

# Eigenlogic

Zeno Toffano

[zeno.toffano@centralesupelec.fr](mailto:zeno.toffano@centralesupelec.fr)

CentraleSupélec, Gif-sur-Yvette, France  
Laboratoire des Signaux et Systèmes, UMR8506-CNRS  
Université Paris-Saclay, France

université  
PARIS-SACLAY



CentraleSupélec



# statement

## facts in quantum physics

---

Quantum Physics started in 1900 with the quantification of radiation (Planck's constant  $h = 6.63 \cdot 10^{-34}$  J.s)

It is the most successful theory in physics and explains: nuclear energy, semiconductors, magnetism, lasers, quantum gravity...

The theory has an increasing influence outside physics: *computer science, AI, communications, biology, cognition...*

Nowadays there is a great quantum revival (second quantum revolution) which addresses principally:

*quantum entanglement* (Bell inequalities, teleportation...), *quantum superposition* (Schrödinger cat), *decoherence non-locality, non-separability, non-contextuality, non-classicality, non-...*

A more basic aspect is *quantification*: a measurement can only give one of the (real) **eigenvalues** of an **observable**.

The associated eigenvector is the resulting *quantum state*.

Measurements on non-eigenstates are indeterminate, the outcome probability is given by the *Born rule*.

The eigenvalues are the *spectrum* (e.g. energies for the Hamiltonian).

In physics it is natural to “work” in different representation *eigenspaces* (of the considered observable) e.g:

semiconductors “make sense” in the reciprocal lattice (momentum space  $\hat{p}$ , spatial Fourier transform of coordinate space),

magnetic materials in topological spin spaces  $\hat{S}$

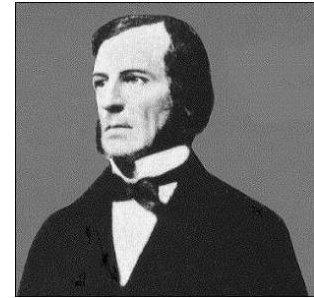
nuclear particles in homogeneous spaces (Lie groups)...

Can logic “fit” in these “natural” pictures ?

Can we, in some way, “adapt” logic to physics ?

## history

# George Boole : the “0” and the “1”



George Boole (1815-1864) gave a mathematical symbolism through the two numbers {0,1} respectively representing the "false" or "true" character of a proposition [a].

Algebra used by Boole was not what today is called "Boolean algebra" but an arithmetic form using idempotent symbols [c].

The symbol  $x$  is idempotent (for the product) if:

$$x^2 = x \quad \text{which can also be written as} \quad x(1 - x) = 0$$

This last equation is verified by only two possible values : 0 and 1,  
“1” meant for Boole the *Universe Class*.

The system is complete:  $x + (1 - x) = 1$

The equation  $x^2 = x$  was considered by George Boole the “fundamental law of thought”! [b]

Boole’s approach is operational as pointed out by Theodore Halperin [c]:  $x$  acts as a selection operator.  
George Boole named these symbols “elective”.

As will be emphasized here the algebra of elective symbols can also be interpreted as an algebra of commuting projection operators and used for developing propositional logic in a linear algebra framework, see also [d].

[a] George Boole, “*The Mathematical Analysis of Logic, being an Essay Towards a Calculus of Deductive Reasoning*” (1847)

[b] George Boole, “*An investigation of Laws of Thought on which are founded the mathematical theories of Logic and Probabilities*” (1854).

[c] Theodore Halperin, “Boole’s Algebra isn’t Boolean Algebra. A Description Using Modern Algebra, of What Boole Really Did Create” J. Gasser (ed.), *A Boole Anthology*, 61-77, Kluwer, (2000)

[d] Zeno Toffano, “Eigenlogic in the spirit of George Boole”, [ArXiv:1512.06632](https://arxiv.org/abs/1512.06632) (2015)

# history

## truth tables : Pierce, Wittgenstein and Post

(excerpts from [\*])

“Truth tables were introduced by Charles Sanders Peirce (1839-1914) in the early 1880's [a] which attracted little attention at the time”

“Truth tables were rediscovered and tautologies discovered, simultaneously and independently by Ludwig Wittgenstein (1889-1951) [b]

and by Emil Leon Post (1897-1954) [c] ”

Post also established the *consistency* and *completeness* of propositional calculus.

Logical truth tables (*semantics*  $\models$ ) are axiomatic at the same level as logical connective canonical forms (*syntax*  $\vdash$ ).



**Wittgenstein's Tractatus : first complete formulation of the truth tables for the 16 logical connectives for two elementary propositions ( $p, q$ ),**

5.101 The truth-functions of a given number of elementary propositions can always be set out in a schema of the following kind:

|        |            |                                                                                                    |
|--------|------------|----------------------------------------------------------------------------------------------------|
| (TTTT) | ( $p, q$ ) | Tautology (If $p$ then $p$ , and if $q$ then $q$ ). ( $p \supset p \cdot q \supset q$ )            |
| (FTTT) | ( $p, q$ ) | In words: Not both $p$ and $q$ . ( $\sim(p \cdot q)$ )                                             |
| (TFTT) | ( $p, q$ ) | „ „ : If $q$ then $p$ . ( $q \supset p$ )                                                          |
| (TTFT) | ( $p, q$ ) | „ „ : If $p$ then $q$ . ( $p \supset q$ )                                                          |
| (TTTF) | ( $p, q$ ) | „ „ : $p$ or $q$ . ( $p \vee q$ )                                                                  |
| (FFTT) | ( $p, q$ ) | „ „ : Not $q$ . ( $\sim q$ )                                                                       |
| (FTFT) | ( $p, q$ ) | „ „ : Not $p$ . ( $\sim p$ )                                                                       |
| (FTTF) | ( $p, q$ ) | „ „ : $p$ or $q$ , but not both. ( $p \cdot \sim q \vee q \cdot \sim p$ )                          |
| (TFFT) | ( $p, q$ ) | „ „ : If $p$ then $q$ , and if $q$ then $p$ . ( $p \equiv q$ )                                     |
| (TFTF) | ( $p, q$ ) | „ „ : $p$                                                                                          |
| (TTFF) | ( $p, q$ ) | „ „ : $q$                                                                                          |
| (FFFT) | ( $p, q$ ) | „ „ : Neither $p$ nor $q$ . ( $\sim p \cdot \sim q$ or $p q$ )                                     |
| (FFTF) | ( $p, q$ ) | „ „ : $p$ and not $q$ . ( $p \cdot \sim q$ )                                                       |
| (FTFF) | ( $p, q$ ) | „ „ : $q$ and not $p$ . ( $q \cdot \sim p$ )                                                       |
| (TFFF) | ( $p, q$ ) | „ „ : $q$ and $p$ . ( $q \cdot p$ )                                                                |
| (FFFF) | ( $p, q$ ) | Contradiction ( $p$ and not $p$ , and $q$ and not $q$ ). ( $p \cdot \sim p \cdot q \cdot \sim q$ ) |

[a] Charles Sanders Peirce, “On the Algebra of Logic: A Contribution to the Philosophy of Notation”, *Journal of Mathematics*, Volume 7, (1885).

[b] Ludwig Wittgenstein “*Tractatus Logico-Philosophicus*”, 5.101, Wien (1918), translated and published in Cambridge, (1921).

[c] Emil Post, “Introduction to a General theory of Elementary Propositions”, *American Journal of Mathematics* 43: 163–185, (1921).

[\*] Karl Menger, “Reminiscences of the Vienna Circle and the Mathematical Colloquium”, Editors: L. Golland, B.F. McGuinness, Sklar, Abe, Springer 1994, (1942).

## history

# projectors as propositions: Von Neumann, Stone and Birkhoff

John von Neumann (1903-1957) considered measurement projectors as propositions in 1932 [a]

To each property  $\mathcal{E}$  we can assign a quantity which we define as follows: each measurement which distinguished between the presence or absence of  $\mathcal{E}$  is considered as a measurement of this quantity, such that its value is 1 if  $\mathcal{E}$  is verified, and zero in the opposite case. This quantity which corresponds to  $\mathcal{E}$  will also be denoted by  $\mathcal{E}$ .

Marshall Harvey Stone (1903-1989) [b] (quoted by Von Neumann in [a]) gave the conditions for operations on projectors and commutativity. He also established that each binary logical proposition corresponds by duality to the set of all its true valuations (*Stone Duality*).

Von Neumann also introduced the formalism of the *density matrix* in quantum mechanics:

a pure quantum state  $|\psi\rangle$  can be represented by a rank-1 projection operator :  $\hat{\rho} = |\psi\rangle\langle\psi|$

This has led to more “adapted” logical systems. The work by Garret Birkhoff and John von Neumann [c] proposed the replacement of Boolean algebras with the lattice of closed subspaces of a (finite) Hilbert space:

*quantum logic*



[a] John von Neumann.: “Mathematical Foundations of Quantum Mechanics”, in German, Eng. Transl. (1955), p.249: “Projectors as Propositions”, (1932).

[b] Marshall Harvey Stone.: “Linear Transformations in Hilbert Space and Their Applications to Analysis”, p.70: “Projections”, (1932),.

[c] Garret Birkhoff, John von Neumann: “The Logic of Quantum Mechanics”. The Annals of Mathematics, 2nd Ser., 37 (4), 823-843 (1936)

# Eigenlogic method

---

Eigenlogic: a logical method using operators in linear algebra [\*]

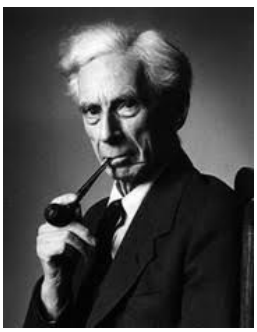
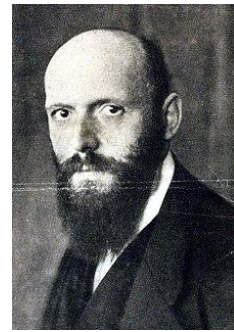
logical operators  $\Leftrightarrow$  logical connectives

eigenvalues of logical operators  $\Leftrightarrow$  truth values

eigenvectors of logical operators  $\Leftrightarrow$  interpretations (atomic propositional cases)

[\*] Zeno Toffano, "Eigenlogic in the spirit of George Boole", [ArXiv:1512.06632](https://arxiv.org/abs/1512.06632) (2015)

## logical coherence theory of truth and quantum



According to the conception of *truth as coherence*, a proposition is true if it is *compatible* with the other propositions of the *system*.

In logic propositions are compatible if and only if they admit at least one interpretation that satisfies them together.

Compatible propositions form a coherent system.

This conception has the merit of making a radical critique of the naive realism that there is a world accessible to immediate knowledge. One of the proponents, **Otto Neurath**, rejected any idea of direct experience of the world.

Also, according to **Bertrand Russel**, one can construct several systems of the world, coherent though incompatible with each other. A necessary condition of truth, coherence would not be a sufficient condition.

The parallel with quantum mechanics seems natural through the concepts of coherence and compatibility.

In Eigenlogic each basis choice constitutes a complete coherent system, and the different logical systems are incompatible.

One can choose, for example, a seed operator given by the operator  $\sigma_z = \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  giving a complete logical family and another by  $\sigma_x = \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

These two generated complete logical systems are coherent but incompatible.



# Eigenlogic

## the seed projector

Let's consider the two-dimensional (one qubit) rank-1 *seed projector*  $\Pi$  :

$$\Pi = |1\rangle\langle 1|, \text{ idempotent: } \Pi \cdot \Pi = |1\rangle\langle 1| |1\rangle\langle 1| = |1\rangle\langle 1| = \Pi$$

What is the result when applying this projector on its eigenvector  $|1\rangle$  ?  $\Pi|1\rangle = 1|1\rangle$  , eigenvalue: 1.

The orthogonal (complementary) vector is  $|0\rangle$  also an eigenvector of  $\Pi$  ,  $\Pi|0\rangle = 0|0\rangle$  , eigenvalue: 0.

In matrix form in the canonical basis:  $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  ,  $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  ,  $\Pi_0 = \mathbb{I} - \Pi = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  ,  $\Pi_1 = \Pi = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

As for *Boole's elective functions* we express the logical operators as a development:

$$F = f(0)\Pi_0 + f(1)\Pi_1 = \begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix} = \text{diag}(f(0), f(1))$$

The cofactors  $f(0)$  and  $f(1)$  are the eigenvalues and correspond to the truth values of the logical connective.  
So the 4 one-argument logical observables are:

$$\begin{aligned} F_A &= 0 \Pi_0 + 1 \Pi_1 = \Pi = \text{diag}(0,1) \\ F_{\bar{A}} &= 1 \Pi_0 + 0 \Pi_1 = \mathbb{I} - \Pi = \text{diag}(1,0) \\ F_{\perp} &= 0 \Pi_0 + 0 \Pi_1 = \mathbb{O} = \text{diag}(0,0) \\ F_T &= 1 \Pi_0 + 1 \Pi_1 = \mathbb{I} = \text{diag}(1,1) \end{aligned}$$



# Eigenlogic

## scaling up to higher arity

The *two-argument* logical operators can be developed using projectors and the Kronecker product (composition rule used for qubits). The four projectors are:

$$\Pi_{00} = |00\rangle\langle 00|, \quad \Pi_{01} = |01\rangle\langle 01|, \quad \Pi_{10} = |10\rangle\langle 10|, \quad \Pi_{11} = |11\rangle\langle 11| = |1\rangle\langle 1| \otimes |1\rangle\langle 1| = \Pi \otimes \Pi$$

Using Boole's *elective development*, with the truth values  $f(x, y)$  the logical operator for two inputs  $x, y \in \{0, 1\}$  is:

$$F = f(0,0)\Pi_{00} + f(0,1)\Pi_{01} + f(1,0)\Pi_{10} + f(1,1)\Pi_{11} = \text{diag}(f(0,0), f(0,1), f(1,0), f(1,1))$$

In binary *propositional* (or *sentential*) *logic* one defines the *elementary* (or *atomic*) *propositions*  $p$  and  $q$  in a *well-formed-formula*. In Eigenlogic these correspond to the rank-2 logical projectors and are function of the unique seed projector  $\Pi$  :

$$P = \Pi \otimes \mathbb{I} = \text{diag}(0,0,1,1), \quad Q = \mathbb{I} \otimes \Pi = \text{diag}(0,1,0,1)$$

This extension of  $\Pi$  ensures the *independence* of the elementary projection operators  $P$  and  $Q$ .

Is a major difference with quantum logic where atomic propositions are state density matrices (rank-1), as e.g.  $|11\rangle\langle 11| = \Pi \otimes \Pi$ .

All the *projector logical observables* can then be obtained from  $P$  and  $Q$  :

$$F_{AND} = P \cdot Q = \Pi \otimes \Pi = \text{diag}(0,0,0,1)$$

$$F_{OR} = P + Q - P \cdot Q = \text{diag}(0,1,1,1)$$

$$F_{\Rightarrow} = \mathbb{I} - P + P \cdot Q = \text{diag}(1,1,0,1)$$

$$F_{XOR} = P + Q - 2P \cdot Q = \text{diag}(0,1,1,0)$$

Negation is obtained by complementation e.g. :  $F_{NAND} = \mathbb{I} - F_{AND} = \mathbb{I} - P \cdot Q = \text{diag}(1,1,1,0)$

# Eigenlogic

## logical isometric self-inverse operators

Using the previously defined “seed” projector  $\Pi$  another “seed” isometric operator is the *Pauli operator*  $\sigma_z = \mathbf{Z}$  :

$$\mathbf{Z} = |00\rangle\langle 00| - |11\rangle\langle 11| = \mathbb{I} - 2|1\rangle\langle 1| = \mathbb{I} - 2\Pi = (-1)^\Pi = e^{i\frac{\pi}{2}} e^{-i\frac{\pi}{2}\mathbf{Z}} = \text{diag}(1, -1)$$

An isomorphism from the projection operators  $\mathbf{F}$  to reversible self-inverse isometric operators  $\mathbf{G}$  is ( *Householder Transform*) :

$$\mathbf{G} = \mathbb{I} - 2\mathbf{F} = (-1)^{\mathbf{F}} = e^{i\pi\mathbf{F}} = e^{i\frac{\pi}{2}} e^{-i\frac{\pi}{2}\mathbf{G}}$$

practically in order to obtain  $\mathbf{G}$  from  $\mathbf{F}$  one replaces the eigenvalue 0 with +1 and 1 with -1.

This formulation is commonly used in *spin-glass* and *Ising* models and in *neural networks*.

$$+1 : \text{spin up} \leftrightarrow 0 : \text{“False”} \quad , \quad -1 : \text{spin down} \leftrightarrow 1 : \text{“True”}$$

In the alphabet  $\{+1, -1\}$  the *logical dictators*  $\mathbf{U}$  and  $\mathbf{V}$  (equivalent of  $\mathbf{P}$  and  $\mathbf{Q}$  for the alphabet  $\{0,1\}$ ) are function of  $\mathbf{Z}$  :

$$\mathbf{U} = \mathbf{Z} \otimes \mathbb{I} = \text{diag}(+1, +1, -1, -1) \quad , \quad \mathbf{V} = \mathbb{I} \otimes \mathbf{Z} = \text{diag}(+1, -1, +1, -1)$$

Negation in the alphabet  $\{+1, -1\}$  is simply obtained by multiplying by -1:  $\bar{\mathbf{G}} = -\mathbf{G}$

This formulation leads to the same results as for the Fourier transform model of quantum Boolean functions developed in [a]

All logical observables in the alphabet  $\{+1, -1\}$  can be expressed as a function of  $\mathbf{U}$  and  $\mathbf{V}$  [b].

[a] Montanaro, A.; Osborne, T.J. “Quantum Boolean Functions”, Chicago J. Theor. Comput. Sci. **41**, (2010).

[b] Dubois, F.; Toffano, Z., “Eigenlogic: A Quantum View for Multiple-Valued and Fuzzy Systems”, QI 2016. In Lecture Notes in Computer Science; Springer: Berlin, Germany, 2017; Volume 10106, (2017).

# Eigenlogic

## operator truth tables

There are  $2^{2^n}$  binary logical connectives for a  $n$ -argument (*arity*) system.

For  $n = 2$  one has 16 binary logical connectives:

- conjunction ( $\wedge$ , *AND*)
- disjunction ( $\vee$ , *OR*)
- exclusive disjunction ( $\oplus$ , *XOR*)
- logical projectors ( $A$ ,  $B$ )
- material implication  $\Rightarrow$
- Sheffer stroke ( $\uparrow$ , *NAND*)
- tautology ( $T$ )

The remaining ones can be derived by the classical theorems of logic (negation and De Morgan duality).

All logical connectives are uniquely characterized by their truth table i.e., by their logical *semantics*.

| logical connective<br>for P, Q | truth table {F, T}:<br>{0, 1} or {+1, -1} | {0, 1} projective<br>logical operator | {+1, -1} isometric<br>logical operator |
|--------------------------------|-------------------------------------------|---------------------------------------|----------------------------------------|
| False F                        | F F F F                                   | 0                                     | + I                                    |
| NOR                            | F F F T                                   | $I - P - Q + PQ$                      | $(1/2) (+I - U - V - UV)$              |
| $P \neq Q$                     | F F T F                                   | $Q - PQ$                              | $(1/2) (+I - U + V + UV)$              |
| $\neg P$                       | F F T T                                   | $I - P$                               | - U                                    |
| $P \neq Q$                     | F T F F                                   | $P - PQ$                              | $(1/2) (+I + U - V + UV)$              |
| $\neg Q$                       | F T F T                                   | $I - Q$                               | - V                                    |
| XOR ; $P \oplus Q$             | F T T F                                   | $P + Q - 2 PQ$                        | $UV = Z \otimes Z$                     |
| NAND ; $P \uparrow Q$          | F T T T                                   | $I - PQ$                              | $(1/2) (-I - U - V + UV)$              |
| AND ; $P \wedge Q$             | T F F F                                   | $PQ = \Pi \otimes \Pi$                | $(1/2) (+I + U + V - UV)$              |
| $P \equiv Q$                   | T F F T                                   | $I - P - Q + 2 PQ$                    | - UV                                   |
| Q                              | T F T F                                   | $Q = I \otimes \Pi$                   | $V = I \otimes Z$                      |
| $P \Rightarrow Q$              | T F T T                                   | $I - P + PQ$                          | $(1/2) (-I - U + V - UV)$              |
| P                              | T T F F                                   | $P = \Pi \otimes I$                   | $U = Z \otimes I$                      |
| $P \Leftarrow Q$               | T T F T                                   | $I - Q + PQ$                          | $(1/2) (-I + U - V - UV)$              |
| OR ; $P \vee Q$                | T T T F                                   | $P + Q - PQ$                          | $(1/2) (-I + U + V + UV)$              |
| True T                         | T T T T                                   | I                                     | - I                                    |

*Eigenlogic logical operators for the 16 binary logical connectives for arity n=2*

# Eigenlogic

## considerations on logical universality: quantum semantics

In binary propositional logic the following logical connectives when combined with negation (NOT) are universal :  
AND, OR, NOR, NAND,  $\Rightarrow$  ,  $\nRightarrow$ ,

But others are not:

the logical projectors P and Q and their negations  $\neg P$  and  $\neg Q$ , equivalence  $\equiv$  , XOR and contradiction F and tautology T.

For two arguments only the binary logical connectives with an **odd number of true and false values** are **universal**

| P | Q | F <sup>0</sup> | NOR <sup>1</sup> | $P \nRightarrow Q$ <sup>2</sup> | $\neg P$ <sup>3</sup> | $P \nRightarrow Q$ <sup>4</sup> | $\neg Q$ <sup>5</sup> | XOR <sup>6</sup> | NAND <sup>7</sup> | AND <sup>8</sup> | $P \equiv Q$ <sup>9</sup> | Q <sup>10</sup> | $P \Rightarrow Q$ <sup>11</sup> | $P$ <sup>12</sup> | $Q \Rightarrow P$ <sup>13</sup> | OR <sup>14</sup> | T <sup>15</sup> |
|---|---|----------------|------------------|---------------------------------|-----------------------|---------------------------------|-----------------------|------------------|-------------------|------------------|---------------------------|-----------------|---------------------------------|-------------------|---------------------------------|------------------|-----------------|
| + | + | +              | -                | +                               | -                     | +                               | -                     | +                | -                 | +                | -                         | +               | -                               | +                 | -                               | +                | -               |
| + | - | +              | +                | -                               | -                     | -                               | +                     | -                | -                 | +                | +                         | -               | -                               | +                 | +                               | -                | -               |
| - | + | +              | +                | +                               | +                     | +                               | -                     | -                | -                 | +                | +                         | +               | +                               | -                 | -                               | -                | -               |
| - | - | +              | +                | +                               | +                     | +                               | +                     | +                | +                 | -                | -                         | -               | -                               | -                 | -                               | -                | -               |

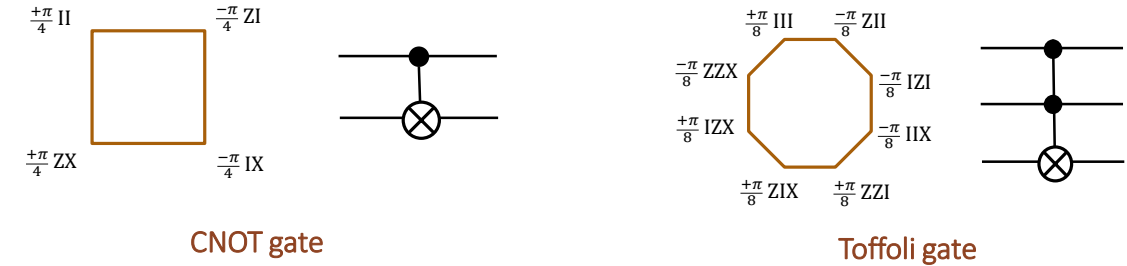
In Eigenlogic this fact can be transposed to eigenvalues.

For self-inverse observables (eigenvalues  $\{+1, -1\}$ ) these are the non-separable quantum gates that cannot be expressed as a Kronecker product. In other words these correspond to the entangling gates.

This states clearly the correspondence between universality and entanglement (a common knowledge in quantum computing).

# Eigenlogic

## CNOT and Toffoli quantum gates



The **CNOT gate**  $\mathbf{C}$  can be expressed, in a polynomial form on the operators :  
using the **Eigenlogic reversible self-inverse conjunction operator** (alphabet  $\{+1, -1\}$ ) :

$$\mathbf{Z}_1 = \mathbf{Z} \otimes \mathbb{I} \quad \text{and} \quad \mathbf{X}_0 = \mathbb{I} \otimes \mathbf{X}$$

$$\mathbf{C} = \mathbb{I} - 2(\mathbf{P}_Z \cdot \mathbf{Q}_X) = \mathbb{I} - 2(\mathbf{\Pi}_Z \otimes \mathbf{\Pi}_X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = (-1)^{\mathbf{\Pi} \otimes \mathbf{\Pi}_X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{C} = \frac{1}{2}(\mathbb{I} + \mathbf{Z}_1 + \mathbf{X}_0 - \mathbf{Z}_1 \cdot \mathbf{X}_0) = e^{i\frac{\pi}{4}} e^{-i\frac{\pi}{4}\mathbf{Z}_1} e^{-i\frac{\pi}{4}\mathbf{X}_0} e^{i\frac{\pi}{4}\mathbf{Z}_1 \cdot \mathbf{X}_0}$$

3 qubit **Toffoli gate**  $\mathbf{TO}$  :  $\mathbf{TO} = \mathbb{I} - 2(\mathbf{\Pi}_Z \otimes \mathbf{\Pi}_Z \otimes \mathbf{\Pi}_X) = \frac{1}{4}(3\mathbb{I} + \mathbf{Z}_2 + \mathbf{Z}_1 + \mathbf{X}_0 - \mathbf{Z}_2 \cdot \mathbf{Z}_1 - \mathbf{Z}_2 \cdot \mathbf{X}_0 - \mathbf{Z}_1 \cdot \mathbf{X}_0 + \mathbf{Z}_2 \cdot \mathbf{Z}_1 \cdot \mathbf{X}_0)$

It is practically difficult to realize a sum of operators, but a product form can always be derived (properties of Paulis :  $\sigma^2 = \mathbb{I}$ )

$$\mathbf{TO} = e^{+i\frac{\pi}{8}} e^{-i\frac{\pi}{8}\mathbf{Z}_1} e^{-i\frac{\pi}{8}\mathbf{Z}_2} e^{-i\frac{\pi}{8}\mathbf{X}_0} e^{i\frac{\pi}{8}\mathbf{Z}_2 \cdot \mathbf{Z}_1} e^{i\frac{\pi}{8}\mathbf{Z}_2 \cdot \mathbf{X}_0} e^{i\frac{\pi}{8}\mathbf{Z}_1 \cdot \mathbf{X}_0} e^{-i\frac{\pi}{8}\mathbf{Z}_2 \cdot \mathbf{Z}_1 \cdot \mathbf{X}_0}$$

Using **T gates** [a]:

$$\mathbf{T} = \mathbf{Z}^{\frac{1}{4}} = e^{i\frac{\pi}{8}} e^{-i\frac{\pi}{8}\mathbf{Z}} = \text{diag}\left(1, e^{i\frac{\pi}{4}}\right)$$

given the arithmetic expression (Reed-Muller form):  $4xyz = x + y + z - x \oplus y - x \oplus z - y \oplus z + x \oplus y \oplus z$

with a decomposition using the **Eigenlogic T-XOR** operator :  $\mathbf{T}_{x \oplus y} = \text{diag}(1, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{4}}, 1)$

The double Control-Z gate  $\mathbf{C}_{CZ}$  is derived :  $\mathbf{C}_{CZ} = \mathbf{T}_0 \cdot \mathbf{T}_1 \cdot \mathbf{T}_2 \cdot (\mathbf{T}_{x \oplus y})^\dagger \cdot (\mathbf{T}_{x \oplus z})^\dagger \cdot (\mathbf{T}_{y \oplus z})^\dagger \cdot (\mathbf{T}_{x \oplus y \oplus z})$

using the Hadamard gate  $\mathbf{H} = \frac{1}{2}(\mathbf{X} + \mathbf{Z})$  and  $\mathbf{H}_0 = \mathbb{I} \otimes \mathbb{I} \otimes \mathbf{H}$ , finally the Toffoli gate :  $\mathbf{TO} = \mathbf{H}_0 \cdot \mathbf{C}_{CZ} \cdot \mathbf{H}_0$

[a] Selinger, P., "Quantum circuits of T-depth one", Phys. Rev. A, 87, 252–259, (2013).

# Eigenlogic

## from Grover towards first order-logic

Example of a Grover search algorithm quantum circuit with a phase oracle [a].

The Grover amplification gate turns out to be a selfinverse Eigenlogic disjunction operator in the X system (in the circuit we have  $\mathbf{G}_{\text{NOR}} = -\mathbf{G}_{\text{OR}}$ ).

The oracle (phase) being here a **double control Z gate** (Eigenlogic  $\mathbf{G}_{\text{AND}}$ ).

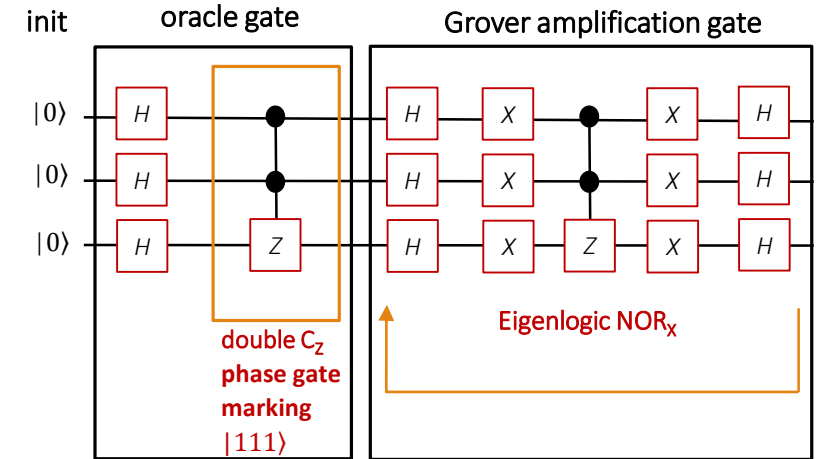
The **Grover search algorithm** looks for an element  $a$  (here the state vector  $|a_2 a_1 a_0\rangle = |111\rangle$ ) in a finite set satisfying a certain property  $P$  (**oracle**).

The circuit for the oracle  $P$  followed by the Grover gate, interpreted as an existential logical quantifier  $\exists$ , becomes the predicate proposition:

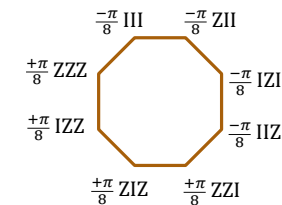
$$\exists a P(a)$$

A justification can be found in **Herbrand's fundamental theorem** [b] that provides a constructive characterization of derivability in first-order predicate logic by means of propositional (sentential) logic.

$\exists a P(a)$  decomposes, using **Skolemization** methods into a succession of "disjunction  $\vee$ " connectives.

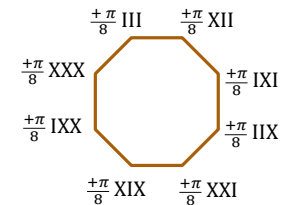


$$P_Z \wedge Q_Z \wedge R_Z$$



Eigenlogic 3 input  $\text{AND}_2$   
(double  $C_2$ )

$$\neg(P_X \vee Q_X \vee R_X)$$



Eigenlogic 3 input  $\text{NOR}_x$

[a] C. Figgatt, et al., "Complete 3-Qubit Grover search on a programmable quantum computer", Nature Communications volume 8, Article n°: 1918, (2017).

[b] Herbrand, J., "Recherches sur la théorie de la démonstration", Thèses présentées à la faculté des sciences de Paris, Paris, (1930).

# fuzzy Eigenlogic

## when the input state is not an eigenstate

Fuzzy logic deals with truth values that can take values between 0 and 1, so the truth of a proposition can “lie” between “completely true” and “completely false” [a].

$|00\rangle$ ,  $|01\rangle$ ,  $|10\rangle$  and  $|11\rangle$  form a complete orthonormal basis for the two inputs of a logical proposition. These 4 vectors correspond to the 4 *interpretations of the Eigenlogic system*.

What happens when the input state  $|\psi\rangle$  is not one of the eigenvectors of the logical system ?

In quantum mechanics one can always express a state vector as a decomposition on a complete orthonormal basis.

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

The fuzziness of logic can be interpreted by the probabilistic nature of quantum measurements (Born rule). For a projective observable  $\hat{P}$  measured in the context of a quantum state  $|\psi\rangle$  we have the probability (Gleason’s theorem [b]):

$$p_{|\psi\rangle} = \langle\psi|\hat{P}|\psi\rangle = \text{Tr}(\hat{\rho} \cdot \hat{P}) \quad \text{with} \quad \hat{\rho} = |\psi\rangle\langle\psi| \quad \text{the “density matrix”}.$$

The mean value of an Eigenlogic projection operator  $\mathbf{F}$  provides a fuzzy measure of the corresponding logical connective given by a

$$\text{fuzzy membership function :} \quad \mu = \langle\psi|\mathbf{F}|\psi\rangle.$$

[a] Zadeh, L.A.: “Fuzzy sets. Information and Control”, 8 (3), 338-353, (1965).

[b] Andrew M. Gleason, “Measures on the closed subspaces of a Hilbert space”. Indiana U. Mathematics Journal, 6, 885–893, (1957).



# fuzzy Eigenlogic

## projection, conjunction and disjunction for compound states

Considering a generic quantum state in Hilbert space on the Bloch sphere:  $|\phi\rangle = \sin\frac{\theta}{2}|0\rangle + e^{i\varphi}\cos\frac{\theta}{2}|1\rangle$ ,

We can perform the quantum mean value (Born rule) by:  $\mu(A) = \langle\phi|\mathbf{\Pi}|\phi\rangle = \cos^2\frac{\theta}{2}$ .

the same operation can be done for the complement:  $\mu(\bar{A}) = \langle\phi|(\mathbb{I} - \mathbf{\Pi})|\phi\rangle = \sin^2\frac{\theta}{2} = 1 - \cos^2\frac{\theta}{2}$

The last expression satisfies the condition of fuzzy logic for the complement (negation) of a fuzzy set.

A compound quantum state can be built by taking the Kronecker product :  $|\psi\rangle = |\phi_p\rangle \otimes |\phi_q\rangle$

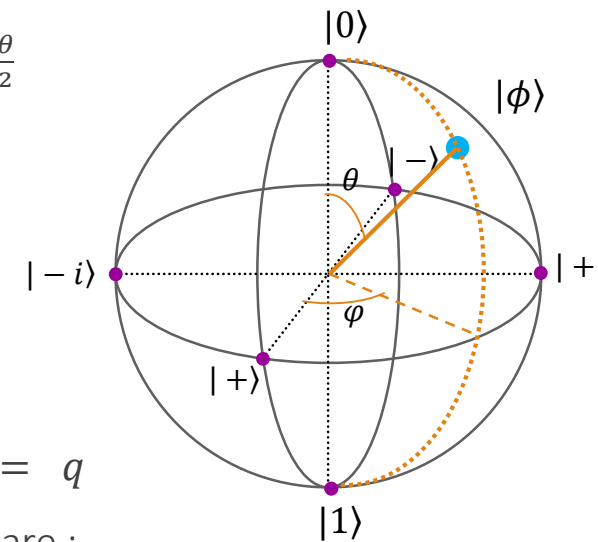
One can calculate the corresponding fuzzy membership function for the logical projectors  $\mathbf{P}$  and  $\mathbf{Q}$  by performing the quantum average with  $p = (\cos\frac{\theta_p}{2})^2$  and  $q = (\cos\frac{\theta_q}{2})^2$ :

$$\mu(P) = \langle\psi|\mathbf{P}|\psi\rangle = \langle\psi|\mathbf{\Pi} \otimes \mathbb{I}|\psi\rangle = p(1 - q) + p \cdot q = p \quad ; \quad \mu(Q) = \langle\psi|\mathbf{Q}|\psi\rangle = \langle\psi|\mathbb{I} \otimes \mathbf{\Pi}|\psi\rangle = q$$

The fuzzy measure of conjunction,  $\mathbf{P} \wedge \mathbf{Q}$ , and disjunction,  $\mathbf{P} \vee \mathbf{Q}$  using the seed projector observable are :

$$\begin{aligned} \mu(P \wedge Q) &= \langle\psi|\mathbf{P} \cdot \mathbf{Q}|\psi\rangle = \langle\psi|\mathbf{\Pi} \otimes \mathbf{\Pi}|\psi\rangle = p \cdot q = \mu(P) \cdot \mu(Q) \\ \mu(P \vee Q) &= p + q - p \cdot q = \mu(P) + \mu(Q) - \mu(P) \cdot \mu(Q) \end{aligned}$$

In agreement with recent results in quantum cognition [a]



**Bloch sphere in Hilbert space**

[a] Diederik Aerts, Sandro Sozzo, Tomas Veloz: "Quantum structure of negation and conjunction in human thought". *Front Psychol.* 6:1447, (2015).

# fuzzy Eigenlogic

## quantum robots as Braitenberg Vehicles

Paul Benioff was the first to propose the idea of a quantum Turing machine in 1980 [a]  
He introduced another paradigm: the theoretical principle of a *quantum robot* [b] as a first step towards a quantum mechanical description of systems that are aware of their environment and make decisions.

The research team led by Marek Perkowski has designed robots based on *Braiteneberg Vehicles* (BV) [c], using quantum gates and introducing control by fuzzy logic and also multivalued logic [d].

Our goal here was to test the multiple combinations of quantum gates used in the control of the two-wheel ( $ML$ ,  $MR$ ) motion of BV in response to fuzzy input signals ( $P$ ,  $Q$ ) from two light sensors ( $SL$ ,  $SR$ ) by analyzing their complex behavior.

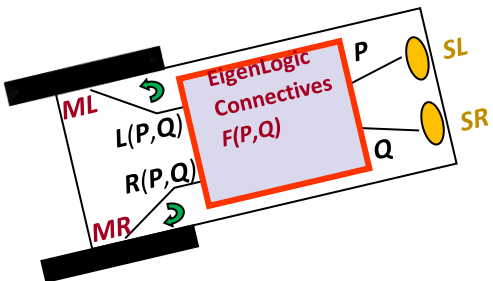


The fuzzy quantities for left and right wheel control, are:

$$\mu_L = \langle \psi | L | \psi \rangle \quad , \quad \mu_R = \langle \psi | R | \psi \rangle$$

the input state vector representing the light intensity on  $SL$  and  $SR$

$$|\psi\rangle = |\phi_{SL}\rangle \otimes |\phi_{SR}\rangle$$



[a] Benioff, P., "Quantum Mechanical Models of Turing Machines That Dissipate No Energy", *Phys. Rev. Lett.*, Vol. 48, pp. 1581–1585, (1982).

[b] Benioff, P., "Quantum Robots and Environments", *Physical Review A*, Vol. 58, No.2, pp. 893–904, (1998).

[c] Braitenberg, V., *Vehicles - Experiments in Synthetic Psychology*, MIT Press; Cambridge USA, (1986).

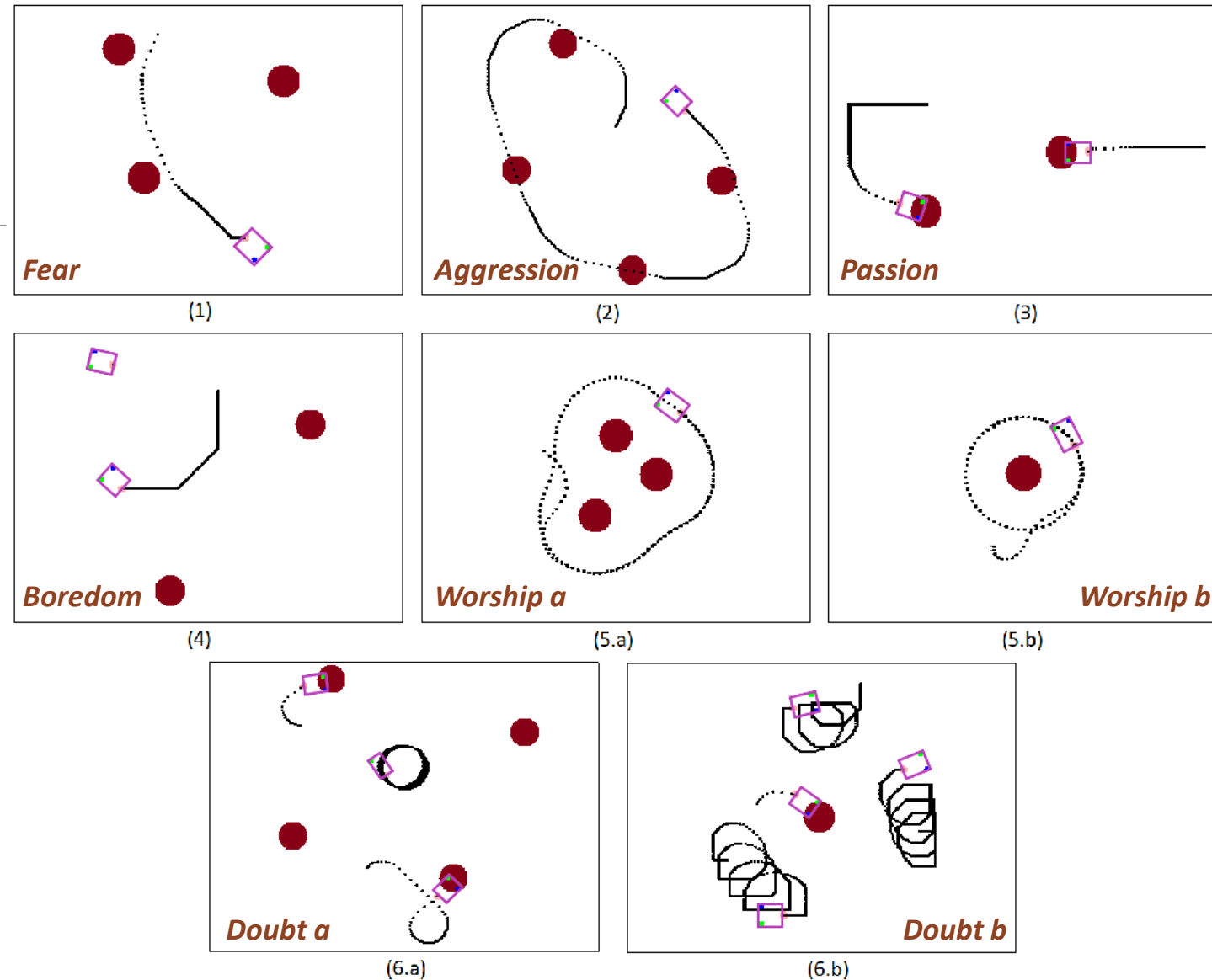
[d] Raghuvanshi, A. and Perkowski, M., "Fuzzy quantum circuits to model emotional behaviors of humanoid robots", *Evolutionary Computation (CEC), IEEE*, (2010).

# fuzzy Eigenlogic quantum robot behavior

Simulations of emotional states using different configurations of Eigenlogic operators in the vehicle's computational unit.

By changing and combining the quantum control gates one can tune small changes in the vehicle's behavior, around the basic BV emotions.

Further extensions to this project can be imagined: when the vehicles collide, their control operators could change in order to reflect a quantum-like entanglement behavior due to interaction.



# multivalued Eigenlogic more than binary



Quantum physics and modern theory of many-valued logics (Jan Łukasiewicz [a] and Emil Post [b], ) were born nearly simultaneously in the second decade of the twentieth century.

The total number of logical connectives for a system of  $m$  values and  $n$  arguments is the combinatorial number  $m^{m^n}$ . For a one-argument system with 3 values the will be  $3^{3^1} = 27$  connectives and for two arguments  $3^{3^2} = 19683$ .

By **Lagrange-Cayley-Hamilton matrix interpolation** using a seed operator  $\mathbf{A}$  with  $m$  non-degenerate eigenvalues  $\lambda_i$ . The projector of each eigenstate is given uniquely by:

$$|\lambda_i\rangle\langle\lambda_i| = \Pi_{\lambda_i}(\mathbf{A}) = \prod_{j=1, j \neq i}^m \frac{\mathbf{A} - \lambda_j \mathbb{I}}{\lambda_i - \lambda_j}$$

This expression is a polynomial form of the seed operator up to powers  $\mathbf{A}^{m-1}$ .

A logical operator for arity-1 is then given by the spectral decomposition :  $\mathbf{F}_L = \sum_{i=1}^m f(\lambda_j) \Pi_{\lambda_i}(\mathbf{A})$

For higher arity multivariate interpolation by Kornecker product gives operators corresponding to all the possible logical connectives [c].

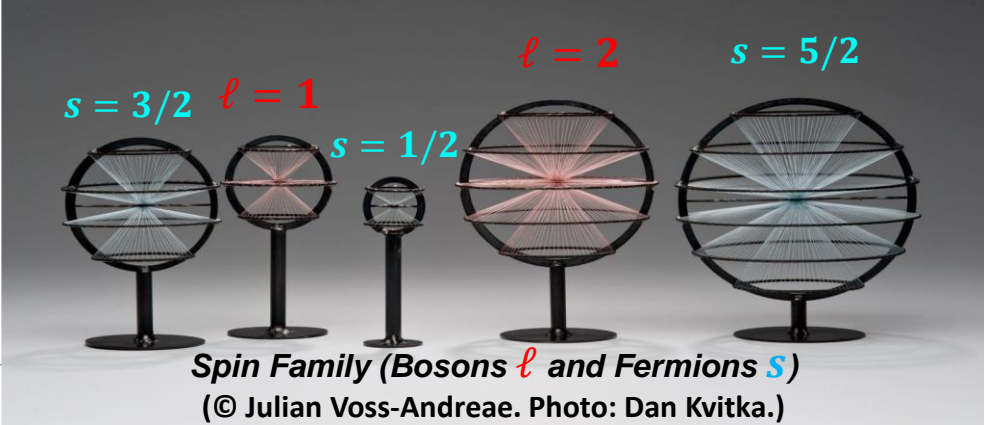
[a] Jan Łukasiewicz, "Selected Works", North-Holland, (1970), pp. 87–88, "On three-valued logic" (1921)

[b] Emil Post, "Introduction to a General theory of Elementary Propositions", American Journal of Mathematics 43: 163–185 (1921)

[c] Dubois, F.; Toffano, Z. "Eigenlogic: A Quantum View for Multiple-Valued and Fuzzy Systems". QI 2016. In Lecture Notes in Computer Science; Springer: Berlin, Germany, 2017; Volume 10106, (2017).

# multivalued Eigenlogic

## identification with angular momentum



Logical observables can be identified with quantum angular momentum.

Intrinsic angular momentum spin  $s = 1/2$  was formally identified with the self-inverse Eigenlogic operators on values  $\{+1, -1\}$ .

Multivalued logic is also naturally associated to higher value quantum angular momentum.

The balanced ternary logic  $\{+1,0,-1\}$  has observables equivalent to orbital angular momentum (OAM) with  $\ell = 1$ .

The z component of the this orbital angular momentum gives is the observable :

$$L_z = \hbar \Lambda = \hbar \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Logical observables can be expressed as spectral decompositions over the three rank-1 projectors:

$$\Pi_{+1} = \frac{1}{2}\Lambda(\Lambda + \mathbb{I}) \quad , \quad \Pi_0 = \mathbb{I} - \Lambda^2 \quad , \quad \Pi_{-1} = \frac{1}{2}\Lambda(\Lambda - \mathbb{I})$$

The dictators,  $\mathbf{U}$  and  $\mathbf{V}$ , are then:  $\mathbf{U} = \Lambda \otimes \mathbb{I}$ ,  $\mathbf{V} = \mathbb{I} \otimes \Lambda$

For example Min and Max are defined by their truth tables:

| Min U \ V          | F  | N  | T  | Max U \ V          | F  | N  | T  |
|--------------------|----|----|----|--------------------|----|----|----|
| False $\equiv +1$  | +1 | +1 | +1 | False $\equiv +1$  | +1 | 0  | -1 |
| Neutral $\equiv 0$ | +1 | 0  | 0  | Neutral $\equiv 0$ | 0  | 0  | -1 |
| True $\equiv -1$   | +1 | 0  | -1 | True $\equiv -1$   | -1 | -1 | -1 |

Using the developments found previously and logical reduction rules, due to the completeness of projection operators, we get the following logical observables corresponding to the universal multivalued logical connectives Min and Max (AND , OR for binary logic) :

$$\begin{aligned} \mathbf{Min}(\mathbf{U}, \mathbf{V}) &= \frac{1}{2} (\mathbf{U} + \mathbf{V} + \mathbf{U}^2 + \mathbf{V}^2 - \mathbf{U} \cdot \mathbf{V} - \mathbf{U}^2 \cdot \mathbf{V}^2) = \text{diag}(+1, +1, +1, +1, 0, 0, +1, 0, -1) \\ \mathbf{Max}(\mathbf{U}, \mathbf{V}) &= \frac{1}{2} (\mathbf{U} + \mathbf{V} - \mathbf{U}^2 - \mathbf{V}^2 + \mathbf{U} \cdot \mathbf{V} + \mathbf{U}^2 \cdot \mathbf{V}^2) = \text{diag}(+1, 0, -1, 0, 0, -1, -1, -1, -1) \end{aligned}$$

# Eigenlogic

## work in progress and acknowledgements

---

on development of interpolation operator models for Fourier Transform in relation to quantum multivalued logic and para-Fermion models with **François Dubois** of CNAM Paris and Dep. Math U. Psud Orsay, (France).

on research projects with students on quantum circuits and quantum programming with **Benoît Valiron** of CentraleSupélec and LRI (Lab. Recherche Informatique) (France).

on quantum models for Information Retrieval and Semantic Web with **Bich-Lien Doan** of CentraleSupélec and LRI (Lab. Recherche Informatique) (France).

on semantic and semiotic quantum models for text mining and language processing and on Braitenberg Vehicles with **Francesco Galofaro** of Politecnico Milano and Libera Università di Bolzano (Italy).

with François Dubois we organize a **monthly seminar : "Modélisation Quantique"** at ISC-PIF (Institut des Systèmes Complexes Paris-Ile-de-France) in Paris. It aims to bring concepts issued from quantum mechanics outside physics.

If you want to join the reflections and discussions on this topic you are welcome !

[zeno.toffano@centralesupelec.fr](mailto:zeno.toffano@centralesupelec.fr)

---

merci