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Navigation in a multi-obstacle environment. From partition of the space to a zonotopic-based MPC.

Daniel Ioan¹, Sorin Olaru¹, Ionela Prodan², Florin Stoican³, Silviu-Iulian Niculescu¹

Abstract—This paper pertains to the navigation in a multi-obstacle environment and advocates the use of local zonotopic approximations within the obstacle and collision avoidance problem. The design problem is commonly stated in the literature in terms of a constrained optimization problem over a non-convex domain. Firstly, it will be shown that a partition of the navigation space can be obtained using the notion of convex liftings. This partition will offer the foundation for the generation of a path from the current position to the destination point. In order to efficiently describe the navigation on this path, the feasible domain is described using zonotopes. The structural properties of zonotopes with respect to the generic polyhedral sets represents an advantage from the computational point of view. The current paper treats the zonotopic approximations from a control perspective, providing a set of conditions able to safeguard the initial domain topology. Globally, an adaptation of the generic collision avoidance problem is considered, aiming to guarantee the feasibility and highlighting through simulations and proof of concepts illustrations the advantages offered by the use of a zonotopic representation.

I. INTRODUCTION

The recent proliferation of the real-world applications involving unmanned vehicles has led to a growing interest of the control community on developing reliable and adequate collision-free control strategies within multi-obstacle environment [1], [2].

There exist in the literature two main approaches dealing with the path/trajectory planning in a non-convex feasible space. The optimization-based methods state the problem as a constrained optimization one either over a non-convex domain (e.g. mixed-integer formulation [3], [4]) or with a non-linear cost (e.g. potential field formulation [5], [6]). The second class of approaches, the sampled (or graph) -based methods, replaces the search of feasible paths with the search of the shortest (w.r.t. a predefined criterion) path within a graph whose nodes are randomly selected samples [7], the most popular ones: PRM (probabilistic roadmaps) [8], RRT (rapidly-exploring random tree) [9] and their variants. Moreover, there are some techniques, like tunnel MILP [10], which combines the previously two classes of methods.

Generally, regardless of the approach, a polytopic representation of the obstacles is used. This allows, especially for the first type of methods, an efficient characterization of the non-convex and non-connected feasible domain as

a union of convex regions through the corresponding hyperplane arrangements in a mixed-integer formalism. The main drawback of this formalism is that the complexity of the control problem is strongly dependent on the number of obstacles. Nonetheless, various technical procedures were developed recently in order to manage the complexity of the formulation: cell merging [11], logarithmic formulations [12] or the use of zonotopic approximations [13].

The later procedure provides rigorous complexity bounds but may lead to infeasibility in the control problem. Therefore, a question appears “*How can we approximate the obstacles with zonotopic sets while, simultaneously, safeguarding the feasible paths of the initial problem?*”. We propose to address this question from two different perspectives. We handle the approximation such that separation among obstacles is preserved (thus avoiding changes in the domain topology and subsequently, infeasibility in the motion planning procedure). In order to identify the obstacles which need to be separated, the problem of finding a feasible trajectory is decomposed into a two-phase algorithm capable to generate feasible trajectories based on a partition of the navigation space. The convex liftings will be instrumental in the construction of a partition while the vertices and faces of the partition will generate a path from the initial state to the destination.

The use of zonotopes is motivated by their intrinsic properties (here, but also in other control areas like: collision detection [14], reachability analysis [15], fault diagnosis [16] or guaranteed state estimation [17]). As well, there are toolboxes like CORA [18] able to manage their representation.

Specifically, the main contributions of this paper are the following:

- i) provide a partitioning of the feasible space;
- ii) obtain a feasible path;
- iii) ensure separation among the zonotopic approximations, i.e., guarding the original space topology;
- iv) propose a navigation strategy with obstacle avoidance guarantees using local zonotopic approximations of the obstacles.

The remaining of this paper is divided as follows: Section II presents some basic set-theoretic tools and briefly introduces the formulation of optimization-based control problem, Section III provides a mechanism able to generate feasible geometric paths through the multi-obstacle environment, and Section IV presents a prototype control strategy and treats the zonotopic approximations from the control perspective. Furthermore, Section IV-D proposes a collision

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avoidance strategy and evaluates the results on different scenarios, while Section V draws the conclusions.

Notation: The Minkowski sum of two sets is denoted as $A \oplus B = \{x : x = a + b, a \in A, b \in B\}$. $\mathcal{C}_X(S)$ denotes the complement of the set S over $X \in \mathbb{R}^d$, and $\text{int}(S)$ its interior. For $x \in \mathbb{R}^d$ we denote $\|x\|_Q^2 = x^\top Q x$

II. PRELIMINARIES

A. Geometric prerequisites and the basic control problem

Typically, convex obstacles are modeled using polyhedral sets [4]. Any polytope (i.e. a bounded polyhedron) has a dual representation in terms of intersection of half-spaces:

$$P = \{x \in \mathbb{R}^d : s_i^\top x \leq r_i, \forall i\}. \quad (1)$$

or, equivalently, a convex hull of extreme points:

$$P = \{x \in \mathbb{R}^d : x = \sum \alpha_j v_j, \sum \alpha_j = 1, \alpha_j \geq 0, \forall j\}, \quad (2)$$

For further use, consider the collection of obstacles (depicted in Fig. 3) as a finite union of polytopes:

$$\mathbb{P} = \bigcup_{j=1}^{N_o} P_j. \quad (3)$$

In order to obtain a collision free trajectory for an agent described by LTI dynamics through a multi-obstacle environment, a MPC (Model Predictive Control) strategy is considered:

$$\min_{\mathbf{u}} \left(\|x_{k+N_p|k} - \bar{x}_{\text{ref}|k}\|_{\mathbf{P}}^2 + \sum_{l=1}^{N_p-1} \|x_{k+l|k} - \bar{x}_{\text{ref}|k}\|_{\mathbf{Q}}^2 + \sum_{l=1}^{N_p-1} \|\Delta u_{k+l|k}\|_{\mathbf{R}}^2 \right) \quad (4)$$

$$\text{s.t. } x_{k+l|k} = A x_{k+l-1|k} + B u_{k+l-1|k}, \quad (5a)$$

$$x_{k+l|k} \in \mathcal{X}, u_{k+l|k} \in \mathcal{U}, \quad (5b)$$

$$x_{k+l|k} \notin \mathbb{P}. \quad (5c)$$

The agent dynamics are referenced by (5a) with $x \in \mathbb{R}^d$ the state vector, $u \in \mathbb{R}^{d_u}$ the input vector and the matrices A, B of appropriate dimension. Furthermore, in (4) – (5) N_p is the prediction horizon, the weight matrices \mathbf{P} (terminal cost penalty), \mathbf{Q} (output error penalty) and \mathbf{R} (control move penalty) are positive semi-definite and of appropriate dimensions. The sets \mathcal{X} and \mathcal{U} are compact sets from \mathbb{R}^d and \mathbb{R}^{d_u} , respectively.

While the constraints (5a)-(5b) are inherent in any control problem, the particularity of the anti-collision strategy comes from the non-convex constraints (5c), whose formulation has a substantial impact on the performances and formulation of the control strategy. Hyperplane arrangements are used to characterize these regions. Consider the collection of the support hyperplanes from \mathbb{R}^d associated with (3):

$$\mathbb{H} = \{\mathcal{H}_i\}_{i \in \mathcal{I}}, \quad (6)$$

with $\mathcal{H}_i = \{x \in \mathbb{R}^d : s_i^\top x = r_i\}$. Each of these hyperplanes divides the space in two disjoint regions:

$$\mathcal{R}_i^+ = \{x \in \mathbb{R}^d : s_i^\top x \leq r_i\}, \quad (7a)$$

$$\mathcal{R}_i^- = \{x \in \mathbb{R}^d : -s_i^\top x \leq -r_i\}. \quad (7b)$$

Hence, the space can be described as a collection of cells using the hyperplane arrangement notion.

Definition 1 (Hyperplane arrangements – [19]): The collection \mathbb{H} partitions the space into a union of disjoint cells $\mathcal{A}(\sigma)$, characterized by a sign tuple $\sigma \in \{-, +\}^N$:

$$\mathcal{A}(\sigma) = \bigcap_{i \in \mathcal{I}} \mathcal{R}_i^{\sigma(i)}. \quad (8)$$

The hyperplane arrangement of cells covering the entire space is described by the collection of all feasible sign tuples:

$$\mathcal{A}(\mathbb{H}) = \bigcup_{l=1 \dots \gamma(N)} (\sigma_l), \quad (9)$$

where $\sigma_l \in \{-, +\}^N$ is the sign tuple resulting from a non-empty intersection of half-spaces and $\gamma(N)$ is the number of feasible cells. \blacklozenge

Labeling the feasible cells (8) into interdicted $\Sigma_{\mathbb{P}} = \{\sigma : \mathcal{A}(\sigma) \cap \mathbb{P} \neq \emptyset\}$ or allowed $\Sigma_{X \setminus \mathbb{P}} = \{\sigma : \mathcal{A}(\sigma) \cap \mathbb{P} = \emptyset\}$ we rewrite the MPC problem (4), replacing the constraints (5c) with:

$$x_{k+l|k} \in \mathcal{A}(\sigma_a), \quad \forall \sigma_a \in \Sigma_{X \setminus \mathbb{P}}. \quad (10)$$

The constraints (10) have the advantage of a inclusion type in comparison with (5c) but brings the complexity of the enumeration of the components.

B. Motivation and problem statement

Generally speaking, the performance of the MPC problem (4) covers several aspects:

- (a) dynamic performance,
- (b) convergence and recursive feasibility,
- (c) complexity of the representation and anti-collision guarantees.

These aspects can be dealt with by properly adjusting the tuning parameters appearing in the MPC formulation:

- i) N_p , the prediction horizon,
- ii) the value(s) of $\bar{x}_{\text{ref}|k}$,
- iii) the weight matrices \mathbf{P} , \mathbf{Q} and \mathbf{R} .

While the dynamic performance is affected, mainly, by the choice of parameters iii), the aspect (b) strongly depends on the way we choose the parameters i) and ii). Ultimately, complexity and anti-collision feature (c) is correlated with the formal description of the non-convex and non-connected constraints (5c).

In what follows we focus on providing a control strategy, tackling the control problem (4) from the perspective of objectives (b) and (c). Therefore, we propose a collision avoidance strategy which is built on:

- i) construction of a partition of $\mathcal{X} \subseteq \mathbb{R}^d$ such that we have a one-to-one relationship with obstacles (Section III-A),

- ii) generation of reference trajectories with feasibility guarantees (Section III-B),
- iii) a real-time MPC, enhanced by a zonotopic representation of the multi-obstacle environment (Section IV).

III. FEASIBLE PATH GENERATION

A. Space partitioning

In what follows, we build a workspace partition which relies on the obstacles. We introduce a couple of definitions.

Definition 2: A family of sets $\{X_i\}_{i \in \mathcal{I}}$ verifying:

- i) $\mathcal{X} = \bigcup_{i=1}^{N_o} X_i$,
- ii) $\text{int}(X_i) \cap \text{int}(X_j) = \emptyset, \forall i \neq j \in \mathcal{I}$,

is called partition of $\mathcal{X} \subseteq \mathbb{R}^d$. \blacklozenge

Definition 3: If the sets \mathcal{X} and $X_i, \forall i$ in Def. 2 are polyhedral, then $\mathcal{X} = \bigcup_{i=1}^{N_o} X_i$ is called a polyhedral partition. \blacklozenge

With these definitions our objective is formulated in terms of constructing $\mathcal{X} = \bigcup_{i=1}^{N_o} X_i$ with $P_i \subset X_i$, through a convex lifting method [20].

Definition 4 (Convex lifting [20]): For a polyhedral partition $\{X_i\}_{i \in \mathcal{I}}$ of a polyhedron \mathcal{X} , a piecewise affine lifting described by the function:

$$z(x) = a_i^\top x + b_i, x \in X_i, \quad (11)$$

is called a convex (piecewise affine) lifting, if $z(x)$ is continuous and convex over partition $\{X_i\}_{i \in \mathcal{I}}$ of \mathcal{X} . \blacklozenge

Remark 1: The lifting $z(x)$ satisfies the continuity condition if $a_i^\top x + b_i = a_j^\top x + b_j, \forall x \in X_i \cap X_j$.

Remark 2: The function $z(x)$ being convex over $\{X_i\}_{i \in \mathcal{I}}$ means that $z(x) > a_j^\top x + b_j, x \in X_i \setminus X_j, \forall i \neq j \in \mathcal{I}_N$. \blacklozenge

For further use, we denote by $\{X_i\}_{i \in \mathcal{I}}$ the partition obtained through convex lifting, and, for the ease of the presentation, the following assumption is in order.

Assumption 1: The compact set \mathcal{X} is a polyhedron.

The actual construction of the convex lifting is done by solving the convex optimization problem which follows:

$$\min_{a_i, b_i} \sum_{i=1}^{N_o} J(a_i, b_i) = \sum_{i=1}^{N_o} [a_i^\top \quad b_i] \begin{bmatrix} a_i \\ b_i \end{bmatrix} \quad (12a)$$

$$\text{s.t. } a_i^\top v + b_i \leq M, \forall v \in \mathcal{V}(P_i), \forall i, \quad (12b)$$

$$a_i^\top v + b_i \geq a_j^\top v + b_j + \epsilon, \forall v \in \mathcal{V}(P_i), \forall i \neq j, \quad (12c)$$

$$a_j^\top v + b_j \geq a_i^\top v + b_i + \epsilon, \forall v \in \mathcal{V}(P_j) \forall i \neq j. \quad (12d)$$

where M is a sufficiently large ("big-M") constant, $\epsilon \geq 0$ a given constant, and $\mathcal{V}(P_i)$ stands for the set of the vertices of $P_i, \forall i = 1 : N_o$ in (3).

Remark 3: Compared to the standard construction from [20], the problem (12) uses the vertices of the obstacle P_i instead of vertices of the partition cell X_i . Moreover, while the constraint (12b) represents an upper bound for the lifting, the other two constraints are given by the characteristics of the convex lifting, according to Def. 4. \blacklozenge

Next, we proceed to obtain the partition $\{X_i\}_{i=1:N_o}$ from its convex lifting. Therefore, gathering the solutions of (12), we define the following "d+1"-dimensional polyhedron:

$$\mathcal{P} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{d+1} : [a_i^\top \quad 1] \begin{bmatrix} x \\ z \end{bmatrix} \leq -b_i, \quad i = 1 : N_o \right\}. \quad (13)$$

Projecting the facets of \mathcal{P} on \mathcal{X} , we obtain the polyhedral partition $\{X_i\}_{i=1:N_o}$:

$$X_i = \text{proj}(\text{facet}_i(\mathcal{P}), \mathcal{X}), \forall i \quad (14)$$

where $\text{facet}_i(\mathcal{P})$ stands for the i -th facet of the polyhedron \mathcal{P} , i.e. the inequalities describing \mathcal{P} remains the same except the i -th one which is converted in an equality. The properties of the partition obtained in this manner are summarized by the following proposition.

Proposition 1: If problem (12) is feasible, then the partition $\{X_i\}_{i=1:N_o}$ has the following properties:

- i) $P_i \subset X_i, \forall i$
- ii) $X_i \cap P_j = \emptyset, \forall j \neq i$.

Proof:

- i) The feasibility of (12) implies that (12c) is checked. Thus, considering (2), it means that for all $x \in P_i$ we have: $\begin{bmatrix} x \\ z(x) \end{bmatrix} = \begin{bmatrix} x \\ a_i^\top x + b_i \end{bmatrix} \in \mathcal{P}$. In other words, the lifting corresponding to P_i is included in $\text{facet}_i(\mathcal{P})$. By projecting over X , it directly leads to the conclusion.
- ii) By reduction to the absurd, let us suppose that there is a value $y \in X_i \cap P_j$. According to i) we have $P_j \subset X_j$. That leads to $y \in X_j$ and, from feasibility of (12c):

$$a_j^\top y + b_j \geq a_i^\top y + b_i + \epsilon. \quad (15)$$

Moreover, $y \in X_j$ means that $y \in X_i \cap X_j$ and (X_i, X_j) -neighboring cells. By consequence, as stated in Remark 1, we have:

$$a_i^\top y + b_i = a_j^\top y + b_j. \quad (16)$$

Combining (15) and (16), gives $\epsilon \leq 0$, which contradicts the initial condition $\epsilon > 0$. \blacksquare

Remark 4: The feasibility of (12) implies the existence of a *convex liftable* partition of the workspace \mathcal{X} and provides a versatile tool for the construction of a partition based on the obstacles. It should be mentioned that the infeasibility of (12) does not exclude the existence of either a *convex non-liftable* or *non-convex liftable* partition [20]. \blacklozenge

Illustrative examples

Fig. 1 depicts the partition cells for two different multi-obstacle environments. We note that there exist a one-to-one relationship between obstacles and partition cells. As a side remark, the construction based on lifting is extremely fast and, thus, can be evaluated for moving obstacles (however, this case is out of the scope of the present paper).

B. Geometric path generation

The constructive result from the previous subsection holds for \mathbb{R}^d . For the clarity of the presentation, in this subsection we restrict the scope to the $d = 2$ case and aim to guarantee a path for any pair of initial and final points in the workspace.

For the purpose of obtaining collision-free geometric path through a multi-obstacle environment, we define a weighted graph $\Gamma = (\mathcal{N}, \mathcal{E}, f)$, $f : \mathcal{E} \rightarrow \mathbb{R}$, based on the partition

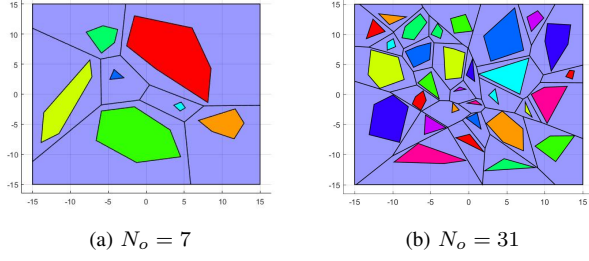


Fig. 1: Space-partitioning as in (14).

$\{V_i\}_{i=1:N_o}$ of the workspace \mathcal{X} . Hence, the nodes of Γ are the vertices of the polyhedral cells:

$$\mathcal{N} = \bigcup_{i=1}^{N_o} \mathcal{V}(X_i),$$

the edges \mathcal{E} are the hyperplanes connecting the selected vertices (i.e. the facets of the partition cells), and the function f is given by the Euclidean distance between the incident nodes of the edge. For illustration, we have depicted in Fig. 2, the graph corresponding to the polyhedral partition in Fig. 1b.

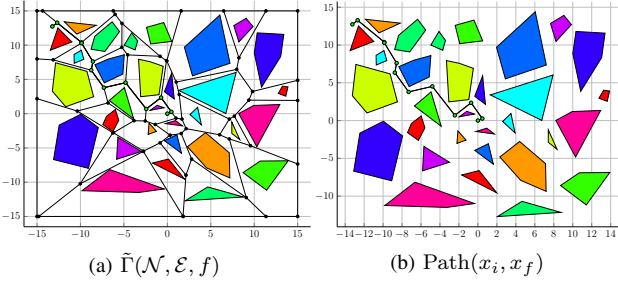


Fig. 2: Graph Γ and the shortest path $\text{Path}(x_i, x_f)$.

Proposition 2: Γ is a connected planar graph.

Proof: Since $\{X_i\}_{i=1:N_o}$ is a partition of \mathcal{X} having the properties described in Def. 2, it results that there exists no intersection among the facets of partitions cells V_i , except the vertices, and, by consequence, among edges of the graph (i.e. the graph is planar). Moreover, there exists a sequence of facets which links any two vertices of the partition $\{X_i\}$. Hence, there exists a path through the graph between any two nodes in \mathcal{N} (i.e. the graph is connected). ■

Finding the shortest path through the graph between the start and final points $x_i, x_f \in \mathcal{X} \setminus \mathbb{P}$ means adding them to the graph and finding the closest edge such that the connection to it does not intersect any obstacles:

$$(j^\perp, x^\perp)(x_i) = \arg \min_{j \in \mathcal{I}_{f_i}} \min_{x \in \text{facet}_j(X_i)} \|x - x_i\| \quad (17a)$$

$$\text{s.t. } \alpha x + (1 - \alpha)x_i \notin P_i, \quad x_i \in X_i, \quad (17b)$$

where \mathcal{I}_{f_i} is the number of facets of X_i . These projections of the points over the closest edges are also new nodes of the graph. Thus, the graph Γ properties are preserved by the new graph $\tilde{\Gamma}(x_i, x_f)$. Afterward, on account of

$\tilde{\Gamma}(x_i, x_f)$ properties, a graph search algorithm (e.g. Dijkstra's Algorithm [8]) is employed and the shortest path between x_i and x_f is obtained. For example, in Fig. 2, the green nodes are the components of the shortest path.

Remark 5: The existence of a shortest path through the graph $\tilde{\Gamma}(x_i, x_f)$ between any pair $(x_i, x_f) \in (\mathcal{X} \setminus \mathbb{P}) \times (\mathcal{X} \setminus \mathbb{P})$ is guaranteed for the reason that the the graph $\tilde{\Gamma}(x_i, x_f)$ is connected for any pair (x_i, x_f) . For further use, we denote the shortest path between x_i and x_f as $\text{Path}(x_i, x_f) = (\bar{x}_0 = x_i, \bar{x}_1 = x^\perp(x_i), \dots, \bar{x}_n = x^\perp(x_f), \bar{x}_{n+1} = x_f)$, the ordered set of points forming the shortest path. ◆

IV. FROM POLYHEDRAL TO ZONOTOPIC MPC FOR COLLISION AVOIDANCE

Let us consider the feasible path $\text{Path}(x_i, x_f)$ through the environment given by the method detailed in Section III. Our aim is to incorporate it in a optimization-based collision avoidance strategy, e.g. the MPC problem (4).

A. Prototype polytopic MPC for collision avoidance

Firstly, we propose a motion planning strategy which exploits, besides the existence of a feasible path, the polytopic representation of the obstacles and offers feasibility guarantees. To this end, we rewrite (4) in a simpler form:

$$u_{MPC_3}^* = \arg \min (\|x_{k+1|k} - \bar{x}_{1|k}\|_{Q_1}^2 + \|x_{k+2|k} - \bar{x}_{2|k}\|_{Q_2}^2 + \|x_{k+3|k} - \bar{x}_{3|k}\|_{Q_3}^2) \quad (18)$$

$$\text{s.t. } x_{k+l|k} = Ax_{k+l-1|k} + Bu_{k+l-1|k}, \quad (19a)$$

$$x_{k+l|k} \in \mathcal{X}, u_{k+l|k} \in \mathcal{U}, \quad (19b)$$

$$x_{k+l|k} \notin \tilde{\mathbb{P}}(k) \quad (19c)$$

where $\tilde{\mathbb{P}}(k) = \{P_i \in \mathbb{P} : i \text{ s.t. } \text{Path}(x_k, x_f) \cap X_i \neq \emptyset\}$ is a significantly reduced set of obstacles. Form (19) provides feasibility guarantees but does not certify the convergence. This feature can be enforced if we consider two more MPC problems with the same constraints as in (19):

$$u_{MPC_2}^* = \arg \min (\|x_{k+1|k} - \bar{x}_{1|k}\|_{Q_1}^2 + \|x_{k+2|k} - \bar{x}_{2|k}\|_{Q_2}^2), \quad (20)$$

$$u_{MPC_1}^* = \arg \min (\|x_{k+1|k} - \bar{x}_{1|k}\|_{Q_1}^2), \quad (21)$$

and select the input to be applied via optimization (22):

$$u_{MPC}^* = \max_{i=1:3} i \quad (22a)$$

$$\text{s.t. } (x^\perp(x_{k+1}^*), \text{proj}(x^\perp(x_{k+1}^*), \text{facet}_{i^\perp}(X_k))) \cap P_i \neq \emptyset \quad (22b)$$

where $x_{k+1}^* = Ax_k + Bu_{MPC_i}^*$.

Theorem 1: If $\text{Path}(x_k, x_f)$ exists and the control law u_{MPC} in (22) is recursive feasible then it guarantees the convergence $x_k \rightarrow x_f$.

Sketch of Proof: All the points $\bar{x}_{i|k}$ are either on the boundary of X_k or outside of it. The cost function is designed to minimize the tracking error with respect to three points and, thus, in the absence of constraints (19b)-(19c), the trajectory will leave X_k , crossing the boundary in finite time as long as $x_f \notin X_k$. The recursive feasibility

assumption ensures the compatibility of (19b). However, (19c) can threaten the convergence if the state converges to a point on the boundary of P_k . The existence of a trajectory converging to the borders of X_k is guaranteed by the existence of $\text{Path}(x_k, x_f)$. Nevertheless, this trajectory might not be generated by $u_{MPC_3}^*$ which weights the tracking at multiple stages and, thus, violating (22b). In order to enforce the satisfaction of (22b), the receding optimization will be enhanced with $u_{MPC_2}^*$ and $u_{MPC_1}^*$. The last one represents an explicit choice of a control converging to the boundary of X_k . Thus, the feasibility of (22b) preserves the subsequence of edges in $\text{Path}(x_k, x_f)$ and the path length (in terms of Euclidean distance) will play the role of a candidate Lyapunov function.

Remark 6: Constraint $x \notin \tilde{\mathbb{P}}(k)$, is reformulated as an inclusion, as in (10). Thus, the complexity of enumeration is substantially reduced, i.e. the number of cells generated by the reduced set of hyperplanes is significantly smaller compared to the one generated in the general case (6). ♦

Hereinafter, the number of anti-collision constraints is reduced in term of number of obstacles and the only computational advantage can be obtained by decreasing the complexity of the obstacles' representation. Thus, in the next subsections we will use a particular class of polytopes, commonly known as zonotopes, which are endowed with a third representation due to their generic symmetry property.

B. Zonotopic approximations and complexity bounds.

Definition 5 (Zonotopes-[21]): A zonotope is a centrally symmetric polytope, which can be described as a Minkowski sum of line segments. In its *generator representation* a zonotope $\mathcal{Z}(G, c)$ is described by *center* $c \in \mathbb{R}^d$ and *generators* $G = [g_1 \dots g_m] \in \mathbb{R}^{d \times m}$:

$$\mathcal{Z}(G, c) = \{c + \sum_{i=1}^m \xi_i g_i : \|\xi\|_\infty \leq 1\}. \quad (23)$$

Zonotopes own several properties of practical interest [22]:

i) are closed under Minkowski sum:

$$\mathcal{Z}(G_1, c_1) \oplus \mathcal{Z}(G_2, c_2) = \mathcal{Z}([G_1 \ G_2], c_1 + c_2); \quad (24)$$

ii) are symmetric, up to their center:

$$-\mathcal{Z}(G_1, c_1) = \mathcal{Z}(G_1, -c_1). \quad (25)$$

The half-space representation can be obtained from the generator form of a zonotope (23), as follows [15]: to each sequence of $d - 1$ generators $1 \leq j_1 < j_2 \dots j_{d-1} \leq m$ corresponds the pair $(h_i, k_i) \in \mathbb{R}^d \times \mathbb{R}$, where:

$$h_i \perp g_{j_l}, \forall j_l \in \{j_1 \dots j_{d-1}\}, k_i = \sum_{j_l \notin \{j_1 \dots j_{d-1}\}} |h_i^\top g_{j_l}|. \quad (26)$$

Adding the center from generator representation, a zonotope is formulated as:

$$\mathcal{Z}(G, c) = \bigcap_{1 \leq j_1 < \dots < j_{d-1} \leq m} \{x \in \mathbb{R}^d : |h_i(x - c)| \leq k_i\}. \quad (27)$$

Moreover, using the support function [23] (and its interesting application regarding the inclusion condition) in combination

with the definition of a zonotope, the inclusion $\mathcal{Z}(c, G) \subseteq P$, with P defined as in (1), is checked iff:

$$s_i^\top c + \sum_j |s_i^\top g_j| \leq r_i \quad \forall i, j. \quad (28)$$

Moreover, the inclusion of a polytopic set P , defined as in (2), into a zonotope $\mathcal{Z}(G, c)$ holds iff

$$|h_i^\top (v_j - c)| \leq k_i \quad \forall i, j. \quad (29)$$

Considering Def. 5 and (23) we refer to a family of zonotopes parametrized after their centers $c_\ell \in \mathbb{R}^d$ and scaling factors $\Delta_\ell \in \mathbb{R}^{m \times m}$ applied to a common generator "seed" (an a priori given matrix $G \in \mathbb{R}^{d \times m}$):

$$\mathbb{Z} = \{\mathcal{Z}(G\Delta_j, c_j), \quad j = 1 \dots N_o\}. \quad (30)$$

Δ_ℓ is a diagonal matrix whose diagonal elements are equal / distinct¹. The k -th diagonal element is noted as δ_{j_k} . This approach threads the line between complexity and feasibility.

Replacing g_k with $g_k \cdot \delta_{j_k}$ in (26), the half-space representation of the j -th zonotope from (30) is given by:

$$h_i \text{ s.t. } h_i \perp g_k, \forall k \in \{k_1 \dots k_{d-1}\}, \quad (31a)$$

$$k_i(\Delta_j) = \sum_{k \notin \{k_1 \dots k_{d-1}\}} |h_i^\top g_k| \delta_{j_k}, \quad (31b)$$

where i enumerates the

$$p(d, m) = \binom{m}{d-1} \quad (32)$$

combinations of $d - 1$ distinct generators selected from the list of m available ones (i.e., $1 \leq k_1 < \dots < k_{d-1} \leq m$).

For further use we gather the support hyperplanes resulted from (31) into the collection

$$\tilde{\mathbb{H}} = \{\mathcal{H}(h_i, \pm k_i(\Delta_j))\}, \forall j = 1 \dots N_o, i = 1 \dots p(d, m). \quad (33)$$

Parametrization (31), with set P defined as before (1)-(2), allows to reformulate the inclusion conditions (28), (29) in a linear form with respect to parameters c_j, Δ_j :

$$\mathcal{Z}(G\Delta_j, c_j) \subseteq P : s_i^\top c_j + \sum_{k=1}^m |s_i^\top g_k| \cdot \delta_{j_k} \leq r_i, \quad \forall i, \quad (34a)$$

$$P \subseteq \mathcal{Z}(G\Delta_j, c_j) : |h_i^\top (v_k - c_j)| \leq k_i(\Delta_j), \quad \forall k. \quad (34b)$$

The overall goal is to provide adequate over-approximations (30) for the multi-obstacle environment (3). That is, seek a (inherently symmetric) zonotope $\mathcal{Z}(G\Delta_j, c_j)$ enclosing the known (usually non-symmetric) polytope P_j such that a specific measure parametrized after c_j, Δ_j is minimized:

$$(\Delta_j, c_j)^* = \arg \min_{\Delta_j, c_j} \mathcal{C}(\Delta_j, c_j) \quad (35a)$$

$$\text{s.t. } P_j \subseteq \mathcal{Z}(G\Delta_j, c_j). \quad (35b)$$

Although several measures can be considered [] with different performances, in this paper we opted to use the measure

¹If not explicitly stated otherwise, we consider the later case since the former is a simplification of the later.

which provides the most accurate (tightest) approximation, specifically the one given by the volume (see the Appendix):

$$\mathcal{C}(\Delta_j, c_j) = \sum_{1 \leq k_1 < \dots < k_d \leq m} |\det(G^{k_1 \dots k_d})| \cdot \prod_{k \in \{k_1, \dots, k_d\}} \delta_{j_k}. \quad (36)$$

Considering the problem (35) and the collection of obstacles (3), we define the operator:

$$\mathbb{Z}^*(\mathbb{P}) = \{\mathcal{Z}(G\Delta_j^*, c_j^*), \quad j = 1 \dots N_o\}. \quad (37)$$

Collection (33), based on (30) induces the hyperplane arrangement² $\mathcal{A}(\mathbb{H})$. Preliminary work provides a cell count with explicit dependence on m , the number of generators and N_o , the number of obstacles:

$$r(\mathcal{A}) = \sum_{k=0}^d \binom{p(d, m)}{k} \cdot (2N_o)^k, \quad (38a)$$

$$b(\mathcal{A}) = \left| \sum_{k=0}^d (-1)^k \binom{p(d, m)}{k} \cdot (2N_o)^k \right|. \quad (38b)$$

where $r(\mathcal{A})$ is the total number of regions, and $b(\mathcal{A})$ the number of bounded regions. Furthermore, these explicit expressions allow to derive a bound for the number of generators in the zonotopic representation.

Corollary 1: Assuming n_o^* support hyperplanes in (3), for any $m \in \mathbb{N}^+$ which verifies:

$$\sum_{k=0}^d \binom{p(d, m)}{k} \cdot (2N_o)^k \leq \sum_{k=0}^d \binom{n_o^*}{k}, \quad (39)$$

the arrangement $\mathcal{A}(\mathbb{H})$ has fewer cells than $\mathcal{A}(\mathbb{H})$.

While Corollary 1 holds for \mathbb{R}^d , for $d = 2$ and $d = 3$ there exists a particularization of the results which leads to an analytic formula for the largest number of the generators m such that $\mathcal{A}(\mathbb{H})$ has fewer cells than $\mathcal{A}(\mathbb{H})$. For example, in Fig. 3a we have $N_o = 7$ hyperplanes and $n_o^* = 34$ supporting hyperplanes. Computing the bounds for the number of cells (38) and using Corollary 1, we obtain the largest value of m for which the approximation is less complex : $m = \lfloor 2.92 \dots \rfloor = 2$. This leads to values $r(\mathcal{A}) = 225$, significantly less than 407 if the $2m \cdot N_o = 28$ hyperplanes would have been in general position and less than 419, the number of cells when considering the initial representation³.

C. Zonotopic approximations with corridors

However, all the improvements in the complexity, presented in the above subsection, are related only to the geometry and they were not analyzed from the dynamics perspective. For the sake of illustration, we proceed to find the zonotopic approximations \mathbb{Z} for the collection (3) solving the problem (35) for the criterion (36) with an empirically chosen value of the common seed: $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In Fig. 3 we remark that the zonotopic approximations corresponding to P_3 and P_4 intersect which may lead to infeasibility in the

²For compactness, whenever clear from the context we use notation \mathcal{A} .

³The differences are more significant for $d = 3$ but for the clarity of the presentation we tackled here exclusively the $d = 2$ case.

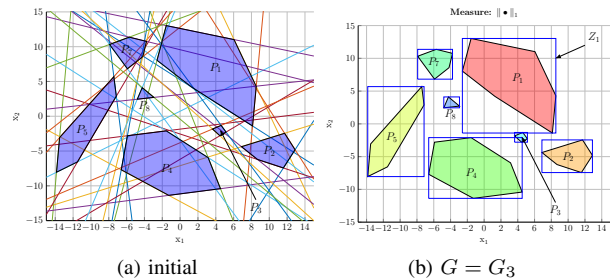


Fig. 3: Zonotopic approximation for the multi-obstacle environment

control strategy. Hereinafter, we treat the problem of closing corridors or passageways.

Definition 6: A corridor between i -th and j -th obstacles is called obstructed iff:

$$P_i \cap P_j = \emptyset \quad \text{and} \quad Z_i \cap Z_j \neq \emptyset, \quad (40)$$

where Z_i, Z_j represent the zonotopic approximations of P_i and P_j ($P_i \subseteq Z_i$ and $P_j \subseteq Z_j$). ♦

There are two possible ways to elude the overlapping and, thus, to maintain the corridors open. Both share a common feature: a separating hyperplane constraint in the optimization problem (35). In the sequel, the hyperplane separating the polytopes P_i and P_j is denoted as $\mathcal{H}_{ij} = \{x \in \mathbb{R}^d : h_{ij}^\top x = k_{ij}\}$, with its corresponding half-spaces \mathcal{R}_{ij}^+ and \mathcal{R}_{ij}^- . It is obtained from the edges of the partition $\{X_i\}_{i=1:N_o}$ in (14), i.e., \mathcal{H}_{ij} represents the common facet between any two neighbor cells of the partitioning (X_i, X_j) .

The first approach imposes an additional constraint: the approximation to stay above or below a separation hyperplane. The decision between *above* or *below* is made based on the position of the center of the polytope relative to this given separation hyperplane. In other words, we impose:

$$\mathcal{Z}(G_j \Delta_j, c_j) \subset \mathcal{R}_{ij}^+ \text{ or } \mathcal{Z}(G_j \Delta_j, c_j) \subset \mathcal{R}_{ij}^-, \quad \forall i. \quad (41)$$

The reformulation of (41) as an lp is straightforward: all vertices of the zonotopic approximation have to stay on the same side of the separation hyperplane as its center.

Remark 7: The vertices of the zonotopes can be readily obtained (28) and the complexity of the dual representation is manageable with respect to the polytopes. ♦

Thus, the optimization problem (35) takes the form:

$$\begin{aligned} \min_{\Delta_j, c_j} \quad & \mathcal{C}(\Delta_j, c_j) \\ \text{s.t.} \quad & P_j \subseteq \mathcal{Z}(G\Delta_j, c_j) \end{aligned} \quad (42a)$$

$$\mathcal{Z}(G_j \Delta_j, c_j) \subset \mathcal{R}_{ij}^\pm, \quad \forall i. \quad (42b)$$

As was stated above, the choice between \mathcal{R}_{ij}^- and \mathcal{R}_{ij}^+ (42b) depends on position of the center. The constraint (42b) is formulated recalling the inclusion condition of a parametrized zonotope into a polytope (34).

Theorem 2: Corridor constraints are convex with respect to the zonotopic over-approximations.

Proof: Taking into account that the separation constraints (42b) are linear, the convexity of the feasible space regarding zonotopic over-approximation is straightforward ■

Remark 8: Certainly, (42b) enforces the corridors but it represents an additional structural constraint and impacts the feasibility of the zonotopic over-approximations. For example, let us consider the case depicted in the Fig. 4, where we take a triangle in \mathbb{R}^2 and proceed to over-approximate with a parametrized zonotope with G given by the Euclidean unit vectors. The problem (42) is infeasible. ♦

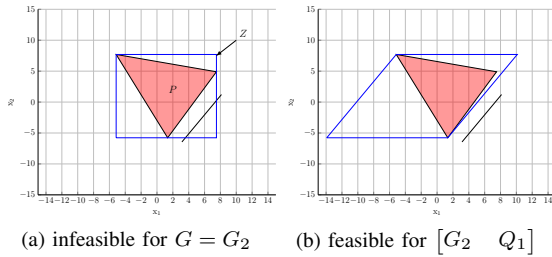


Fig. 4: The feasibility of (42).

Despite the limitation highlighted in Remark 8, there exists a strong constructive result for the existence of a zonotopic over-approximation which satisfies the corridor constraints.

Proposition 3: Any polyhedral set (1) can be approximated by a zonotopic set (23) with corridor type of constraints (41) by an adequate choice of generators G .

Proof: The idea is to generate a face of the zonotopic over-approximation aligned to the separating hyperplane. This is done by spanning the separation hyperplane with a subset of the generators. More precisely, we should obtain any basis for the subspace described by the separating hyperplane. This basis represents a valid subset of generators for the zonotopic set. The result is constructive if we consider a full rank orthogonal matrix Q with $Q = [h_{sep}^\top \quad Q_1]$. If G includes a basis of the space perpendicular on separating hyperplane, i.e. Q_1 then the problem (42) is feasible (as in Fig. 4b). ■

A key issue with this approach is that we cannot guarantee the monotonicity of the over-approximations with respect to the number of the generators. By adding a generator as in the proof of Corollary 3, we solve an obstructed corridor and, at the same time, we may obstruct others which were initially unobstructed. An important theoretical result which guarantees the separation in a d -dimensional space for a number of “ $d + 1$ ” over-approximations is presented next.

Theorem 3: In \mathbb{R}^d , the maximum number of joint constraints for corridors with feasibility guarantees is $d + 1$. ♦

Proof: Let P_1, \dots, P_{d+1} be the obstacles and P_i a generic element of this collection. If $P_i \cap P_j = \emptyset, \forall i \neq j$, then there exist d separation hyperplanes h_{ij} and a zonotope satisfying $w_{ij} \leq \|h_{ij}^\top x\| \leq \bar{w}_{ij}, \forall x \in P_i$. Obviously, w_{ij} and \bar{w}_{ij} can be chosen in order to avoid the intersection with one side of the separation hyperplane.

Corollary 2: For any “ $d + 1$ ” obstacles, there exists a “box” approximation which guarantees the separation.

Proof: Firstly, we construct the matrix G such that the resulting over-approximation to have the facets parallel with the “ $d + 1$ ” separating hyperplanes (for each separating hyperplane we choose the vectors which span the space perpendicular on that hyperplane). There exists a zonotopic over-approximation which can be adjusted either through a proper scaling factor or through a displacement of the center.

Illustrative example

We depicted in Fig. 5 the results of a procedure which combines the advantages given by Theorem 2 and Corollary 3. This procedure performs well in simulations, but it cannot provide theoretical guarantees on the separation among the zonotopic over-approximations within a generic multi-obstacle environment.

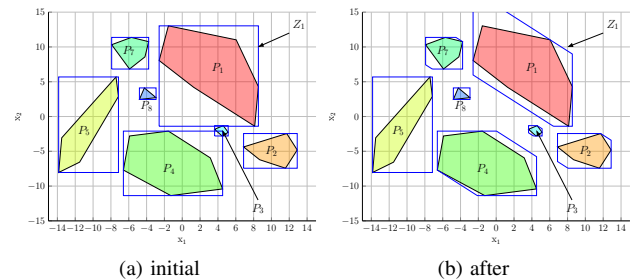


Fig. 5: Maintaining corridors procedure over example in Fig. 3

As was stated above, the strong result from Theorem 3 is restricted to $d + 1$ obstacles. In order to cope with a generic obstacles distribution (in terms of their number and positions) we use a strategy able to select these $d + 1$ obstacles for each given state based on the results from Section III and on the prototype MPC strategy from Section IV-A.

D. Generic collision avoidance strategy and examples

The proposed motion planning strategy through a multi-obstacle environment can be summarized by the following steps:

Algorithm 1 The motion planning strategy.

- 1: {OFF-LINE} Compute $\{X_i\}_{i=1:N_o}$ in (14).
 - 2: {OFF-LINE} Construct the graph Γ based on $\{X_i\}_{i=1:N_o}$
 - 3: Add start and goal to the graph and find the shortest path through the graph $\rightarrow \text{Path}(x_i, x_f)$.
 - 4: Select the points $\bar{x}_{i|k}, i = 1 : 3$ and the corridors they defined with the corresponding local zonotopic approximation ensuring the corridors.
 - 5: Solve the MPC problem (18) by replacing (19c) with $x_{k+l|k} \notin \mathbb{Z}^*(\hat{\mathbb{P}}(k))$.
-

For illustration, let us consider an agent described by the dynamics (5a) in \mathbb{R}^d :

$$A = \begin{bmatrix} O_d & I_d \\ O_d & -\frac{\mu}{M} I_d \end{bmatrix}, \quad B = \begin{bmatrix} O_d \\ M I_d \end{bmatrix},$$

where $\mu = 3$ and $M = 60$. The agent's state and input are constrained: $\mathcal{X} = \{x : -15 \leq x_i \leq 15, \forall i = 1 \dots 2d\}$ and $\mathcal{U} = \{u : -1 \leq u_i \leq 1, \forall i = 1 \dots 2d\}$.

In order to obtain a collision free trajectory for the agent through the complex⁴ multi-obstacle environment depicted in Fig. 2 we firstly employed the strategy (4) with enhancement (10), considering the entire set of obstacles and imposing a large prediction horizon ($N_p = 40$). The resulting trajectory (the red one from Fig. 6a) did not converge to the final position, the agent remaining on the boundary of an obstacle. However, applying the Algorithm 1 we obtain a trajectory, which attains x_f . Moreover, Fig. 6b depicts the local zonotopic approximation corresponding to the first sequence of $\bar{x}_{i|k}, i = 1 : 3$, and the highlighted obstacles represent the union of all sets $\tilde{\mathbb{P}}(k)$ whose approximations were used in the control problem at a certain moment.

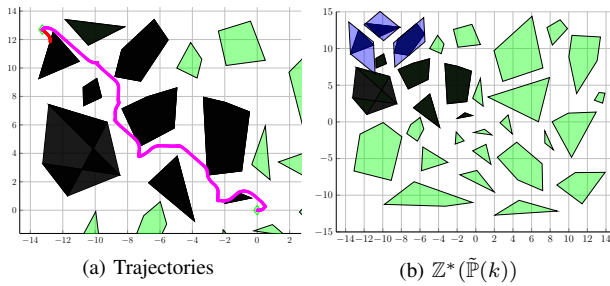


Fig. 6: Simulation results for the environment Fig. 2

V. CONCLUSIONS

The obstacle avoidance problem was treated aiming to ensure the convergence and feasibility and, concurrently, to reduce the complexity of the original problem. Based on convex lifting and graph theory we have generated feasible reference trajectories which were further employed in the collision avoidance strategy. Additionally, zonotopic over-approximations were used to cover efficiently the non-convex region of interest and a series of theoretical results were proposed to address the issues related to their impact on the control problem.

APPENDIX

Volume of the parametrized zonotopes

The zonotopes volume has an explicit formulation [24]:

$$\text{Vol}(\mathcal{Z}(G, c)) = \sum_{1 \leq j_1 < j_2 \dots j_d \leq m} |\det(G^{j_1 \dots j_d})| \quad (43)$$

where $G^{j_1 \dots j_d}$ denotes the matrix composed from columns of indices $j_1 \dots j_d$ from G . Replacing g_k with $g_k \cdot \delta_{j_k}$ in (43) leads to

$$\text{Vol}(\mathcal{Z}(G\Delta_j, c_j)) = \sum_{1 \leq k_1 < \dots < k_d \leq m} \left| \det(G^{k_1 \dots k_d} \cdot \Delta_j^{k_1 \dots k_d}) \right|.$$

Noting that $\det(M \cdot N) = \det(M) \cdot \det(N)$ and that Δ_j is a diagonal matrix with positive elements gives (36).

⁴in the sense of the number of the obstacles, $N_o = 33$

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