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Abstract

This paper discusses the comparison between the class of spherically invariant processes and a particular class of Gaussian compound processes. We give a simple expression for the probability distribution and calculate some expectation values. The comparison shows that spherically invariant processes are slightly more general.

I. Introduction

The concept of spherically invariant processes (SIP) has been recently introduced by Vershik [l], and in an interesting paper by Blake and Thomas [2] some important properties of such processes have been explored. In particular they have calculated the multivariate characteristic function and distribution function and studied the relationships to the Gaussian processes. Spherically invariant processes are

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defined by their family of finite dimensional distribution functions. The multivariate characteristic function of these distributions are

$$\phi(\{u_i\}) = \phi(u_1, \ldots, u_n) = f\left[(1/2)\sum_{ij} c_{ij} u_i u_j\right], \tag{1}$$

and Gaussian stochastic processes appear as a particular case for which $f(s) = exp(--s)$.

Evidently the function f in (1) is not completely arbitrary because $\phi(\mathbf{u})$ must be a characteristic function, i.e., satisfy some positive definiteness property. This point was not discussed in [2] and will appear in the following.

However, the main idea of this paper is that (1) is also the characteristic function of a particular case of Gaussian compound stochastic processes (GCP), and therefore it is interesting to explore the relationship between this class of processes and the spherically invariant one. This comparison will allow us to give another form of probability density than that in [2]. This form is simpler and with it we can very easily understand some fundamental properties of such processes

II. On a particular case of Gaussian Compound Process

The concept of compound processes has not yet been used very often in problems of communication theory, and there are only a few results concerning such processes in standard textbooks [3], [4].

Nevertheless, Poisson compound processes are used in some problems of statistical optics [5], [6]. In particular, the point process formed by the time instants t_i where a photoelectron is emitted by a cathode immersed in an optical field is generally a Poisson compound process. Such a process can be described as a Poisson process whose density is a random function of time. Given this function, or in a posteriori distribution, the process is a nonstationary true Poisson process, but in the a priori distribution, it is evidently no longer a Poisson process. Many properties of such processes have been studied, particularly to describe experiments of photon counting or photon coincidences, and there is generally good agreement with experimental results.

General Gaussian compound processes have not yet been completely defined. They would be Gaussian processes whose covariance function is random. Some GCP can appear as a limit of compound Poisson processes by an extension of the central limit theorem, and we will discuss this problem in another paper. For our purpose of comparison with SIP, it is sufficient to consider the simplest classes of GCP that are Gaussian with only a random standard deviation.

More precisely, let us consider a stationary Gaussian random process $G(t)$ defined by its correlation function

$$E|G(t)G(t-\tau)] = \sigma^2\ \tilde{\gamma}(\tau) \tag{2}$$

where σ^2 is the standard deviation of $G(t)$ and $\tilde{\gamma}(\tau)$ its normalized correlation function. If we assume now that σ^2 is a positive random variable A, $G(t)$ becomes a new stationary process $X(t)$, which is Gaussian only for a given value of A, i.e., in a posteriori distribution, but is a GCP in the a priori distribution.

A. Characteristic Function

We know that a Gaussian process is completely defined by its familv of finite dimensional distribution functions. Let us n arbitrary time instants $t_1, \ldots, t_n$, and the n dimensional random variable $\mathbf{X} = [X(t_1), \ldots, X(t_n)]^T$. Assuming that $X(t)$ has a zero mean value, the a posteriori characteristic function of $\mathbf{X}$ for a given A is evidently

$$\phi(u_1, \ldots, u_n \,|A) = \exp\left[-\frac{A}{2}\sum_{ij}\gamma_{ij}u_iu_j,\right] \tag{3}$$

where

$$\gamma_{ij} = \tilde{\gamma}_{ij}. \tag{4}$$

and $\tilde{\gamma}(t)$ is a given normalized correlation function.

The a priori characteristic function of $\{X(t_i)\}$ is obtained by averaging (3), and

$$\phi(\mathbf{u}) = \phi_A\left[\frac{i}{2}\sum_{ij}\gamma_{ij}u_iu_j,\right] \tag{5}$$

is evidently obtained where $\phi_A(u)$ is the characteristic function of the random variable A.

The characteristic functions described by (1) and (5) have exactly the same form. Therefore, this kind of GCP is certainly a spherical invariant process. But in (5) we have more information than in (1). In fact, the function $\phi_A(u)$ must satisfy some conditions as being a characteristic function of a positive random variable. The question therefore arises of knowing if the class of spherically invariant processes is more general than that of GCP whose characteristic function is defined by (5). Evidently if A is a nonrandom variable $\phi_A(u) = \exp(iua)$, (5) becomes the characteristic function of a pure Gaussian process.

B. Probability Density Distribution

One important problem discussed by Blake and Thomas was the inversion of the characteristic function defined by (1), which was solved after some calculations by means of the Hankel transform. For the characteristic function defined by (5), this problem, which can always be solved by the same method, has, however, a simpler solution.

The a posteriori probability density $\{f(\mathbf{x})\}$ is the Fourier transform of $\phi(\mathbf{u}|A)$ defined by (3) and is evidently the classical Gaussian distribution function $g_A(\mathbf{x})$ defined by the second-order moments $A\gamma_{ij}$. The a priori distribution is obtained by averaging, and can be written as

$$p(\mathbf{x}) = \int_0^\infty q(a) g_a(\mathbf{x}) da \tag{6}$$

where $q(a)$ is the probability density of the random variable A and

$$g_a(\mathbf{x}) = (2\pi)^{-n/2} |\Gamma_a|^{-1/2} \exp[-(1/2)\mathbf{x}^T \mathbf{\Gamma}_a^{-1} \mathbf{x}]. \tag{7}$$

As the elements of $\mathbf{\Gamma}_a$ are

$$\mathbf{\Gamma}_{a;i,j} = a\gamma_{i,j} \tag{8}$$

this expression can be written

$$g_a(\mathbf{x}) = (2\pi)^{-n/2} |\tilde{\Gamma}|^{-1/2} a^{-n/2} \exp\left[\frac{1}{2a} \mathbf{x}^T \tilde{\mathbf{\Gamma}}^{-1} \mathbf{x}\right] \tag{9}$$

where $\tilde{\mathbf{\Gamma}}$ is the matrix defined by γ_{ij}. For the one-dimensional case, we obtain

$$p(x) = \frac{1}{2\pi} \int_0^\infty q(a) \frac{1}{\sqrt{a}} \exp(-x^2/(2a)) da. \tag{10}$$

C. Conditional Expectation Values

As pointed out by Blake and Thomas, a fundamental property of the SIP for application to estimation problems is that the conditional expectation value $E[X_1|x_2, \ldots, x_n]$ is, as in the Gaussian case, a linear form in $x_2, \ldots, x_n$. Because they are also spherical invariant, this property is evidently true for our class of GCP, but it can be shown by a simpler way. Indeed, this conditional expectation value is defined by

$$E[X_1|x_2, \ldots, x_n] = \frac{\int x_1 p(x_1, \ldots, x_n) dx_1}{\int p(x_1, \ldots, x_n) dx_1} = \frac{\int\int x_1 q(a) g_a(x_1, \ldots, x_n) da dx_1}{\int\int q(a) g_a(x_1, \ldots, x_n) da dx_1} \tag{11}$$

However, we know as a classical result on the Gaussian random variables that

$$\frac{\int x_1 g_a(x_1, \ldots, x_n) dx_1}{\int g_a(x_1, \ldots, x_n) dx_1} = \sum_2^n m_i x_i \tag{12}$$

where the m_i coefficients depend only on the γ_{ij} and not on a. By comparing (11) and (12), we obtain

$$E[X_1|x_2, \ldots, x_n] = \sum_2^n m_i x_i \tag{13}$$

and therefore, we have the same relationship with the same coefficients for Gaussian and this class of GCP.

D. Higher-Order Moments

For some problems it is interesting to calculate the higher order moments of $X(t)$ such $E[X(t_1)\ldots X(t_n)]$. For that it is possible to use directly (5) or (l), but it is more convenient to use (3) and calculate a posteriori moments, which are given by the standard formulas for the Gaussian case

$$E[X_1 \ldots, X(_{2k+1)}|A] = 0 \tag{14}$$

$$E[X_1 \ldots, X_{2k)}|A] = \sum_P \prod_{i,j} A^k \gamma_{i,j}; \tag{15}$$

this is the sum of $(2n-1)!!$ terms obtained by the different permutations of all the different pairs $\gamma_{i,j}$. For the a priori distribution, (14) still holds and (15) becomes

$$E[X_1 \ldots, X_{2k)}] = E[A^k] \sum_P \prod_{i,j} \gamma_{i,j} \tag{16}$$

It is interesting to calculate the Fourier transform of $E[X_1 \ldots, X_{2k)}]$, which can be expressed by using (4) and the spectral density $\gamma(\nu)$, Fourier transform of $\tilde{\gamma}(\nu)$. Thus we obtain

$$\gamma(\nu_1, \ldots, \nu_{2n}) = E[A^n] \sum_P \prod_{i,j} \gamma(\nu_i)\delta(\nu_i + \nu_j). \tag{17}$$

This expression shows that $\gamma(\nu_1, \ldots, \nu_{2n})$ is non zero only on the Gaussian manifolds of the space $\nu_1 \times \ldots \times \nu_{2n}$.

In this space there are $(2n-1)!!$ distinct such manifolds [7] obtained by permutations of the ν_i in

$$\nu_1 + \nu_2 = 0; \nu_{2i-1} + \nu_{2i} = 0; \ldots; \nu_{2n-1} + \nu_{2n} = 0. \tag{18}$$

The difference from the pure Gaussian case is that the density on the manifold does not have a Gaussian value that is a product of spectral densities. This density would be a Gaussian density only if

$$E[A^k] = E[A]^k, \ \forall k, \tag{19}$$

which indicates that A is almost surely a nonrandom constant. In this case, $X(t)$ evidently becomes a Gaussian process.

This fact has a strong connection with ergodic properties. Vershik has shown that a spherically invariant process that is ergodic is also Gaussian, and this property can be directly obtained by calculation of the density on the Gaussian manifolds.

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We can also consider the problem of linear filtering. It is well known that a Gaussian process remains Gaussian after linear filtering. For a GCP as defined by (5) this property is still true and we have the same equation as in the pure Gaussian case with different γ_{ij}. Moreover, a GCP cannot become Gaussian only by a linear filtering as in the case of many other non-Gaussian process, which become Gaussian after a very narrow-band linear filtering. As it has already been shown for other processes, this fact is a consequence of the non-Gaussian density on the Gaussian manifolds [7].

III. Relation with Spherically Invariant Processes

As previously noticed, the function $f(s)$ and the coefficients c_{ij} appearing in (I) must satisfy some conditions because $\phi(u)$ is a characteristic function. But these general conditions are not simple. Conversely, the class of functions $f(s)$ and coefficients c_{ij} for which (1) defines a GCP is simple to define. Indeed, if we compare (1) and (5), we obtain for a GCP the relation

$$f(s) = \phi_A(is) = \int_0^\infty \exp(-as)q(a)da \tag{20}$$

where $q(a)$ is nonnegative. Moreover, we have from (1) and (20)

$$c_{ij} = \frac{-1}{f'(0)} E[X_i X_j] = \frac{1}{E(A)} E[X_i X_j] \tag{21}$$

where $E(A)$ is the expectation value of the non negative random variable with probability density $q(a)$. Thus the coefficients c_{ij} are as $E[X_i X_j]$ positive definite.

Therefore, the SIP defined by (1) is a GCP if and only if $f(s)$ is the Laplace transform of the probability density of a nonnegative random variable and the coefficients c_{ij} are positive definite.

Conversely, a stochastic process can be a SIP and not a GCP. To explore this point, let us assume that $f(s)$ can be written by (20) in which $q(a)$ is not nonnegative. From (1) we obtain

$$\phi(u) = \int_0^\infty \exp[(-a/2) \sum_{ij} c_{ij} u_i u_j] q(a)da, \tag{22}$$

which gives

$$E[X_i X_j] = \bar{A} c_{ij} = c_{ij} \int_0^\infty a q(a)da. \tag{23}$$

If $\bar{A} > 0$, the c_{ij}'s are definite positive, and we can write the Fourier transform of $\phi(u)$ by

$$\rho(\mathbf{x}) = \int_0^\infty g_a(\mathbf{x}) q(a)da \tag{24}$$

where $g_a(\mathbf{x})$ is defined by (7).

The only condition is that $\rho(\mathbf{x})$ is a multivariate probability density and we can have $\rho(\mathbf{x}) > 0$ without having $q(a) > 0$. Therefore, the class of SIP is more general than that of CGP studied in Section II.

To show the last point we will only construct one example for the one-dimensional case by using (10). Let us consider the function

$$q(a) = 2(1+\alpha)^{-a/2} - \alpha\delta(a - 1/2\pi) \ , \alpha > 0, \tag{25}$$

which by substituting in (10) gives

$$p(x) = (1+\alpha)\exp(-2|x|) - \alpha\exp(-\pi x^2) \tag{26}$$

If we have

$$\alpha < \frac{1}{\exp(1/\pi) - 1} \tag{27}$$

we obtain $p(x) > 0$, and $q(a)$ is clearly not a probability distribution.

Evidently, it would be interesting to specify precisely the condition on $q(a)$ of (24) such that $p(x)$ is a probability distribution. But this problem appears to be very difficult. The same unsolved problem appears in statistical optics: conditions about the photocounting distribution to obtain a Poisson compound process.

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