Quantum Entanglement and the Lorentz Group

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The Lorentz metric represented by the diagonal matrix $\boldsymbol{G} = \mathrm{diag}(1,-1,-1,-1)$ acts on Minkowski space-time quadrivectors. In the language of Quantum Information the operator $\boldsymbol{G}$ can be viewed as an entangling gate this because it acts in a similar way as the Controlled-Z gate on the computational basis of a 2-qubit separable quantum vector. The entangling power corresponds to the fact that the resulting vector, considered as a 2-qubit vector, cannot be put into a Kronecker product of two 1-qubit vectors. For example considering a uniform positive normalized input vector, which is separable, one has the transformation :

$$\boldsymbol{G} \cdot \tfrac{1}{2}(1,1,1,1)^T = \tfrac{1}{2}(1,-1,-1,-1)^T$$

It can be easily verified that in this case the resulting output vector is completely entangled, for example by calculating the associated quantum concurrence which equals to 1.

One can represent the generators of the Lorentz group by 4×4 matrices. An example is given by the Lorentz rotation matrix $\boldsymbol{Z}(\phi)$ [1] :

$$\boldsymbol{Z}(\phi) = \exp(-i\phi \boldsymbol{J}_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \; , \; \boldsymbol{J}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \boldsymbol{\Pi}_1 \otimes \boldsymbol{\sigma}_y$$

The matrix $\boldsymbol{J}_3$ is the associated Lorentz group generator. $\boldsymbol{J}_3$ can be expressed as the Kronecker product of the qubit logical-1 projector $\boldsymbol{\Pi}_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ with the Pauli matrix $\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and consequently the operator $\boldsymbol{Z}(\phi)$ in a qubit representation acts as a control gate on the unitary double angle rotation operator $\boldsymbol{R}_y(2\phi) = \exp(\text{-}i\phi\boldsymbol{\sigma}_y)$ around the Oy axis. This can be highlighted by developing the exponential and using the idempotent property of the projector and the involution property of the Pauli matrix :

$$\boldsymbol{Z}(\phi) = \exp(-i\phi(\boldsymbol{\Pi}_1 \otimes \boldsymbol{\sigma}_y)) = \boldsymbol{\Pi}_0 \otimes \boldsymbol{I}_2 + \boldsymbol{\Pi}_1 \otimes \boldsymbol{R}_y(2\phi)$$

where $\boldsymbol{\Pi}_0 = \boldsymbol{I}_2 - \boldsymbol{\Pi}_1 = |0\rangle\langle 0|$ is the qubit logical-0 projector. The form can be compared with the more-known entangling gate Control-NOT which can be expressed also as a function of projectors :

$$\boldsymbol{C}_{NOT} = \boldsymbol{\Pi}_0 \otimes \boldsymbol{I}_2 + \boldsymbol{\Pi}_1 \otimes \boldsymbol{\sigma}_x = \boldsymbol{\Pi}_0 \otimes \boldsymbol{I}_2 + \boldsymbol{\Pi}_1 \otimes i\boldsymbol{R}_x(\pi)$$

which is a control gate on the unitary Pauli operator $\boldsymbol{\sigma}_x = \boldsymbol{X}$ (the NOT gate) and corresponds geometrically to a reflection or equivalently a rotation around the Ox axis of angle π multiplied by the imaginary number i.

The following questions arise : is there a link between the Lorentz space-time structure and entanglement ? Do they have a common origin ? These questions could be related with the lack of mechanical understanding of the nature of a relativistic spinor. Spinors are often identified with qubits, for example the complex 1-qubit column vector $|\psi\rangle = (\psi_1, \psi_2)^T$ transforms under left-multiplication with matrices in the $SU(2)$ special-unitary group like a 2-spinor [2] and Dirac 4-spinors can be associated with 2-qubit states. The physics related to the information content of the Lorentz group is often overlooked, not being considered for actual applications in Quantum Information. Spinor algebra arises by the Lorentz invariance constraint in the quantum Hilbert space framework and conversely spinor algebra implies Lorentz invariance. Therefore the Lorentz group can be considered as a bridging algebraic structure between Quantum Information and Relativity Theory. Also the logic content of the associated linear algebra structures [3] could provide a new perspective to further explore this matter.

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