On Qualitative Properties of Low-Degree Quasipolynomials: Further remarks on the spectral abscissa and rightmost-roots assignment

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On Qualitative Properties of Low-Degree Quasipolynomials: Further remarks on the spectral abscissa and rightmost-roots assignment

by

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Abstract

The present study concerns a frequency domain approach in the stability analysis and stabilization via reduced-order controller design for linear time-invariant retarded Time-delay systems. More precisely, we address the problem of the spectral abscissa characterization and the coexistence of non oscillating modes for such functional differential equations. The design approach we propose is merely a delayed-output-feedback where the candidates' parameters result from the manifold defined by the coexistence of an exact number of negative spectral values, which guarantees the asymptotic stability of the system's solutions. The dominancy of such non oscillating modes is analytically shown for the considered reduced order Time-delay systems. Finally, using the W-Lambert function, further description of the spectrum distribution is presented.

Key Words: Time-delay systems, Stability, Spectral abscissa, Control design, Pole assignment, Non oscillation
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1 Introduction

Investigation of dynamical systems with time-delay is an active research area that connects a wide range of scientific disciplines including mathematics, physics, engineering, biology, economics etc. The present paper focuses on stability and stabilizing-controllers design for linear time-invariant retarded time-delay systems. The study of conditions on the equation parameters that guarantees the exponential stability of solutions is a question of ongoing interest and remains an open problem especially when the systems are of high order or having multiple and/or distributed delays. In particular, in frequency-domain, the problem reduces to the analysis of the distribution of the roots of the corresponding characteristic equation, see for instance [2, 12, 30, 27, 15, 19, 21, 26].

The starting point of the present work is an interesting property, discussed in recent studies, called *multiplicity induced-dominancy.* As a matter of fact, it is shown that multiple spectral values for Time-delay systems can be characterized using a Birkhoff/Vandermonde-based approach; see for instance [5, 4, 3, 10]. More precisely, in [4], it is shown that the admissible multiplicity of the zero spectral value is bounded by the generic *Polya and Szegö bound* denoted PS_B, which is nothing but the *degree* of the corresponding quasipolynomial (i.e the number of the involved polynomials plus their degree minus one), see for instance [22]. In [5], it is shown that a given *crossing imaginary roots* with non vanishing frequency never reaches PS_B and a sharper bound for its admissible multiplicities is established. Moreover, in [10], the variety corresponding to a multiple root for scalar Time-delay equations defines a stable variety for the steady state. The multiplicity of a root itself is not important as such but its connection with the dominancy of this root is a meaningful tool for control synthesis. An example of a scalar retarded equation with two delays is studied in [5] where it is shown that the multiplicity of real spectral values may reach the PS_B. In addition, the corresponding system has some further interesting properties: (i) it is asymptotically stable, (ii) its spectral abscissa (rightmost root) corresponds to this maximal allowable multiple root located on the imaginary axis. Such observations enhanced the outlook of further exhibiting the existing links between the maximal allowable multiplicity of some negative spectral value reaching the quasipolynomial degree and the stability of the trivial solution of the corresponding dynamical system. This property induced from multiplicity appears also in optimization problems since such a multiple spectral value is nothing but the rightmost root, see also [28, 18]. Also notice that the property was already observed in [24], where a tuning strategy is proposed for the design of a delayed Proportional-Integral controller by placing a triple real dominant root for the closed-loop system. However, the dominancy is only checked using a Mikhailov curve and QPmR toolbox, see for instance [29]. To the best of our knowledge, the first time an analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented in [17]. The dominancy property is further explored and analytically shown in the case of second-order systems and a rightmost root assignment based design using delayed state-feedback is proposed in [7, 9] where its applicability in damping active vibrations for a piezo-actuated beam is proved. See also [8, 6] which exhibit an analytical proof for the dominancy of the spectral value with maximal multiplicity for second-order systems controlled via a delayed proportional-derivative controller.

By this paper, we would like to extend such an analytical characterization of the spectral abscissa for retarded time-delay system with real spectral values which are not necessarily multiple. The effect of the coexistence of such non oscillatory modes on the asymptotic stability of the trivial solution will be explored. In particular, the coexistence of PS_B real spectral values makes them rightmost-roots of the corresponding quasipolynomial. Furthermore, if they are negative, this guarantees the asymptotic stability of the trivial solution. The study of oscillatory/non oscillatoty solutions of delay differential equation has been the focus of numerous contributions, see for instance [1, 2, 13], and [14] where the strongest deal is with linear delay differential equations with constant coefficients and constant delay. In this case necessary and sufficient conditions are given for all solutions to be oscillatory, for the case of non constant coefficient and non constant delay and for nonlinear equations, typical results give conditions sufficient for all solutions to be oscillatory, or conditions sufficient for some solutions to be non oscillatory see for instance [31] and [16].

The remaining paper is organized as follows; in Section 2 we recall some important facts on the spectrum distribution for retarded Time-delay systems. Section 3 is dedicated to investigate the coexistence non oscillatory modes for the scalar differential equation with a single delay. In Section 4 second order delay equation with three real spectral values is considered. In Section 5, some concluding remarks ends the paper.

2 Prerequisites and Problem Statement

In this section, we present some important facts and properties of the spectrum distribution of Time-delay systems. Let consider the generic n-order system with a single time delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau). \tag{2.1}$$

Here τ is a positive constant delay and the matrices $A_j \in M_n(\mathbb{R})$ for $j = 0 \dots 1$. It is well known that the asymptotic behavior of the solutions of (2.1) is determined from the spectrum designating the set of the roots of the associated characteristic function (denoted in the sequel $\Delta(s,\tau)$). Namely, the characteristic function corresponding to system (2.1) is a quasipolynomial $\Delta : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ of the form:

$$\Delta(s,\tau) = \det(sI - A_0 - A_1 \mathrm{e}^{-\tau s}). \tag{2.2}$$

Asymptotic stability of the trivial solution and oscillatory behavior of (2.1) are known. In particular, the zero solution of this equation is asymptotically stable if and only if all roots of (2.2) lie in the left half plane, and all solutions of (2.1) are non oscillatory if and only if (2.2) have a real roots.

We start by introducing a proposition which plays an important role in the study of continuity properties of the spectrum of retarded Time-delay systems. For more details see [19, page 10].

Proposition 1 ([19]). *If s is a spectral value corresponding to system* (2.1) *then it satisfies*

$$|s| \leq \|A_0 + A_1 e^{-s\tau}\|_2. \tag{2.3}$$

The above proposition allows to construct an envelope curve around the characteristic roots of the quasipolynomial (2.2).

The following result was first introduced and claimed in the problems collection published in 1925 by G. Pólya and G. Szegö. In the fourth edition of their book [22, Problem 206.2, page 144 and page 347], G. Pólya and G. Szegö emphasize that the proof was obtained by N. Obreschkoff in 1928 using the principle argument, see [20]. Such a result gives a bound for the number of quasipolynomial's roots in any horizontal strip. As a consequence, a bound for the number of quasipolynomial's real roots can be easily deduced.

Theorem 1 ([22]). *Let $\tau_1, \dots, \tau_N$ denote real numbers such that $\tau_1 < \tau_2 < \dots < \tau_N$ and $d_1, \dots, d_N$ positive integers such that $d_1 + d_2 + \dots + d_N = D$. Let $f_{i,j}(s)$ stand for the function $f_{i,j}(s) = s^{i-1}\exp(\tau_j s)$, for $1 \leq i \leq d_j$ and $1 \leq j \leq N$. Let $\sharp$ be the number of zeros of the function*

$$f(s) = \sum_{\substack{1 \leq j \leq N \\ 1 \leq i \leq d_j}} c_{i,j} f_{i,j}(s) \tag{2.4}$$

that are contained in the horizontal strip $\alpha \leq \mathrm{Im}(z) \leq \beta$. Assuming that

$$\sum_{1 \leq k \leq d_1} |c_{k,1}| > 0 \quad \textit{and} \quad \sum_{1 \leq k \leq d_N} |c_{k,N}| > 0$$

then

$$\frac{(\tau_N - \tau_1)(\beta - \alpha)}{2\pi} - D + 1 \leq \sharp \leq \frac{(\tau_N - \tau_1)(\beta - \alpha)}{2\pi} + D + N - 1. \tag{2.5}$$

Setting $\alpha = \beta = 0$, the above theorem yields $\sharp_{PS} \leq D + N - 1$ where D stands for the sum of the degrees of the polynomials involved in the quasipolynomial function f and N designates the associated number of polynomials. This gives a sharp bound for the number of fs real roots.

3 On the coexistence of real spectral value and their dominancy for scalar delay equation

Consider the simple scalar differential equation with one delay representing a biological model discussed by K.L.Cooke in [11]. It describes a vector disease dynamics where the infected host population $x(t)$ is governed by:

$$\dot{x}(t) + ax(t) + bx(t - \tau) = 0, \tag{3.1}$$

here $b > 0$ designates the contact rate between infected and uninfected populations and it is assumed that the infection of the host recovery proceeds exponentially at a rate $b > 0$, see also [25] for more insights on the modeling and stability results.
The characteristic equation associated to (3.1) is as follows:

$$\Delta(s, \tau) := s + a + b\exp(-s\tau) = 0. \tag{3.2}$$

Theorem 2. *For a given delay $\tau > 0$, the system* (3.1) *admits two distinct real spectral values at $s = s_2$ and $s = s_1$, with $s_2 < s_1$, if and only if*

$$\begin{cases} a = a(s_1, s_2, \tau) & := & \dfrac{s_2 \exp(-s_1\tau) - s_1 \exp(-s_2\tau)}{\exp(-s_2\tau) - \exp(-s_1\tau)}; \\ b = b(s_1, s_2, \tau) & := & \dfrac{s_1 - s_2}{\exp(-s_2\tau) - \exp(-s_1\tau)}. \end{cases} \tag{3.3}$$

- *Moreover, both spectral values s_2 and s_1 of* (3.1) *are negative, if and only if equation*

$$a(s_1, s_2, \tau) = 0$$

  is solvable with respect to $\tau > 0$. Furthermore, the zero solution of (3.1) *is asymptotically stable.*

- *The spectral value s_1 is nothing but the spectral abscissa corresponding to* (3.1).

Proof. According to the Theorem 1, see also [22], the number of real roots for (3.2) is two, hence we are interested to investigate the existence of two distinct negative spectral values, s_2 and s_1 with $s_2 < s_1$. The values of a and b are calculated by solving the system:

$$\begin{cases} s_2 + a + b\exp(-s_2\tau) = & 0; \\ s_1 + a + b\exp(-s_1\tau) = & 0. \end{cases} \tag{3.4}$$

We obtain immediately the values of a and b given in (3.3), as functions of s_2 and s_1 and τ. These values are unique for each fixed $\tau > 0$. Observe that $b(s_1, s_2, \tau) > 0$ for every $\tau > 0$, while the parameter $a(s_1, s_2, \tau)$ changes the sign according to the delay τ and the position of s_1 and s_2 on the real line. On the other hand, both $s_1 + a$ and $s_2 + a$ are not null, otherwise, $b = 0$ and then $s_1 = s_2$. Which is impossible. Now, from (3.4), we have

$$\frac{s_1 + a}{s_2 + a} = \exp(-\tau(s_1 - s_2)).$$

This means that $s_2 + a$ and $s_1 + a$ have necessarily same sign, and the following expression of τ:

$$\tau = -\frac{\ln(s_1 + a) - \ln(s_2 + a)}{s_1 - s_2} \tag{3.5}$$

is well-defined. Applying the Mean Value Theorem to the function $t \mapsto \ln(t + a)$, $t \in [s_2, s_1]$. This ensures the existence of $c \in]s_2, s_1[$ such that

$$\tau = -\frac{1}{c + a}. \tag{3.6}$$

Since $\tau > 0$ so $c + a < 0$. Consequently $s_2 + a < a + c < 0$. This means that $s_2 < -a$. Likewise for s_1, since $\mathrm{sign}(s_1 + a) = \mathrm{sign}(s_2 + a)$, we get $s_1 < -a$.

To show the negativeness of s_1, we use the variations of the mapping $\tau \mapsto a(s_1, s_2, \tau)$. An easy calculation shows that a is a continuous function with respect to τ, and increasing from $-\infty$ to $-s_1$. So, if equation $a(s_1, s_2, \tau) = 0$ admits a root $\tau^{**} > 0$, then such root is necessarily

$$\tau^{**} = \frac{\ln|s_2| - \ln|s_1|}{s_1 - s_2} \tag{3.7}$$

and $a(s_1, s_2, \tau) \geq 0$ holds for all $\tau \geq \tau^{**}$. Consequently, we get $s_1 < 0$. The converse implication is obvious.

The study of the stability of the system (3.1) is based on the dominancy of s_1. To do this, we use an adequate factorization of the quasipolynomial $\Delta(s, \tau)$ from the characteristic equation (3.2). This later can be written as follows:

$$\begin{aligned}\Delta(s, \tau) &= (s - s_1)\left(1 - \frac{(a + s_1) - \exp(-(s - s_1)\tau)}{s - s_1}\right) \\ &= (s - s_1)\left(1 - (a + s_1)\Big(\frac{-\exp(-(s - s_1)\tau)}{s - s_1} + \frac{1}{s - s_1}\Big)\right).\end{aligned} \tag{3.8}$$

Thus,

$$\Delta(s, \tau) = (s - s_1)\left(1 - \tau(a + s_1)\int_0^1 \exp(-\tau(s - s_1)t)dt\right). \tag{3.9}$$

To prove that s_1 is the right most root (dominancy), we assume that there exists some

$s_0 = \zeta + j\eta$, a root of (3.2) such that $\zeta > s_1$. Then

$$\begin{aligned}
1 &= \tau(a+s_1)\int_0^1 \exp(-\tau(s_0-s_1)t)dt \\
&= \mathrm{Re}\left(\tau(a+s_1)\int_0^1 \exp(-\tau(\zeta+j\eta-s_1)t)dt)\right) \\
&\leq \tau|a+s_1|\left|\int_0^1 \exp(-\tau(\zeta+j\eta-s_1)t)dt\right| \\
&\leq \tau|a+s_1|\int_0^1 \exp(-\tau(\zeta-s_1)t)dt. \qquad (3.10)
\end{aligned}$$

From (3.6), we have

$$\tau < \frac{1}{-(a+s_1)} = \frac{1}{|a+s_1|}. \qquad (3.11)$$

Thus

$$\tau \mid a+s_1 \mid < 1. \qquad (3.12)$$

Moreover, since $\zeta - s_1 > 0$, we have

$$\int_0^1 \exp(-\tau(\zeta-s_1)t)dt < 1. \qquad (3.13)$$

Inequality (3.10) can not be satisfied simultaneously with (3.12) and (3.13). Which proves that the hypothesis $\zeta > s_1$ is inconsistence. So s_1 is the spectral abscissa corresponding to (3.1), guarantying the asymptotic stability of the system (3.1). ∎

Remark 1. *Theorem 2 shows that the co-existence of two spectral negative values s_1 and s_2 is guaranteed by the positivity of the mapping $\tau \mapsto a(s_1, s_2, \tau)$, in some interval. Inequality $a(s_1, s_2, \tau) \geq 0$ is satisfied if and only if $\tau \geq \tau^{**}$. From a control theory point of view, the assignment of such real negative spectral values subjects to choose them in the interval $]-\infty, -a[$.*

Remark 2.

1. *In recent results [3, 5], the question of the effect of multiple spectral values on the dynamics of time-delay systems is investigated. So, consider s_0 to be the corresponding double real root, we are interested in the behavior and the geometric structure of the corresponding envelope. In such case, the coefficients a and b satisfy the following relations $a(s_0, \tau) = -s_0 - \frac{1}{\tau}$; and $b(s_0, \tau) = \frac{1}{\tau}\exp(\tau s_0)$. Furthermore, there exists a critical value $\tilde{\tau} = -\frac{1}{s_0}$, which corresponds to the vanishing of $a(s_0, \tau)$. The connected structure of the envelope is then observed, with the appearance of an invariant node with respect to the delay, at the spectral abscissa s_0, for every $\tau \geq \tilde{\tau}$. see Figure 1.*

2. *Interestingly, when considering two real distinct spectral values, varying the value of the delay τ may induce a change in the geometry of the envelope curve. The connected structure of the envelope may be lost. Indeed, as τ increases reaching some critical*

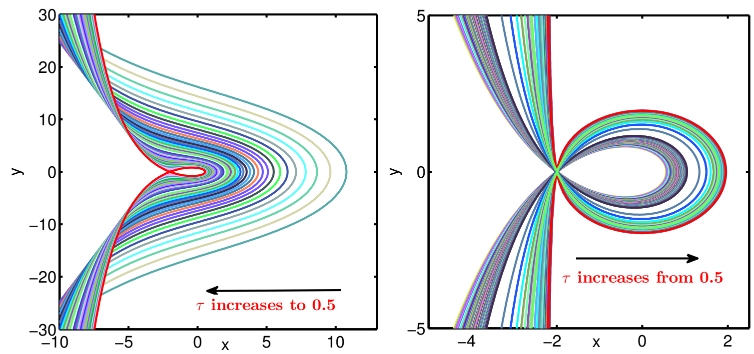

Figure 1: Envelope curve of the characteristic equation (3.2). Case when $s_0 = -2$ is a double root. The critical delay $\tau = 0.5$ corresponds to the value that vanishes the coefficient $a = 2 - \frac{1}{\tau}$, see [10].

delay τ^*, the connected structure of the envelope is preserved, only in a single point (a node), see Figure 2 (left). Exceeding this critical value τ^*, there is appearance of two disconnected components, which moving away more and more until a constant distance that is reached for every delay $\tau \geq \tau^{**}$, see Figure 2 (right).

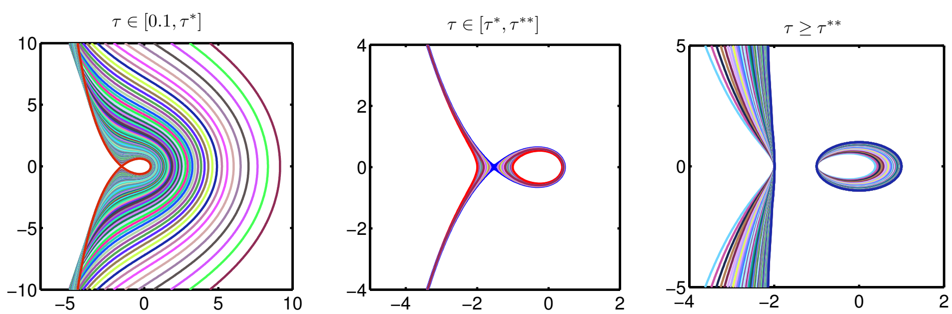

Figure 2: Envelope curve of the characteristic equation (3.2). Case of co-existence of two simple real roots $s_1 = -1$, $s_2 = -2$.

3. *Such a geometry is encountered in [23], where an analytical study and synthesis of rightmost eigenvalues of $\dot{x}(t) = Ax(t - \tau)$ is considered.*

The description of this kind of curve, when considering two simples spectral values s_2, s_1, with $s_2 < s_1 < 0$, is given in the following theorem.

Theorem 3. *Let*

$$C^{\tau} := C\left(s_2, s_1, \tau\right) = \left\{(x, y)\,;\ \sqrt{x^2+y^2} = \left(\left|a\left(s_2, s_1, \tau\right)\right| + \left|b\left(s_2, s_1, \tau\right)\right| \exp(-x\tau)\right)\right\} \tag{3.14}$$

be the envelope curve associated to the quasipolynomial (3.2)*.*

By browsing the delay τ from a small value until a large value, the following assertions holds for (3.2)*:*

1. *For any $\tau > 0$, the envelope curve C^{τ} intersects one times the real axis, in a positive abscissa, $x^{+}(\tau)$.*
2. *There exists $\tau^{*}, \tau^{**} > 0$, with $\tau^{*} < \tau^{**}$, such that:*
   - *When $\tau = \tau^{*}$, the curve $C^{\tau^{*}}$ contains a node, $(x(\tau^{*}), 0)$, in the region $x < 0$.*
   - *For $\tau < \tau^{*}$, there is only one intersection of the envelope curve with the real axis.*
   - *For $\tau \in]\tau^{**}, \tau^{*}[$, the curve C^{τ} splits into two disjoint segments C_1^{τ} and C_2^{τ}, which intersect the the region $]s_2, s_1[$ at points $(x_1^{-}(\tau), 0)$ and $(x_2^{-}(\tau), 0)$, respectively.*
   - *There exists $\overline{x}\left(\tau^{**}\right) < s_2$ such that for every $\tau < \tau^{**}$, $\overline{\overset{\circ}{C}{}^{\tau^{**}}} \subset \overline{\overset{\circ}{C}{}^{\tau}}$ in the region $x > \overline{x}\left(\tau^{**}\right)$, and for every $\tau > \tau^{**}$, $\overline{\overset{\circ}{C}{}^{\tau}} \subset \overline{\overset{\circ}{C}{}^{\tau^{**}}}$.*
   - *For $\tau \geq \tau^{**}$, the curve C^{τ} splits into two disjoint segments C_1^{τ} and C_2^{τ}, with the property that $(s_2, 0) \in \bigcap_{\tau \geq \tau^{**}} C_1^{\tau}$ and $(s_1, 0) \in \bigcap_{\tau \geq \tau^{**}} C_2^{\tau}$ and*

     $$d(C_1^{\tau}, C_2^{\tau}) = |s_2 - s_1|.$$
3. *As $\tau \to \infty$,*
   - *$x_1^{-}(\tau) \to s_2$, $x_2^{-}(\tau) \to s_1$ and $x^{+}(\tau) \to -s_1$. The two first limits are reached from $\tau = \tau^{**}$, see Figure 4.*
   - *C^{∞}, the envelope limit, is the circle centered at the origin with the rayon $r = |s_1|$.*

Proof.

To find the point of intersection of the curve with the x-axis, we solve this system of two equation obtained from (3.14) using the W-Lambert function [32]

$$\left\{ \begin{array}{l} |a| + |b|e^{-x\tau} - x = 0 \\ |a| + |b|e^{-x\tau} + x = 0. \end{array} \right. \tag{3.15}$$

Using a change of variable $z = -x\tau$ we obtain

$$|s_1 - s_2|e^{z} - \frac{z}{\tau}|e^{-s_2\tau} - e^{-s_1\tau}| + |s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}| = 0 \tag{3.16}$$

$$|s_1 - s_2|e^{z} + \frac{z}{\tau}|e^{-s_2\tau} - e^{-s_1\tau}| + |s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}| = 0. \tag{3.17}$$

The above two equation are under the following form $\gamma e^{z} + \beta + \sigma = 0$, where $\gamma = |s_1 - s_2|$, $\beta = \pm\frac{e^{-s_2\tau} - e^{-s_1\tau}}{\tau}$ and $\sigma = |s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|$ The existence of real solutions depends on the sign of the discriminant $\Delta(\tau) = \frac{\gamma}{\beta} e^{-\frac{\sigma}{\beta}}$ and its position with respect to $-e^{-1}$.

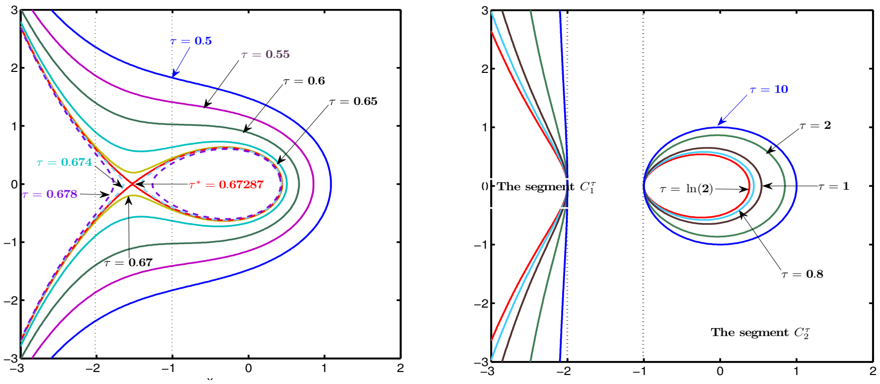

Figure 3: (Left) Decreasing behavior of the envelope curve versus $\tau < \tau^{**}$. (Right) Increasing behavior of the envelope curve versus $\tau \geq \tau^{**}$. Case $s_1 = -1$ and $s_2 = -2$

1. For $\tau > 0$, we search the positive solution, so we study the equation (3.17). The discriminant of this equation is given by:

$$\Delta^+(\tau) = \frac{\tau|s_1 - s_2|}{|e^{-s_2\tau} - e^{-s_1\tau}|} \exp\Big(- \frac{\tau|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}\Big). \tag{3.18}$$

Hence equation (3.17) has a positif real solution

$$z_1^+(\tau) = -W_0(\Delta^+) - \frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|},$$

thus

$$x^+(\tau) = \frac{1}{\tau} W_0(\Delta^+) + \frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}, \tag{3.19}$$

which implies that the envelope curve intersects only one times the real axis in the right half plan at $x^+(\tau)$.

2. Now we study the equation (3.16), where the discriminant is written as follows:

$$\Delta^-(\tau) = -\frac{\tau|s_1 - s_2|}{|e^{-s_2\tau} - e^{-s_1\tau}|} \exp\Big(\frac{\tau|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}\Big). \tag{3.20}$$

Since $\Delta^-(\tau) < 0$, we need to evaluate the sign of $\Delta^-(\tau) + e^{-1}$. Note that the function $\tau \mapsto \Delta^-(\tau) + \exp(-1)$ is continuous on $\mathbb{R}^+$. In addition, as $\tau \to 0$, $\Delta^-(\tau) + \exp(-1) \to -\exp(1) + \exp(-1) < 0$. We have then

$$\Delta^-(\tau^{**}) = -\ln\left(\frac{s_1}{s_2}\right)\left(\left(\frac{s_1}{s_2}\right)^{-\frac{s_1}{s_2 - s_1}} - \left(\frac{s_1}{s_2}\right)^{-\frac{s_2}{s_2 - s_1}}\right)^{-1} + \mathrm{e}^{-1}.$$

The variations of

$$F(t) := -\ln(t)\left(t^{-\frac{t}{1-t}} - t^{-(1-t)^{-1}}\right)^{-1} + \mathrm{e}^{-1},\ t \in \mathbb{R}_+^* - \{1\}$$

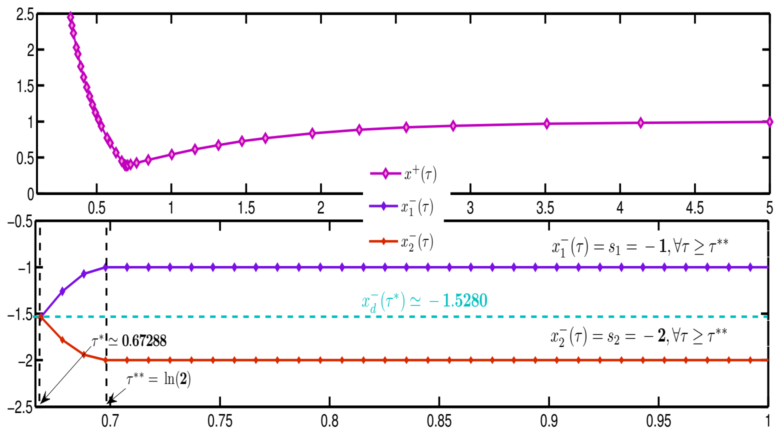

Figure 4: Asymptotic behavior of the roots of the envelope curve. Case $s_1 = -1$, $s_2 = -2$

show that $\Delta^-(\tau^{**}) + e^{-1} > 0, \forall t \in \mathbb{R}_+^* - \{1\}$. Thanks to the intermediate values theorem, there exists $\tau^* \in]0, \tau^{**}[$ such that $\Delta^-(\tau^*) + e^{-1} = 0$.

- For $\tau = \tau^*$, $\Delta^-(\tau^*) = -e^{-1}$, so (3.16) admits a (double) negative real solution

$$x_d^-(\tau^*) = \frac{1}{\tau^*} W_0(-e^{-1}) - \frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|} = -\frac{1}{\tau^*} - \frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|},$$

  which allows to the appearance of a node at the point $(x(\tau^*), 0)$. This equation admits also a positive solution $(x^+(\tau), 0)$ given by (3.19), from which we deduce that there are two intersections with the real axis.

- For $\tau < \tau^*$, there exists one positive solution $(x^+(\tau), 0)$ given by (3.19), which corresponds to unique intersection with the half real axis, $x > 0$.

- For $\tau \in]\tau^*, \tau^{**}[$, we have $\Delta^-(\tau) + e^{-1} > 0$, then equation (3.16) admits two negative real solutions

$$\begin{aligned} z_1(\tau) &= -W_0\left(\Delta^-\right) + \frac{\tau\,|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}, \\ z_2(\tau) &= -W_{-1}\left(\Delta^-\right) + \frac{\tau\,|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}, \end{aligned}$$

  thus

$$x_1^-(\tau) = \frac{1}{\tau} W_0\left(\Delta^-\right) - \frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-\tau s_2} - e^{-s_1\tau}|}, \tag{3.21}$$

$$x_2^-(\tau) = \frac{1}{\tau} W_{-1}\left(\Delta^-\right) - \frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}, \tag{3.22}$$

  this implies that the curve C^τ intersects two times the real axis, from which we deduce that the curve splits into two disjoints segments C_1^τ and C_2^τ which intersect the region $]s_2, s_1[$ at points $(x_1^-(\tau), 0)$ and $(x_2^-(\tau), 0)$ respectively. Details about the property $C_2^\tau \cap C_2^\tau = \emptyset$ will be presented bellow.

- Here we consider the case where $\tau < \tau^{**}$. Recall that the sign of $a(s_2, s_1, \tau)$ depends on the parameter $\tau > 0$, while $b(s_2, s_1, .) > 0$. So

$$|s_2 \exp(-s_1\tau) - s_1 \exp(-s_2\tau)| = \begin{cases} 0 & \text{if} \quad \tau = \tau^{**} \\ s_2 \exp(-s_1\tau) - s_1 \exp(-s_2\tau) & \text{if} \quad \tau > \tau^{**} \\ s_1 \exp(-s_2\tau) - s_2 \exp(-s_1\tau) & \text{if} \quad \tau < \tau^{**} \end{cases}$$

Hence

$$\frac{d}{d\tau}\left(|a\left(s_2, s_1, \tau\right)|\right) = \begin{cases} e^{-\tau s_2} e^{-\tau s_1} \dfrac{\left(s_2 - s_1\right)^2}{\left(e^{-\tau s_2} - e^{-\tau s_1}\right)^2} & \text{if} \quad \tau > \tau^{**} \\ -e^{-\tau s_2} e^{-\tau s_1} \dfrac{\left(s_2 - s_1\right)^2}{\left(e^{-\tau s_2} - e^{-\tau s_1}\right)^2} & \text{if} \quad \tau < \tau^{**} \end{cases}$$

$$\frac{d}{d\tau}\left(|b\left(s_2, s_1, \tau\right)| \exp(-x\tau)\right) = e^{-x\tau} \frac{\left(s_1 - s_2\right)}{\left(e^{-\tau s_2} - e^{-\tau s_1}\right)} \left(\frac{s_2 e^{-\tau s_2} - s_1 e^{-\tau s_1}}{e^{-\tau s_2} - e^{-\tau s_1}} - x\right)$$

Note that $s_2 e^{-\tau s_1} - s_1 e^{-\tau s_2} = \left(1 - \theta\tau\right) e^{-\theta\tau} \left(s_2 - s_1\right)$, for some $\theta \in \left]s_2, s_1\right[$, so

$$\bar{x}(\tau) \triangleq \frac{s_2 e^{-\tau s_1} - s_1 e^{-\tau s_2}}{e^{-\tau s_2} - e^{-\tau s_1}} < 0, \, \forall \tau > 0.$$

In addition, $\tau \mapsto \bar{x}(\tau)$ is continuous and increasing on $]0, \tau^{**}[$, with

$$\lim_{\tau \to \tau^{**}} \overline{x}\left(\tau\right) = s_2 - \left(s_1 - s_2\right) \frac{\left(\dfrac{s_2}{s_1}\right)^{\dfrac{s_1}{s_2 - s_1}}}{\left(\dfrac{s_2}{s_1}\right)^{\dfrac{s_2}{s_2 - s_1}} - \left(\dfrac{s_2}{s_1}\right)^{\dfrac{s_1}{s_2 - s_1}}} = \overline{x}\left(\tau^{**}\right) < s_2.$$

Thus $\tau \mapsto |b\left(s_2, s_1, \tau\right)| \exp(-x\tau)$ is decreasing on $\mathbb{R}^{+*}$, for every $x > \overline{x}\left(\tau^{**}\right)$. This means that when $x > \overline{x}\left(\tau^{**}\right)$, the function

$$\tau \mapsto |a\left(s_2, s_1, \tau\right)| + |b\left(s_2, s_1, \tau\right)| \exp(-x\tau)$$

is continuous and decreasing on $]0, \tau^{**}[$. We deduce that

$$\forall \tau \in]0, \tau^{**}[\,; \; C\left(s_2, s_1, \tau^{**}\right) \subset C\left(s_2, s_1, \tau\right).$$

On the other hand, the variations of $x \mapsto |a(s_2, s_1, \tau)| + |b(s_2, s_1, \tau)| \exp\left(-x\tau\right)$ show that for all $x \in \mathbb{R} - [s_2, s_1]$ and for all $\tau > \tau^{**}$

$$|b(s_2, s_1, \tau^{**})| \exp\left(-x\tau^{**}\right) - \big(|a(s_2, s_1, \tau)| + |b(s_2, s_1, \tau)| \exp\left(-x\tau\right)\big) < 0, \quad (3.23)$$

from which we deduce that

$$\forall \; \tau > \tau^{**}; \; C\left(s_2, s_1, \tau^{**}\right) \subset C\left(s_2, s_1, \tau\right). \quad (3.24)$$

- Now we study the case of $\tau > \tau^{**}$. We have

$$\begin{aligned}\lim_{\tau \to \tau^{**}} C\left(s_2, s_1, \tau\right) &= \bigcap_{\tau < \tau^{**}} C\left(s_2, s_1, \tau\right) = C\left(s_2, s_1, \tau^{**}\right) \\ &= \left\{(x, y)\,;\ \sqrt{x^2+y^2} = M(s_2, s_1, x)\right\}\end{aligned}$$

where

$$M(s_2, s_1, x) = \frac{s_1 - s_2}{\left(\dfrac{s_2}{s_1}\right)^{-\dfrac{s_2 - x}{s_1 - s_2}} - \left(\dfrac{s_2}{s_1}\right)^{-\dfrac{s_1 - x}{s_1 - s_2}}}.$$

Observe that

$$\begin{aligned}M(s_2, s_1, s_2) &= -s_2; \\ M(s_2, s_1, s_1) &= -s_1.\end{aligned}$$

Hence the boundary of $C\left(s_2, s_1, \tau^{**}\right)$ contains both $(s_1, 0)$ and $(s_2, 0)$. We get more, that is

$$(s_2, 0) \in \bigcap_{\tau \geq \tau^{**}} C_1^{\tau} \quad \text{and} \quad (s_1, 0) \in \bigcap_{\tau \geq \tau^{**}} C_2^{\tau}. \tag{3.25}$$

On the other hand, since

$$M\left(s_2, s_1, \alpha s_1 + (1-\alpha)\, s_2\right) = -s_2 \left(\frac{s_1}{s_2}\right)^{\alpha}$$

for every $\alpha \in \left]0, 1\right[$, we can show that

$$C\left(s_2, s_1, \tau^{**}\right) \cap \left]s_2, s_1\right[= \emptyset.$$

Indeed, if not there exists $z \in \left]s_2, s_1\right[$, such that $|z| \leq M\left(s_2, s_1, z\right)$. Writing $z = \alpha s_1 + (1-\alpha)\, s_2$, for some $\alpha \in \left]0, 1\right[$, we have

$$-s_1 < |z| \leq -s_2 \left(\frac{s_1}{s_2}\right)^{\alpha} < -s_2 \left(\frac{s_1}{s_2}\right) = -s_1.$$

This implies that

$$-s_1 < -s_1$$

allowing to a contradiction.
This reasoning allows us to show that

$$C\left(s_2, s_1, \tau^{**}\right) \cap \left\{(x, y) \in \mathbb{R}^2,\ s_2 < x < s_1\right\} = \emptyset. \tag{3.26}$$

In fact, in the contrary case, we will get

$$-s_1 \leq \sqrt{s_1^2 + y^2} < \sqrt{x^2 + y^2} = |z| \leq -s_2 \left(\frac{s_1}{s_2}\right)^{\alpha} < -s_1.$$

Impossible. This means that the envelope curve $C(s_2, s_1, \tau^{**})$ is splits into two disjointed parts $C^{x<s_2}(s_2, s_1, \tau^{**})$ and $C^{x>s_1}(s_2, s_1, \tau^{**})$, separated by the band $\left\{(x,y) \in \mathbb{R}^2,\ s_2 < x < s_1\right\}$. As a consequence, we get that distance

$$\begin{aligned} d\left(C^{x<s_2}(s_2, s_1, \tau^{**}), C^{x>s_1}(s_2, s_1, \tau^{**})\right) &= d\left(C^{x<s_2}(s_2, s_1, \tau), C^{x>s_1}(s_2, s_1, \tau)\right) \\ &= \inf_{\substack{z_1 \in C^{x<s_2}(s_2,s_1,\tau^{**}) \\ z_2 \in C^{x>s_1}(s_2,s_1,\tau^{**})}} d(z_1, z_2) \\ &= s_1 - s_2,\ \forall \tau > \tau^{**}. \end{aligned}$$

d being the Euclidean distance.

3. Now, we deal with the asymptotic behavior of the envelope curve, $x_1^-(\tau)$, $x_2^-(\tau)$, and $x^+(\tau)$ as $\tau \to \infty$.

   - Since $\Delta^-(\tau) \in \left]-e^{-1}, 0\right[$, we have $W_0(\Delta^-(\tau))$ satisfies:

     $$-1 = W_0\left(-e^{-1}\right) < W_0\left(\Delta^-(\tau)\right) < W_0(0) = 0,$$

     thus

     $$\lim_{\tau\to\infty} \frac{1}{\tau} W_0\left(\Delta^-\right) = 0.$$

     From this we deduce that

     $$\lim_{\tau\to\infty} x_1^-(\tau) = s_1.$$

     On the other hand, since we have

     $$d(C_1^\tau, C_2^\tau) = s_1 - s_2,\ \forall \tau > \tau^{**},$$

     we deduce that

     $$\lim_{\tau\to\infty} x_2^-(\tau) = s_2.\text{[1]} \tag{3.27}$$

     Similarly, we prove that $\lim\limits_{\tau\to\infty} x^+(\tau) = -s_1$. Indeed, there is always $\tau_0 > 0$ such that

     $$\frac{\left|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}\right|}{\left|e^{-\tau s_2} - e^{-s_1\tau}\right|} > -\frac{s_1}{2},\ \text{ for all } \tau \geq \tau_0.$$

     This means that $x^+(\tau)$ in (3.19) given via the WLambert function satisfies

     $$x^+(\tau) > -\frac{s_1}{2}, \quad \forall \tau \geq \tau_0.$$

     Using the fact that

     $$0 < \frac{e^{-\tau x^+(\tau)}}{\left|e^{-\tau s_2} - e^{-s_1\tau}\right|} < \frac{e^{\frac{s_1\tau}{2}}}{\left|e^{-\tau s_2} - e^{-s_1\tau}\right|}, \quad \forall \tau \geq \tau_0.$$

[1] As a consequence, using (3.22), we get the following property of the WLambert function: $\lim_{\tau\to\infty} \frac{1}{\tau} W_{-1}\left(\Delta^-(\tau)\right) = 0$.

we deduce that $\dfrac{e^{-\tau x^+(\tau)}}{|e^{-\tau s_2} - e^{-s_1 \tau}|} = 0$. Letting τ tend to ∞ in the following expression:

$$x^+(\tau) = \frac{e^{-\tau x^+(\tau)}}{|e^{-\tau s_2} - e^{-s_1 \tau}|} + \frac{|s_1 e^{-\tau s_2} - s_2 e^{-s_1 \tau}|}{|e^{-\tau s_2} - e^{-s_1 \tau}|} \tag{3.28}$$

we obtain

$$\lim_{\tau \to \infty} x^+(\tau) = |s_1|.$$ [2]

- Let us consider the equation of the envelope curve (3.14). In the region $x > s_1$, we have

$$\lim_{\tau \to \infty} |b(s_1, s_2, \tau)| \exp(-x\tau) = 0,$$

and

$$\lim_{\tau \to \infty} |a(s_1, s_2, \tau)| = |s_1|.$$

Hence, we deduce that as $\tau \to \infty$ the envelope curve (3.14) becomes

$$x^2 + y^2 = s_1^2,$$

which corresponds to a circle C^∞ with the ray $r = |s_1|$. Concerning the region $x < s_2$, simulation results show that for τ sufficiently large, the segment C_1^τ disappears. The question of finding the upper bound of τ for which $\overline{C_1^\tau}$ is exactly the half-plan $x \leq s_2$ remains open.

□

4 Second-order systems

Second-order linear systems capture the dynamic behavior of many natural phenomena, and have found wide applications in a variety of fields, such as vibration and structural analysis. The equation of this system is given as follows:

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + \alpha x(t - \tau) = 0. \tag{4.1}$$

The characteristic equation associated to (4.1) is given by:

$$\Delta(s, \tau) = s^2 + as + b + \alpha \exp(-s\tau) = 0. \tag{4.2}$$

[2] As a consequence, we get the following property $\lim_{\tau \to \infty} \frac{1}{\tau} W_0(\Delta^+) = 0$.

Theorem 4. *The system* (4.1) *admits three distinct real spectral values* s_3, s_2 *and* s_1 *with* $s_3 < s_2 < s_1$ *if and only if the parameters* a, b *and* α *satisfy*

$$\begin{cases} a = a(s_1, s_2, s_3, \tau) := \dfrac{1}{Q} \displaystyle\sum_{\substack{i,j,k\in\Lambda \\ i<j, i\neq j\neq k}} (-1)^{i+j} \left(s_i^2 - s_j^2\right) \exp(-s_k\tau) \\ b = b(s_1, s_2, s_3, \tau) := -\dfrac{1}{Q} \displaystyle\sum_{\substack{i,j,k\in\Lambda \\ i<j, i\neq j\neq k}} (-1)^{i+j} s_i s_j (s_i - s_j) \exp(-s_k\tau) \\ \alpha = (s_1, s_2, s_3, \tau) := -\dfrac{1}{Q} \displaystyle\prod_{\substack{i,j\in\Lambda \\ i<j}} (s_i - s_j) \end{cases} \tag{4.3}$$

where

$$Q := Q(s_1, s_2, s_3, \tau) = \sum_{\substack{i,j,k\in\Lambda \\ i<j, k\neq i,j}} (-1)^{i+j} (s_i - s_j) \exp(-s_k\tau).$$

In this case, α *is necessarily negative.*

- *The spectral value* s_1 *is negative if and only if there exists* $\tau_0 > 0$ *such*

$$a(s_1, s_2, s_3, \tau_0) + s_2 = 0.$$

  This guarantees the asymptotic stability of the system.

- *The root* s_1 *is the spectral abscissa of* (4.1).

Proof. The parameters given in (4.3) can be easily obtained by solving the system

$$\begin{cases} s_3^2 + a s_3 + b + \alpha \mathrm{e}^{-s_3\tau} = 0 \\ s_2^2 + a s_2 + b + \alpha \mathrm{e}^{-s_2\tau} = 0 \\ s_1^2 + a s_1 + b + \alpha \mathrm{e}^{-s_1\tau} = 0. \end{cases} \tag{4.4}$$

Now, let us study the sign of α. Recall that

$$\alpha = -\frac{(s_1 - s_2)(s_2 - s_3)(s_1 - s_3)}{Q(s_1, s_2, s_3, \tau)}. \tag{4.5}$$

We apply twice the Mean Value Theorem to the denominator $Q(s_1, s_2, s_3, \tau)$, we get:

$$\begin{aligned} Q(s_1, s_2, s_3, \tau) &= s_2 e^{-\tau s_3} - s_1 e^{-\tau s_3} + s_1 e^{-\tau s_2} - s_3 e^{-\tau s_2} + s_3 e^{-\tau s_1} - s_2 e^{-\tau s_1} \\ &= (s_2 - s_1)\left(e^{-\tau s_3} - e^{-\tau s_1}\right) + (s_3 - s_1)\left(e^{-\tau s_1} - e^{-\tau s_2}\right) \\ &= -\tau (s_2 - s_1)(s_3 - s_1) \int_0^1 e^{-\tau(t s_1 + (1-t)s_3)} dt \\ &\quad + \tau (s_3 - s_1)(s_2 - s_1) \int_0^1 e^{-\tau(t s_1 + (1-t)s_2)} dt \\ &= -\tau (s_2 - s_1)(s_3 - s_1) \int_0^1 \left(e^{-\tau(t s_1 + (1-t)s_3)} - e^{-\tau(t s_1 + (1-t)s_2)}\right) dt \\ &= \tau^2 (s_2 - s_1)(s_3 - s_1)(s_3 - s_2) \int_0^1 \int_0^1 (1-t)\, e^{-\tau(t s_1 + (1-t)(\theta s_3 + (1-\theta)s_2))} d\theta dt. \end{aligned}$$

This means that

$$\alpha\left(s_1, s_2, s_3, \tau\right) = \frac{-1}{\tau^2 \widetilde{\mathcal{Q}}\left(s_1, s_2, s_3, \tau\right)} \tag{4.6}$$

where

$$\widetilde{\mathcal{Q}}\left(s_1, s_2, s_3, \tau\right) = \int_0^1 \int_0^1 (1-t)\, e^{-\tau(ts_1+(1-t)(\theta s_3+(1-\theta)s_2))} d\theta dt > 0.$$

Now, let us show the negativeness of $s_1,\ s_2,\ s_3$. From (4.4) and (4.6), we get that

$$\begin{cases} (s_3 - s_2)\left[s_3 + s_2 + a\right] = -\alpha\left[e^{-\tau s_3} - e^{-\tau s_2}\right] > 0 \\ (s_2 - s_1)\left[s_2 + s_1 + a\right] = -\alpha\left[e^{-\tau s_2} - e^{-\tau s_1}\right] > 0 \\ (s_3 - s_1)\left[s_3 + s_1 + a\right] = -\alpha\left[e^{-\tau s_3} - e^{-\tau s_1}\right] > 0. \end{cases}$$

Since $s_3 < s_2 < s_1$, we obtain the following equations:

$$\begin{cases} s_3 + s_2 < -a \\ s_2 + s_1 < -a \\ s_3 + s_1 < -a \end{cases} \tag{4.7}$$

System (4.7) is reduced to the single inequality $s_2 + s_1 < -a$, from which we deduce that $s_3 < s_2 < -\frac{a}{2}$, and $s_1 < -a - s_2$. Since the mapping $\tau \mapsto a(s_1, s_2, s_3, \tau) + s_2$ is continuous and increasing from $-\infty$ to $-s_1$ when τ varies in $\mathbb{R}^{+*}$, this means that $a(s_1, s_2, s_3, \tau) + s_2$ takes positive values if and only if $s_1 < 0$. This means that the equation $a(s_1, s_2, s_3, \tau) + s_2 = 0$ has a unique root, $\tau_0 > 0$.

Now, to study the stability of the system, we need to study the dominancy of s_1 by using an adequate factorization of the quasipolynimial (4.2), so we have:

$$\begin{aligned} s^2 + as + b + \alpha \exp\left(-\tau s\right) &= \left(s - s_2\right)\left(s - s_1\right)\left[\frac{s^2 + as + b + \alpha \exp\left(-\tau s\right)}{\left(s - s_2\right)\left(s - s_1\right)}\right] \\ &= \left(s - s_2\right)\left(s - s_1\right) P(s, s_1, s_2, \tau). \end{aligned}$$

The factor $P(s) := P(s, s_1, s_2, \tau)$ can be rewritten under the more suitable form

$$\begin{aligned} P\left(s\right) &= 1 + \frac{\left(a + s_2 + s_1\right)s + b - s_2 s_1}{\left(s - s_2\right)\left(s - s_1\right)} + \frac{\alpha \exp\left(-\tau s\right)}{\left(s - s_2\right)\left(s - s_1\right)} \\ &= 1 - \frac{s_2^2 + as_2 + b}{\left(s_1 - s_2\right)\left(s - s_2\right)} + \frac{s_1^2 + as_1 + b}{\left(s_1 - s_2\right)\left(s - s_1\right)} + \frac{\alpha \exp\left(-\tau s\right)}{\left(s_2 - s_1\right)\left(s - s_2\right)} - \frac{\alpha \exp\left(-\tau s\right)}{\left(s_2 - s_1\right)\left(s - s_1\right)} \\ &= 1 + \frac{\alpha \exp\left(-\tau s\right)}{\left(s_2 - s_1\right)\left(s - s_2\right)} - \frac{\alpha \exp\left(-\tau s_2\right)}{\left(s_2 - s_1\right)\left(s - s_2\right)} + \frac{\alpha \exp\left(-\tau s_1\right)}{\left(s_1 - s_2\right)\left(s - s_1\right)} - \frac{\alpha \exp\left(-\tau s\right)}{\left(s_2 - s_1\right)\left(s - s_1\right)}. \end{aligned}$$

Using the integral form of the remainder in Taylor's Theorem, we get

$$P(s) = 1 + \frac{\alpha e^{-\tau s_2}\left(e^{-\tau(s-s_2)}-1\right)}{(s_2-s_1)(s-s_2)} + \frac{\alpha e^{-\tau s_1}\left(1-e^{-\tau(s-s_1)}\right)}{(s_2-s_1)(s-s_1)} \tag{4.8}$$

$$= 1 + \frac{\alpha e^{-\tau s_2}\left(-\tau(s-s_2)\int_0^1 e^{-\tau(s-s_2)t}dt\right)}{(s_2-s_1)(s-s_2)} - \frac{\alpha e^{-\tau s_1}\left(-\tau(s-s_1)\int_0^1 e^{-\tau(s-s_1)t}dt\right)}{(s_2-s_1)(s-s_1)}$$

$$= 1 + \frac{\alpha\tau}{(s_2-s_1)}\int_0^1 e^{-\tau ts}\left(e^{-\tau(1-t)s_1} - e^{-\tau(1-t)s_2}\right)dt.$$

Using again the following integral representation

$$\exp(-\tau(1-t)s_1) - \exp(-\tau(1-t)s_2) = -\tau(s_1-s_2)\int_0^1 \exp(-\tau(1-t)(\theta s_2 + (1-\theta)s_1))\,d\theta$$

we get

$$P(s,s_1,s_2,\tau) = 1 + \alpha\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t(s-s_1))\exp(-\tau[s_1+\theta(1-t)(s_2-s_1)])\,d\theta dt.$$

To prove the dominancy property, namely s_1 is the rightmost root of (4.2), let us assume that there exists some $s_0 = \zeta + j\eta$ a root of (4.2) such that $\zeta > s_1$. Then $P(s_0,s_1,s_2,\tau) = 0$ and for any $t > 0$, one has:

$$\exp(-\tau[t(\zeta - s_1)]) < 1.$$

So

$$1 = -\alpha\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t(s_0-s_1))\exp(-\tau[s_1+\theta(1-t)(s_2-s_1)])\,d\theta dt$$

$$= \mathrm{Re}\left(-\alpha\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t(s_0-s_1))\exp(-\tau[s_1+\theta(1-t)(s_2-s_1)])\,d\theta dt\right)$$

$$\leq |\alpha|\,\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t(\zeta-s_1))\exp(-\tau[s_1+\theta(1-t)(s_2-s_1)])\,d\theta dt$$

$$< |\alpha|\,\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau[(1-\theta(1-t))s_1+\theta(1-t)s_2])\,d\theta dt. \tag{4.9}$$

Using the relation $1-\theta(1-t) = t + (1-t)(1-\theta)$, we get

$$\exp(-\tau(1-\theta(1-t))s_1) = \exp(-\tau ts_1)\exp(-\tau(1-t)(1-\theta)s_1).$$

Moreover, since $s_3 < s_1$, we obtain, for every $t \in (0,1)$ and every $\theta \in (0,1)$, the following estimation

$$\exp(-\tau(1-t)(1-\theta)s_1) < \exp(-\tau(1-t)(1-\theta)s_3).$$

Summing up, using (4.6) , inequality (4.9) becomes

$$1 < |\alpha|\,\tau^2 \int_0^1 \int_0^1 (1-t)\exp(-\tau t s_1)\exp\left(-\tau\,(1-t)\,[(1-\theta)\,s_3 + \theta s_2]\right) d\theta dt = |\alpha|\,\tau^2 \widetilde{\mathcal{Q}} = 1,$$

which is inconsistent. This proves the dominancy of s_1. The proof of Theorem 4 is achieved.
□

Remark 3. *The geometric structure of envelope curve of the quasipolynomial* (4.2)*, defined by*

$$\sqrt{x^2+y^2} - \|A_0\|_2 - \|A_1\|_2 e^{-\tau x} = 0$$

, with $A_0 = \begin{pmatrix} -a & 1 \\ -b & 0 \end{pmatrix}$ *and* $A_1 = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}$*, may loose its connection as observed in the first order equation, depending on the distribution of the roots* s_1*,* s_2 *and* s_3*. More precisely, three cases can be observed according to the distance of the root* s_3 *with respect to the centered circle, of radius* R*, with*

$$R^2 = \|A_0\|_2^2 = 1/2(\sqrt{(a^2+(b-1)^2)(a^2+(b+1)^2)} + a^2 + b^2 + 1).$$

1. *If* $s_3 < -R$*, exceeding some critical value of the delay,* $\tau = \tilde{\tau}$*, gives rise to the birth of two disconnected components, see Figure 1, in which the arrows indicate the motion direction of the envelope as* τ *increases. For* τ *sufficiency large, the right component is the centered circle, of radius* R*, while the left component approaches the half-plane delimited by* $x = s_3$*. The two components are are separated by the distance* $-R + s_3$*.*

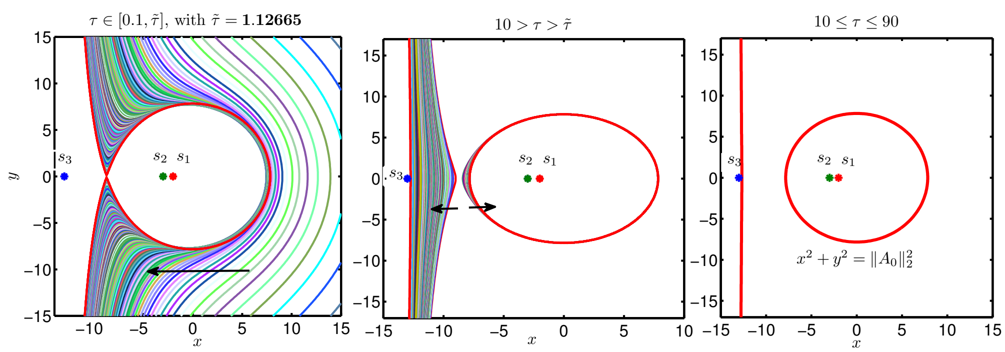

Figure 5: Envelope curve of the characteristic equation (4.2). Case $s_1 = -2$, $s_2 = -3$, $s_3 = -13$.

2. *If* $s_3 > -R$*, the connected structure of the envelope is preserved independently of the delay* τ*. For* τ *sufficiency large, the envelope takes the shape of a "vertical Omega"* ℧ *never closing, positioned in* s_3*, and which is carried by the centered circle of radius* R*, see Figure 6 (left).*

3. If $s_3 = -R$, the connected structure of the envelope is still preserved as in the second case. Here the "vertical Omega" structure is also observed. This structure is deformed when the delay increases. Indeed, the angular points are getting closer and closer, until forming a node at s_3, see Figure 6 (right).

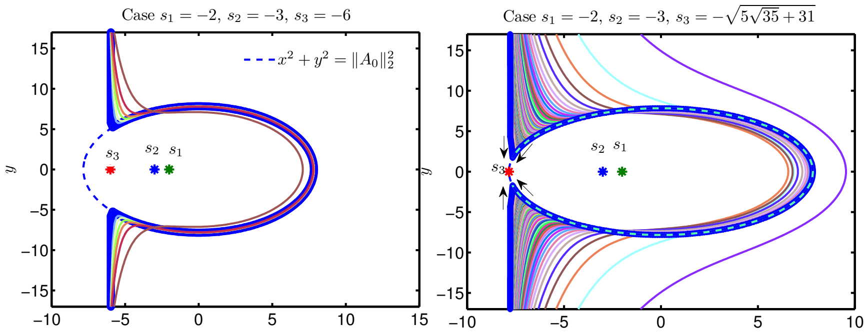

Figure 6: Envelope curve of the characteristic equation (4.2). Case $s_1 = -2$, $s_2 = -3$, and $s_3 \geq -R$.

5 Concluding remarks

The property of *multiplicity induced-dominancy* for spectral value of time-delay systems of retarded type is extended through this work. More precisely, it is shown, for reduced order Time-delay systems, that assigning PS_B real spectral values (not necessarily multiple root) make them the rightmost roots of the corresponding quasipolynomial. Furthermore, if they are set to be negative, this guarantees the asymptotic stability of the trivial solution. This new result emphasizes a new delayed controller-design based on the trivial solution's decay rate assignment.

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