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Abstract: This paper presents a systematic frequency domain approach to analyse the stability of reduced-order linear systems with single delay. More precisely, we address the problem of the spectral abscissa characterization and the coexistence of non oscillating modes for such functional differential equations. The dominancy of such non oscillating modes is analytically shown for the considered reduced order Time-delay systems.

Keywords: Time-delay systems, Stability, Spectral abscissa, Control design, Pole assignment, Non oscillation

1. INTRODUCTION

Investigation of dynamical systems with time-delay is an active research area that connects a wide range of scientific disciplines including mathematics, physics, engineering, biology, economics etc. The present paper focuses on stability and stabilizing-controllers design for linear time-invariant retarded time-delay systems. The study of conditions on the equation parameters that guarantees the exponential stability of solutions is a question of ongoing interest and remains an open problem especially when the systems are of high order or having multiple and/or distributed delays. In particular, in frequency-domain, the problem reduces to the analysis of the distribution of the roots of the corresponding characteristic equation, see for instance (Bellman and Cooke, 1963; Cooke and van den Driessche, 1986; Walton and Marshall, 1987; Stépán, 1989; Hale and Lunel, 1993; Michiels and Niculescu, 2007; Sipahi et al., 2011).

The starting point of the present work is an interesting property, discussed in recent studies, called *Multiplicity-Induced-Dominancy* denoted in the sequel (MID). As a matter of fact, it is shown that multiple spectral values for Time-delay systems can be characterized using a Birkhoff/Vandermonde-based approach; see for instance (Boussaada and Niculescu, 2016b,a, 2014; Boussaada et al., 2016). More precisely, in the previous works, it is emphasized that the admissible multiplicity of the zero spectral value is bounded by the generic *Polya and Szegö bound* denoted PS_B, which is nothing but the *degree* of the corresponding quasipolynomial (i.e the number of the involved polynomials plus their degree minus one), see for instance (Pólya and Szegő, 1972). The multiplicity of a root itself is not important as such but its connection with the dominancy of this root is a meaningful tool for control synthesis. To the best of our knowledge, the first time an analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented in (Hayes, 1950). The dominancy property is further explored and analytically shown in the case of second-order systems and a rightmost root assignment based design using delayed state-feedback is proposed in (Boussaada et al., 2017, 2018b) where its applicability in damping active vibrations for a piezo-actuated beam is proved. See also (Boussaada et al., 2018a; Boussaada and Niculescu, 2018) which exhibit an analytical proof for the dominancy of the spectral value with maximal multiplicity for second-order systems controlled via a delayed proportional-derivative controller.

By this paper, we would like to extend such an analytical characterization of the spectral abscissa for retarded time-delay system with real spectral values which are not necessarily multiple. The effect of the coexistence of such non oscillatory modes on the asymptotic stability of the trivial solution will be explored. In particular, the coexistence of PS_B real spectral values makes them rightmost-roots of the corresponding quasipolynomial. Furthermore, if they are negative, this guarantees the asymptotic stability of the trivial solution.

The remaining paper is organized as follows; in Section 2 we summarize some important facts on the coexistence non oscillatory modes for the first and second order time-delay differential equation with a single delay. Section 3 contains the main result of this paper. We address the problem of the spectral abscissa characterization and the coexistence of non oscillating modes for the third-order differential equation with a single delay. Section 4 is dedicated to illustrate the main result of the paper on Mach number regulation in a wind tunnel by the assignment of four dominant equidistributed real roots. In Section 5, some concluding remarks end the paper.

2. PRELIMINARIES

Let consider the generic n-order system with a single time delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau). \tag{1}$$

Here τ is a positive constant delay and the matrices $A_j \in M_n(\mathbb{R})$ for $j = 0 \ldots 1$. It is well known that the asymptotic behavior of the solutions of (1) is determined from the spectrum designating the set of the roots of the associated characteristic function (denoted in the sequel $\Delta(s,\tau)$). Namely, the characteristic function corresponding to system (1) is a quasipolynomial $\Delta : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ of the form:

$$\Delta(s,\tau) = \det(sI - A_0 - A_1 \mathrm{e}^{-\tau s}). \tag{2}$$

Asymptotic stability of the trivial solution and oscillatory behavior of (1) are known. In particular, the zero solution of this equation is asymptotically stable if and only if all roots of (3) lie in the left half plane, and a given solution of (1) is said to be non oscillatory if it corresponds to a real root of (3).

This section summarizes the main findings reported in (Amrane et al., 2018), concerning the question of coexistence of PS_B negative real roots for (3), when in particular

$$\Delta(s,\tau) = P_0(s) + P_1(s)\mathrm{e}^{-\tau s}. \tag{3}$$

where $\deg(P_0) = n, \deg(P_1) = 0$ for $n = 1$ and $n = 2$.

Let's consider the following systems

$$\dot{x}(t) + ax(t) + bx(t-\tau) = 0, \tag{4}$$

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + \alpha x(t-\tau) = 0. \tag{5}$$

The characteristic equation associated to (4) and (5) are respectively as follows:

$$\Delta_1(s,\tau) := s + a + b\exp(-s\tau) = 0, \tag{6}$$

$$\Delta_2(s,\tau) = s^2 + as + b + \alpha\exp(-s\tau) = 0. \tag{7}$$

Theorem 1. For a given delay $\tau > 0$, the system (4) admits two distinct real spectral values at $s = s_1$ and $s = s_2$, with $s_2 < s_1$, if and only if

$$\begin{cases} a = a(s_1, s_2, \tau) \;:=\; \dfrac{s_2\exp(-s_1\tau) - s_1\exp(-s_2\tau)}{\exp(-s_2\tau) - \exp(-s_1\tau)}; \\ b = b(s_1, s_2, \tau) \;:=\; \dfrac{s_1 - s_2}{\exp(-s_2\tau) - \exp(-s_1\tau)}. \end{cases} \tag{8}$$

- Moreover, both spectral values s_1 and s_2 of (4) are negative, if and only if equation
$$a(s_1, s_2, \tau) = 0$$
admits a positive solution in τ. Furthermore, in such a case the zero solution of (4) is asymptotically stable.
- The spectral value s_1 is nothing but the spectral abscissa corresponding to (4).

Theorem 2. The system (5) admits three distinct real spectral values s_3, s_2 and s_1 with $s_3 < s_2 < s_1$ if and only if the parameters a, b and α satisfy

$$\begin{cases} a(\tau) := \dfrac{1}{Q(\tau)} \displaystyle\sum_{\substack{i,j,k\in\Lambda \\ i<j, i\neq j\neq k}} (-1)^{i+j}\,(s_i^2 - s_j^2)\exp(-s_k\tau) \\ b(\tau) = -\dfrac{1}{Q(\tau)} \displaystyle\sum_{\substack{i,j,k\in\Lambda \\ i<j, i\neq j\neq k}} (-1)^{i+j}\, s_i s_j (s_i - s_j) e^{-s_k\tau} \\ \alpha(\tau) = -\dfrac{1}{Q(\tau)} \displaystyle\prod_{\substack{i,j\in\Lambda \\ i<j}} (s_i - s_j) \end{cases} \tag{9}$$

where

$$Q(\tau) = \sum_{\substack{i,j,k\in\Lambda \\ i<j, k\neq i,j}} (-1)^{i+j}\,(s_i - s_j)\exp(-s_k\tau).$$

In this case, α is necessarily negative.

- The spectral value s_1 is negative if and only if there exists $\tau_0 > 0$ such that
$$a(\tau_0) + s_2 = 0.$$
This guarantees the asymptotic stability of the system.
- The root s_1 is the spectral abscissa of (5).

2.1 Application to a control problem

Let us focus on the second order system

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = u(t), \tag{10}$$

where u is the unknown control and a and b are known parameters. Assume that the system (10) is unstable in the uncontrolled case, namely when $u(t) = 0$. This arises for instance if $a < 0$. Our aim is to design a control u under the form:

$$u(t) = -\alpha x(t) - \beta x(t-\tau), \tag{11}$$

that stabilizes the closed loop system:

$$\ddot{x}(t) + a\dot{x}(t) + (b+\alpha)x(t) + \beta x(t-\tau) = 0, \tag{12}$$

by pole assignment method. It is therefore a question to assign three negative spectral values, s_1, s_2 and s_3, to the characteristic equation associated to (12). This pole assignment is then interpreted by the conditions (9), in which case $b := b + \alpha$, allowing the computation of the parameters α, β and $\tau > 0$ of the control (11). By choosing s_1, s_2 and s_3 equi-distributed ($s_1 - s_2 = s_2 - s_3 = d > 0$), we obtain, in view of (12), the following values of the parameters τ, α and β:

$$\begin{cases} b = 1/2\,a^2 + as_1 - ad - 2\,s_1 d + {s_1}^2 + 1/2\,d^2, \\ \beta = -1/2\,\left(-8\,s_1 d + 3\,d^2 - 4\,ad + 4\,{s_1}^2 + 4\,as_1 + a^2\right) \times \\ \left(\dfrac{-3\,d + 2\,s_1 + a}{2\,s_1 - d + a}\right)^{-\frac{-s_1+d}{d}}, \\ \tau = \ln\left(\dfrac{-3\,d + 2\,s_1 + a}{2\,s_1 - d + a}\right) d^{-1} \end{cases}$$

The asymptotic stability of the nontrivial solution of the closed-loop system (12) is guaranteed by the dominance property of s_1 established in Theorem 2, and the positivity of the delay τ.

2.2 About the geometric structure of the envelope curve associated to (6) *and* (7)

(1) It is well known that the classical envelope curve has a connected geometric structure (see for instance

(Niculescu et al., 2010)). Interestingly, when considering two real distinct spectral values, the connected structure of the envelope may be lost, see Fig. 1.

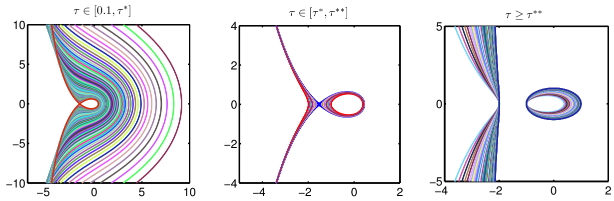

Fig. 1. Envelope curve of the characteristic equation (6). Case of co-existence of two simple real roots $s_1 = -1$, $s_2 = -2$. Here $\tau^* \approx 0.67288$ and $\tau^{**} = \ln(2)$.

(2) Such a geometry is encountered in (Qiao and Sipahi, 2013), where an analytical study and synthesis of rightmost eigenvalues of $\dot{x}(t) = Ax(t-\tau)$ is considered.

(3) Likewise, the geometric structure of the envelope curve of the quasipolynomial (7), defined by

$$\sqrt{x^2+y^2} - \|A_0\|_2 - \|A_1\|_2 e^{-\tau x} = 0,$$

with $A_0 = \begin{pmatrix} -a & 1 \\ -b & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}$, may loose its connection as observed in the first order equation, depending on the distribution of the roots s_1, s_2, s_3 and the delay τ. More precisely, three cases can be observed according to the distance of the root s_3 with respect to the centered circle, of radius $R = \|A_0\|_2$.

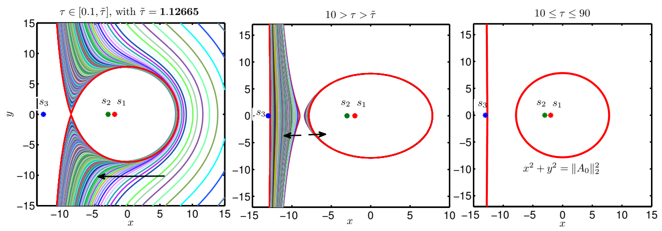

Fig. 2. Envelope curve of the characteristic equation (7). Case $s_1 = -2$, $s_2 = -3$, $s_3 = -13$.

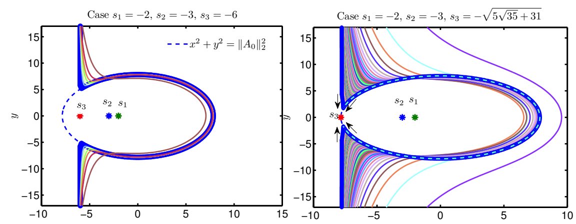

Fig. 3. Envelope curve of the characteristic equation (7). Case $s_1 = -2$, $s_2 = -3$, and $s_3 = -R$ (right), and $0 > s_3 > -R$ (left).

3. ON POLE-PLACEMENT FOR THIRD-ORDER EQUATIONS WITH A SINGLE DELAY

3.1 Four real poles assignment is possible

Let consider the generic 3-order equation with a single time delay

$$x^{(3)}(t) + a_2 x^{(2)}(t) + a_1 x^{(1)}(t) + a_0 x(t) + \alpha x(t-\tau) = 0. \quad (13)$$

The characteristic equation associated to (13) is given by:

$$\Delta_3(s,\tau) = s^3 + a_2 s^2 + a_1 s + a_0 + \alpha \exp(-s\tau) = 0. \quad (14)$$

To simplify some formulas, let us introduce the notation

$$[s_1, s_2]_t := t s_1 + (1-t)\, s_2.$$

Theorem 3. i) System (13) admits four distinct real spectral values s_4, s_3, s_2 and s_1 with $s_4 < s_3 < s_2 < s_1$ if and only if

$$Q(\tau) := \sum_{k=1}^{4} (-1)^k \exp(-s_k \tau) \prod_{\substack{i,j=1 \\ i<j, k\neq i,j}}^{4} (s_i - s_j) \neq 0.$$

In this case, the coefficients a_i, $i = 1, \ldots, 3$, and α are uniquely determined as a continuous function with respect to the delay $\tau > 0$. The parameter variable $\alpha(\tau)$ is necessarily positive for every $\tau > 0$, and satisfies

$$\alpha(\tau) = \frac{1}{Q(\tau)} \prod_{i<j, i,j=1}^{4} (s_i - s_j) = \frac{1}{\tau^3 \iiint_{[0,1]^3} (1-t)^2 (1-\theta)\, e^{-\tau\left[s_2, \left[s_3, [s_4, s_1]_\lambda\right]_\theta\right]_t} d\lambda\, d\theta\, dt}$$

ii) The spectral value s_1 is negative if and only if there exists $\tau_0 > 0$ such that

$$a_2(\tau_0) + s_2 + s_3 = 0.$$

This guarantees the asymptotic stability of the system.

Sketch of the proof

First, let us find conditions on the coefficients a_i, for $i = 0, \cdots, 2$, and α ensuring the coexistence of PS_B real spectral values (here $PS_B = 4$) for the quasipolynomial (14), recall that PS_B is *Polya and Szegö bound*. For, assume that (14) admits four real spectral values $s_1 > s_2 > s_3 > s_4$. This means that

$$s_i^3 + a_2 s_i^2 + a_1 s_i + a_0 + \alpha \exp(-s_i \tau) = 0, \ \text{ for all } i = 1, \cdots, 4. \quad (15)$$

From which we deduce the following values of the coefficients a_i, for $i = 0, \cdots, 2$, and α:

$$a_2(\tau) = \frac{1}{Q(\tau)} \sum_{k=1}^{4} (-1)^{k+1} e^{-s_k \tau} \prod_{\substack{i,j=1 \\ i<j \\ k\neq i,j}}^{4} (s_i - s_j) \sum_{\substack{i=1 \\ i\neq k}}^{4} s_i;$$

$$a_1(\tau) = \frac{1}{Q(\tau)} \sum_{k=1}^{4} (-1)^{k} e^{-s_k \tau} \prod_{\substack{i,j=1 \\ i<j \\ k\neq i,j}}^{4} (s_i - s_j) \sum_{\substack{i,j=1 \\ i<j \\ k\neq i,j}}^{4} s_i s_j;$$

$$a_0(\tau) = \frac{1}{Q(\tau)} \sum_{k=1}^{4} (-1)^{k+1} e^{-s_k \tau} \prod_{\substack{i,j=1 \\ i<j \\ k\neq i,j}}^{4} (s_i - s_j) \prod_{\substack{i,j=1 \\ i<j \\ k\neq i,j}}^{4} s_i s_j;$$

$$\alpha(\tau) = \frac{1}{Q(\tau)} \prod_{\substack{i,j=1 \\ i<j}}^{4} (s_i - s_j);$$

Clearly, a_i, for $i = 0, \cdots, 2$, and α are well-defined, for any value of the delay τ, if and only if the following condition

$$Q(\tau) \neq 0, \quad \forall \tau > 0. \quad (16)$$

is fulfilled.

So, let us check the condition (16) . By rearranging the terms in Q, and using the mean value theorem, we get

$$Q(\tau) = \tau^3 \prod_{\substack{i<j \\ i,j=1}}^{4} (s_i - s_j)\, \tilde{Q}(\tau) > 0 \tag{17}$$

with

$$\tilde{Q}(\tau) = \iiint\limits_{[0,1]^3} (1-t)^2 (1-\theta)\, e^{-\tau\left[s_2,\left[s_3,[s_4,s_1]_\lambda\right]_\theta\right]_t} d\lambda d\theta dt.$$

The existence and uniqueness of the coefficients $a_2(\tau)$, $a_1(\tau)$, $a_0(\tau)$, $\alpha(\tau)$) is then proved. The positiveness of α is provided by (17) and the sign of $\tilde{Q}$.

Since the mapping $\tau \mapsto a_2(\tau) + s_2 + s_3$ is continuous and increasing from $-\infty$ to $-s_1$ when τ varies in $\mathbb{R}^{+*}$, this means that the mapping $\tau \mapsto a_2(\tau)+s_2+s_3$ takes positive values if and only if $s_1 < 0$. Also , if and only if, there exists (a unique) root $\tau_0 > 0$ to equation $a_2(\tau) + s_2 + s_3 = 0$.

3.2 The dominancy of s_1 for (13)

To study the stability of the system (13), we need to study the dominancy of s_1 by using an adequate factorization of the quasi-polynomial Δ_3 in (14).

Theorem 4. The root s_1 is the spectral abscissa of (13).

Sketch of the proof
Rewrite the quasipolynomial Δ_3 as:

$$\Delta_3(s,\tau) = (s-s_1)(s-s_2)(s-s_3)\, P(s,\tau)$$

with

$$P(s,\tau) = \frac{s^3 + a_2 s^2 + a_1 s + a_0 + \alpha \exp(-\tau s)}{(s-s_1)(s-s_2)(s-s_3)}.$$

Define the quantities: $b_0 := a_0 + s_1 s_2 s_3$; $b_1 := a_1 - \sum_{\substack{i,j=1 \\ i\neq j}}^{3} s_i s_j$; and $b_2 := a_2 + \sum_{i=1}^{3} s_i$. Some tedious algebraic manipulations allows to write $P(s,\tau)$

$$P(s,\tau) = 1 - \tau^3 \alpha \tilde{P}(s,\tau)$$

where

$$\tilde{P}(s,\tau) = \iiint\limits_{[0,1]^3} (1-t)^2 (1-\theta)\, e^{-\tau\left[s,\left[s_2,[s_1,s_3]_\lambda\right]_\theta\right]_t} d\lambda d\theta dt.$$

To prove dominancy property for s_1, let us assume that there exists some $s_0 = \zeta + j\eta$ a root of (14) such that $\zeta > s_1$. This means that $P(s_0, \tau) = 0$. Combining this fact and the positiveness of α, we get

$$\begin{aligned} 1 &= \tau^3 \alpha \tilde{P}(s,\tau) \\ &\le \tau^3 \alpha \iiint\limits_{[0,1]^3} (1-\theta) \left| e^{-\tau\left[s,\left[s_1,[s_2,s_3]_\lambda\right]_\theta\right]_t} \right| d\lambda d\theta dt \qquad (18) \\ &< \tau^3 \alpha \iiint\limits_{[0,1]^3} (1-t)^2 (1-\theta)\, e^{-\tau\left[s_1,\left[[s_2,s_3]_\lambda,s_4\right]_\theta\right]_t} d\lambda d\theta dt \\ &= \tau^3 \alpha Q(\tau) = 1 \end{aligned}$$

which is inconsistent.

4. MACH NUMBER REGULATION IN A WIND TUNNEL: EQUIDISTRIBUTED DOMINANT-ROOTS ASSIGNMENT

Roughly speaking, the Mach number regulation in a wind tunnel is based on the Navier-Stokes equations for unsteady flow and contains control laws for temperature and pressure regulation.

As an illustrative example for the applicative potential of the proposed main result, let revisit the following simplified model of Mach number regulation proposed in (Manitius, 1984) and consists of a system of three state equations with a delay in one of the state variables. It is stressed that in steady-state operating conditions, the dynamic response of the Mach number perturbations ξ_1 to small perturbations in the guide vane angle actuator ξ_2 are governed by:

$$\begin{cases} \dot{\xi}_1(t) = -a\xi_1(t) + k\, a\, \xi_2(t-\tau) \\ \dot{\xi}_2(t) = \xi_3(t) \\ \dot{\xi}_3(t) = -\omega^2 \xi_2(t) - 2\zeta\omega\xi_3(t) + \omega^2 u(t) \end{cases} \tag{19}$$

where a, ω, ζ, k and τ are parameters depending on the operating point and presumed constant when the perturbations ξ_i are small. Moreover, following the experimental parameter values of the wind tunnel developed at NASA Langley Research Center, the parameters a, ω, ζ, τ are positive.

In (Manitius, 1984), a feedback consisting of a linear combination of state variables and weighted integrals of some of the state variables over a period equal to the time delay, where the spectrum of the closed-loop system is finite (consists of three eigenvalues). However, our method does not render the closed-loop system finite dimensional but only involves controlling its rightmost root. In (Boussaada et al., 2018a) the control law $u(t) = -\frac{\alpha}{\omega^2}\xi_2(t) - \frac{\beta_0}{\omega^2}\xi_2(t-\tau) - \frac{\beta_1}{\omega^2}\xi_3(t-\tau)$ is proposed allowing to the closed-loop quasipolynomial function:

$$\Delta(s,\tau) = (s+a)\left((s\beta_1+\beta_0)e^{-s\tau} + s^2 + 2\, s\, \zeta\, \omega + \omega^2 + \alpha\right). \tag{20}$$

Thanks to such a factorization and since a is a positive parameter, the aim in (Boussaada et al., 2018a) were to establish conditions on parameters such that the rightmost root of the second factor of (20) has a negative real part and the MID property for second order systems were exploited. Here, we propose the control law: $u(t) = \alpha\xi_1(t) + \beta\xi_2(t) + \gamma\xi_3(t)$ which gives in closed-loop the following quasipolynomial function $\Delta(s,\tau) =$

$$\begin{aligned} &s^3 + (2\,\zeta\omega - \omega^2\gamma + a)s^2 + \left((1-\beta-a\Gamma)\omega^2 + 2a\zeta\omega\right)s \\ &+ (1-\beta)\, a\omega^2 - \omega^2\alpha\, e^{-s\tau} ka \end{aligned} \tag{21}$$

Using Theorem 3 and assuming that $s_l = s_1 - (l-1)\, d$ for $l = 2, \ldots, 4$ one obtains the following values of the controller gains:

$$\begin{cases} \alpha = -3\, \dfrac{d^3 e^{-a\tau}}{\omega^2 ka},\ \beta = 1 + \dfrac{-a^2 - 3\, ad - 8\, d^2}{\omega^2}, \\ \gamma = 2\, \dfrac{\zeta}{\omega} + \dfrac{-2\, a - 3\, d}{\omega^2}, \end{cases} \tag{22}$$

as well as the precise value of the spectral abscissa and the distance between two successive assigned roots:

$$s_1 = -\frac{a\tau + \ln(2)}{\tau}, \quad d = \frac{\ln(2)}{\tau}.$$

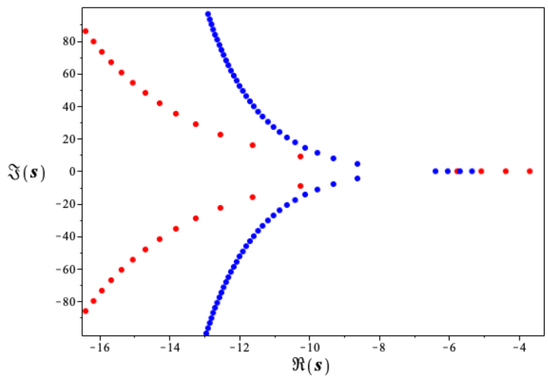

Fig. 4. The spectrum distribution of (21) satisfying (22) exhibiting the four-roots placement in two configurations. In blue, the spectrum corresponding to $a = 5, \tau = 2$. In red, the spectrum corresponding to $a = 3, \tau = 1$. In both cases, the dominancy of the four assigned equidistributed real rootsis underlined.

5. CONCLUSION

In this note, we extended some recent results by the authors on pole-placement for Time-delay systems. This new result emphasizes a new delayed controller-design based on the trivial solution's decay rate assignment. The potential applicability of the approach is illustrated through the regulation of Mach number in a wind tunnel. Further insights on the applicability of the presented method in damping active vibrations can be found in (Tliba et al., 2019).

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