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## Comments on “Automatic Target Detection for Sparse Hyperspectral Images” by Ahmad W. Bitar et al.

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Comments on «Automatic Target Detection for Sparse Hyperspectral  
Images [1]» by Ahmad W. Bitar et al.

AMERICAN UNIVERSITY OF BEIRUT

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# Technical Report

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## Summary

In this technical report, we explain how our proposed sparse and low-rank matrix decomposition method for hyperspectral target detection, provided in our work «Automatic Target Detection for Sparse Hyperspectral Images [1]», can be extended to the  $l_q$  norm ( $0 < q \leq 1$ ). Since the use of the  $l_1$  norm is still too far away from the ideal  $l_0$  norm, many non-convex regularizers, interpolated between the  $l_0$  norm and the  $l_1$  norm, have been proposed to better approximate the  $l_0$  norm.



## Main Notations

Throughout this report, we depict vectors in lowercase boldface letters and matrices in uppercase boldface letters. The notation  $(\cdot)^T$  and  $\text{Tr}(\cdot)$  stand for the transpose and trace of a matrix, respectively. In addition,  $\text{rank}(\cdot)$  is for the rank of a matrix. A variety of norms on matrices will be used. For instance,  $\mathbf{M}$  is a matrix, and  $[\mathbf{M}]_{:,j}$  is the  $j$ th column. The matrix  $l_{2,0}$  and  $l_{2,q}$  ( $0 < q \leq 1$ ) norms are defined by  $\|\mathbf{M}\|_{2,0} = \#\{j : \|[ \mathbf{M} ]_{:,j} \|_2 \neq 0\}$ , and  $\|\mathbf{M}\|_{2,q} = \left( \sum_j \|[ \mathbf{M} ]_{:,j} \|_2^q \right)^{(1/q)}$ , respectively. The Frobenius norm and the nuclear norm (the sum of singular values of a matrix) are denoted by  $\|\mathbf{M}\|_F$  and  $\|\mathbf{M}\|_* = \text{Tr}(\mathbf{M}^T \mathbf{M})^{(1/2)}$ , respectively.

# 1 Main contribution

Consider the following minimization problem:

$$\min_{\mathbf{L}, \mathbf{C}} \left\{ \tau \text{rank}(\mathbf{L}) + \lambda \|\mathbf{C}\|_{2,0} + \|\mathbf{D} - \mathbf{L} - (\mathbf{A}_t \mathbf{C})^T\|_F^2 \right\}, \quad (1)$$

where  $\mathbf{D}, \mathbf{L}, (\mathbf{A}_t \mathbf{C})^T \in \mathbb{R}^{e \times p}$ ,  $\mathbf{A}_t \in \mathbb{R}^{p \times N_t}$ ,  $\mathbf{C} \in \mathbb{R}^{N_t \times e}$ ,  $\tau$  controls the rank of  $\mathbf{L}$ , and  $\lambda$  the sparsity level in  $\mathbf{C}$ .

We relax the rank term and the  $l_{2,0}$  norm to their convex proxies [1, 2, 3, 4, 5]. More precisely, we use the nuclear norm  $\|\mathbf{L}\|_*$  as a surrogate for the  $\text{rank}(\mathbf{L})$  term, and the  $l_{2,1}$  norm  $\|\mathbf{C}\|_{2,1}$  as a surrogate for the  $l_{2,0}$  norm  $\|\mathbf{C}\|_{2,0}$ .

$$\min_{\mathbf{L}, \mathbf{C}} \left\{ \tau \|\mathbf{L}\|_* + \lambda \|\mathbf{C}\|_{2,1} + \|\mathbf{D} - \mathbf{L} - (\mathbf{A}_t \mathbf{C})^T\|_F^2 \right\}. \quad (2)$$

Problem (2) can be re-written as

$$\min_{\mathbf{L}, \mathbf{C}} \left\{ \tau \sum_{i=1}^{\min(e,p)} \sigma_i(\mathbf{L}) + \lambda \sum_{j=1}^e \|\mathbf{C}_{:,j}\|_2 + \|\mathbf{D} - \mathbf{L} - (\mathbf{A}_t \mathbf{C})^T\|_F^2 \right\}, \quad (3)$$

where  $\{\sigma_i\}_{i=1}^{\min(e,p)}$  are the singular values of  $\mathbf{L}$ .

## Extension to the $l_q$ norm ( $0 < q \leq 1$ )

We replace the nuclear norm and the  $l_{2,1}$  norm by their  $q$ -norm proxies in Eq. (2), with  $0 < q \leq 1$ . More precisely [6, 7, 8]

$$\min_{\mathbf{L}, \mathbf{C}} \left\{ \tau \sum_{i=1}^{\min(e,p)} (\sigma_i(\mathbf{L}) + \epsilon)^q + \lambda \sum_{j=1}^e \|\mathbf{C}_{:,j}\|_2^q + \|\mathbf{D} - \mathbf{L} - (\mathbf{A}_t \mathbf{C})^T\|_F^2 \right\}, \quad (4)$$

where  $0 < \epsilon \ll 1$ . Problem (4) is recasted into two sub-problems, and thus, at each iteration  $k$  we have

$$\min_{\mathbf{L}} \left\{ \left\| \mathbf{L} - \left( \mathbf{D} - (\mathbf{A}_t \mathbf{C}^{(k-1)})^T \right) \right\|_F^2 + \tau \sum_{i=1}^{\min(e,p)} (\sigma_i(\mathbf{L}) + \epsilon)^q \right\}, \quad (5a)$$

$$\min_{\mathbf{C}} \left\{ \left\| (\mathbf{D} - \mathbf{L}^{(k)})^T - \mathbf{A}_t \mathbf{C} \right\|_F^2 + \lambda \sum_{j=1}^e \|\mathbf{C}_{:,j}\|_2^q \right\}. \quad (5b)$$

### 1.1 Providing an optimal solution to sub-problem (5a)

For ease of notation, we consider the matrix  $\mathbf{E}^{(k-1)} = (\mathbf{A}_t \mathbf{C}^{(k-1)})^T$ . Let us suppose  $g_i(\mathbf{L}) = \sigma_i(\mathbf{L}) + \epsilon$ ,  $f(g_i(\mathbf{L})) = (\sigma_i(\mathbf{L}) + \epsilon)^q$ , and  $h(l_{m,j}) = \left( l_{m,j} - (d_{m,j} - e_{m,j}^{(k-1)}) \right)^2$ , with  $i \in [1, \min(e, p)]$ ,  $j \in [1, e]$ , and  $m \in [1, p]$ .

The function  $f(g_i(\mathbf{L}))$  is concave, and thus,  $-f(g_i(\mathbf{L}))$  is convex. According to the definition of the subgradient of a convex function, we can write [6, 7, 8]

$$-f(g_i(\mathbf{L})) \geq -f(g_i(\mathbf{L}^{(k-1)})) + \langle -w_i^{(k-1)}, g_i(\mathbf{L}) - g_i(\mathbf{L}^{(k-1)}) \rangle, \quad (6)$$

with  $-w_i^{(k-1)} = \partial(-f(g_i(\mathbf{L}^{(k-1)})))$  or  $w_i^{(k-1)} = -\partial(-f(g_i(\mathbf{L}^{(k-1)})))$ .

We can re-write Eq. (6) as

$$f(g_i(\mathbf{L})) \leq f(g_i(\mathbf{L}^{(k-1)})) + \langle w_i^{(k-1)}, g_i(\mathbf{L}) - g_i(\mathbf{L}^{(k-1)}) \rangle. \quad (7)$$

The loss function  $h(l_{m,j})$  has a Lipschitz continuous gradient, and thus, we can surrogate it as

$$h(l_{m,j}) \leq h(l_{m,j}^{(k-1)}) + \langle \nabla h(l_{m,j}^{(k-1)}), l_{m,j} - l_{m,j}^{(k-1)} \rangle + \frac{\mu}{2} (l_{m,j} - l_{m,j}^{(k-1)})^2, \quad (8)$$

with  $\mu > 0$ . By combining Eqs. (7) and (8), the sub-problem (5a) is approximated as

$$\begin{aligned} & \min_{\mathbf{L}} \left\{ \tau \sum_{i=1}^{\min(e,p)} \left[ (\sigma_i(\mathbf{L}^{(k-1)}) + \epsilon)^q + \langle w_i^{(k-1)}, \sigma_i(\mathbf{L}) - \sigma_i(\mathbf{L}^{(k-1)}) \rangle \right] \right. \\ & \quad \left. + \sum_{m=1}^p \sum_{j=1}^e \left[ h(l_{m,j}^{(k-1)}) + \langle \nabla h(l_{m,j}^{(k-1)}), l_{m,j} - l_{m,j}^{(k-1)} \rangle + \frac{\mu}{2} (l_{m,j} - l_{m,j}^{(k-1)})^2 \right] \right\}, \\ & \implies \min_{\mathbf{L}} \left\{ \tau \sum_{i=1}^{\min(e,p)} \left[ \langle w_i^{(k-1)}, \sigma_i(\mathbf{L}) \rangle \right. \right. \\ & \quad \left. \left. + \frac{\mu}{2} \sum_{m=1}^p \sum_{j=1}^e \left[ l_{m,j} - \left( l_{m,j}^{(k-1)} - \frac{1}{\mu} \nabla h(l_{m,j}^{(k-1)}) \right) \right]^2 \right] \right\}, \\ & \implies \min_{\mathbf{L}} \left\{ \tau \sum_{i=1}^{\min(e,p)} \left[ \langle w_i^{(k-1)}, \sigma_i(\mathbf{L}) \rangle \right. \right. \\ & \quad \left. \left. + \frac{\mu}{2} \sum_{m=1}^p \sum_{j=1}^e \left[ l_{m,j} - \left( l_{m,j}^{(k-1)} - \frac{2}{\mu} (l_{m,j}^{(k-1)} - (d_{m,j} - e_{m,j}^{(k-1)})) \right) \right]^2 \right] \right\}, \\ & \implies \min_{\mathbf{L}} \left\{ \tau \sum_{i=1}^{\min(e,p)} \left[ \langle w_i^{(k-1)}, \sigma_i(\mathbf{L}) \rangle \right. \right. \\ & \quad \left. \left. + \frac{\mu}{2} \left\| \mathbf{L} - \left( \mathbf{L}^{(k-1)} - \frac{2}{\mu} (\mathbf{L}^{(k-1)} - (\mathbf{D} - \mathbf{E}^{(k-1)})) \right) \right\|_F^2 \right] \right\}, \quad (9) \end{aligned}$$

with  $w_i^{(k-1)} = -\partial(-(\sigma_i(\mathbf{L}^{(k-1)}) + \epsilon)^q) = q(\sigma_i(\mathbf{L}^{(k-1)}) + \epsilon)^{q-1} = \frac{q}{(\sigma_i(\mathbf{L}^{(k-1)}) + \epsilon)^{1-q}}$ .

Let us consider that  $\mathbf{X}^{(k-1)} = \mathbf{L}^{(k-1)} - \frac{2}{\mu} \left( \mathbf{L}^{(k-1)} - (\mathbf{D} - \mathbf{E}^{(k-1)}) \right)$ . Given  $\mathbf{X}^{(k-1)} \in \mathbb{R}^{e \times p}$ ,  $0 \leq w_1^{(k-1)} \leq \dots \leq w_{\min(e,p)}^{(k-1)}$ , and according to Theorem 2.3 in [9], the global optimal “unique” solution (if  $\mathbf{X}^{(k-1)}$  has a unique singular value decomposition (SVD)) to the above optimization problem (9) is given by the adaptive SVD soft-thresholding operator

$$\mathbf{L}^{(k)} = \mathcal{S}_{\frac{\tau \mathbf{w}^{(k-1)}}{\mu}}(\mathbf{X}^{(k-1)}) = \mathbf{U}^{(k-1)} \mathcal{S}_{\frac{\tau \mathbf{w}^{(k-1)}}{\mu}}(\boldsymbol{\Sigma}^{(k-1)}) \mathbf{V}^{(k-1)T}$$

with  $\mathbf{X}^{(k-1)} = \mathbf{U}^{(k-1)} \boldsymbol{\Sigma}^{(k-1)} \mathbf{V}^{(k-1)T}$ , and

$$\mathcal{S}_{\frac{\tau \mathbf{w}^{(k-1)}}{\mu}}(\boldsymbol{\Sigma}^{(k-1)}) = \text{Diag} \left\{ \left( \sigma_i(\mathbf{X}^{(k-1)}) - \frac{\tau w_i^{(k-1)}}{\mu} \right)_+, i \in [1, \min(e, p)] \right\}.$$

*Proof.* Let  $\mathbf{g} = \{g_i\}_{i=1}^{\min(e,p)} = \boldsymbol{\sigma}(\mathbf{L})$ . According to Theorem 2.3 in [9], the optimization problem (9) can be equivalently written as

$$\mathbf{g}: g_1 \geq \dots \geq g_{\min(e,p)} \geq 0 \left\{ \min_{\substack{\mathbf{L} \in \mathbb{R}^{e \times p} \\ \boldsymbol{\sigma}(\mathbf{L}) = \mathbf{g}}} \left\{ \frac{\mu}{2} \|\mathbf{L} - \mathbf{X}^{(k-1)}\|_F^2 \right\} + \tau \sum_{i=1}^{\min(e,p)} w_i^{(k-1)} g_i \right\}. \quad (10)$$

For the inner minimization, we have the inequality

$$\begin{aligned} \frac{\mu}{2} \|\mathbf{L} - \mathbf{X}^{(k-1)}\|_F^2 &= \frac{\mu}{2} \text{Tr} \left[ (\mathbf{L} - \mathbf{X}^{(k-1)}) (\mathbf{L} - \mathbf{X}^{(k-1)})^T \right] \\ &= \frac{\mu}{2} \text{Tr} \left[ \mathbf{L}\mathbf{L}^T - \mathbf{L}\mathbf{X}^{(k-1)T} - \mathbf{X}^{(k-1)}\mathbf{L}^T + \mathbf{X}^{(k-1)}\mathbf{X}^{(k-1)T} \right] \\ &= \frac{\mu}{2} \text{Tr}(\mathbf{L}\mathbf{L}^T) + \frac{\mu}{2} \text{Tr}(\mathbf{X}^{(k-1)}\mathbf{X}^{(k-1)T}) - \mu \text{Tr}(\mathbf{X}^{(k-1)}\mathbf{L}^T) \\ &= \frac{\mu}{2} \sum_{i=1}^{\min(e,p)} g_i^2 + \frac{\mu}{2} \sum_{i=1}^{\min(e,p)} \sigma_i^2(\mathbf{X}^{(k-1)}) - \mu \text{Tr}(\mathbf{X}^{(k-1)}\mathbf{L}^T) \\ &\geq \frac{\mu}{2} \sum_{i=1}^{\min(e,p)} g_i^2 + \frac{\mu}{2} \sum_{i=1}^{\min(e,p)} \sigma_i^2(\mathbf{X}^{(k-1)}) - \mu \sum_{i=1}^{\min(e,p)} g_i \sigma_i(\mathbf{X}^{(k-1)}). \end{aligned}$$

The optimization problem (10) is re-written as

$$\begin{aligned} \mathbf{g}: g_1 \geq \dots \geq g_{\min(e,p)} \geq 0 &\sum_{i=1}^{\min(e,p)} \left( \frac{\mu}{2} g_i^2 + \frac{\mu}{2} \sigma_i^2(\mathbf{X}^{(k-1)}) - \mu g_i \sigma_i(\mathbf{X}^{(k-1)}) + \tau w_i^{(k-1)} g_i \right), \\ \mathbf{g}: g_1 \geq \dots \geq g_{\min(e,p)} \geq 0 &\sum_{i=1}^{\min(e,p)} \left( \frac{\mu}{2} g_i^2 + [-\mu \sigma_i(\mathbf{X}^{(k-1)}) + \tau w_i^{(k-1)}] g_i + \frac{\mu}{2} \sigma_i^2(\mathbf{X}^{(k-1)}) \right). \end{aligned} \quad (11)$$

By computing the derivative w.r.t.  $g_i$  and setting it to zero, we have

$$\mu g_i - \mu \sigma_i(\mathbf{X}^{(k-1)}) + \tau w_i^{(k-1)} = 0,$$

and thus, the optimal solution to Eq. (11) is given by

$$g_i = \left( \sigma_i(\mathbf{X}^{(k-1)}) - \frac{\tau w_i^{(k-1)}}{\mu} \right)_+.$$

Hence, the global optimal unique solution to the optimization problem (9) is given by  $\mathbf{L}^{(k)} = \mathbf{U}^{(k-1)} \text{Diag} \left( \left\{ \left( \boldsymbol{\sigma} \left( \mathbf{X}^{(k-1)} \right) - \frac{\tau \mathbf{w}^{(k-1)}}{\mu} \right)_+ \right\} \right) \mathbf{V}^{(k-1)T}$ , and which concludes the proof. ■

## 1.2 Providing an optimal solution to sub-problem (5b)

Eq. (5b) can be solved by various methods, among which we adopt the alternating direction method of multipliers (ADMM) [10]. More precisely, we introduce an auxiliary variable  $\mathbf{F}$  into sub-problem (5b) and recast it into the following form

$$\left( \mathbf{C}^{(k)}, \mathbf{F}^{(k)} \right) = \underset{\mathbf{C}=\mathbf{F}}{\text{argmin}} \left\{ \left\| \left( \mathbf{D} - \mathbf{L}^{(k)} \right)^T - \mathbf{A}_t \mathbf{C} \right\|_F^2 + \lambda \sum_{j=1}^e \left\| [\mathbf{F}]_{:,j} \right\|_2^q \right\}. \quad (12)$$

Problem (12) is then solved as

$$\mathbf{C}^{(k)} = \underset{\mathbf{C}}{\text{argmin}} \left\{ \left\| \left( \mathbf{D} - \mathbf{L}^{(k)} \right)^T - \mathbf{A}_t \mathbf{C} \right\|_F^2 + \frac{\rho^{(k-1)}}{2} \left\| \mathbf{C} - \mathbf{F}^{(k-1)} + \frac{1}{\rho^{(k-1)}} \mathbf{Z}^{(k-1)} \right\|_F^2 \right\}, \quad (13a)$$

$$\mathbf{F}^{(k)} = \underset{\mathbf{F}}{\text{argmin}} \left\{ \lambda \sum_{j=1}^e \left\| [\mathbf{F}]_{:,j} \right\|_2^q + \frac{\rho^{(k-1)}}{2} \left\| \mathbf{C}^{(k)} - \mathbf{F} + \frac{1}{\rho^{(k-1)}} \mathbf{Z}^{(k-1)} \right\|_F^2 \right\}, \quad (13b)$$

$$\mathbf{Z}^{(k)} = \mathbf{Z}^{(k-1)} + \rho^{(k-1)} \left( \mathbf{C}^{(k)} - \mathbf{F}^{(k)} \right). \quad (13c)$$

where  $\mathbf{Z} \in \mathbb{R}^{N_t \times e}$  is the Lagrangian multiplier matrix, and  $\rho$  is a positive scalar.

### 1.2.1 Solving sub-problem (13a)

$$\begin{aligned} -2 \mathbf{A}_t^T \left( \left( \mathbf{D} - \mathbf{L}^{(k)} \right)^T - \mathbf{A}_t \mathbf{C} \right) + \rho^{(k-1)} \left( \mathbf{C} - \mathbf{F}^{(k-1)} + \frac{1}{\rho^{(k-1)}} \mathbf{Z}^{(k-1)} \right) &= \mathbf{0}, \\ \Rightarrow \left( 2 \mathbf{A}_t^T \mathbf{A}_t + \rho^{(k-1)} \mathbf{I} \right) \mathbf{C} &= \rho^{(k-1)} \mathbf{F}^{(k-1)} - \mathbf{Z}^{(k-1)} + 2 \mathbf{A}_t^T \left( \mathbf{D} - \mathbf{L}^{(k)} \right)^T. \end{aligned}$$

This implies:

$$\mathbf{C}^{(k)} = \left( 2 \mathbf{A}_t^T \mathbf{A}_t + \rho^{(k-1)} \mathbf{I} \right)^{-1} \left( \rho^{(k-1)} \mathbf{F}^{(k-1)} - \mathbf{Z}^{(k-1)} + 2 \mathbf{A}_t^T \left( \mathbf{D} - \mathbf{L}^{(k)} \right)^T \right)$$

### 1.2.2 Solving sub-problem (13b)

According to Lemma 3.3 in [11] and Lemma 4.1 in [12], problem (13b) admits the following closed-form solution:

$$[\mathbf{F}]_{:,j}^{(k)} = \max \left( \left\| [\mathbf{C}]_{:,j}^{(k)} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)} \right\|_2^{2-q} - \frac{\lambda}{q \rho^{(k-1)}}, 0 \right) \left( \frac{[\mathbf{C}]_{:,j}^{(k)} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)}}{\left\| [\mathbf{C}]_{:,j}^{(k)} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)} \right\|_2^{2-q}} \right)$$



*Proof.* At the  $j$ th column, sub-problem (13b) refers to

$$[\mathbf{F}]_{:,j}^{(k)} = \underset{[\mathbf{F}]_{:,j}}{\operatorname{argmin}} \left\{ \lambda \left\| [\mathbf{F}]_{:,j} \right\|_2^q + \frac{\rho^{(k-1)}}{2} \left\| [\mathbf{C}]_{:,j}^{(k)} - [\mathbf{F}]_{:,j} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)} \right\|_2^2 \right\}.$$

By finding the derivative w.r.t  $[\mathbf{F}]_{:,j}$  and setting it to zero, we obtain

$$\begin{aligned} -\rho^{(k-1)} \left( [\mathbf{C}]_{:,j}^{(k)} - [\mathbf{F}]_{:,j} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)} \right) + \frac{\lambda \partial \left( \left\| [\mathbf{F}]_{:,j} \right\|_2^q \right)}{\partial [\mathbf{F}]_{:,j}} &= \mathbf{0} \\ \Rightarrow [\mathbf{C}]_{:,j}^{(k)} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)} &= [\mathbf{F}]_{:,j} + \frac{\lambda \partial \left( \left\| [\mathbf{F}]_{:,j} \right\|_2^q \right)}{\rho^{(k-1)} \partial [\mathbf{F}]_{:,j}}. \end{aligned} \quad (14)$$

Let  $[\mathbf{F}]_{:,j} = [f_{1,j}, \dots, f_{N_t,j}]^T \in \mathbb{R}^{N_t}$ . We have

$$\begin{aligned} \frac{\partial}{\partial f_{t,j}} \left\| [\mathbf{F}]_{:,j} \right\|_2^q &= \frac{\partial}{\partial f_{t,j}} \left( \left( \sum_{s=1}^{N_t} |f_{s,j}|^2 \right)^{1/2} \right)^q \\ &= \frac{\partial}{\partial f_{t,j}} \left( \sum_{s=1}^{N_t} |f_{s,j}|^2 \right)^{q/2} \\ &= \frac{q}{2} \left( \sum_{s=1}^{N_t} |f_{s,j}|^2 \right)^{\frac{q-2}{2}} \times \frac{\partial}{\partial f_{t,j}} \left( \sum_{s=1}^{N_t} |f_{s,j}|^2 \right) \\ &= \frac{q}{2} \left( \left( \sum_{s=1}^{N_t} |f_{s,j}|^2 \right)^{1/2} \right)^{q-2} \times \sum_{s=1}^{N_t} \left( 2|f_{s,j}| \times \frac{\partial}{\partial f_{t,j}} |f_{s,j}| \right) \\ &= q \left\| [\mathbf{F}]_{:,j} \right\|_2^{q-2} \times \sum_{s=1}^{N_t} |f_{s,j}| \delta_{s,t} \frac{f_{s,j}}{|f_{s,j}|} = q \left\| [\mathbf{F}]_{:,j} \right\|_2^{q-2} \times f_{t,j} \\ &= \frac{f_{t,j}}{q \left\| [\mathbf{F}]_{:,j} \right\|_2^{2-q}}, \quad t \in [1, N_t]. \end{aligned}$$

This implies  $\frac{\partial}{\partial [\mathbf{F}]_{:,j}} \left\| [\mathbf{F}]_{:,j} \right\|_2^q = \frac{[\mathbf{F}]_{:,j}}{q \left\| [\mathbf{F}]_{:,j} \right\|_2^{2-q}}$ . Hence, Eq. (14) is re-written as

$$[\mathbf{C}]_{:,j}^{(k)} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)} = [\mathbf{F}]_{:,j} + \frac{\lambda [\mathbf{F}]_{:,j}}{q \rho^{(k-1)} \left\| [\mathbf{F}]_{:,j} \right\|_2^{2-q}}. \quad (15)$$

By computing the  $\|\cdot\|_2^{2-q}$  norm of (15), we obtain

$$\left\| [\mathbf{C}]_{:,j}^{(k)} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)} \right\|_2^{2-q} = \left\| [\mathbf{F}]_{:,j} \right\|_2^{2-q} + \frac{\lambda}{q \rho^{(k-1)}}. \quad (16)$$

From Eqs. (15) and (16), we have

$$\frac{[\mathbf{C}]_{:,j}^{(k)} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)}}{\left\| [\mathbf{C}]_{:,j}^{(k)} + \frac{1}{\rho^{(k-1)}} [\mathbf{Z}]_{:,j}^{(k-1)} \right\|_2^{2-q}} = \frac{[\mathbf{F}]_{:,j}}{\left\| [\mathbf{F}]_{:,j} \right\|_2^{2-q}}. \quad (17)$$

Consider that

$$[\mathbf{F}]_{:,j} = \|[\mathbf{F}]_{:,j}\|_2^{2-q} \times \frac{[\mathbf{F}]_{:,j}}{\|[\mathbf{F}]_{:,j}\|_2^{2-q}}. \quad (18)$$

By replacing  $\|[\mathbf{F}]_{:,j}\|_2^{2-q}$  from Eq. (16) into Eq. (18), and  $\frac{[\mathbf{F}]_{:,j}}{\|[\mathbf{F}]_{:,j}\|_2^{2-q}}$  from Eq. (17) into Eq. (18), we conclude the proof.  $\blacksquare$

### 1.3 Some Initializations and Convergence Criterion

We initialize  $\mathbf{L}^{(0)} = \mathbf{0}$ ,  $\mathbf{F}^{(0)} = \mathbf{C}^{(0)} = \mathbf{Z}^{(0)} = \mathbf{0}$ ,  $\rho^{(0)} = 10^{-4}$  and update  $\rho^{(k)} = 1.1\rho^{(k-1)}$ . The criteria for convergence of sub-problem (5b) is  $\|\mathbf{C}^{(k)} - \mathbf{F}^{(k)}\|_F^2 \leq 10^{-6}$ .

For Problem (4), we stop the iteration when the following convergence criterion is satisfied:

$$\frac{\|\mathbf{L}^{(k)} - \mathbf{L}^{(k-1)}\|_F}{\|\mathbf{D}\|_F} \leq \epsilon \quad \text{and} \quad \frac{\|(\mathbf{A}_t \mathbf{C}^{(k)})^T - (\mathbf{A}_t \mathbf{C}^{(k-1)})^T\|_F}{\|\mathbf{D}\|_F} \leq \epsilon,$$

where  $\epsilon > 0$  is a precision tolerance parameter.

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