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Abstract: A further extension of a result on maximal multiplicity induced-dominancy for spectral values is analytically derived for generic retarded second-order systems with a single delay in the parameter space. Several examples illustrate the applicative perspectives of the result, towards a rightmost spectral value assignment approach.

Keywords: Time-delay systems, stability analysis, spectral abscissa, rightmost spectral values.

1. INTRODUCTION

The present study concerns a frequency domain approach in the stability analysis of linear time-invariant retarded Time-delay systems. The investigation of conditions on the equation parameters that guarantee the exponential stability of solutions is a question of ongoing interest. In the Laplace domain, where a number of effective methods have been proposed, the stability analysis amounts to studying the distribution of the characteristic quasipolynomial function's roots, see for instance (Cooke and van den Driessche, 1986; Walton and Marshall, 1987; Stépán, 1989; Michiels and Niculescu, 2007; Olgac and Sipahi, 2002; Sipahi et al., 2011). The delay effect in controllers design was first introduced in (Suh and Bien, 1979) where is shown that delayed proportional controller performs an averaged derivative action and thus can replace the proportional-derivative controller, see also (Atay, 1999). Also, under appropriate conditions the Time-delay may have a stabilizing effect in the control design, see for instance (Niculescu et al., 2010). Indeed, that the closed-loop stability is guaranteed precisely by the existence of the delay. In the context of mechanical engineering problems, the effect of Time-delay was emphasized in (Stépán, 1989) where concrete applications are studied, such as, the machine tool vibrations and robotic systems.

In recent works, it is shown that the multiple spectral values for Time-delay systems can be characterized using a Birkhoff/Vandermonde-based approach; see for instance (Boussaada and Niculescu, 2016a,b, 2014; Boussaada et al., 2016). More precisely, in (Boussaada and Niculescu, 2016b), it is shown that the admissible multiplicity of the zero spectral value is bounded by the generic *Polya and Szegö bound* denoted PS_B, which is nothing but the *degree* of the corresponding quasipolynomial, see for instance (Pólya and Szegő, 1972). In (Boussaada and Niculescu, 2016a), it is shown that a given CIR with non vanishing frequency never reaches PS_B and a sharper bound for its admissible multiplicities is established.

Moreover, in (Boussaada et al., 2016), the variety corresponding to a multiple root for scalar Time-delay equations defines a stable variety for the steady state. The multiplicity of a root itself is not important as such but its connection with the dominancy of this root is a meaningful tool for control synthesis. An example of a scalar retarded equation with two delays is studied in (Boussaada and Niculescu, 2016a) where it is shown that the multiplicity of real spectral values may reach the PS_B. In addition, the corresponding system has some further interesting properties: (i) it is asymptotically stable, (ii) its spectral abscissa (rightmost root) corresponds to this maximal allowable multiple root located on the imaginary axis. Such observations enhance the outlook of further exhibiting the existing links between the maximal allowable multiplicity of some negative spectral value reaching the *quasipolynomial degree* (i.e the number of the involved polynomials plus their degree minus one) and the stability of the trivial solution of the corresponding dynamical system. This property induced from multiplicity appears also in optimization problems since such a multiple spectral value is nothing but the rightmost root, see also (Vanbiervliet et al., 2008; Michiels et al., 2000). Also notice that the property was already observed in (Ramirez et al., 2015), where a tuning strategy is proposed for the design of a delayed Proportional-Integral controller by placing a triple real dominant root for the closed-loop system. However, the dominancy is only checked using a Mikhailov curve and QPmR toolbox, see for instance (Vyhlídal and Zitek, 2009). To the best of our knowledge, the first time an

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analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented in (Hayes, 1950). The dominancy property is further explored and analytically shown in the case of second-order systems and a rightmost root assignment based design using delayed state-feedback is proposed in (Boussaada et al., 2017, 2018) where its applicability in damping active vibrations for a piezo-actuated beam is proved.

The multiplicity of a root itself is not important as such but its connection with the dominancy of this root is a meaningful tool for control synthesis. This work, further explores such a connexion and gives an analytical proof for the dominancy of the spectral value with maximal multiplicity for second-order systems controlled via a delayed proportional-derivative controller.

2. PREREQUISITES AND MOTIVATING EXAMPLES

Second-order linear systems capture the dynamic behavior of many natural phenomena, and have found wide applications in a variety of fields, such as vibration and structural analysis. In the sequel we recall some hints, recent results and examples motivating the use of delay in controller design for stabilizing the steady state solution corresponding to such a class of systems. Consider the generic second-order system with a single time delay:

$$\dot{x} = A_0 x(t) + A_1 x(t-\tau) \tag{1}$$

with the state-vector $x = (x_1, x_2) \in \mathbb{R}^2$, under appropriate initial conditions belonging to the Banach space of continuous functions $\mathcal{C}([-\tau_N, 0], \mathbb{R}^2)$. Here τ is a positive constant delay and the matrices $A_j \in \mathcal{M}_2(\mathbb{R})$ for $j = 0 \ldots 1$. It is well known that the asymptotic behavior of the solutions of (1) is determined from the spectrum $\aleph$ designating the set of the roots of the associated characteristic function (denoted in the sequel $\Delta(s,\tau)$). Namely, the characteristic function corresponding to system (1) is a quasipolynomial $\Delta : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ of the form:

$$\Delta(s,\tau) = \det\left(s\, I - A_0 - A_1\, e^{-\tau s}\right). \tag{2}$$

To start with, let us recall a generic result on the location of spectral values corresponding to (2). The proof of the proposition below can be found in (Michiels and Niculescu, 2007).

Proposition 1. If s is a characteristic root of the system (1), then it satisfies

$$|s| \leq ||A_0 + A_1\, e^{-\tau s}||_2. \tag{3}$$

The above proposition provides a generic *envelope curve* around the characteristic roots corresponding to (1).

In particular, the present work is focused on Time-delay systems characterized by the quasipolynomial function of the form

$$\Delta(s,\tau) = P_0(s) + P_1(s)\, e^{-\tau s}, \tag{4}$$

and we are concerned by the problem of the analytical characterization of the rightmost root. More precisely, equation (4) is written as:

$$\Delta(s,\tau) = s^2 + c_1 s + c_0 + (\beta_0 + s\beta_1)\, e^{-\tau\, s}. \tag{5}$$

The case $\beta_1 = 0$, yields a rightmost root with maximal multiplicity as characterized in (Boussaada et al., 2018).

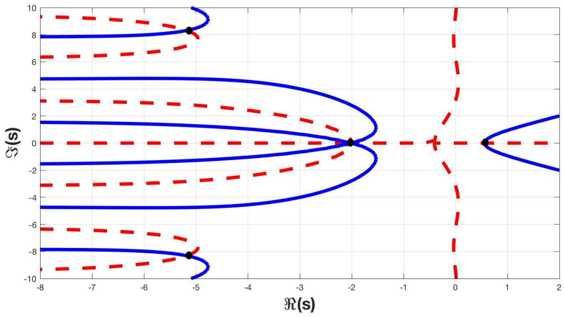

Fig. 1. Sparsity-induced loss of dominancy for the multiple spectral value. Each intersection between the blue/red curves corresponds to a spectral value of (8). The roots' distribution is illustrated using QPmR toolbox from (Vyhlídal and Zitek, 2009)

2.1 Multiple spectral values for Time-delay systems are not necessarily dominant

The problem of stabilization of a chain of integrators is considered in (Niculescu and Michiels, 2004) where a single integrator can be stabilized by a single delay state-feedback. It is asserted that either 2 distinct delays or a proportional+delay are sufficient to stabilize a chain including integrators. In (Kharitonov et al., 2005), a like assertion is shown to be also necessary to stabilize the double integrator. In conclusion, there exists at least a spectral value for (6) with positive real part. As a matter of fact, consider the following quasipolynomial function:

$$\Delta(s,\tau) = s^2 + \alpha\, e^{-\tau\, s}. \tag{6}$$

It can be checked that the maximal admissible multiplicity is 2 and it can be attained if, and only if,

$$\alpha = -4\,\frac{e^{-2}}{\tau^2},\ s = -\frac{2}{\tau}. \tag{7}$$

As a result, $s_0 = -\frac{2}{\tau}$, while being a multiple root it cannot be dominant. Figure 1 illustrates the particular quasipolynomial (6)-(7) with $\tau = 1$, that is

$$\Delta(s,1) = s^2 - 4e^{-(s+2)}. \tag{8}$$

where the dominancy property of the multiple root is lost since $s_1 \approx 0.557$ is a root of (8). This is structurally explained by the sparsity of the corresponding quasipolynomial.

2.2 Sunflower Equation

In (Boussaada et al., 2016) the well-known Sunflower model is considered. Namely, the helical movement of a growing plant is governed by the following delay equation:

$$\ddot{x} + \frac{a}{\tau}\dot{x} + \frac{b}{\tau}\sin(x(t-\tau)) = 0 \tag{9}$$

This model is known to reproduce the dynamics of the upper part of the stem of the plant, which performs a rotating movement. Here the state $x(t)$ designates the angle of the plant with respect to the vertical line, the delay τ corresponds to a geotropic reaction time, and a and b some positive parameters. The corresponding linearized system with $\alpha = a/\tau$ and $\beta = b/\tau$ is given by:

$$\ddot{x} + \alpha\dot{x} + \beta x(t-\tau) = 0 \tag{10}$$

In (Boussaada et al., 2016) it is shown that equation (10) admits a spectral value at z with multiplicity 2 if, and only if, $(\alpha, z) = (\alpha_+, z_+)$ or $(\alpha, z) = (\alpha_-, z_-)$ where:

$$\begin{cases} \alpha_- = -\left(2+\sqrt{4+\tau^2\beta^2}\right) \mathrm{e}^{1/2\,\tau\left(-\beta-\frac{2+\sqrt{4+\tau^2\beta^2}}{\tau}\right)}\tau^{-2}, \\ z_- = -\dfrac{\beta}{2} - \dfrac{2+\sqrt{4+\tau^2\beta^2}}{2\tau} \\ \alpha_+ = \left(-2+\sqrt{4+\tau^2\beta^2}\right) \mathrm{e}^{1/2\,\tau\left(-\beta+\frac{-2+\sqrt{4+\tau^2\beta^2}}{\tau}\right)}\tau^{-2}, \\ z_+ = -\dfrac{\beta}{2} + \dfrac{-2+\sqrt{4+\tau^2\beta^2}}{2\tau} \end{cases} \tag{11}$$

It is also proved that if $(z, \alpha) = (z_+, \alpha_+)$ (respectively (z_-, α_-)) and $\tau\beta > 2\sqrt{3}$ ($\tau\beta < 2\sqrt{3}$) then z_+ (respectively z_-) is the rightmost root and the corresponding steady state solution is asymptotically stable. The next section enunciates the main contribution, which extends the above results.

3. MAIN RESULT

The following result generalizes the result from in (Boussaada et al., 2018) which is restricted to $\beta_1 = 0$.

Theorem 2. Consider the quasipolynomial function (5):

$$\Delta(s,\tau) = s^2 + c_1 s + c_0 + (\beta_0 + s\beta_1)\, e^{-\tau\, s},$$

The following assertions hold:

i) The multiplicity of any given root of the quasipolynomial function (5) is bounded by 4, it can be attained only on the real axis.
ii) The quasipolynomial (5) admits a real spectral value at $s = s_\pm$ with algebraic multiplicity 4 if, and only if, either

$$\begin{cases} s_+ = \dfrac{-2+\sqrt{-2+c_0\tau^2}}{\tau}, \\ \beta_0 = 2\,\dfrac{\mathrm{e}^{-2+\sqrt{-2+c_0\tau^2}}\left(-5 \pm \sqrt{-2+c_0\tau^2}\right)}{\tau^2}, \\ \beta_1 = -2\,\dfrac{\mathrm{e}^{-2+\sqrt{-2+c_0\tau^2}}}{\tau},\; c_1 = -2\,\dfrac{\sqrt{-2+c_0\tau^2}}{\tau} \end{cases} \tag{12}$$

or

$$\begin{cases} s_- = \dfrac{-2-\sqrt{-2+c_0\tau^2}}{\tau}, \\ \beta_0 = 2\,\dfrac{\mathrm{e}^{-2-\sqrt{-2+c_0\tau^2}}\left(-5 - \sqrt{-2+c_0\tau^2}\right)}{\tau^2}, \\ \beta_1 = -2\,\dfrac{\mathrm{e}^{-2-\sqrt{-2+c_0\tau^2}}}{\tau}, c_1 = 2\,\dfrac{\sqrt{-2+c_0\tau^2}}{\tau} \end{cases} \tag{13}$$

where τ is arbitrarily chosen satisfying $c_0\tau^2 \geq 2$.
iii) If either (12) or (13) is satisfied then $s = s_\pm$ is the rightmost root of (4).

A complete proof of the main result will be presented in an extended version of the paper. Its sketch is summarized below.

Sketch of the Proof: The degree of the quasipolynomial function is equal to 4 as defined above. First, the vanishing of the quasipolynomial Δ yields the elimination of the exponential term as a rational function in s. The substitution of the obtained equality in the first three derivatives gives a system of algebraic equations. Solving them, one obtains the two solutions (12) and (13). Next using *the argument principle* one shows the dominancy of $s_\pm$; see Figure 4. Further explanation can be found in the next section. For an effective implementation for complex integral computations we refer the reader to (Xu et al., 2016). □

4. ILLUSTRATIVE EXAMPLE

Consider the damping-free oscillator controlled by a delayed proportional-derivative controller

$$\begin{cases} \ddot{\xi}(t) + \gamma\xi(t) = u(t), \\ u(t) = -\tilde{\beta}\xi(t-\tau) - \tilde{\alpha}\dot{\xi}(t-\tau) \end{cases} \tag{14}$$

where γ a real parameter. The corresponding quasipolynomial function is given by

$$\Delta(s,\tau) = s^2 + \gamma + (\beta + \alpha\, s)\, \mathrm{e}^{-s\,\tau} \tag{15}$$

If $\gamma = 0$ then the control problem (14) reduces to the stabilization of the double integrator using delayed PD controller. Otherwise, using a linear transformation, it is sufficient to study the two cases $\gamma = 1$ and $\gamma = -1$ to get a complete picture of the effect of the parameter γ on the dominancy of admissible multiple roots.

4.1 Double integrator stabilized by delayed PD-Controller

A result from (Niculescu and Michiels, 2004; Kharitonov et al., 2005), mentioned in Section 2.1, asserts that a delayed proportional controller (with a single delay) is not able to stabilize a double integrator. Here we investigate the effect of the additional delayed derivative term and explore its stabilizing effect through the multiplicity induced-dominancy property. Consider the quasipolynomial function

$$\Delta(s,\tau) = s^2 + (\beta + \alpha\, s)\, \mathrm{e}^{-s\,\tau} \tag{16}$$

where $\alpha \neq 0$.

Proposition 3. The following assertions hold for (16):

i) The multiplicity of any given root of the quasipolynomial function (16) is bounded by 3, it can be attained only on the real axis.
ii) The quasipolynomial (16) admits a real spectral value at $s = s_\pm$ with algebraic multiplicity 3 if, and only if,

$$\begin{cases} \alpha = 2\,\dfrac{\left(-1-\sqrt{2}\right)\mathrm{e}^{-2-\sqrt{2}}}{\tau}, \\ \beta = 2\,\dfrac{\mathrm{e}^{-2-\sqrt{2}}\left(-7-5\sqrt{2}\right)}{\tau^2}, \\ s_- = \dfrac{-2-\sqrt{2}}{\tau} \end{cases} \tag{17}$$

or

$$\begin{cases} \alpha = 2\,\dfrac{\left(\sqrt{2}-1\right)\mathrm{e}^{-2+\sqrt{2}}}{\tau}, \\ \beta = 2\,\dfrac{\mathrm{e}^{-2+\sqrt{2}}\left(-7+5\sqrt{2}\right)}{\tau^2}, \\ s_+ = \dfrac{-2+\sqrt{2}}{\tau} \end{cases} \tag{18}$$

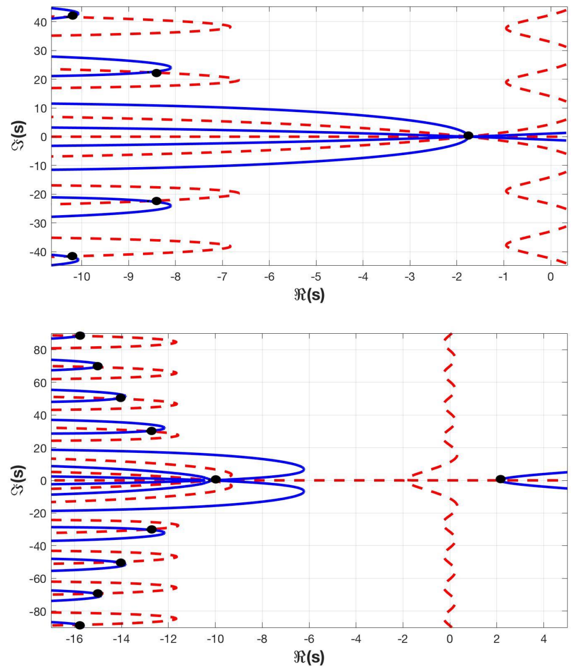

Fig. 2. (Up) Spectrum distribution corresponding to (16) for $\tau = 1/3$ and (18) is satisfied. (Down) Spectrum distribution corresponding to (16) for $\tau = 1/3$ and (17) is satisfied, the dominancy property is lost.

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iii) A spectral value of (16) with maximal multiplicity (multiplicity 3) is dominant if and only if $s = s_+$.

Sketch of the Proof: The degree of the quasipolynomial function is equal to 4 as defined in Section 2. First, the vanishing of the quasipolynomial Δ yields the elimination of the exponential term as a rational function in s. The substitution of the obtained equality in the first three derivatives gives a system of algebraic equations. Solving them, one shows that the solutions set is empty. Thus, the maximal multiplicity is less or equal to 3. Solving the two first derivatives yields to the solutions (17) and (18).

The dominancy of s_+ proof follows the same steps as that of Theorem 2. First, using Proposition 1, one establishes a generic supremum bound for the real and imaginary parts of roots of (21)-(23). Then define an integration contour $\gamma = \cup_{k=1}^{6} C_k$ which is taken as a counterclockwise closed curve, then an integral over γ is defined as the sum of the integrals over the directed smooth curves that make γ up, as illustrated in Figures 4. Elementary calculations give parametrization of γ on each C_k. Since Δ is analytic then *the argument principle* asserts:

$$\frac{1}{2i\pi}\oint_{\gamma}\frac{\partial_s\Delta(s,\tau)}{\Delta(s,\tau)}ds = \mathcal{Z}, \tag{19}$$

where $\mathcal{Z}$ designates the number of the quasipolynomial roots enclosed by γ. Furthermore, the left-hand side of (24) gives:

$$\oint_{\gamma}\frac{\partial_s\Delta(s,\tau)}{\Delta(s,\tau)}ds = \lim_{\epsilon\to 0}\sum_{k=1,k\neq 4}^{6}\int_0^1 \dot{s}^k(t)\frac{\partial_s\Delta(s^k(t),\tau)}{\Delta(s^k(t),\tau)}dt + \lim_{\epsilon\to 0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{s}^4(t)\frac{\partial_s\Delta(s^4(t),\tau)}{\Delta(s^4(t),\tau)}dt \tag{20}$$

where $s^k(t)$ designates the parametrization of s along C_k for $k \in \{1,\ldots,6\}$. Some tedious but elementary computations lead to $\mathcal{Z} = 0$ in case (18), but $\mathcal{Z} = 1$ in case (17). Finally, Figure 2 illustrate the result. □

4.2 Harmonic oscillator stabilized by delayed PD-controller

Consider the problem of stabilization of a classical harmonic oscillator using PD controller:

$$\Delta(s,\tau) = s^2 + 1 + (\beta + \alpha\, s)\,\mathrm{e}^{-s\,\tau} \tag{21}$$

Proposition 4. Consider the quasipolynomial function (21) for which the following assertions hold:

i) The multiplicity of any given root of the quasipolynomial function (5) is bounded by 4, it can be attained only on the real axis.

ii) The quasipolynomial (5) admits a real spectral value at $s = -\sqrt{2}$ with algebraic multiplicity 4 if, and only if,

$$\alpha = -\sqrt{2}\mathrm{e}^{-2},\ \beta = -5\,\mathrm{e}^{-2},\ \tau = \sqrt{2}. \tag{22}$$

iii) If (22) is satisfied then $s = -\sqrt{2}$ is the rightmost root of (4).

Sketch of the Proof: Following Theorem 2, $s_0 = -\sqrt{2}$ is root of (21) of multiplicity 4 with parameters values:

$$\alpha_0 = -\sqrt{2}\mathrm{e}^{-2}, \beta_0 = -5\,\mathrm{e}^{-2},\ \tau_0 = \sqrt{2} \tag{23}$$

Furthermore, s_0 is the rightmost root of (21)-(23). The dominancy proof follows the same steps as that of Theorem 2. First, using Proposition 1, one establish a generic supremum bound for the real part as well as the imaginary part of roots of (21)-(23). Then define an integration contour $\gamma = \cup_{k=1}^{6} C_k$ which is taken as a counterclockwise closed curve, then an integral over γ is defined as the sum of the integrals over the directed smooth curves that make γ up, as illustrated in Figures 4. Since Δ is analytic function then *the principle of argument* asserts:

$$\frac{1}{2i\pi}\oint_{\gamma}\frac{\partial_s\Delta(s,\tau)}{\Delta(s,\tau)}ds = \mathcal{Z}, \tag{24}$$

where $\mathcal{Z}$ designates the number of the quasipolynomial roots enclosed by γ. Furthermore, the left-hand side of (24) gives:

$$\oint_{\gamma}\frac{\partial_s\Delta(s,\tau)}{\Delta(s,\tau)}ds = \lim_{\epsilon\to 0}\sum_{k=1,k\neq 4}^{6}\int_0^1 \dot{s}^k(t)\frac{\partial_s\Delta(s^k(t),\tau)}{\Delta(s^k(t),\tau)}dt + \lim_{\epsilon\to 0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{s}^4(t)\frac{\partial_s\Delta(s^4(t),\tau)}{\Delta(s^4(t),\tau)}dt \tag{25}$$

Some tedious but elementary computations allows to $\mathcal{Z} = 0$.

Figures 3 and 4 illustrate the distribution of the spectrum of (21)-(23). □

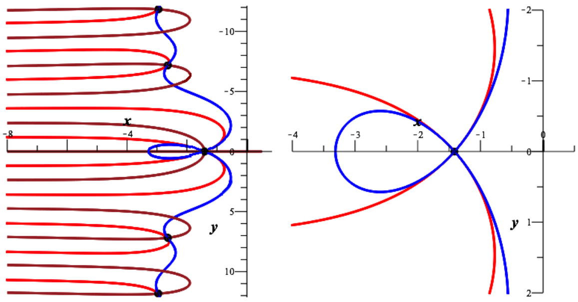

Fig. 3. (Left) Spectrum distribution corresponding to (21)-(23). The black points correspond to the spectral values, the solid blue line is the zero-modulus-manifold, the solid brown line is the zero-real part-manifold, the solid red line is the zero-imaginary part-manifold where x and y represent respectively the real part and the imaginary part of the complex variable s. (Right) Zoom on the dominant non resonant spectral value corresponding to (21)-(23) located at $\lambda = -\sqrt{2}$. The solid blue line is the zero-modulus-manifold and the solid red line is the zero-imaginary part-manifold of (21)-(23).

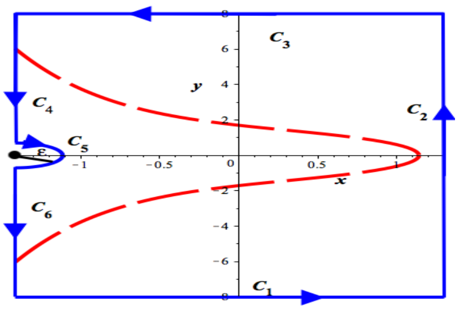

Fig. 4. The used contour for applying the argument principle to investigate the dominancy of the multiple root in blue solid line. The dashed red line gives the generic spectrum envelope established from Proposition 1 .

4.3 Controlling unstable second-order system via a delayed Proportional-Derivative Controller

Consider the problem of stabilization of the second-order sparse polynomial (having two real roots with different signs):

$$\Delta(s,\tau) = s^2 - 1 + (\beta + \alpha\, s)\, e^{-s\,\tau}. \tag{26}$$

Proposition 5. The following assertions hold for (26):

i) The multiplicity of any given root of the quasipolynomial function (26) is bounded by 3, it can be attained only on the real axis.

ii) The quasipolynomial (26) admits a real spectral value at $s = s_\pm$ with algebraic multiplicity 3 if and only if either

$$\begin{cases} s_+ = \dfrac{-2+\sqrt{\tau^2+2}}{\tau}, \\ \beta = 2\,\dfrac{e^{-2+\sqrt{\tau^2+2}}\left(-7+5\sqrt{\tau^2+2}-\tau^2\right)}{\tau^2}, \\ \alpha = 2\,\dfrac{\left(-1+\sqrt{\tau^2+2}\right)e^{-2+\sqrt{\tau^2+2}}}{\tau} \end{cases} \tag{27}$$

or

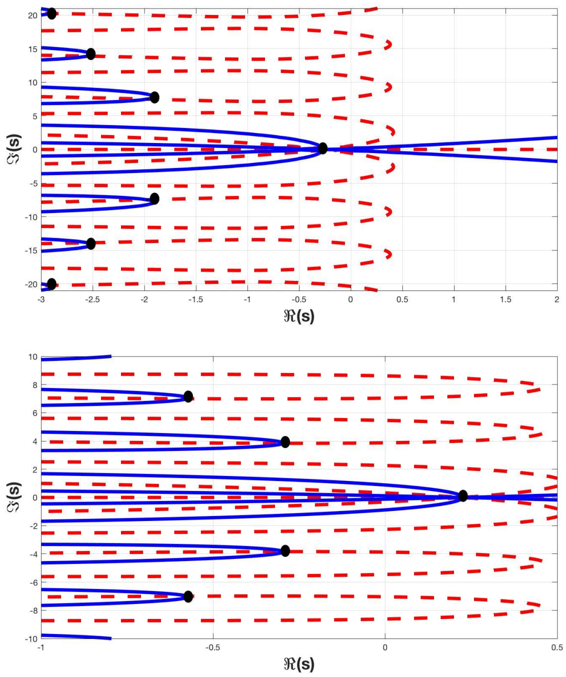

Fig. 5. (Up)The spectral distribution of (26) under the conditions (27) with $\tau = 1$. The multiple spectral value is dominant and stable. (Down) The spectral distribution of (26) under the conditions (27) with $\tau = 2$. The multiple spectral value is dominant but unstable.

$$\begin{cases} s_- = \dfrac{-2-\sqrt{\tau^2+2}}{\tau}, \\ \alpha = 2\,\dfrac{\left(-1-\sqrt{\tau^2+2}\right)e^{-2-\sqrt{\tau^2+2}}}{\tau}, \\ \beta = 2\,\dfrac{e^{-2-\sqrt{\tau^2+2}}\left(-7-5\sqrt{\tau^2+2}-\tau^2\right)}{\tau^2}, \end{cases} \tag{28}$$

Throughout the above result, one can illustrate various scenarios. In the first one, the multiple root is dominant and stable, for instance when condition (27) is satisfied and $\tau = 1$, see Figure 5 (Left). In the second, the multiple root is dominant and unstable which occurs when condition (27) is satisfied and $\tau = 2$, see Figure 5 (Right). In the last, the multiple spectral value is not dominant which is illustrated in Figure 6.

5. CONCLUSION

A new extension of a dominancy result based on the maximal multiplicity of spectral value is analytically shown for generic second-order systems with a single delay. The dominancy property is parametrically analyzed in the case of a damping-free oscillator. Also examples of multiple roots loss of dominancy are provided. Unlike methods based on finite spectrum assignment, the method proposed in this work does not render the closed loop system finite dimensional but consists in controlling its rightmost spectral value, see (Boussaada and Niculescu, 2018) for further applications of the approach.

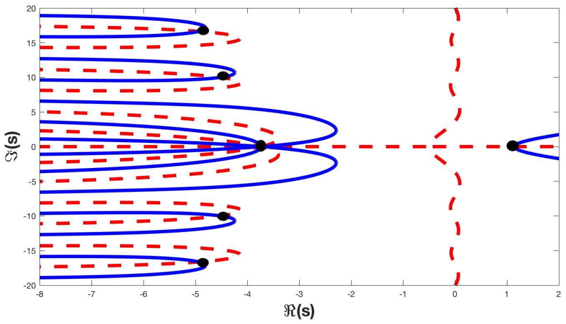

Fig. 6. The spectral distribution of (26) under the conditions (28) with $\tau = 1$. The multiple spectral value is not dominant

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