\alpha^2[\tilde{\gamma}_F(\nu_1) + \tilde{\gamma}_F(\nu_3)] + \rho^2, \tag{21}$$

where $h(\nu_1, \nu_3)$ is the contribution on this manifold of the Fourier transform of $E[\tilde{F}(\theta_1) \ldots \tilde{F}(\theta_4)]$. This function $h(\nu_1, \nu_3)$ is called "Gaussian density" if it has the particular structure

$$h(\nu_1, \nu_3) = \tilde{\gamma}_F(\nu_1)\tilde{\gamma}_F(\nu_3), \tag{22}$$

i.e., a product of power spectra of $F(t)$. If Eq. (22) holds we obtain

$$g(\nu_1, \nu_3) = [\rho + \alpha\tilde{\gamma}_F(\nu_1)][\rho + \alpha\tilde{\gamma}_F(\nu_3)], \tag{23}$$

and, from Eq. (14) and the definition (22), we see that $g(\nu_1, \nu_3)$ is also a "Gaussian density." Evidently, if $h(\nu_1, \nu_3)$ is not a "Gaussian density," i.e., if Eq.22) does not hold, $g(\nu_1, \nu_3)$ cannot be a "Gaussian density." The same results can be easily found for the two other Gaussian manifolds of the space $\nu_1 \times \nu_2 \cdots \times \nu_4$, and by longer calculations extended for the higher-order moments of $dN(\theta)$ [23].

As an example, we can study the form of the function $h(\nu_i, \nu_j)$ for the thermal light, i.e., when the optical field is a Gaussian quasimonochromatic field. In this case, the function $F\theta)$ defined by Eq. (4)] can be written as

$$F\theta) = Z\theta)Z^*\theta), \tag{24}$$

where $Z\theta)$ is the analytic signal of a zero mean, Gaussian, and quasimonochromatic real rf. Therefore, we have

$$E[F(\theta_1) \ldots F(\theta_4)] = E[Z(\theta_1) \ldots Z(\theta_4)Z^*(\theta_1) \ldots Z^*(\theta_4)], \tag{25}$$

which can be expressed only with $\Gamma_Z(\tau) = E[Z(t)Z^*(t-\tau)]$ by classical expressions for the Gaussian case. From this expression we can prove that the distribution of the Fourier transform on the Gaussian manifold $(\nu_1 + \nu_2) = 0, (\nu_3 + \nu_4) = 0$ is given by Eq. (22) in which $\tilde{\gamma}_F(\nu)$ is the Fourier transform of $|\Gamma_F(\tau)|^2$, which for a Gaussian field is equal to $|\Gamma_Z(\tau)|^2$. Therefore, for the non-Gaussian random function $F(t)$, we have a "Gaussian density" on the Gaussian manifolds, and this result is true for higher-order moments.

The results of this section have a direct application to the deterministic shot noise described by Eq. (1). For instance, the general moment of $X(t)$ can be written

$$E[X(t_1)\ldots X(t_n)] = \int \cdots \int_{-\infty}^{\infty} R(t_1 - \theta_1)\ldots R(t_n - \theta_n)E[dN(\theta_1)\ldots dN(\theta_n)]. \qquad (26)$$

Particularly by using Eq. (6), we obtain

$$E[X(t)] = \rho G(0), \qquad (27)$$

where $G(\nu)$ is the frequency response , Fourier transform of $R(t)$. Similarly, we obtain from Eq. (11) the power spectrum of $X(t)$ by

$$\gamma_X(\nu) = G^2(0)\rho^2\delta(\nu) + \rho|G(\nu)|^2 + \alpha^2\tilde{\gamma}_F(\nu)|G(\nu)|^2. \qquad (28)$$

In this expression the first term is due to the nonzero mean value of $X(t)$, the second describes the shot noise of a Poisson process with density ρ, and the third, owing to the fluctuations of $F(t)$, describes the Hanbury Brown and Twiss effect. A similar expression, obtained by a different method, was already presented by Mandel. [24]

Moreover, the structure of the higher-order moments of $dN(\theta)$ can be used for the determination of some asymptotic properties of $X(t)]$ [22]. For a given mean density ρ of the point process, let us consider the limit of $X(t)$ when the time constant T of the linear system becomes very large ($\rho T >> 1$). The asymptotic properties of $X(t)$, after such very narrow band filtering, depend only on the structure of the Fourier transform of the higher-order moments of $dN(\theta)$ in the space $\nu_1 \times \cdots \times \nu_n$ If we have a "Gaussian density" on the "Gaussian manifolds" of this space, the asymptotic shot noise is described by a Gaussian stochastic process. That is the case for a stationary Poisson process [21], and for a compound Poisson process we have seen that the result depends on the properties of the random function $F(t)$ which describes the light intensity. Nevertheless, we have shown that for thermal light the result is still true and therefore the asymptotic shot noise is Gaussian.

III. Photoelectron Random Shot Noise

In this section, we study the statistical properties of the "random" shot noise which, as we have seen, describes more precisely the output of a detector of such a photomultiplier than the deterministic one

defined by Eq. (1). Evidently, the "deterministic" shot noise is a particular case of "random" shot noise, and therefore we will obtain general expression valid in the two cases [25].

The random function describing this noise can be written

$$X(t) = \int_{-\infty}^{\infty} dX(t; \theta), \tag{29}$$

in which the $dX(t; \theta)$ are random increments depending on the fixed parameter t. The increments $dX(t; \theta)$ are connected with the $dN(\theta)$ defined previously by the fact that $dX(t; \theta) = 0$ if $dN(\theta) = 0$ and $dX(t; \theta) = R_\theta(t - \theta; \omega)$ if $dN(\theta) = 1$, where $R_\theta(t; \omega)$ are a sequence of nonstationary random functions defined on the same probability space and depending on the parameter θ. They are assumed independent and with the same probability distribution. This assumption means that the random impulses due to different photoelectrons are independent, which can be considered in a first time as a good physical approximation. In the a posteriori distribution, i.e., for a given $F(t)$, the $dN(\theta)$ are independent (Poisson distribution) and our assumption on the $R_\theta(t; \omega)$ functions means that the increments $dX(t, \theta)$ are also independent. This allows the calculation of the a posteriori characteristic function. For that, let us consider n arbitrary-time instants $t_1, \ldots, t_n$ .. The a posteriori characteristic function is

$$\varphi[\{u_i\}|F] = E \left\{ \exp \left[i \sum_i^n u_i X(t_i) \right] \mid F \right\}. \tag{30}$$

We can write

$$\sum_{i=1}^n u_i X(t_i) = \int_{-\infty}^{\infty} \sum_{i=1}^n u_i dX(t_i; \theta) = \int_{-\infty}^{\infty} dX'(t_1, \ldots, t_n; \theta), \tag{31}$$

in which the increments dX' are still independent. This allows us to write the second characteristic function by the expression

$$\Psi[\{u_i\} \mid F] = \ln \varphi[\{u_i\} \mid F] = \int_{-\infty}^{\infty} \ln E[\exp(idX'(\theta))]. \tag{32}$$

From the structure of $dX(t; \theta)$, we deduce that

$$E[\exp(idX'(\theta))] = 1 + [A(\{u_i\}; \{t_i\}; \theta) - 1]\alpha F(\theta)d\theta, \tag{33}$$

where

$$A(\{u_i\}; \{t_i\}; \theta) = E \exp \left[i \sum_1^n u_i R_\theta(t_i - \theta) \right]. \tag{34}$$

By using Eq. (32) we obtain finally

$$\varphi[\{u_i\} \mid F] = \exp \alpha \int_{-\infty}^{\infty} [A(\{u_i\}; \{t_i\}; \theta) - 1]F(\theta)d\theta, \tag{35}$$

and the *a priori* characteristic function is obtained by taking the ensemble average on F, which gives

$$\varphi[\{u_i\}] = E \left\{ \exp \alpha \int_{-\infty}^{\infty} [A(\{u_i\}; \{t_i\}; \theta) - 1]F(\theta)d\theta \right\}. \tag{36}$$

As previously noticed, this expression is still valid in the case of the nondeterministic shot noise and the only difference is that we do not have to take the ensemble average in Eq. (34) because R is not random. For this case, a similar result was already presented by Hellstrom [26].

By a limited expansion of the characteristic function we can obtain the moments of $X(t)$. Thus, in the stationary case, the mean value of $X(t)$ is

$$E[X(t)] = \rho \int_{-\infty}^{\infty} E[R(t - \theta)]d\theta = \rho h(0), \tag{37}$$

in which $h(\nu)$ is the Fourier transform of

$$H(t) = E[R(t)]. \tag{38}$$

The result given by the Eq. (37) has the same form as the mean value of the deterministic shot noise given by a stationary Poisson process with the mean density ρ and a linear filter with impulse function $H(t)$, mean value of the random function $R(t)$. Thus, $H(t)$ can be considered as an equivalent impulse function.

The covariance function of $X(t)$ is given by an expansion limited to the second order and we obtain

$$E[X(t_1)X((t_2)] = \rho^2 h^2(0) + \rho \int_{-\infty}^{\infty} E[R(t_1 - \theta)R(t_2 - \theta)]d\theta + \alpha^2 \int \int_{-\infty}^{\infty} H(t_1 - \theta_1)R(t_2 - \theta_2)\tilde{\Gamma}_F(\theta_2 - \theta_1)d\theta_1 d\theta_2. \tag{39}$$

By Fourier transformation we obtain the power spectrum of $X(t)$ which can be written

$$\gamma_X(\nu) = \rho^2 h^2(0)\delta(\nu) + \rho|g(\nu)|^2 + \alpha^2\tilde{\gamma}_F(\nu)|h(\nu)|^2, \tag{40}$$

where $g(\nu)$ is the Fourier transform of $\int_{-\infty}^{\infty} E[R(t)R(t - \tau)]dt$.

It is interesting to compare this expression with Eq. (28), which is the power spectrum of the deterministic shot noise. For the random shot noise, $R(t)$, and, therefore, its Fourier transform $G(\nu)$, are random, and we have

$$|g(\nu)|^2 = E[|G(\nu)|^2]. \tag{41}$$

Moreover, from Eq. (38) we obtain

$$|h(\nu)|^2 = |E[G(\nu)]|^2. \tag{42}$$

and, therefore, $|g(\nu)|^2$ and $|h(\nu)|^2$ are in general quite different. In particular the last term can disappear completely if $E[G(\nu)] = E[R(t)] = 0$. Evidently, that is not in general the case, if the randomness of $R(t)$ only due to the fluctuations of secondary emission in a photomultiplier.

Now we will consider some asymptotic properties of the shot noise $X(t)$. In the previous section, we have seen that the deterministic shot noise becomes Gaussian for very large time constant of the linear system and with some conditions on the light intensity. Now, we will study the asymptotic problem

appearing if α, i.e., the mean light intensity, or the mean density of points [see Eq. (3)], becomes very large.

For this discussion let us introduce the two random functions $Y_\alpha(t)$ and $Z_\alpha(t)$ defined from $X(t)$ by

$$Y_\alpha(t) = \tilde{X}(t) - E[X((t)], \tag{43}$$

$$Z_\alpha(t) = X(t) - \alpha F_R(t), \tag{44}$$

where $F_R(t)$ is

$$F_R(t) = \int_{-\infty}^{\infty} H(t - \theta)F(\theta)d\theta = \int_{-\infty}^{\infty} E[R(t - \theta)]F(\theta)d\theta. \tag{45}$$

At first, let us suppose that $F(t)$ is a nonrandom function, i.e., that the point process is a pure nonstationary Poisson process. In this case, we obtain from Eq. (35) $E[X(t)] = \alpha F_R(t)$, and, therefore,

$$Y_\alpha(t) \equiv Z_\alpha(t). \tag{46}$$

Moreover, the standard deviation of $Y_\alpha(t)$ is

$$E[Y_\alpha^2(t)] = \alpha \int E[R^2(t - \theta)]F(\theta)d\theta). \tag{47}$$

If we assume that $F(\theta)$ is bounded and $R(t)$ square integrable, the integral in Eq. (47) is finite and we obtain, therefore,

$$\lim_{\alpha \to \infty} \text{q.m.} \frac{Y_\alpha(t)}{\alpha} = 0, \tag{48}$$

where the quadratic mean limit is understood. This means that

$$\lim_{\alpha \to \infty} \text{q.m.} \frac{X(t)}{\alpha} = F_R(t), \tag{49}$$

This well-known result means that for very large α the fluctuations of the shot noise are suppressed. Evidently as convergence in the quadratic mean gives convergence in distribution, the characteristic function of $X(t)/\alpha$ converges to the characteristic function of $F_R(t)$.

But for the deterministic shot noise there is also a well-known result concerning the fluctuations of $Y_\alpha(t)$ [27]. For large α, $Y_\alpha(t)/\alpha^{1/2}$ converges in distribution to a Gaussian random function defined by the covariance

$$\Gamma(t_1, t_2) = \int_{-\infty}^{\infty} R(t_1 - \theta)R(t_2 - \theta)F(\theta)d\theta. \tag{50}$$

By using Eq. (35) it is possible to show that this particular form of the central limit theorem is still valid for the random shot noise and the covariance of the limit process is now

$$\Gamma(t_1, t_2) = \int_{-\infty}^{\infty} E[R(t_1 - \theta)R(t_2 - \theta)]F(\theta)d\theta. \tag{51}$$

Now let us consider the same problem for a stationary Poisson compound process, i.e., for the case where $F(t)$ is a stationary random function. In this case $Y_\alpha(t)$ and $Z_\alpha(t)$ are different and Eq. (46) must be replaced by

$$Y_\alpha(t) = Z_\alpha(t)T + \alpha \tilde{F}_R(t), \tag{52}$$

where $\tilde{F}_R(t) = F_R(t) - E[F_R(t)]$.

It is easy to show that Eq. (47) and Eq. (48) are no longer valid, because of the term in α^2 in Eq. (9). Therefore, we will calculate $E[Z_\alpha^2(t)]$ and by using Eqs. (29), (39), and (45) we obtain

$$E[Z_\alpha^2(t)] = \alpha E[F] \int_{-\infty}^{\infty} E[R^2(t-\theta)]d\theta. \tag{53}$$

From this equation we deduce that

$$\lim_{\alpha \to \infty} \text{q.m.} \frac{Z_\alpha(t)}{\alpha} = 0. \tag{54}$$

Thus we obtain from Eq. (44)

$$\lim_{\alpha \to \infty} \text{q.m.} \frac{X(t)}{\alpha} = F_R(t). \tag{55}$$

but this equation is quite different from Eq. (49), because here $F_R(t)$ is a random function. Evidently the interpretation is the same and as previously the fluctuations of the shot noise are suppressed. Likewise the characteristic function of $X(t)/\alpha$ converges to the characteristic function of $F_R(t)$. Moreover, we obtain from Eq. (52) that

$$\lim_{\alpha \to \infty} \text{q.m.} \frac{Y_\alpha(t)}{\alpha} = \tilde{F}_R(t), \tag{56}$$

which is to compare with Eq. (48), because for a nonrandom $F(t)$, $\tilde{F}_R(t) = 0$.

Now let us study as previously the fluctuations of the shot noise. Evidently they are no longer described by $Y_\alpha(t)$, because we see from Eq. (56) that $Y_\alpha/\alpha^{1/2}$ is not finite when $\alpha \to \infty$. But we see from Eq. (53) that $Z_\alpha/\alpha^{1/2}$ remains finite for $\alpha \to \infty$, and therefore it is interesting to study the stochastic limit of this random function. The characteristic function $\varphi_\alpha[\{u_i\}]$ of $Z_\alpha/\alpha^{1/2}$ is obtained directly from Eq. (36) and after simple calculations we obtain that

$$\lim_{\alpha \to \infty} \varphi_\alpha[\{u_i\}] = \varphi_\infty[\{u_i\}], \tag{57}$$

where

$$\varphi_\infty[\{u_i\}] = E \exp \left[-(1/2) \sum_{ij} u_i u_j \int_{-\infty}^{\infty} E[R(t_i - \theta)R(t_j - \theta)F(\theta)d\theta \right]. \tag{58}$$

If $F(t)$ is nonrandom, we do not have to take the first expectation value in this equation and therefore, as we have seen previously, $\varphi_\infty[\{u_i\}]$ is the characteristic function of a Gaussian random function. If $F(t)$ is random, which.is generally the case in optical problems, $\varphi_\infty[\{u_i\}]$ is the characteristic function

of a compound Gaussian random function. Therefore, the asymptotic fluctuations of the shot noise are no longer Gaussian, and the central-limit theorem is not strictly valid. The same kind of situation appears in the problem of the limit of a random number of independent random variables [28].

IV. PROPERTIES OF THE ASYMPTOTIC FLUCTUATIONS OF THE SHOT NOISE

COMPOUND GAUSSIAN PROCESSES

The random function $M(t)$ which describes the asymptotic fluctuations of the shot noise is completely defined by the characteristic functions of $M(t_1), \ldots, M(t_n)$ given by Eq. (58). This function can be written

$$\varphi[\{u_i\}] = E \exp\left[-(1/2)\sum_{i,j} u_i u_j \Gamma_{ij}\right],$$ (59)

where Γ_{ij} is a random variable defined by

$$\Gamma_{ij} = \int_{-\infty}^{\infty} F(\theta) E[R(t_i - \theta)R(t_j - \theta)]d\theta.$$ (60)

The random function $M(t)$ is called a compound Gaussian process because if $F(\theta)$ is nonrandom, $M(t)$ is a pure zero mean Gaussian random function. From the probability distribution of the random function $F(\theta)$ which describes the fluctuations of the light intensity, we can in principle obtain the probability distribution of the n^2 random variables Γ_{ij} of which evidently only $n(n-1)$ are different. This distribution is defined by the characteristic function

$$\varphi_\Gamma[\{v_{i,j}\}] = E\left[\exp i \sum_{i,j} v_{i,j}\Gamma_{ij}.\right],$$ (61)

With this function, Eq. (59) can be written

$$\varphi[\{u_i\}] = \varphi_\Gamma\left[\{(i/2)u_i u_j\}\right].$$ (62)

This equation defines completely $M(t)$, and we will at first consider the I-dimensional case. Thus the

$$\varphi(u) = \varphi_\Gamma[(i/2)u^2],$$ (63)

where $\varphi_\Gamma(u)$ is the characteristic function of the random variable

$$\Gamma = \int_{-\infty}^{\infty} F(\theta) E[R^2(t - \theta)]d\theta.$$ (64)

The probability distribution of $M(t)$ is obtained by taking the Fourier transform of $\varphi(u)$, and we obtain

$$p(x) = (2\pi)^{-1/2} \int_{0}^{\infty} [1/y^{(1/2)}] \exp -(x^2/2y) \, p_\Gamma(y)dy,$$ (65)

where $p_\Gamma(y)$ is the probability distribution of Γ. We have evidently $p_\Gamma(y) = 0$ for $y < 0$, because Γ is a positive random variable. We have the same kind of equation for the probability distribution of the

photon counting, in which the kernel is a Poisson kernel instead of a Gaussian one, and therefore we can apply the same kinds of methods. Particularly, it is in general very difficult to calculate the probability distribution of the random variable Γ, integral of a random function. Thus we will suppose that the correlation time of $F(t)$ is much greater than the average time constant of the random filter of IR $R(t)$. In this case the integration can be omitted, and Γ has the same statistical distribution as the light intensity F.

To perform the calculations we will consider some examples of optical fields.

At first, let us suppose that the optical field has a constant light intensity (which is for example the case of the ideal amplitude stabilized laser light with only phase fluctuations). Thus $\Gamma = a$ and $\varphi_\Gamma(u) = \exp(iau)$. Therefore $\varphi(u) = \exp[-(1/2)au^2]$, which is quite obvious, because if $F(\theta)$ is constant, the compound Gaussian process becomes a pure Gaussian process.

If now the optical field is created by a thermal source (natural light, or pseudothermal light), it is well known that the probability distribution of the light intensity is $a^{-1}\exp(-x/a)$ and the characteristic function $1/(1 - iau)$. Thus from Eq. (63) we obtain

$$\varphi(u) = \frac{1}{1 + (1/2)au^2} \tag{66}$$

and the probability distribution is

$$p(x) = (2a)^{-1/2}\exp[-(2/a)^{1/2}|x|]. \tag{67}$$

Finally we will consider the superposition of the two previous fields. In this case the probability distribution of the light intensity is [29]

$$p_Y(y) = (m+1)\exp-[(m+1)y + m]\mathrm{I}_0[2(m(m+)y)^{1/2}]. \tag{68}$$

where $m = I_l/I_t$, ratio of the mean light intensities of the laser and the thermal fields, and y is a reduced variable, ratio of the instantaneous intensity and the total mean intensity $I_l + I_t$. The characteristic function of this distribution is obtained by Fourier transformation

$$\varphi_\Gamma(u) = \frac{m+1}{m+1-iu}\exp\left(\frac{-imu}{m+1-iu}\right). \tag{69}$$

and therefore the characteristic function of the asymptotic shot noise is

$$\varphi(u) = \frac{m+1}{m+1+u^2/2}\exp\left(\frac{-mu^2}{2(m+1+u^2/2)}\right). \tag{70}$$

We have unfortunately no simple explicit form of the Fourier transform of $\varphi(u)$.

Applying the same procedure similar but much more complicated calculations can be presented in od=rder to obtain the multidimensional probability distributions. This paper was more devoted to the principles of the method that to explicit algebraic calculations of some particular cases.

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