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# Conditions on Shifted Passivity of Port-Hamiltonian Systems

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## Abstract

In this paper, we examine the shifted passivity property of port-Hamiltonian systems. Shifted passivity accounts for the fact that in many applications the desired steady-state values of the input and output variables are nonzero, and thus one is interested in passivity with respect to the shifted signals. We consider port-Hamiltonian systems with strictly convex Hamiltonian, and derive conditions under which shifted passivity is guaranteed. In case the Hamiltonian is quadratic and state dependency appears in an affine manner in the dissipation and interconnection matrices, our conditions reduce to negative semidefiniteness of an appropriately constructed constant matrix. Moreover, we elaborate on how these conditions can be extended to the case when the shifted passivity property can be enforced via output feedback, thus paving the path for controller design. Stability of forced equilibria of the system is analyzed invoking the proposed passivity conditions. The utility and relevance of the results are illustrated with their application to a 6th order synchronous generator model as well as a controlled rigid body system.

*Keywords:* Passivity, shifted passivity, incremental passivity, port-Hamiltonian systems, stability theory

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## 1. Introduction

Passive systems are a class of dynamical systems in which the rate at which the energy flows into the system is not less than the increase in storage. In other words, starting from any initial condition, only a finite amount of energy can be extracted from a passive system. This, together with the invariance under negative feedback interconnection, has promoted passivity as a basic building block for control of dynamical and interconnected systems. Interested readers are referred to [1, 2, 3] for a tutorial account of the applications of passivity in control theory.

Passivity of state-space systems is commonly defined as an input-output property for systems whose desired equilibrium state is the origin and the input and output variables are zero at this equilibrium [4, 1, 2]. If several such systems are interconnected—for instance, a plant with a controller—the origin is an equilibrium point of the overall system whose stability may be assessed using the tools of passivity theory. In many applications, however, the desired equilibrium is not at the origin and the input and output variables of the system take nonzero values at steady-state. A standard procedure to describe the dynamics in these cases is to generate a so-called incremental model with inputs and outputs the deviations with respect to their value at the equilibrium. A natural question that arises is whether passivity of the original system is inherited by its incremental model, a property that we refer in this paper as *shifted passivity*. Following [5], we use a shifted storage function to address this issue, see also the shaped Hamiltonian in [6]. This shifted function is closely related to the notion of availability function used in thermodynamics [7, 8]. A byproduct of the construction of shifted storage functions is a passivity property which is uniform for a range of steady-state solutions. This is particularly advantageous in flow networks, distribution, and

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electrical networks where loads/demands are not precisely known and are treated as constant disturbances, [9, 10, 11, 12, 13, 14]; see also [15, 16, 17] where the term “equilibrium-independent passivity” has been used to refer to the aforementioned uniform passivity property.

We study in this paper shifted passivity of port-Hamiltonian (pH) systems that, as is well-known, provide an attractive energy-based modeling framework for nonlinear physical systems [18, 19, 20]. The Hamiltonian readily serves as a storage function certifying passivity of a pH system, however, proving its shifted passivity is in general nontrivial. In [5] it is shown that pH systems with convex Hamiltonian are also shifted passive provided the input, dissipation and interconnection matrices are all constant. Conditions for shifted passivity of pH systems with state-dependent matrices have been reported in [6] and [21]. In the former case, quite conservative, integrability conditions, are imposed while the latter ones are too general and thus can be difficult to verify. The main contribution of the present paper is to give easily verifiable conditions—*i.e.*, monotonicity of a suitably defined function—to ensure shifted passivity of pH systems with strictly convex Hamiltonian and *state-dependent* dissipation and interconnection matrices. Similarly to [5] our candidate storage function is the shifted Hamiltonian, which is associated with the Bregman distance of the Hamiltonian with respect to an equilibrium of the system [22].

Notably, for the case of affine pH systems with quadratic Hamiltonian, our conditions reduce to negative semidefiniteness of an appropriately constructed *constant* matrix. The proposed conditions are exploited to certify local and global stability of *forced* pH systems, *i.e.*, under constant external inputs, see [6]. An additional contribution of our work is that the proposed conditions provide an estimate of the excess and shortage of passivity that serves as a tool for controller design, see *e.g.* [23].

The structure of the paper is as follows. The problem formulation is provided in Section 2. The main results are given in Section 3, and are specialized to quadratic affine pH systems in Section 4. The results are illustrated with a synchronous generator and a rigid body model in Section 5. The paper closes with conclusions in Section 6.

**Notation** All functions are assumed to be sufficiently smooth. For mappings  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\mathcal{C} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  we denote the transposed gradient as  $\nabla H := \left(\frac{\partial H}{\partial x}\right)^\top$  and the transposed Jacobian matrix as  $\nabla \mathcal{C} := \left(\frac{\partial \mathcal{C}}{\partial x}\right)^\top$ . The Jacobian  $(\nabla \mathcal{C}(\cdot))^\top$  is simply denoted by  $\nabla \mathcal{C}(\cdot)^\top$ . An  $n \times m$  matrix of zeros is denoted by  $0_{nm}$ . For a vector  $x \in \mathbb{R}^n$ , we denote its Euclidean norm by  $\|x\|$ .

## 2. Problem Formulation

Consider the pH system

$$\dot{x} = (J(x) - R(x))\nabla H(x) + Gu \tag{1a}$$

$$y = G^\top \nabla H(x), \tag{1b}$$

with state  $x \in \mathbb{R}^n$ , input  $u \in \mathbb{R}^m$ , and output  $y \in \mathbb{R}^m$ . The constant matrix  $G \in \mathbb{R}^{n \times m}$  has full column rank, and  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  is the Hamiltonian of the system. The matrix  $J$  is skew-symmetric, *i.e.*,  $J(x) + J^\top(x) = 0$ , and

$$R(x) \geq R^*, \quad \forall x \in \mathbb{R}^n \tag{2}$$

for some constant positive semidefinite matrix  $R^*$ .

Define the steady-state relation

$$\mathcal{E} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid (J(x) - R(x))\nabla H(x) + Gu = 0\}.$$

Fix  $(\bar{x}, \bar{u}) \in \mathcal{E}$  and the corresponding output  $\bar{y} := G^\top \nabla H(\bar{x})$ . We are interested in finding conditions under which the mapping  $(u - \bar{u}) \rightarrow (y - \bar{y})$  is passive. We refer to this property as shifted passivity, which is formally defined next:

**Definition 1** Consider the pH system (1). Let  $(\bar{x}, \bar{u}) \in \mathcal{E}$  and define  $\bar{y} := G^\top \nabla H(\bar{x})$ . The pH system (1) is *shifted passive* if the mapping  $(u - \bar{u}) \rightarrow (y - \bar{y})$  is passive, *i.e.*, there exists a function  $\mathcal{H} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\dot{\mathcal{H}} = (\nabla \mathcal{H})^\top \dot{x} \leq (u - \bar{u})^\top (y - \bar{y}) \quad (3)$$

for all  $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ .

**Remark 2** Note that shifted passivity is different from the classical incremental passivity property [24]. In fact, the latter is much more demanding as the word “incremental” refers to two arbitrary input-output pairs of the system, whereas in the former only one input-output pair is arbitrary and the other one is fixed to a constant.

### 3. Main Results

In this section, we provide our main results concerning shifted passivity, stability, and shifted feedback passivity of the pH system (1).

#### 3.1. Shifted passivity

Here, we provide conditions under which the pH system (1) is shifted passive in the sense of Definition 1. Towards this end, we make two assumptions:

**Assumption 1** The Hamiltonian  $H$  is strictly convex.

Given the strictly convex function  $H$  we define the *Legendre transform*, sometimes called Legendre-Fenchel transform, of  $H$  as the function

$$H^*(p) := \max_x \{x^\top p - H(x)\},$$

where the domain of  $H^*$  is the set of all  $p$  for which the expression is well-defined (*i.e.*, the maximum is attained). We list the following properties of the Legendre transform  $H^*$ ; see *e.g.* [25], [26].

1. The domain of  $H^*$  is equal to the convex range of  $\nabla H$ .
2.  $H^*$  is strictly convex.
3.  $H^{**} = H$ .
4.  $\nabla H^*(\nabla H(x)) = x$ , for all  $x$ .
5.  $\nabla H(\nabla H^*(p)) = p$ , for all  $p$  in the convex range of  $\nabla H$ .

Let  $F(x) := J(x) - R(x)$ . Leveraging the Legendre transform above, the function  $F(x)$  can be restated in terms of co-energy variables  $s := \nabla H(x)$  as

$$F(x) = F(\nabla H^*(s)) =: \mathcal{F}(s). \quad (4)$$

We denote the domain of  $H^*$ , which is equal to the range of  $\nabla H$ , by  $S$ . Let  $\bar{s} := \nabla H(\bar{x})$ . We impose the following assumption on  $\mathcal{F}$ :

**Assumption 2** The mapping  $\mathcal{F}$  verifies

$$\nabla(\mathcal{F}(s)\bar{s}) + \nabla(\mathcal{F}(s)\bar{s})^\top - 2R^* \leq 0, \quad \forall s \in S. \quad (5)$$

Note that the choice of  $R^*$  is important in feasibility of (5), and is best to choose the lower bound in (2) as tight as possible. Now, we have the following result:

**Proposition 3** *Let Assumptions 1 and 2 hold. Then, the pH system (1) is shifted passive, namely*

$$\dot{\mathcal{H}} \leq (u - \bar{u})^\top (y - \bar{y}) \quad (6)$$

is satisfied with

$$\mathcal{H}(x) := H(x) - (x - \bar{x})^\top \nabla H(\bar{x}) - H(\bar{x}). \quad (7)$$

**Proof.** First, note that  $\mathcal{H}$  is nonnegative as the Hamiltonian  $H$  is (strictly) convex [22, 5]. Substituting (4) into (1) yields

$$\dot{x} = \mathcal{F}(s)s - \mathcal{F}(\bar{s})\bar{s} + G(u - \bar{u}),$$

where we have subtracted  $0 = \mathcal{F}(\bar{s})\bar{s} + G\bar{u}$ . Noting that  $\nabla \mathcal{H}(x) = \nabla H(x) - \nabla H(\bar{x})$ , the time derivative of  $\mathcal{H}(x)$  is computed as

$$\begin{aligned} \dot{\mathcal{H}} &= (\nabla \mathcal{H})^\top \dot{x} = (s - \bar{s})^\top (\mathcal{F}(s)s - \mathcal{F}(\bar{s})\bar{s}) + (y - \bar{y})^\top (u - \bar{u}) \\ &= (s - \bar{s})^\top (\mathcal{F}(s) - \mathcal{F}(\bar{s}))\bar{s} \\ &\quad + (s - \bar{s})^\top F(x) (s - \bar{s}) + (y - \bar{y})^\top (u - \bar{u}) \\ &\leq (s - \bar{s})^\top (\mathcal{F}(s) - \mathcal{F}(\bar{s}))\bar{s} \\ &\quad - (s - \bar{s})^\top R^* (s - \bar{s}) + (y - \bar{y})^\top (u - \bar{u}), \end{aligned} \quad (8)$$

where we used (1b) in the first identity, added and subtracted the term  $(s - \bar{s})^\top (\mathcal{F}(s)\bar{s})$  and used (4) to write the second equality, while the bound is obtained invoking (2). Now, let

$$\mathcal{M}(s) := \mathcal{F}(s)\bar{s} - R^* s. \quad (9)$$

Then  $\dot{\mathcal{H}}$  can be written as

$$\dot{\mathcal{H}} = (s - \bar{s})^\top (\mathcal{M}(s) - \mathcal{M}(\bar{s})) + (y - \bar{y})^\top (u - \bar{u}). \quad (10)$$

By (5), we have that  $\nabla \mathcal{M}(s) + (\nabla \mathcal{M}(s))^\top \leq 0$ , for all  $s \in \mathcal{S}$ , which ensures that the map  $\mathcal{M}(\cdot)$  is monotone [27]. The proof is completed noting that by monotonicity

$$(s - \bar{s})^\top (\mathcal{M}(s) - \mathcal{M}(\bar{s})) \leq 0. \quad \blacksquare$$

**Remark 4** By Assumptions 1 and 2, both the strict convexity and the monotonicity property must hold for the whole sets  $\mathbb{R}^n$  and  $\mathcal{S}$ , respectively, which results in “global” shifted passivity of (1). For local shifted passivity,<sup>1</sup> we can restrict to a subset  $\mathcal{X} \subseteq \mathbb{R}^n$ , with Assumptions 1 and 2 modified to

1. The Hamiltonian is strictly convex in  $\mathcal{X} \subseteq \mathbb{R}^n$ .
2. Inequality (5) holds for all  $s \in \mathcal{S} := \{\nabla H(x) \mid x \in \mathcal{X}\}$ ,

while  $R^*$  is any matrix satisfying, instead of (2),  $R(x) \geq R^*$ ,  $\forall x \in \mathcal{X}$ .

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<sup>1</sup>By “local” we mean that there exist open neighborhoods  $\mathcal{X} \subseteq \mathbb{R}^n$  and  $\mathcal{U} \subseteq \mathbb{R}^m$  of  $(\bar{x}, \bar{u}) \in \mathcal{X} \times \mathcal{U}$  such that (6) holds for all  $(x, u) \in \mathcal{X} \times \mathcal{U}$ .

### 3.2. Stability of the forced equilibria

Lyapunov stability of the equilibrium of (1) with  $u = \bar{u}$ , immediately follows from Proposition 3, with the Lyapunov function being the shifted Hamiltonian  $\mathcal{H}$ . Moreover, asymptotic stability follows by imposing the condition that  $\dot{\mathcal{H}}$  is negative definite. Below, we provide the results concerning stability of the forced pH-system (1a) with  $u = \bar{u}$ . Although deducing stability properties from passivity is well-known [1], we provide the proof for the sake of completeness.

**Proposition 5** *Consider the pH system (1a) for some constant input  $u = \bar{u}$ , and let  $(\bar{x}, \bar{u}) \in \mathcal{E}$ . Then, we have*

1. *The equilibrium is asymptotically stable if  $\nabla^2 H(\bar{x}) > 0$  and there exists  $\epsilon > 0$  such that the inequality*

$$\nabla(\mathcal{F}(s)\bar{s}) + \nabla(\mathcal{F}(s)\bar{s})^\top - 2R^* \leq -2\epsilon I_n, \quad (11)$$

*holds at  $s = \bar{s}$ .<sup>2</sup>*

2. *The equilibrium is globally asymptotically stable if the Hamiltonian  $H$  is strongly convex and (11) holds for all  $s \in \mathbb{R}^n$ .*

**Proof.** Set  $u = \bar{u}$  and take the shifted Hamiltonian  $\mathcal{H}$  as the Lyapunov candidate, and suppose that (11) holds at the point  $s = \bar{s}$ . Then there exists an open neighborhood  $\mathcal{S}$  of  $\bar{s}$  and some  $0 < \epsilon' \leq \epsilon$  such that

$$\nabla(\mathcal{F}(s)\bar{s}) + \nabla(\mathcal{F}(s)\bar{s})^\top - 2R^* \leq -2\epsilon' I_n, \quad (12)$$

for all  $s \in \mathcal{S}$ . Let  $\mathcal{X} := \{x \mid \nabla H(x) \in \mathcal{S}\}$ . By  $\nabla H^*(\bar{s}) = \bar{x}$ , and continuity of  $\nabla H^*$ <sup>3</sup>, the set  $\mathcal{X} \in \mathbb{R}^n$  defines an open neighborhood of the equilibrium  $\bar{x}$ . It is easy to see that (12) implies (local) strong monotonicity of the map  $\mathcal{M}$  in (9), see [27], and the dissipation inequality (10) gives

$$\dot{\mathcal{H}} = -\epsilon' \|s - \bar{s}\|^2, \quad \forall x \in \mathcal{X}.$$

Now, noting that  $H$  is locally strictly convex, the function  $S$  is locally nonnegative and is equal to zero whenever  $x = \bar{x}$  [22]. Hence, there exists a compact subset of  $\mathcal{X}$  which is forward invariant along the solutions of the system (see also [5, Prop. 2]). By invoking LaSalle's invariance principle, on the invariant set we have  $s = \bar{s}$ , which results in  $x = \bar{x}$  by strict convexity of  $H$ .

To prove the second statement, it suffices to show that  $S$  is radially unbounded. This follows from strong convexity of  $H$  noting that [28, Ch.2]

$$\mathcal{H}(x) = H(x) - (x - \bar{x})^\top \nabla H(\bar{x}) - H(\bar{x}) \geq \mu \|x - \bar{x}\|^2,$$

for some  $\mu \in \mathbb{R}^+$ . ■

**Remark 6** The identity matrix in the right hand side of (11) can be replaced by a positive semidefinite matrix  $C^\top C$ , with  $C \in \mathbb{R}^{m \times n}$ , if the equilibrium is “observable” from the input-output pair  $(\bar{u}, C\nabla H(\bar{x}))$ , namely if

$$\dot{x} = F(x)\nabla H + g\bar{u}, \quad C\nabla H(x) = C\nabla H(\bar{x}) \implies x = \bar{x}.$$

<sup>2</sup>This means that the Jacobian of  $\mathcal{F}(s)\bar{s}$  in (11) has to be evaluated at  $s = \bar{s}$ .

<sup>3</sup>The map  $\nabla H^*$  is still well-defined as  $H$  is locally strictly convex, *i.e.*,  $\nabla^2 H(\bar{x}) > 0$ , and  $\mathcal{S}$  can be chosen as a convex set.

### 3.3. Enforcing shifted passivity via output feedback

We complete this section by considering the case where the condition (5) does not hold, which means that the system (1) may not be shifted passive, but it can be rendered shifted passive via output feedback.

**Proposition 7** *Consider the pH system (1) verifying Assumption 1 and such that*

$$\nabla(\mathcal{F}(s)\bar{s}) + \nabla(\mathcal{F}(s)\bar{s})^\top - 2R^* \leq 2\gamma GG^\top,$$

for some  $\gamma \in \mathbb{R}$ . Then, the shifted Hamiltonian (7) satisfies the following dissipation inequality

$$\dot{\mathcal{H}} \leq (u - \bar{u})^\top (y - \bar{y}) + \gamma \|y - \bar{y}\|^2.$$

**Proof.** The proof is analogous to that of Proposition 3, by adding and subtracting the term  $\gamma GG^\top (s - \bar{s})$  in (8), and modifying the map  $\mathcal{M}$  as

$$\widetilde{\mathcal{M}}(s) := \mathcal{M}(s) - \gamma GG^\top s.$$

■

**Remark 8** Note that a negative  $\gamma$  proves that the pH system is (output-strictly) shifted passive. On the other hand, a positive  $\gamma$  indicates the shortage of shifted passivity. Notice that the simple proportional controller

$$u = \bar{u} - K_P(y - \bar{y}) + v,$$

with  $K_P \geq \gamma I$ , ensures that the interconnected system is passive from the external input  $v$  to output  $y - \bar{y}$ . Analogously, Proposition 7 can be used to design dynamic passive controllers to stabilize the closed-loop system, see [23] for an application to control of permanent magnet synchronous motors.

## 4. Application to Quadratic Affine Systems

In this section we specialize our results to the case where

$$F(x) = F_0 + \sum_{i=1}^n F_i x_i, \quad (13)$$

with  $F_j \in \mathbb{R}^{n \times n}$ ,  $j = 0, \dots, n$ , constant and

$$H(x) = \frac{1}{2} x^\top Q x, \quad (14)$$

with  $Q \in \mathbb{R}^{n \times n}$  being positive definite. We call these systems quadratic affine pH systems.

In order to satisfy (2) and state the global version of our results, we need to assume that  $R(x) = R_0$  for some constant matrix  $R_0$ . This is due to the fact that in the affine case the inequality  $R(x) \geq R^*$ , for all  $x \in \mathbb{R}^n$ , implies that the matrix  $R$  is constant. In Remark 12, we elaborate on how this assumption is relaxed to obtain local results. Note that, in this case,  $F_0 + F_0^\top = -2R_0 \leq 0$  and  $F_j + F_j^\top = 0$  for each  $j \geq 1$ .

**Proposition 9** *Consider the quadratic affine pH system (1) with (13) and (14). Fix  $(\bar{x}, \bar{u}) \in \mathcal{E}$  and define the  $n \times n$  constant matrix*

$$B := \sum_{i=1}^n F_i Q \bar{x} e_i^\top Q^{-1}, \quad (15)$$

with  $e_i \in \mathbb{R}$  the  $i$ -th element of the orthogonal basis. If

$$B + B^\top - 2R_0 \leq 0, \quad (16)$$

then (1) is shifted passive, namely

$$\dot{\mathcal{H}} \leq (y - \bar{y})^\top (u - \bar{u}),$$

where  $\mathcal{H}$  is the quadratic shifted Hamiltonian function

$$\mathcal{H}(x) := \frac{1}{2}(x - \bar{x})^\top Q(x - \bar{x}).$$

**Proof.** The proof follows by verifying the conditions of Proposition 3. In this case,  $H^*(p) = \frac{1}{2}p^\top Q^{-1}p$ ,

$$\nabla H^*(s) = Q^{-1}s, \quad s = Qx,$$

and

$$\mathcal{F}(s) = F_0 + \sum_{i=1}^n F_i(e_i^\top Q^{-1}s). \quad (17)$$

Hence,

$$\mathcal{F}(s)\bar{s} = (F_0 + \sum_{i=1}^n F_i(e_i^\top Q^{-1}s))Q\bar{x}.$$

Now, by rewriting the last expression in the equivalent form

$$\mathcal{F}(s)\bar{s} = F_0Q\bar{x} + \sum_{i=1}^n F_iQ\bar{x}e_i^\top Q^{-1}s,$$

we obtain that

$$\nabla(\mathcal{F}(s)\bar{s}) = \sum_{i=1}^n F_iQ\bar{x}e_i^\top Q^{-1}.$$

Finally, using (15), condition (5) takes the form

$$\nabla(\mathcal{F}(s)\bar{s}) + \nabla(\mathcal{F}(s)\bar{s})^\top - 2R_0 = B + B^\top - 2R_0 \leq 0. \quad \blacksquare$$

**Remark 10** The stability condition in Proposition 9 can equivalently be stated in terms of the co-energy variables  $s = \nabla H(x)$ , which in certain cases decreases the computational effort. To this end, note that by (17), the function  $\mathcal{F}(s)$  can be written in the affine form:

$$\mathcal{F}(s) = F_0 + \sum_{i=1}^n \mathcal{F}_i s_i,$$

where  $\mathcal{F}_i := \sum_{j=1}^n F_j Q_{ij}^{-1}$ . Hence, the matrix  $B$  in (15) can be equivalently written as

$$B := \sum_{i=1}^n \mathcal{F}_i Q\bar{x}e_i^\top.$$

Noting that  $H$  is strongly convex and the condition (16) is state independent, Proposition 5 yields the following result:

**Corollary 11** Consider the quadratic affine pH system (1a) with (13) and (14) under some constant input  $u = \bar{u}$ , and let  $(\bar{x}, \bar{u}) \in \mathcal{E}$ . The equilibrium  $\bar{x}$  is globally asymptotically stable if (16) holds with strict inequality.

**Remark 12** Analogous to the previous section, local variations of Proposition 9 and Corollary 11 can be obtained by restricting  $x$  in a domain  $\mathcal{X} \in \mathbb{R}^n$  with  $\bar{x} \in \mathcal{X}$ . In that case, the matrix  $R_0$  in (16) is replaced by  $R^*$ , where  $R(x) \geq R^* \geq 0$  for all  $x \in \mathcal{X}$ .

## 5. Case Studies

In this section, we apply the proposed method to two physical systems. Both systems are affine and have quadratic Hamiltonian.

### 5.1. Synchronous generator (6<sup>th</sup>-order model) connected to a resistor

The state variables of the six-dimensional model of the synchronous generator comprise of the stator fluxes on the  $dq$  axes  $\psi_d \in \mathbb{R}, \psi_q \in \mathbb{R}$ , rotor fluxes  $\psi_r \in \mathbb{R}^3$  (the first component of  $\psi_r$  corresponds to the field winding and the remaining two to the damper windings), and the angular momentum of the rotor  $p$ . The Hamiltonian  $H$  (total stored energy of the synchronous generator) is the sum of the magnetic energy of the generator and the kinetic energy of the rotating rotor. More precisely, the Hamiltonian takes the form  $H(x) = \frac{1}{2}x^\top Qx$  with  $x = [\psi_d \ \psi_q \ \psi_r \ p]^\top$  and

$$Q = \begin{bmatrix} L^{-1} & 0_{51} \\ 0_{15} & m^{-1} \end{bmatrix} > 0, \quad L = \begin{bmatrix} L_d & 0 & kL_{afd} & kL_{akd} & 0 \\ 0 & L_q & 0 & 0 & -kL_{akq} \\ kL_{afd} & 0 & L_{ffd} & L_{akd} & 0 \\ kL_{akd} & 0 & L_{akd} & L_{kkd} & 0 \\ 0 & -kL_{akq} & 0 & 0 & L_{kkq} \end{bmatrix},$$

where  $m \in \mathbb{R}$  is the total moment of inertia of the turbine and the rotor. Note that the elements of the inductance matrix  $L$  are all constant parameters, see [29] for more details. The system dynamics is then given by the pH system [29]

$$\dot{x} = (J(x) - R)\nabla H(x) + G \begin{bmatrix} V_f \\ \tau \end{bmatrix},$$

with

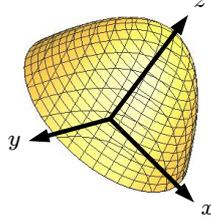
$$J(x) = \begin{bmatrix} 0_{22} & 0_{23} & \begin{bmatrix} -\psi_q \\ \psi_d \end{bmatrix} \\ 0_{32} & 0_{33} & 0_{31} \\ \begin{bmatrix} \psi_q & -\psi_d \end{bmatrix} & 0_{13} & 0 \end{bmatrix}, \quad R = \begin{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} & 0_{23} & 0_{21} \\ 0_{32} & \begin{bmatrix} R_f & 0 & 0 \\ 0 & R_{kd} & 0 \\ 0 & 0 & R_{kq} \end{bmatrix} & 0_{31} \\ 0_{12} & 0_{13} & d \end{bmatrix} > 0, \quad G = \begin{bmatrix} 0_{21} & 0_{21} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & 0_{31} \\ 0 & 1 \end{bmatrix},$$

where  $V_f$  represents the rotor field winding voltage,  $\tau$  is the mechanical torque,  $r$  is the summation of the load and stator resistances,  $R_f, R_{kd}, R_{kq}$  denote the rotor resistances, and  $d$  corresponds to the mechanical friction. We can rewrite the system as

$$Q^{-1}\dot{s} = (\mathcal{J}(s) - R)s + G \begin{bmatrix} V_f \\ \tau \end{bmatrix}, \quad (18)$$

where  $s = Qx = [I_d \ I_q \ I_r \ \omega]^\top$ . Here  $I_d \in \mathbb{R}, I_q \in \mathbb{R}$  are the components of the stator current on the  $dq$  axes, and  $I_r \in \mathbb{R}^3$  and  $\omega \in \mathbb{R}$  are the currents and angular velocity of the rotor, respectively. Note that

$$\mathcal{J}(s) = \begin{bmatrix} 0_{22} & 0_{23} & v_J(s) \\ 0_{32} & 0_{33} & 0_{31} \\ v_J^\top(s) & 0_{13} & 0 \end{bmatrix}, \quad v_J(s) := \begin{bmatrix} -L_q I_q + L_{akq} I_{kq} \\ L_d I_d + L_{afd} I_f + L_{akd} I_{kd} \end{bmatrix}.$$



**Figure 1:** A rigid body with three axes of rotation

Let  $V_f = \bar{V}_f$  and  $\tau = \bar{\tau}$ , for some constant vectors  $\bar{V}_f$  and  $\bar{\tau}$ . Through straightforward calculations, and using Remark 10, the condition (16) reads as

$$\begin{bmatrix} -2r & \bar{\omega}(L_d - L_q) & 0 & 0 & k\bar{\omega}L_{akq} & -\bar{I}_q L_d \\ \bar{\omega}(L_d - L_q) & -2r & k\bar{\omega}L_{afd} & k\bar{\omega}L_{akd} & 0 & \bar{I}_d L_d \\ 0 & k\bar{\omega}L_{afd} & -2R_f & 0 & 0 & -\bar{I}_q L_{afd} \\ 0 & k\bar{\omega}L_{akd} & 0 & -2R_{kd} & 0 & -\bar{I}_q L_{akd} \\ k\bar{\omega}L_{akq} & 0 & 0 & 0 & -2R_{kq} & -\bar{I}_d L_{akq} \\ -\bar{I}_q L_d & \bar{I}_d L_d & -\bar{I}_q L_{afd} & -\bar{I}_q L_{akd} & -\bar{I}_d L_{akq} & -2d \end{bmatrix} \leq 0, \quad (19)$$

where  $\bar{I}_d, \bar{I}_q, \bar{I}_r, \bar{\omega}$  are the associated values of  $s$  at the equilibrium of (18), *i.e.*

$$(\mathcal{J}(\bar{s}) - R)\bar{s} + G \begin{bmatrix} \bar{V}_f \\ \bar{\tau} \end{bmatrix} = 0,$$

with  $\bar{s} = [\bar{I}_d \ \bar{I}_q \ \bar{I}_r \ \bar{\omega}]^\top$ . Hence, by Proposition 9, (18) is shifted passive if (19) holds. Moreover, by Corollary 11, if (19) holds with strict inequality, then the equilibrium  $\bar{x} = Q^{-1}\bar{s}$  is globally asymptotically stable. The stability result is consistent with those of [30, 31]<sup>4</sup>. Note that Corollary 11 is valid for a general quadratic affine pH-system, and the condition (19) is obtained in a systematic manner here, namely by verifying the negative definiteness test in (16). Moreover, if (19) does not hold, then in view of Proposition 7, one can investigate the possibility of designing suitable proportional, PI, or more generally dynamic (input-strictly) passive controllers rendering the equilibrium globally asymptotically stable.

## 5.2. Controlled rigid body under constant disturbances

The equations for the angular momentum of a rigid body (see Figure 1) with external torque  $u \in \mathbb{R}^3$  and disturbance  $d \in \mathbb{R}^3$  reads as [19]

$$\dot{p} = J(p)\nabla H(p) + u + d \quad (20)$$

$$y = \nabla H(p), \quad (21)$$

where  $p = [p_x \ p_y \ p_z]^\top$ ,

$$J(p) = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix},$$

and the Hamiltonian is given by  $H(p) = \frac{1}{2}p^\top M p$ , with

$$M = \begin{bmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & m_z \end{bmatrix} > 0.$$

<sup>4</sup>Notice that there is a typo in [31] as the term  $\bar{\omega}(L_d - L_q)$  is missing.

Here,  $m_x \in \mathbb{R}$ ,  $m_y \in \mathbb{R}$ , and  $m_z \in \mathbb{R}$  are the principal moments of inertia. Consider a constant disturbance  $d = [d_x \ d_y \ d_z]^\top$  and a proportional controller  $u = -Ry$  with  $R = \text{diag}(r_x, r_y, r_z) > 0$ . We can rewrite the system as

$$M\dot{\omega} = \mathcal{J}(\omega) - R\omega + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}. \quad (22)$$

where

$$\mathcal{J}(\omega) = \begin{bmatrix} 0 & -\omega_z m_z & \omega_y m_y \\ \omega_z m_z & 0 & -\omega_x m_x \\ -\omega_y m_y & \omega_x m_x & 0 \end{bmatrix},$$

and  $\omega = M^{-1}p = [\omega_x \ \omega_y \ \omega_z]^\top$  is the vector of angular velocities around the axes  $x$ ,  $y$ , and  $z$ . The point  $(\bar{\omega}_x, \bar{\omega}_y, \bar{\omega}_z)$  is an equilibrium of the system (22) satisfying

$$\mathcal{J}(\bar{\omega}) - R\bar{\omega} + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = 0,$$

with  $\bar{\omega} = [\bar{\omega}_x \ \bar{\omega}_y \ \bar{\omega}_z]^\top$ . Through straightforward calculations, the condition (16) (with strict inequality) reads as

$$\begin{bmatrix} -2r_x & \bar{\omega}_z(m_y - m_x) & \bar{\omega}_y(m_x - m_z) \\ \bar{\omega}_z(m_y - m_x) & -2r_y & \bar{\omega}_x(m_z - m_y) \\ \bar{\omega}_y(m_x - m_z) & \bar{\omega}_x(m_z - m_y) & -2r_z \end{bmatrix} < 0. \quad (23)$$

Hence, by Corollary 11, if (23) holds for the equilibrium point  $(\bar{\omega}_x, \bar{\omega}_y, \bar{\omega}_z)$ , then global asymptotic stability is guaranteed. In the case that there is disturbance actuating only on one axis, *e.g.*  $d_y = d_z = 0$  (without loss of generality), the equilibrium  $(\omega_x, \omega_y, \omega_z) = (\frac{d_x}{r_x}, 0, 0)$  is globally asymptotically stable if

$$r_x^2 r_y r_z > \left( \frac{d_x(m_z - m_y)}{2} \right)^2.$$

## 6. Conclusion

We have examined the shifted passivity property of pH systems with convex Hamiltonian by proposing conditions in terms of the monotonicity of suitably constructed functions. We have leveraged these conditions to study (global) asymptotic stability of forced equilibria of the system. As we observed, for quadratic affine pH system, shifted passivity and (global) asymptotic stability are guaranteed if an appropriately constructed constant matrix is negative semidefinite. We demonstrated the applicability and usefulness of the results on a 6th order synchronous generator model and a controlled rigid body system. Future works include attempting to reduce possible conservatism in the stability conditions as well as investigating the connections of the proposed results to contraction and differential passivity [32, 33].

## 7. References

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