Perturbations in polyhedral partitions and the related fragility of piecewise affine control.

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by

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Abstract

The control design techniques for linear or hybrid systems under constraints lead often to off-line state-space partitions with non-overlapping convex polyhedral regions. This corresponds to a piecewise affine (PWA) state feedback control laws associated to polyhedral partition of the state-space. The aim of this paper is to consider perturbation in the representation of the vertices of the polyhedral regions. The idea behind this is to perform a quantization operation on the representation of the state-space regions and the associated PWA control laws in order to reduce the hardware requirements in terms of processor speed and memory unit. The quantized state-space partitions lose some of the important properties of the explicit controllers: non-overlapping, convexity and invariant characterization. How the perturbation affects the polyhedral regions and invoke overlapping to the modified polyhedral regions is first shown. The major contribution of this work is to analyze to what extend the non-overlapping and the invariance characteristics of the PWA controller can be preserved when perturbation takes place on the vertex representation. We determine two different sets called vertex-sensitivity and sensitivity margin to characterize admissible perturbation preserving the non-overlapping and the invariance property of the controller respectively. Finally, we show how to perturb multiple vertex sequentially and reconfigure the polyhedral regions to the perturbed vertices.

Key Words: PWA control, Predictive Control, Explicit MPC, Robustness Margin.

1 Introduction

Explicit PWA control laws can be easily evaluated and implemented on-line for systems with extremely fast dynamics and as long as the state-space models are of small dimensions. Recently, such control laws have gained popularity for a wide range of real-time control applications [1, 2, 3, 4, 5, 6]. However, the adoption of such control laws are pertained to the numbers of state-space partitions and the piecewise affine control laws associated with those partitions. In order to exploit the computational advantages of the explicit controller, a "*truncation or quantization operation*" must be performed on the representation of the state-space partitions and on their associated PWA controls. The implications of the quantized state partitions and the quantized PWA gains and offsets extend to affect control input accuracy, whose computations are based on point location functions, and the properties of the PWA controller. The quantized state partitions might also adversely affect the non-overlapping and non-emptiness characteristics of the PWA controller. In the recent work [8, 9], a geometrical approach to determine robustness/fragility margins with respect to the invariance characteristics of the PWA controller has been proposed. However their approach does not extend to the quantized state-space partitions. In a recent study [10], the accuracy of the explicit control input for the quantized regions and the quantized PWA control laws are analyzed in general to prove the scale of quantization required in order to obtain a certain degree of control accuracy. However, all these references build the control input analysis on the assumption that the modified state-space regions are non-overlapping and thus they do not address one of the essential characteristics of the representation of the state partitions: the well-posedness and completeness of the polyhedral partition of the feasible domain.

The framework of the present paper is the one of a linear discrete-time system controlled by piecewise affine explicit control law. In this paper, it will be analyzed how the regions or polyhedral partitions change in the event of perturbation on the vertex representation of the partitions occurs

The paper is organized as follows, after introducing some basic notations and definitions, the background of the system description is presented in section 2. In section 3, the motivation of the work is described with graphical illustrations and the problem formulation is stated in a mathematical form. In section 4, the main results concerning with the formulation of overlapping in the change of partition is shown and the vertex sensitivity margin is provided with related theorem, proof and implementation algorithm. In section 5, a method to update the vertices on the frontier of the feasible set for admissible perturbation is provided and, in the same section, the conditions to regain the convexity of the feasible domain is also discussed. Later in section 5, the treatment of an individual inner vertices of the polyhedral partitions with respect to the non-overlapping and invariant properties is provided and an examples for multiple treatment of inner vertices are discussed.

Preliminaries and notations

This section addresses some basic notations and definitions. We denote $\mathbb{R}^n$ a Euclidean space and $x \in \mathbb{R}^n$ a vector with n elements. A matrix $A \in \mathbb{R}^{n\times m}$, $A = [a_{ij}]$ is denoted. The sets $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_+$ denote set of real numbers, set of non-negative real numbers, set of integers, set of non-negative integers, set of positive integers, respectively. For a $N \in \mathbb{N}_+$, $\mathcal{I}_N$ denotes the set of integers, $\mathcal{I}_N := \{\ i \in \mathbb{N}_+ \mid i \leq N\ \}$.

A set $\mathcal{S} \subset \mathbb{R}^n$ is a proper *$\mathcal{C}$-set* if it is convex, closed, compact and contains the origin in its interior. A *polyhedron* is the (convex) intersection of a finite number of open or closed half-spaces. Such a definition is called *half-space representation* or simply *H-representation.* Consider a polyhedron $\mathcal{P}$ whose closed half-spaces can be written as a system of linear inequalities, e.g,

$$\mathcal{P} = \{x \in \mathbb{R}^n | Ax \leq b\}.$$

A polyhedron can also be defined as a *convex hull* of finite set of points $x_1, \cdots, x_d$, e.g,

$$\mathcal{P} = conv\{x_1, x2, \cdots, x_d\}$$

and such a representation is called *vertex representation* or *V-representation.* A closed and bounded polyhedron is called a *polytope.* The set of vertices of a polytope or polyhedron $\mathcal{M} \subset \mathbb{R}^n$ is denoted $\mathcal{V}(\mathcal{M})$. Given two sets $\mathcal{P} \in \mathbb{R}^n$ and $\mathcal{Q} \in \mathbb{R}^n$,

- the intersection, denoted by $\mathcal{P} \cap \mathcal{Q}$, of $\mathcal{P}$ and $\mathcal{Q}$ is defined as

$$\mathcal{P} \cap \mathcal{Q} = \{x \in \mathbb{R}^n | x \in \mathcal{P} \text{ and } x \in \mathcal{Q}\}.$$

– the union, denoted by $\mathcal{P} \cup \mathcal{Q}$, of $\mathcal{P}$ and $\mathcal{Q}$ is defined as

$$\mathcal{P} \cup \mathcal{Q} = \{x \in \mathbb{R}^n | x \in \mathcal{P} \text{ or } x \in \mathcal{Q}\}.$$

– the set difference, denoted by $\mathcal{P} \setminus \mathcal{Q}$, of $\mathcal{P}$ and $\mathcal{Q}$ is defined as

$$\mathcal{P} \setminus \mathcal{Q} = \{x \in \mathcal{P} | x \notin \mathcal{Q}\}.$$

2 System Description

Let us consider a discrete-time linear system given by,

$$x_{k+1} = Ax_k + Bu_k. \tag{2.1}$$

here, $x_k \in \mathbb{R}^n$ is the state vector at time k and $u_k \in \mathbb{R}^m$ is the control input vector. The system states and inputs variables are subject to constraints, with state constraints given by

$$\mathcal{X} = \{x : H_x x \leq h_x, H_x \in \mathbb{R}^{p_x \times n}, h_x \in \mathbb{R}^{p_x}\}, \tag{2.2}$$

and input constraints by,

$$\mathcal{U} = \{u : H_u u \leq h_u, H_u \in \mathbb{R}^{p_u \times m}, h_u \in \mathbb{R}^{p_u}\}. \tag{2.3}$$

Where the matrices H_x, H_u and the vectors h_x, h_u are assumed to be constant, and $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$. The state and input constraints sets $\mathcal{X}$ and $\mathcal{U}$ are proper $\mathcal{C}$-sets.

Definition 2.1. *A closed and bounded set $\mathcal{R} \subset \mathcal{X}$ is called controlled positively invariant with respect to the system* (2.1) *if there exists a control law $u^*(x_k)$, such that $\forall\ x(0) \in \mathcal{R}$, then $x_k \in \mathcal{R}$, $\forall k \in \mathbb{Z}$.*

Definition 2.2. *Consider a set of sets $\mathcal{P}_N(\mathcal{R})$. This will define a polyhedral partition of the $\mathcal{C}$-set $\mathcal{R} \subset \mathbb{R}^n$, with $\mathcal{P}_N(\mathcal{R}) = \{\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_N\}$, $N \in \mathbb{N}_+$ and $\mathcal{R}_i \subset \mathbb{R}^n$ if*

1. *$\mathcal{R}_i$ are polyhedral and $\mathcal{I}_N$ is finite.*
2. *$\mathcal{R} = \cup_{i \in \mathcal{I}_N} \mathcal{R}_i$,*
3. *$int(\mathcal{R}_i) \neq \emptyset$,*
4. *$int(\mathcal{R}_i) \cap int(\mathcal{R}_j) = \emptyset$, $\forall (i,j) \in \mathcal{I}_N^2$, and $i \neq j$.*

Definition 2.3. *Consider for a given $x \in \mathbb{R}^n$ and the polyhedral partitions $\mathcal{R} = \cup_{i=1}^N \mathcal{R}_i$, the point location function of the polyhedral partition of $\mathcal{R}$ is given by,*

$$x \rightarrow i(x) \ \text{with} \ \ i : \mathcal{R} \rightarrow \mathbb{N}_{\leq N}. \tag{2.4}$$

Practically, $i(x)$ indicates (the unique) polyhedral region that contains x within the partition. Whenever x lies on the frontiers, there might be several polyhedral sets containing the point. In such cases, without loss of generality $i(x)$ is selected as the minimal index.

The feedback control law takes the form of a mapping u_{pwa}: $\mathcal{R} \rightarrow \mathbb{R}^m$

$$u_{pwa}(x_k) = F_{i(x)} x_k + g_{i(x)}, \quad x_k \in \mathcal{R}_{i(x)}. \tag{2.5}$$

defined over the polyhedral partition of the set $\mathcal{R} = \cup_{i \in \mathcal{I}_N} \mathcal{R}_i$. With respect to the PWA function, the following assumptions hold

Assumption 2.1. *1. $\mathcal{R}$ is positively invariant with respect to $x_{k+1} = Ax_k + Bu_{pwa}(x_k)$.*

3 Motivation and Problem Formulation

For real-time implementation of the PWA control law, three stages need to be considered:
(A) Off-line: Storage of the polyhedral regions $\mathcal{R}_i$, the PWA control gains F_i and affine components g_i.
(B) On-line: Use of a point location mechanism with respect to the parameter x and the polyhedral partitions $\mathcal{R} = \cup_{i=1}^{N} \mathcal{R}_i$, This can be assimilated to a function $x_k \rightarrow i(x_k)$.
(C) On-line: Evaluation of the PWA control law $u_{pwa}(x_k) = F_{i(x_k)}x + g_{i(x_k)}$ based on the current state x_k and the result of the previous stage of positioning.
In practice this evaluation procedure can fail due to several reasons.
(i) The precision of $\mathcal{R}_i$ representation.
(ii) Due to point location mismatch.
(iii) PWA control accuracy inflicted by the precision of representation of the control gain F_i and offset g_i.

The PWA control accuracy and the fragility issues of the gains and affine terms F_i and g_i have been extensively discussed in [8, 9, 10]. The resulting solution obtained from the EMPC problem is a set of PWA functions defined over the polyhedral partition $\mathcal{P}_N(\mathcal{R})$ and their analysis in the point iii) above can be handled in the respective framework. However, the issues related with the representation and the closely related point location problems (items i) and ii) above) remain largely uncovered and will represent the main goal of the present work. Before entering into the details of the main results, let us motivate the chosen approach by considering a polyhedral region $\mathcal{R}_i \subset \mathcal{R} \in \mathbb{R}^n$, $i \in \mathcal{I}_N$, and its half-space representation given by,

$$\mathcal{R}_i = \{x \mid h_{i,j}x \leq b_{i,j}, \forall i \in \mathcal{I}_N, \quad j = 1, \cdots, r_i\}. \tag{3.1}$$

Here, r_i denotes the number of closed half-spaces of the region $\mathcal{R}_i$. In order to analyze the sensitivity of the polyhedral partition representation and its implication on the PWA control, a perturbation in the representation of the half-space $\{h_{i,r_b} \leq b_{i,r_b}\}$, for one of the indices $r_b \in \mathcal{I}_{r_i}$ of the region $\mathcal{R}_i$ will be considered,

$$\hat{h}_{i,r_b} = h_{i,r_b} + \Delta h_{i,r_b} \text{ and } \hat{b}_{i,r_b} = b_{i,r_b} + \Delta b_{i,r_b} \tag{3.2}$$

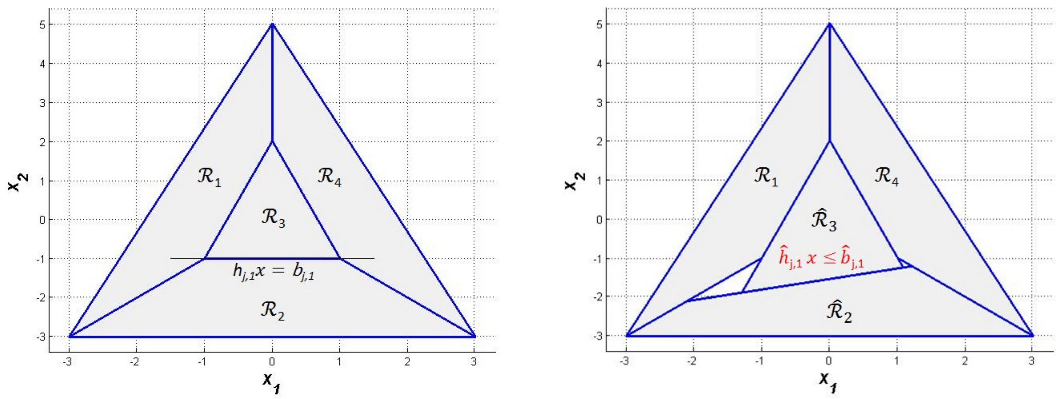
which leads to a new polyhedral set:

$$\hat{\mathcal{R}}_i = \{x \mid \hat{h}_{i,r_b}x \leq \hat{b}_{i,r_b}\}. \tag{3.3}$$

The perturbation of the half-space representation of the region $\mathcal{R}_i$ will concomitantly affect all the neighbor regions $\mathcal{R}_j$ sharing the respective frontier. As several neighboring regions are affected, the analysis of the effects on the partition will encounter structural problems:

1. Invalidation of the polyhedral partitions definition due to the violation of the property: $int(\hat{\mathcal{R}}_i) \cap int(\hat{\mathcal{R}}_j) = \emptyset, \ \forall i \neq j$.
2. $\mathcal{R} \setminus \left\{\cup_{i=1}^{N} \hat{\mathcal{R}}_i\right\} \neq \emptyset$ even if $conv(\mathcal{R}) = conv(\cup_{i \in \mathcal{I}_N} \hat{\mathcal{R}}_i)$ posing an well-possessedness issue in the characterization of the polyhedral partition and subsequently in the PWA function evaluation (2.5).

The first type of problem arise from the asymmetric consideration of the perturbation in between neighboring regions while the second can take place even if the perturbation is treated similarly among the neighboring regions. Moreover, both phenomena lead to invalidation of the PWA control law defined over the partition $\hat{\mathcal{R}} = \cup_{i \in \mathcal{I}_N} \hat{\mathcal{R}}_i$. Particularly the second phenomenon leaves the point location function seemingly untraceable and this case is shown in Figure 1. The drawbacks demonstrated by the perturbation on the half-space



(a) 2-D Polyhedral with four regions $\mathcal{R} = \cup_{i=1}^{4} \mathcal{R}_i$ before perturbation of the half-space representation.

(b) Illustration of regions $\hat{\mathcal{R}}_1$, $\hat{\mathcal{R}}_2$, $\hat{\mathcal{R}}_3$ and $\hat{\mathcal{R}}_4$ after perturbation of $h_{3,1}x = b_{3,1}$. Such a reconfiguration produces holes in the feasible domain that will undermine the well-possessedness characteristics.

Figure 1: 2-D polyhedral representation before and after perturbation on the half-space representation.

representation are the consequence of the fact that the perturbations are not considered jointly for all half-spaces. This is due to the fact that the closed half-spaces of the regions $\mathcal{R}_i$ are treated independently at the level of each neighboring region and addressing perturbation on such representation is missing the interplay between regions in composing the polyhedral partition. These drawbacks lead us to the duality of the polyhedron representation where the problem formulation can be reformulated.

Eq (3.1) can be given with equivalent vertex representation in the virute of Motzkin duality:

$$\mathcal{R}_i = Conv\{v_{i,1}, \cdots, v_{i,r_i}\}, \; \forall i \in \mathcal{I}_N \tag{3.4}$$

here r_i is the number of vertices of $\mathcal{R}_i$. Now, consider a perturbation with respect to the vertex representation $v_{i,j}, \; j \in \mathcal{I}_{r_i}$ of the region $\mathcal{R}_i$,

$$\hat{v}_{i,j} = v_{i,j} + \Delta v_{i,j}, \; i \in \mathcal{I}_N, \; j \in \mathcal{I}_{r_i} \tag{3.5}$$

this will lead to a new polyhedral set:

$$\hat{\mathcal{R}}_i = Conv\{v_{i,1} + \Delta v_{i,1}, \cdots, v_{i,r_i} + \Delta v_{i,r_i}\}. \tag{3.6}$$

It becomes obvious that in this case $\mathcal{R} \setminus \left\{ \cup_{i \in \mathcal{I}_N} \hat{\mathcal{R}}_i \right\} = \emptyset$ if the vertices on the frontier of $\mathcal{R}$ are not perturbed. The loss of continuity is the price to be paid for the loss of precision in the partition representation and can be acceptable as long as the control action is uniquely defined on the interior of the full-dimensional regions within the partition. This possible overlapping due to changes in the vertices of the polyhedral partition represents a critical structural change because the unicity of the control law is lost on a compact full-dimensional region of the state-space. The non-uniqueness of the control action leads to behaviors which are difficult to characterize in terms of determinedness and lose of performance and thus should be avoided in the first place. This issue forms the basis for investigation in the present paper and can be resumed by the need to characterize the limits of the perturbation which preserve the "non-overlapping" property of the polyhedral partition. In order to illustrate the partitions in this framework and present the obvious advantages of considering perturbation on the vertex representation, a similar partition to the one presented in Figure 1 is depicted in the Figure 2. This time it is obvious that using the dual representation of polyhedra and their perturbed version, the completeness of the partition is not lost. In general terms, the case $\mathcal{R} \setminus \left\{ \cup_{i=1}^{N} \hat{\mathcal{R}}_i \neq \right\} \emptyset$ is avoided from the consequences of the perturbations in the polyhedral partition. To resume, starting from the existence of the system in the form (2.1) stabilized by a PWA control law, the main objective is to discuss the impact of perturbations on the vertex representation of the polyhedral region by proposing:

- An analysis of the admissible perturbations with respect to the overlapping characteristics of the PWA controller,
- An analysis of the admissible perturbations with respect to the invariance properties of the PWA controller.

4 Treatment of a vertex considered independently - Polyhedral overlapping

In the following, a formal definition of the vertex sensitivity is provided focusing on the non-overlapping property of the polyhedral regions under the assumption that all the other vertices are fixed and only the vertex under study is subject to perturbations.

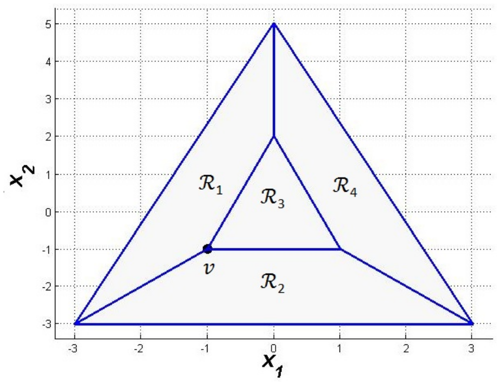
Definition 4.1. *Consider the set of partitions $\mathcal{P}_N(\mathcal{R}) \in \mathbb{R}^n$ with each region given by its vertex representation $\mathcal{R}_i = Conv\{v_{i,1}, \cdots, v_{i,r_i}\},\ i \in \mathcal{I}_N$. Assume $v \in \mathbb{R}^n$ be a vertex within $\mathcal{P}_N(\mathcal{R})$ and denote Θ^v as the set of indexes of polyhedral regions having v as a vertex:*

$$\Theta^v = \{j \in \mathcal{I}_N |\ v \in \mathcal{V}(\mathcal{R}_j)\} \tag{4.1}$$

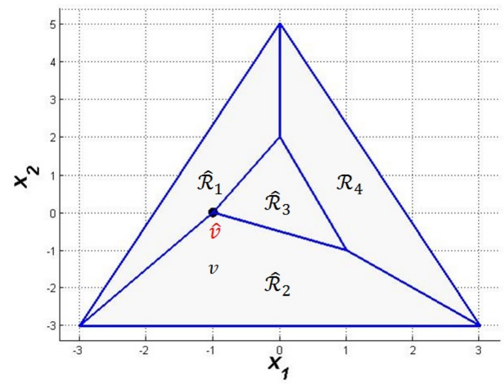
A compact set $\Psi \subset \mathcal{R} \subset \mathbb{R}^n$ is describing a vertex sensitivity for the vertex v if $v \in \Psi$ and for all $(v + \Delta v) \in \Psi$ the collection of sets

$$\begin{cases} \hat{\mathcal{R}}_j = Conv\{\mathcal{V}(\mathcal{R}_j) \setminus \{v\},\ v + \Delta v\},\ \forall j \in \Theta^v, \\ \hat{\mathcal{R}}_j = \mathcal{R}_j, \forall j \in \mathcal{I}_N \setminus \Theta^v \end{cases} \tag{4.2}$$

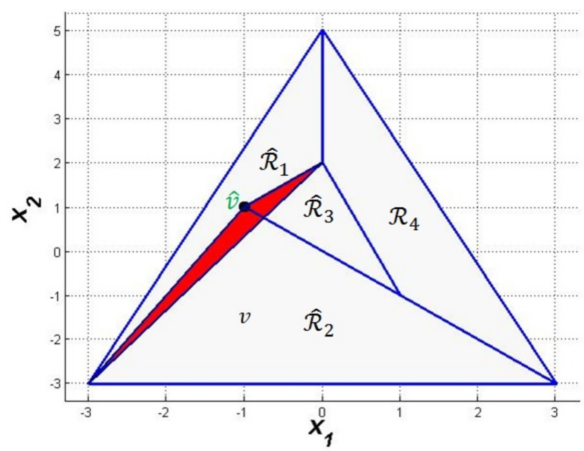
*represents a polyhedral partition: $\hat{\mathcal{P}}_N(\mathcal{R}) = \{\hat{\mathcal{R}}_1, \cdots, \hat{\mathcal{R}}_N\}$. The **sensitivity margin** for the vertex v is defined as the set Ψ^v containing any valid vertex sensitivity $\Psi \subset \Psi^v$.*



(a) 2-D Polyhedral with four regions $\mathcal{R} = \cup_{i=1}^{4}\mathcal{R}_i$ before perturbation of the vertex representation.



(b) Illustration of regions $\hat{\mathcal{R}}_1$, $\hat{\mathcal{R}}_2$, $\hat{\mathcal{R}}_3$ and $\hat{\mathcal{R}}_4 = \mathcal{R}_4$ after perturbation preserving the property, $int(\hat{\mathcal{R}}_i) \cap int(\hat{\mathcal{R}}_j) = \emptyset$, $\forall i, j \in \mathcal{I}_4, i \neq j$. Such a reconfiguration of the partition is admissible from the point of view of point-location.



(c) Illustration of regions $\hat{\mathcal{R}}_1$, $\hat{\mathcal{R}}_2$, $\hat{\mathcal{R}}_3$ and $\hat{\mathcal{R}}_4 == R_4$ after perturbation with the particular configuration leading to: $int(\hat{\mathcal{R}}_i) \cap int(\hat{\mathcal{R}}_j) \neq \emptyset$, $\forall i, j \in \mathcal{I}_4, i \neq j$.

Figure 2: 2-D polyhedral representation before and after perturbation of the vertex representation

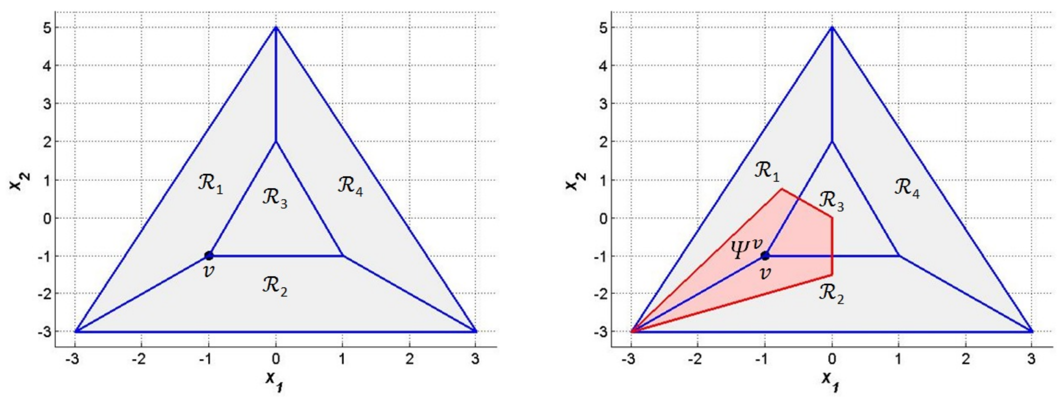
Given this formal definition, we concentrate next on the structural properties of this set and on its practical construction.

4.1 Characterization of the vertex sensitivity

In the next result, the structure of the sensitivity margin is stated while the proof will be constructed in such a way that the two scenarios of infeasible perturbations are enumerated and fully characterized. More than that, the set characterization will be constructive and allows the statement of a finite algorithmic procedure.

Theorem 4.1. *Consider the subset of regions $\mathcal{R}_j,\ j \in \Theta^v$ of $\mathcal{P}_N(\mathcal{R})$ such that $v \in \mathcal{V}(\mathcal{R}_j),\ \forall j \in \Theta^v$, then the vertex sensitivity margin Ψ^v is represented by a polyhedral set.*

In order to illustrate the result Figure 3a presents a polyhedral partition with four regions $\mathcal{R}_i, i = 1, \cdots, 4$ and the vertex of interest $v = [1\ \ -1]^T$ is denoted by a black dot. The vertex v belongs to three regions. In Figure 3b, the vertex sensitivity region Ψ^v is represented by a blue polytope and the vertex v can be settled to any of the points in the polytope Ψ^v in the event of reduced precision in the representation of the polytopic region.

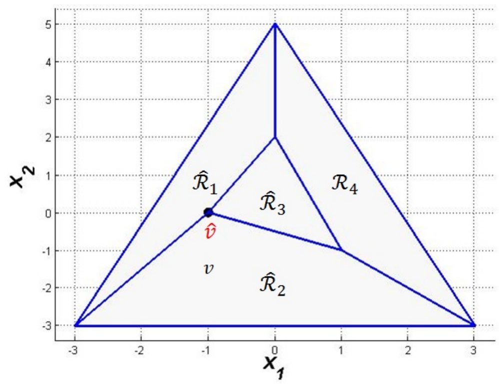


(a) Polyhedral with four regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ and the black dot represents v.

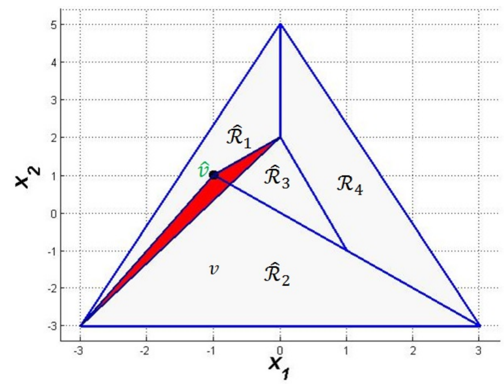
(b) The vertex of interest v and the vertex sensitivity region Ψ^v is shown.

Figure 3: Polyhedral partition with four regions, the vertex of interest v and the vertex sensitivity region Ψ^v are shown.

Following the structural result in Theorem 4.1, the vertex sensitivity region Ψ^v is a polytopic set. For any perturbed vertex $\hat{v}$ or point outside the red polytope in Figure 3, the non-overlapping property of the PWA control law is lost. As expected, the new regions formed with the displaced vertex $\hat{v}$ guarantees the "non-overlapping" property of the polyhedral partition if $\hat{v} \in \Psi^v$. This observation is validated with the help of the Figure 4a and Figure 4b, where the polyhedral regions are recreated by the displacement of vertex $\hat{v}$. In Figure 4a the vertex $v = [-1\ \ -1]^T$ is displaced to vertex $\hat{v} = [-1\ 0]^T \in \Psi^v$ which alters all the four regions in with indices in Θ^v but still preserves the overlapping property i.e., $int(\hat{\mathcal{R}}_i) \cap int(\hat{\mathcal{R}}_j) = \emptyset,\ \forall i, j \in \mathcal{I}_4, i \neq j$. Conversely, in Figure 4b, it is clearly visible that the overlapping of the regions takes place since $\hat{v} = [-1\ 1]^T \notin \Psi^v$.



(a) After perturbation $\hat{v} \in \Psi^v$ the regions are changed and $int(\hat{\mathcal{R}}_i) \cap int(\hat{\mathcal{R}}_j) = \emptyset,\ \forall i,j \in J^v, i \neq j$.



(b) After perturbation $\hat{v} \notin \Psi^v$ $int(\hat{\mathcal{R}}_i) \cap int(\hat{\mathcal{R}}_j) \neq \emptyset,\ \forall i,j \in J^v, i \neq j$.

Figure 4: The vertex v denoted by a black dot in Fig. 3.4 (a) is perturbed to $\hat{v}$ changing the regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ to $\hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2, \hat{\mathcal{R}}_3, \hat{\mathcal{R}}_4$.

5 Impact of Vertex Perturbation on the invariance characterization

In this section, we bring into discussion the set invariance characterization in relationship with the PWA controller. The positive invariance of the closed-loop dynamics will be considered on top of the non-overlapping property of the PWA control function (Theorem 4.1) which retains a well-possessedness structural property. It is important to mention that we preserve the assumption that only one vertex is perturbed at the time, all the other vertices being maintained at the nominal values.

From Theorem 4.1, it is understood that the vertex sensitivity can be analyzed with respect to the admissible perturbation related to the non-overlapping characteristics for any single vertex of the polyhedral partition $\mathcal{P}_N(\mathcal{R})$. In order to incorporate the analysis of vertex sensitivity with respect to the invariance property of the PWA control law, we will have to make a difference among the vertices and the impact of their perturbation. The vertices that represent extreme points of the set $\mathcal{R}$ are particularly sensitive to perturbation taking into account that they characterize the controlled-invariant properties *per se*. Indeed, any perturbation to these vertices will change the topology of the boundary of the set $\mathcal{R}$ and potentially invalidate the positive invariance. The second class of vertices are those that are included in the strict interior of the set $\mathcal{R}$.

5.1 Perturbations of vertices on the frontier of the feasible domain $\mathcal{R}$

In the following we analyze the perturbation of vertices that represent extreme points of the set $\mathcal{R}$ (placed on the frontier of $\mathcal{R}$) and thus, by their repositioning lead to a reconstruction of the polyhedral partition $\mathcal{P}_N(\mathcal{R}) = \{\mathcal{R}_1, \cdots, \mathcal{R}_N\}$.

Consider the set $\mathcal{R} = \cup_{i=1}^{N} \mathcal{R}_i$, with $\mathcal{R}_i = Conv\{v_{i,1}, v_{i,2}, \cdots, v_{i,r_i}\}$. Let us define the set of vertices on the frontier of $\mathcal{R}$ as:

$$V = \{v \in \mathcal{R} : \exists i \text{ such that } v \in \mathcal{V}(\mathcal{R}_i) \text{ and } v \notin int(\mathcal{R})\}. \tag{5.1}$$

For the sake of notation, the set will be represented as, $V = \{v_1, v_2, \cdots, v_r\}$ with r the number of vertices, lying on the frontier of the set $\mathcal{R}$.

The analysis of perturbations in the representation of the set $\mathcal{R}$ all by assuring the non-overlapping and invariance characteristics of $\mathcal{P}_N(\mathcal{R})$ is directly related to the positioning of the frontier vertices and will be considered for each vertex in V taken independently. We start by recalling the closed-loop mapping for any point in the set $\mathcal{R}$ preserving the invariance characteristics of the PWA controller:

$$f_{pwa}(x) = Ax + Bu_{pwa}(x) \in \mathcal{R}. \tag{5.2}$$

Using (5.2), we can represent the image of the set $\mathcal{R}$ by,

$$\mathcal{F}_{pwa}(\mathcal{R}) = \{y \in \mathbb{R}^n | \exists x \in \mathcal{R} \text{ such that } y = f_{pwa}(x)\}. \tag{5.3}$$

In the work of Scibilia et al [13], it has been shown that any approximation of $\mathcal{R}$ denoted by $\mathcal{R}^\alpha$ and which satisfies $\mathcal{R}^\alpha \subseteq \mathcal{R}$ and $\mathcal{R}^\alpha \supseteq \mathcal{F}_{pwa}$ preserves the invariance property of the closed loop. We aim to exploit the same principle in the framework of the vertex perturbations of the PWA control functions. We are interested in guaranteeing that the invariance holds with respect to a set $\mathcal{R}^\alpha$ defined in relationship with the existing PWA controller by perturbation of one of the frontier vertices $v \in V$ towards a point $\hat{v} \in \mathcal{R}$ thus leading to a novel (perturbed) set:

$$\begin{cases} \mathcal{R}_i^\alpha(v, \hat{v}) = conv\{\mathcal{V}(\mathcal{R}_i) \setminus v, \hat{v}\}, \forall i \in \mathcal{I}_N, \\ \mathcal{R}^\alpha(v, \hat{v}) = \cup_{i=1}^{N} \mathcal{R}_i^\alpha(v, \hat{v}), \\ \mathcal{P}_N(\mathcal{R}^\alpha(v, \hat{v})) = \{\mathcal{R}_1^\alpha(v, \hat{v}), \ldots, \mathcal{R}_N^\alpha(v, \hat{v})\}. \end{cases} \tag{5.4}$$

Theorem 5.1. *Let a dynamical system in the form* (2.1) *and the PWA control law* $u_{pwa}(x)$ (2.5) *defined over the set* $\mathcal{R}$ *and assuring its positive invariance in closed-loop. Given a set* $\mathcal{R}^\alpha \subset \mathcal{R}$*, the function* $\hat{u}_{pwa} : \mathcal{R}^\alpha \to \mathcal{U}$ *defined as* $\hat{u}_{pwa}(x) = u_{pwa}(x)$, $\forall x \in \mathcal{R}^\alpha$ *ensures the positive invariance of* $\mathcal{R}^\alpha$ *with respect to* $x_{k+1} = Ax + B\hat{u}_{pwa}(x)$ *if* $\mathcal{R}^\alpha \supseteq \mathcal{F}$.

Unfortunately, the Theorem 5.1 is not offering the appropriate guarantees for the positive invariance of $\mathcal{R}^\alpha$ in closed loop with the perturbed PWA control law. The main reason is that after perturbation of a vertex of the set $\mathcal{R}^\alpha(v, \hat{v})$, the new PWA function is not guaranteed to preserve the relationship $u_{pwa}(x) = \hat{u}_{pwa}(x)$, $\forall x \in \mathcal{R}^\alpha$ as stated in the Theorem above. The new partition $\mathcal{P}_N(\mathcal{R}^\alpha) \neq \mathcal{P}_N(\mathcal{R})$ and it differs in the regions affected by the perturbation of the vertex v as long as $\mathcal{R}_i^\alpha(v, \hat{v}) \neq \mathcal{R}_i, \forall i \in \Theta^v$. Explicitly, after perturbation, we have:

$$\hat{u}_{pwa}(x) = F_i x + g_i \text{ for } x \in \mathcal{R}_i^\alpha(v, \hat{v}), \tag{5.5}$$

and $\hat{u}_{pwa}(x) \neq u_{pwa}(x)$ when $x \in \mathcal{R}_i$ but $x \notin \mathcal{R}_i^\alpha$. This observation leads us to the statement of the main result where the following notation will be used:

$$\tilde{\mathcal{F}}(\mathcal{R}^\alpha) = \{y \in \mathcal{R} | \exists x \in \mathcal{R}^\alpha \text{ such that } y = Ax + B\hat{u}_{pwa}(x)\}. \tag{5.6}$$

Theorem 5.2. *Let $v \in V$ and its perturbation $\hat{v} = (v + \Delta v) \in \Psi^v$. The positive invariance properties of the set $\mathcal{R}^\alpha(v, \hat{v})$ with respect to $x_{k+1} = Ax + B\hat{u}_{pwa}(x_k)$ is guaranteed if $\mathcal{R}^\alpha(v, \hat{v}) \supseteq \mathcal{F}$ and $\tilde{\mathcal{F}}(\mathcal{R}^\alpha) \subset \mathcal{F}$.*

Example

In the Figure 5 (a), there are 8 inner vertices and we choose to manually displace them for this analysis and illustrative purpose. In the subplots from Figure 5, the polyhedral regions $\hat{\mathcal{R}}_i$ are presented with the vertex sensitivity and invariance-vertex sensitivity sets depicted in red and green color respectively, for the vertex that has the smallest Chebychev radius. The symbols dot and $\times$ in the subplots are the vertex candidate and the new position where the candidate will end up after perturbation. The numerical values of the vertices $\bar{v}$ are originally double precision representation but in the table we restricted the values till four decimal places due to space constraint.

Starting from Figure 5a, for the first vertex candidate, we perturb the vertex from $[-1.3314\ 8.1440]^T$ to the position $[-4.0, 1.6]^T$ there by affecting three regions. The next subplot shows the new polyhedral regions after perturbation. After the 8th iteration, the subplot 5i represents the final set $\hat{\mathcal{R}}$. From Figure 5, it is obvious from the subplots that no overlapping took place although a very aggressive perturbation has been tested for illustration. This validates one part of our work. In order to conclude on the closed loop behavior, we simulated for the state trajectories for the PWA controller for the outer vertices as initial states and this is presented in Figure 6. In the second analysis, we assume that the vertices on the frontier of the set $\mathcal{R}$ are fixed. A quantization function $f(\bar{v}_j) = \bar{v}_j + \Delta\bar{v}_j,\ \forall j \in \mathcal{I}_p$ with a random variable $\|\Delta\bar{v}_j\|_\infty \leq 0.2$ is considered for all the inner vertices in the set $\mathcal{R}$.

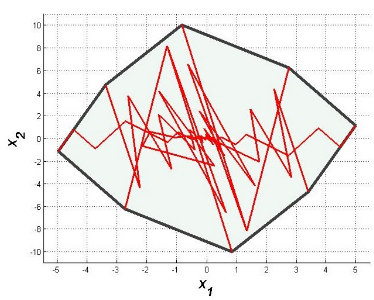


Figure 6: The states trajectories for the polyhedral partition for the vertices that lie on the boundary of the polytope.

6 Conclusion

In this work the analysis on the perturbation of the vertex representation has been presented. The vertex sensitivity that characterize for the admissible perturbation for assuring the non-overlapping properties has been derived. The sensitivity set that preserve the

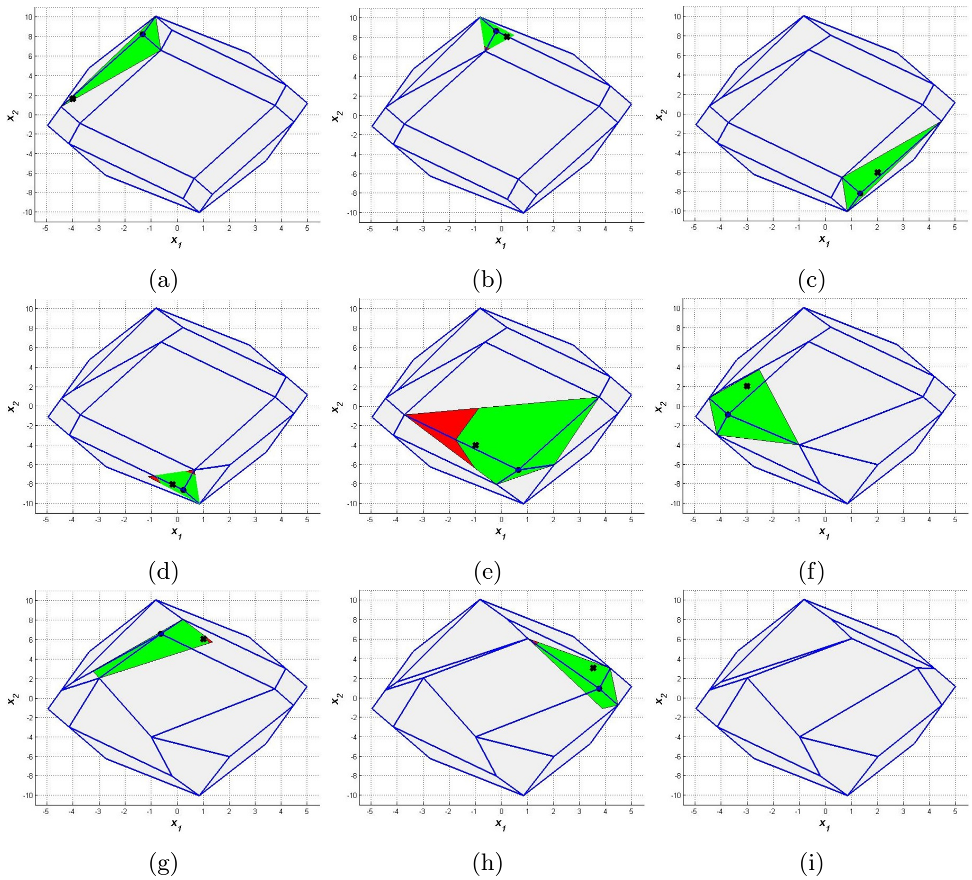


Figure 5: In the subplots, the polyhedral regions $\hat{\mathcal{R}}_i$ are presented with the vertex sensitivity and invariant-vertex sensitivity sets depicted in red and green color respectively. The dot and the $\times$ in the subplots are the vertex candidate and their new positions

invariance characteristics in the event of perturbation for the PWA control has been computed. It was shown that a perturbed polyhedral partition can be constructed by treating sequentially each vertex with a higher priority on those with a small sensitivity margin.

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