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Gaussian process model selection for computer experiments



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Context

- Exploration of black-box numerical simulators $f : \mathbb{X} \subset \mathbb{R}^d \rightarrow \mathbb{R}$ with Gaussian processes
- Given data $D_n = (\mathcal{X}_n, f|_{\mathcal{X}_n})$, a Gaussian process ξ can be used to make probabilistic predictions of f

$$\xi(x)|D_n \sim \mathcal{N}(\hat{\xi}_\theta(x), \hat{\sigma}_\theta^2(x)) \quad (1)$$

- ξ is a prior over functions
- The choice of ξ is critical for good predictions and design-of-experiments techniques

The prior ξ is often chosen within a parametric family.

- Very often: the Matérn covariance functions family is used [7]
- Many procedures have been proposed in the literature for selecting the parameters of a covariance function
- Little is known about their relative benefits

What are the most useful procedures to select the parameters of a Matérn covariance function (including or not regularity)?

- Toms 829 C^k ($k \in \{0, 1, 2\}$, $d \in \{2, 5\}$)

- Rotated Rosenbrock ($d \in \{2, 5\}$)

- Borehole ($d = 8$)

with space-filling designs \mathcal{X}_n , $n \in \{10d, 20d, 50d\}$. For each case:

- We compare model selection procedures using predictions evaluated on a dense test grid

- In particular, we study the influence of the regularity parameter p of a Matérn covariance function, by setting $p \in \{0, 1, 2, 3, 4, d, 2d, +\infty\}$ or automatically selecting its value using the selection criteria

- We present averaged results through repetitions (using random \mathcal{X}_n for instance)

An example

Observe that

- Procedures ML, LOO-MSPE, LOO-NLPD and LOO-CRPS give similar accuracies

- The influence of p on the accuracy is strong

MSPE	ML	LOO MSPE	LOO NLPD	LOO CRPS	KA	GCV	"Best"
0	$1.08 \cdot 10^{-2}$	$1.12 \cdot 10^{-2}$	$1.07 \cdot 10^{-2}$	$1.06 \cdot 10^{-2}$	$2.16 \cdot 10^{-1}$	$1.04 \cdot 10^{-2}$	$9.74 \cdot 10^{-3}$
1	$3.27 \cdot 10^{-5}$	$3.11 \cdot 10^{-5}$	$2.78 \cdot 10^{-5}$	$2.85 \cdot 10^{-5}$	$1.71 \cdot 10^{-1}$	$2.79 \cdot 10^{-5}$	$2.45 \cdot 10^{-5}$
2	$1.17 \cdot 10^{-5}$	$1.23 \cdot 10^{-5}$	$1.29 \cdot 10^{-5}$	$1.14 \cdot 10^{-5}$	$1.59 \cdot 10^{-1}$	$1.66 \cdot 10^{-5}$	$8.68 \cdot 10^{-6}$
3	$1.54 \cdot 10^{-5}$	$1.80 \cdot 10^{-5}$	$1.81 \cdot 10^{-5}$	$1.62 \cdot 10^{-5}$	$1.36 \cdot 10^{-1}$	$2.32 \cdot 10^{-5}$	$1.26 \cdot 10^{-5}$
4	$1.90 \cdot 10^{-5}$	$2.32 \cdot 10^{-5}$	$2.30 \cdot 10^{-5}$	$2.11 \cdot 10^{-5}$	$1.23 \cdot 10^{-1}$	$3.13 \cdot 10^{-5}$	$1.60 \cdot 10^{-5}$
6	$2.36 \cdot 10^{-5}$	$3.07 \cdot 10^{-5}$	$2.90 \cdot 10^{-5}$	$2.68 \cdot 10^{-5}$	$1.12 \cdot 10^{-1}$	$4.00 \cdot 10^{-5}$	$1.99 \cdot 10^{-5}$
12	$2.60 \cdot 10^{-5}$	$3.30 \cdot 10^{-5}$	$3.04 \cdot 10^{-5}$	$2.97 \cdot 10^{-5}$	$1.03 \cdot 10^{-1}$	$4.11 \cdot 10^{-5}$	$2.16 \cdot 10^{-5}$
$+\infty$	$2.94 \cdot 10^{-5}$	$3.77 \cdot 10^{-5}$	$3.33 \cdot 10^{-5}$	$3.18 \cdot 10^{-5}$	$9.23 \cdot 10^{-2}$	$4.31 \cdot 10^{-5}$	$2.43 \cdot 10^{-5}$
\hat{p}	$1.17 \cdot 10^{-5}$	$1.27 \cdot 10^{-5}$	$1.29 \cdot 10^{-5}$	$1.15 \cdot 10^{-5}$	$9.23 \cdot 10^{-2}$	$1.74 \cdot 10^{-5}$	$8.68 \cdot 10^{-6}$

Table: Average MSPE on the validation sets for the different selection procedures and regularity choices.

Influence of the selection criteria

We compare the selection procedures with automatically selected p . Fig. 3: $\log(S_{\text{MSPE}})$ normalized by "Best" values; Fig. 4: interval score [4] defined by

$$S_\alpha^{\text{IS}}(l, u, x) = (u - l) + \frac{2}{\alpha}(l - x)\mathbf{1}_{x \leq l} + \frac{2}{\alpha}(x - u)\mathbf{1}_{x > u}. \quad (3)$$

6 Conclusions

- The regularity parameter has a strong impact on the goodness of fit
- We recommend selecting the regularity from data instead of fixing it to a "standard" value
- The choice of a reasonable selection procedure has second-order impact but ML and LOO CRPS seem to give the best performances
- All procedures have the same numerical complexity, using appropriate computations of the selection criteria and their gradients [6]

1 Maximum-likelihood [5]

- A very popular technique
- Choose the parameters that yield the highest value of the probability density for the observations, or equivalently, minimize

$$z^T K_\theta z + \ln(\det(K_\theta)),$$

where K_θ is the covariance matrix of ξ at points $\mathcal{X}_n = (x_1, \dots, x_n)$ for parameters θ and $z = (z_1, \dots, z_n)^T$ denotes the values of f at \mathcal{X}_n

2 Cross-validation

Leave-one-out (LOO) [3] is a second very popular technique

- Consists in averaging *losses* for predicting one observation using the others
- We suggest using *negatively-oriented scoring rules* [4] for the loss functions
- A (negatively-oriented) scoring rule is a mapping $S : (\mathcal{P}, \mathbb{R}) \rightarrow \mathbb{R}$ where \mathcal{P} is a class of probability distributions, with $S(P, z)$ representing a loss for observing z while predicting P
- Given a scoring rule S the corresponding LOO criterion is

$$L_S^{\text{LOO}}(\theta) = \frac{1}{n} \sum_{i=1}^n S(\mathcal{N}(\hat{\xi}_{\theta, -i}, \hat{\sigma}_{\theta, -i}^2), z_i), \quad (2)$$

where $\mathcal{N}(\hat{\xi}_{\theta, -i}, \hat{\sigma}_{\theta, -i}^2)$ denotes LOO predictive distributions

In this work we consider the following scoring rules [4]:

- $S_{\text{MSPE}}(P, z) = (\mathbb{E}_{Z \sim P}(Z) - z)^2$
- $S_{\text{NLPD}}(P, z) = -\ln(p(z))$, with p the pdf of P
- $S_{\text{CRPS}}(P, z) = \|F - \mathbf{1}_{z \leq \cdot}\|_{L^2(\mathbb{R})}^2$, with F the cdf of P

We shall denote the resulting selection procedures by LOO-MSPE, LOO-NLPD and LOO-CRPS respectively.

3 Generalized cross-validation [1]

- A version of LOO-MSPE that takes the heterogeneity of the design into account

4 Kernel alignment [2]

- Aligns the eigenvector related to the highest eigenvalue of K_θ with the data
- Can also be seen as a similarity between K_θ and the covariance matrix obtained from the kernel $(x, y) \rightarrow f(x)f(y)$

5 Numerical study

We use a set of 36 problems:

- Goldstein-Price ($d \in \{1, 2\}$)
- Mystery ($d = 2$)

Influence of the regularity

We focus on two subsets of problems with different smoothness. Fig. 1: 5-dimensional Toms 829 problems; Fig. 2: 5-dimensional Rosenbrock and Borehole.

We compare $\log S_{\text{MSPE}}$ normalized by "Best" values both with automatically selected or fixed- $p \in \{0, 1, 2, 3, 4, d, 2d, +\infty\}$.

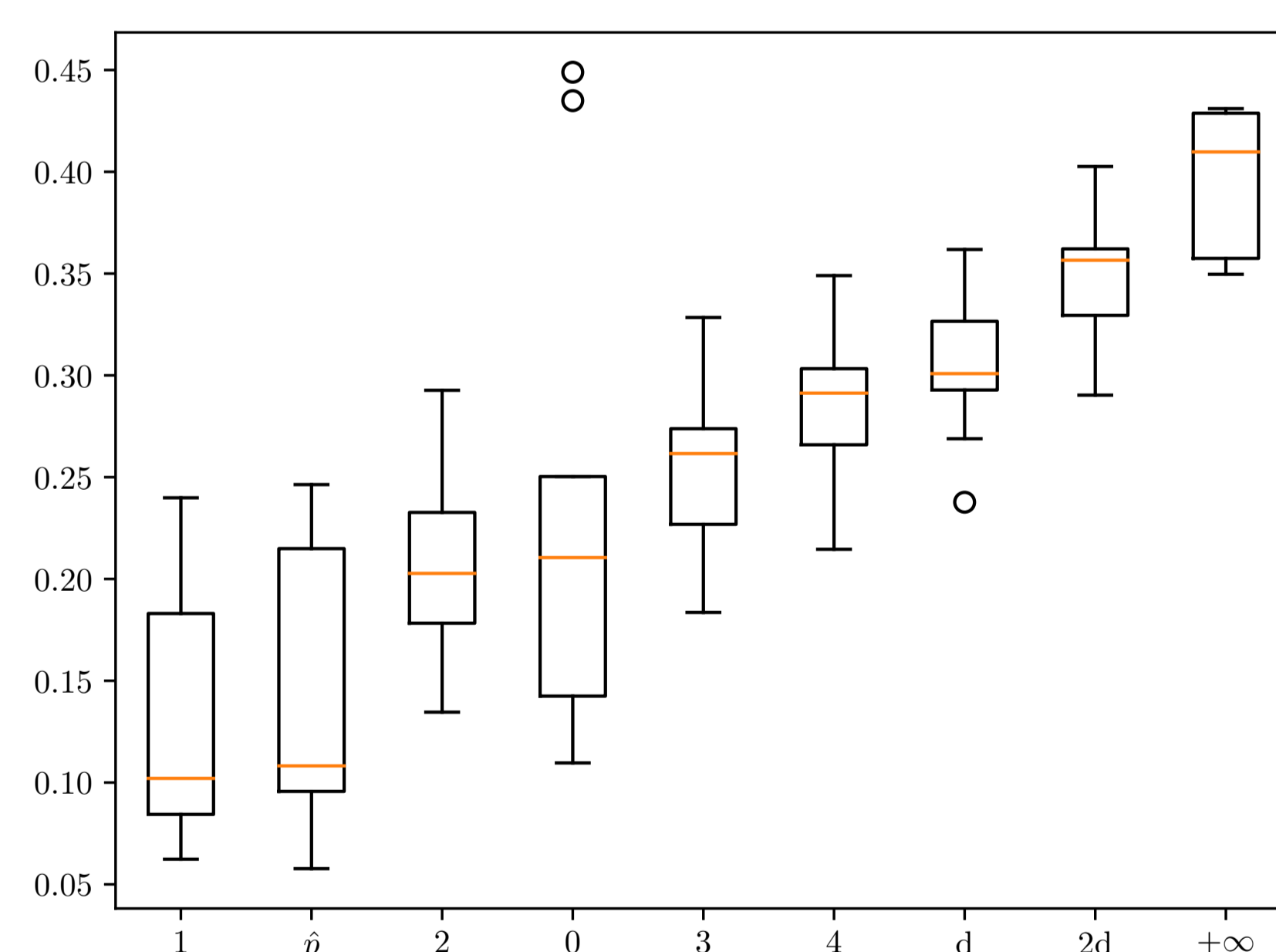


Fig. 1: Influence of the regularity on the loss for non-smooth problems.

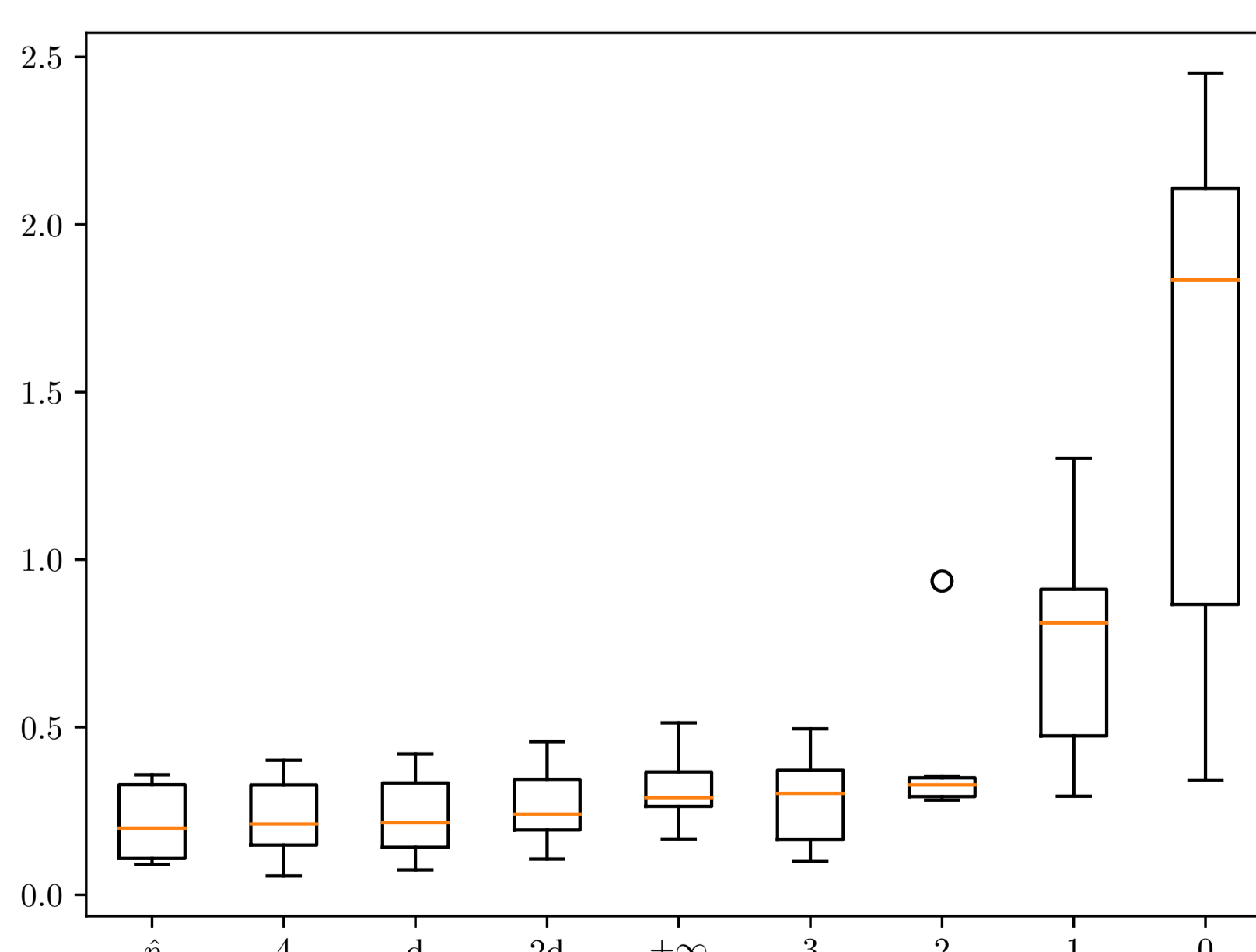


Fig. 2: Influence of the regularity on the loss for smooth problems.

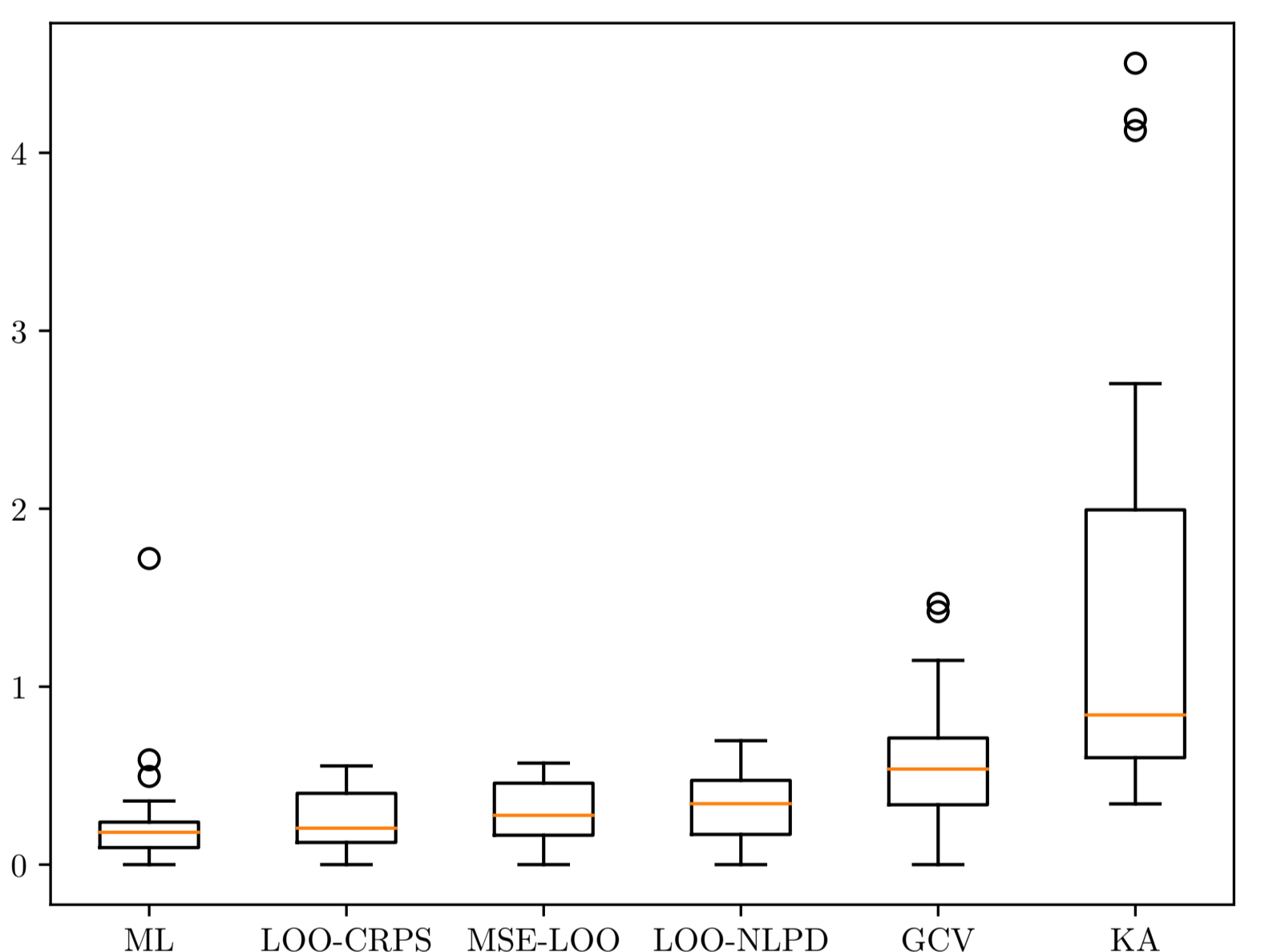


Fig. 3: Influence of the selection criteria on the MSPE.

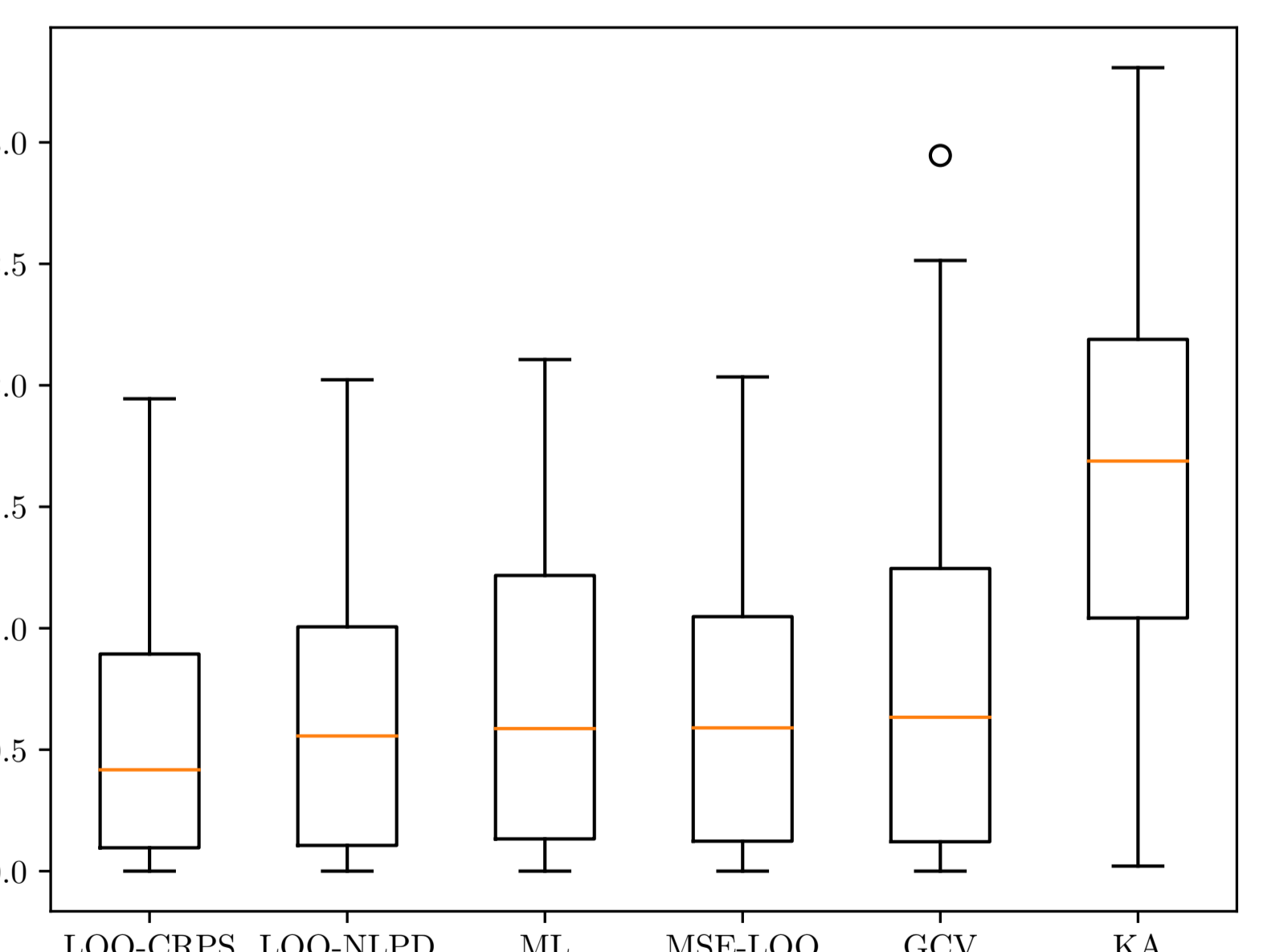


Fig. 4: Influence of the selection criteria on the interval score.

References

- P. Craven and G. Wahba. Smoothing noisy data with spline functions. Estimating the correct degree of smoothing by the method of generalized cross-validation. *Numerische Mathematik*, 1979.
- N. Cristianini, J. Shawe-Taylor, A. Elisseeff, and J. Kandola. On kernel-target alignment. In *Proceedings of the 14th International Conference on Neural Information Processing Systems: Natural and Synthetic*, NIPS'01, Cambridge, MA, USA, 2001. MIT Press.
- C. Currin, T. Mitchell, M. Morris, and D. Yivisaker. A Bayesian approach to the design and analysis of computer experiments. Technical report, Oak Ridge National Lab., TN (USA), 1988.
- T. Gneiting and A. Raftery. Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102:359–378, 2007.
- P. K. Kitanidis. Statistical estimation of polynomial generalized covariance functions and hydrologic applications. *Water Resources Research*, 19(4):909–921, 1983.
- S. J. Petit, J. Bect, S. Da Veiga, P. Feliot, and E. Vazquez. Towards new cross-validation-based estimators for Gaussian process regression: efficient adjoint computation of gradients, 2020.
- M. L. Stein. *Interpolation of Spatial Data: Some Theory for Kriging*. Springer Series in Statistics. Springer New York, 1999.