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## Chance-constrained games with mixture distributions

Shen Peng · Navnit Yadav · Abdel  
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**Abstract** In this paper, we consider an  $n$ -player non-cooperative game where the random payoff function of each player is defined by its expected value and her strategy set is defined by a joint chance constraint. The random constraint vectors are independent. We consider the case when the probability distribution of each random constraint vector belongs to a subset of elliptical distributions as well as the case when it is a finite mixture of the probability distributions from the subset. We propose a convex reformulation for the joint chance constraint of each player and derive the bounds for players' confidence levels and the weights used in the mixture distributions. Under mild conditions on the players' payoff functions, we show that there exists a Nash equilibrium of the game when the players' confidence levels and the weights used in the mixture distributions are within the derived bounds. As an application of these games, we consider the competition between two investment firms on the same set of portfolios. We use a best response algorithm to compute a Nash equilibrium of the randomly generated games of different sizes.

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**Keywords** Chance-constrained game · Mixture of elliptical distributions · Nash equilibrium · Portfolio.

## 1 Introduction

In strategic games, the rational players choose their strategies simultaneously which results in payoffs for all the players. The payoffs of each player depend on the strategies of all the players whereas the strategy set of each player does not depend on other players' strategies. The Nash equilibrium concept is used to analyze the outcome of such strategic interactions. Nash [23] showed that there exists a mixed strategy Nash equilibrium of a game with finite number of players where each player has finite number of actions. Since then, general strategic games have been extensively studied in the literature [1, 10, 11]. The games considered in the above-mentioned papers are deterministic in nature, i.e., the payoffs and the strategy sets of all the players are defined by real valued functions. However, the decision making process is often subject to uncertainties due to the presence of various external factors [9, 22]. The stochastic modelling of uncertainties present in the system depends on the attitude of decision makers. The expected value criterion is used for risk neutral players [25]. For risk averse players, the payoff criterion with the risk measure CVaR [18, 25] and the variance are considered in the literature [8]. A chance constraint based payoff criterion for the case of risk averse players is introduced in [26, 27]. Singh et al. [26] considered a finite strategic games where the payoff vector of each player is a random vector, and each player is interested in payoffs obtained with a given confidence level. They showed that there exists a mixed strategy Nash equilibrium when the payoff vector of each player follows a multivariate elliptical distribution. In [27], the authors showed that the Nash equilibria of chance-constrained games (CCGs) considered in [26] can be obtained from the global optimal solutions of a certain mathematical program. Notice that there are zero-sum CCGs studied in the literature, see for instance [2, 4, 5, 6].

The above-mentioned papers on CCGs considered the case where players' payoffs are random and the strategy sets are deterministic in nature. There is a scarce literature on games with stochastic strategy sets defined by chance constraints [24, 28]. Singh and Lisser [28] showed that a saddle point equilibrium of a zero-sum game with individual chance constraints can be obtained by solving a primal-dual pair of second order cone programs. Peng et al. [24] considered  $n$ -player games with deterministic payoff functions and the strategy sets defined using joint chance constraints. They showed that there exists a Nash equilibrium of the game when the random constraint vectors are independent, and follow a multivariate normal distribution. A normal distribution belongs to the class of elliptical distributions. In practical situations, a normal distribution is often used to model the uncertain parameters, but, sometimes the member distributions from elliptical class other than normal distribution are used to model the uncertainties present in the system. For example, in fi-

financial market the random return vector from a portfolio follows an elliptical distribution [16] as well as a finite mixture of elliptical distributions [3, 13, 17].

In this paper, we consider an  $n$ -player game where the payoff function of each player is random and the strategy set of each player is defined by a joint chance constraint. The players' payoff functions are defined using expected value of their random payoff functions and the random constraint vectors are independent. We identify a list of prominent probability distributions belonging to the class of elliptical distributions. Under mild conditions, we show that there exists a Nash equilibrium of these CCGs if the underlying probability distributions of random constraint vectors belong to the list. Then, we consider the case where the random constraint vectors follow a finite mixture of elliptical distributions from the list and we show that there exists a Nash equilibrium of the CCG. The CCGs considered in this paper are significantly different from the ones considered in [24]. Unlike [24], we consider expected payoff functions for each player and the existence of Nash equilibrium is shown for certain elliptical distributions and their finite mixture. As an application of the CCG under mixture distributions, we consider the case of two investment firms which compete on the same market by investing in the same set of portfolios. The random return vector of each firm follows a finite mixture of elliptical distributions. The firms want to keep their random losses below a certain threshold. The constraints on the random loss function of each player are modelled as a joint chance constraint. We consider the case where each firm incurs transaction cost which depends on the strategies of both the firms. Each firm is interested in maximizing its expected payoffs over the chance-constrained strategy set. We use a best response algorithm to compute the Nash equilibrium of the game. The numerical experiments are performed on randomly generated instances of different sizes.

The rest of the paper is organized as follows. Section 2 contains the definition of the CCGs. In Sections 3 and 4, we show the existence of a Nash equilibrium for a subset of elliptical distributions, and for a finite mixture of elliptical distributions from the subset, respectively. Section 5 contains the application of the CCG in financial market. We conclude our paper in Section 6.

## 2 The model

We consider an  $n$ -player non-cooperative game, where  $I = \{1, 2, \dots, n\}$  denotes the set of players. Let  $X^i \subset \mathbb{R}_{++}^{l_i}$  be a strategy set of player  $i$  which is a convex and compact set;  $\mathbb{R}_{++}^{l_i}$  denotes the positive orthant of  $\mathbb{R}^{l_i}$ . The generic element of  $X^i$  is denoted by  $x^i$  and the vector of strategies of all the players except player  $i$  is denoted by  $x^{-i}$ . The strategy profile of all the players is denoted by  $x = (x^i, x^{-i})$ . Let  $\xi : \Omega \rightarrow \mathbb{R}^d$  be a random vector defined on probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . For each  $\omega \in \Omega$ , let  $v_i(x, \xi(\omega))$  represents a real valued payoff function of player  $i$  which is defined on  $\prod_{i \in I} \mathbb{R}_{++}^{l_i} \times \mathbb{R}^d$ . The players use

expected payoff criterion and the payoff function of player  $i$  is defined as

$$u_i(x) = \mathbb{E}_{\mathbb{P}}[v_i(x, \xi)], \quad (1)$$

where  $\mathbb{E}_{\mathbb{P}}$  denotes the expectation operator associated with probability measure  $\mathbb{P}$ . The strategy set of player  $i$ ,  $i \in I$ , is restricted by the following joint chance constraint

$$\mathbb{P}\{A^i x^i \leq b^i\} \geq \alpha_i, \quad (2)$$

where  $\alpha_i$  is confidence level of player  $i$ ,  $A^i = [A_1^i, A_2^i, \dots, A_{K_i}^i]^T$  is a  $K_i \times l_i$  random matrix defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and  $b^i \in \mathbb{R}^{K_i}$ ;  $T$  denotes the transposition. For each  $k = 1, 2, \dots, K_i$ ,  $A_k^i$  is the  $k^{\text{th}}$  row of  $A^i$ . For a given  $\alpha_i$ , the feasible strategy set of player  $i$  is defined by

$$S_{\alpha_i}^i = \left\{ x^i \in \mathbb{R}_{++}^{l_i} \mid x^i \in X^i, \mathbb{P}\{A^i x^i \leq b^i\} \geq \alpha_i \right\}, \quad \forall i \in I. \quad (3)$$

Let  $\alpha = (\alpha_i)_{i \in I}$  be a confidence level vector. We denote  $S_{\alpha} = \prod_{i \in I} S_{\alpha_i}^i$  and  $S_{\alpha_{-i}}^{-i} = \prod_{j \in I; j \neq i} S_{\alpha_j}^j$ . We assume that the set  $S_{\alpha_i}^i$  is nonempty, and the probability distribution of the random matrix  $A^i$  and the probability level vector  $\alpha$ , and the payoff functions defined by (1) are known to all the players. We call this game a CCG which is a non-cooperative game with complete information. A strategy profile  $x^* = (x^{i*}, x^{-i*}) \in S_{\alpha}$  is a Nash equilibrium of the CCG at confidence level vector  $\alpha$  if and only if for each  $i \in I$ ,

$$u_i(x^{i*}, x^{-i*}) \geq u_i(x^i, x^{-i*}), \quad \forall x^i \in S_{\alpha_i}^i.$$

**Assumption 1** For each player  $i$ ,  $i \in I$ , the following conditions hold.

1.  $v_i(x^i, x^{-i}, \xi)$  is a concave function of  $x^i$  for every  $(x^{-i}, \xi) \in X^{-i} \times \mathbb{R}^d$ .
2.  $v_i(\cdot)$  is a continuous function.
3.  $\mathbb{E}_{\mathbb{P}}[v_i(x, \xi)]$  is finite valued for any  $x \in X$ .

In this paper, we consider the case where the row vectors  $A_k^i$ ,  $k = 1, 2, \dots, K_i$ , of matrix  $A^i$ ,  $i \in I$ , are independent [15]. We select a subset of elliptical distributions given by Table 1, and we show that there exists a Nash equilibrium of CCG. In addition, we show that a Nash equilibrium exists when the probability distribution is a finite mixture of the elliptical distributions listed in Table 1.

**Definition 1** A  $d$ -dimensional random vector  $\zeta$  follows an elliptically symmetric distribution  $Ellip_d(\mu, \Sigma, \varphi)$  with location parameter  $\mu$ , positive definite scale matrix  $\Sigma$ , and characteristic generator  $\varphi$ , if its characteristic function is given by  $\mathbb{E}e^{it^T \zeta} = e^{it^T \mu} \varphi(t^T \Sigma t)$ .

The probability density function (if it exists) of an elliptically symmetric distribution is given by

$$f(\mathbf{u}) = \frac{c}{\sqrt{\det \Sigma}} g \left( \sqrt{(\mathbf{u} - \mu)^T \Sigma^{-1} (\mathbf{u} - \mu)} \right), \quad (4)$$

where  $g(\cdot)$  is a nonnegative function called radial density, and  $c > 0$  is a normalization factor such that  $f$  integrates to one. The elliptical distributions with location parameter  $\mu = \mathbf{0}$  and scale matrix  $\Sigma = I_{d \times d}$  are called as spherical distributions [12];  $\mathbf{0}$  and  $I_{d \times d}$  denote a  $d \times 1$  zero vector and a  $d \times d$  identity matrix, respectively. Table 1 presents a short selection of prominent spherical distributions, together with their radial densities for univariate case [12].

Table 1: List of selected 1-dimensional spherical distributions.

Probability distribution	Radial density	$\bar{\alpha}_{k,i}$	$\bar{\alpha}_i$
normal	$e^{-\frac{1}{2}u^2}$	$\Psi_k^i(1)$	$\max_{k=1,2,\dots,K_i} \bar{\alpha}_{k,i}$
$t$	$(1 + \frac{1}{\nu}u^2)^{-(1+\nu)/2}$ , $\nu > 0, \nu$ integer	$\Psi_k^i(1)$	
Cauchy	$(1 + u^2)^{-1}$	$\Psi_k^i(1)$	
Laplace	$e^{- u }$	$\Psi_k^i(1)$	
Kotz type	$u^{2(N-1)} e^{-ru^{2s}}$ , $r, s > 0, N > \frac{1}{2}$	$\Psi_k^i \left( \left( \frac{2N-1}{2sr} \right)^{\frac{1}{2s}} \right)$	
Pearson Type VII distribution	$\left(1 + \frac{u^2}{m}\right)^{-N}$ , $m > 0, N > \frac{1}{2}$	$\Psi_k^i \left( \sqrt{\frac{m}{2N-1}} \right)$	

### 3 Existence of Nash equilibrium under elliptical distribution

The convexity of the feasibility set of joint chance constraint (2) plays an important role in the existence of Nash equilibrium for the CCG. When there is only one random linear constraint, i.e.,  $K_i = 1$ , the chance constraint (2) is equivalent to a second-order cone constraint [14]. When the row vectors  $A_k^i$ ,  $k = 1, 2, \dots, K_i$ , are independent and follow a multivariate normal distribution, then, Strugarek and Henrion [15] showed that the joint chance constraint (2) is convex for large values of  $\alpha_i$  and Shen et al. [24] proposed a new convex reformulation of (2). Sometimes, in practical situations the distribution of uncertain parameters belongs to the class of elliptical distribution [16]. We identify a list of prominent elliptical distributions as given in Table 1 and propose a convex reformulation of (2) when the row vectors  $A_k^i$ ,

$k = 1, 2, \dots, K_i$  are independent and follow  $Ellipl_i(\mu_k^i, \Sigma_k^i; \varphi_k^i)$  from Table 1. Under independence assumption, the joint chance constraint (2) can be equivalently written as  $\prod_{k=1}^{K_i} \mathbb{P}\{A_k^i x^i \leq b_k^i\} \geq \alpha_i$ , or alternatively with nonnegative auxiliary variables  $z_k^i$  [7]

$$\mathbb{P}\left\{A_k^i x^i \leq b_k^i, k = 1, 2, \dots, K_i, \sum_{k=1}^{K_i} z_k^i = 1, z_k^i \geq 0, k = 1, 2, \dots, K_i.\right\} \quad (5)$$

The chance constraint given in (5) can be treated as an individual chance constraint with  $\alpha_i^{z_k^i}$  confidence level. Using the reformulation of individual chance constraint from [14], we have the following reformulation of joint chance constraint (2) [7]

$$Q_{\alpha_i}^i = \begin{cases} (i) (\mu_k^i)^T x^i + \Psi_k^{i-1}(\alpha_i^{z_k^i}) \left\| (\Sigma_k^i)^{\frac{1}{2}} x^i \right\| \leq b_k^i, \forall k = 1, 2, \dots, K_i, \\ (ii) \sum_{k=1}^{K_i} z_k^i = 1, \\ (iii) z_k^i \geq 0, \forall k = 1, 2, \dots, K_i, \end{cases} \quad (6)$$

where  $\Psi_k^{i-1}(\cdot)$  is a quantile function of an 1-dimensional spherical distribution, and  $\|\cdot\|$  is the Euclidean norm. The deterministic reformulation (6) of (2) is not convex due to constraints (i) from (6). When  $\alpha_i \geq 0.5$ , we have  $\Psi_k^{i-1}(\alpha_i^{z_k^i}) \geq 0$ , where  $0 \leq z_k^i \leq 1$ . Then, under logarithmic transformation [20]  $y_j^i = \log x_j^i, j = 1, 2, \dots, l_i$ , we have the following reformulation for (6)

$$\tilde{Q}_{\alpha_i}^i = \begin{cases} (i) (\mu_k^i)^T e^{y^i} + \left\| (\Sigma_k^i)^{\frac{1}{2}} e^{\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i}) \mathbb{1}_{l_i} + y^i} \right\| \leq b_k^i, \forall k = 1, 2, \dots, K_i, \\ (ii) \sum_{k=1}^{K_i} z_k^i = 1, \\ (iii) z_k^i \geq 0, \forall k = 1, 2, \dots, K_i, \end{cases} \quad (7)$$

where  $\mathbb{1}_{l_i}$  is an  $l_i \times 1$  vector of ones, and  $e^{y^i} = (e^{y_1^i}, \dots, e^{y_{l_i}^i})^T$  and

$$e^{\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i}) \cdot \mathbb{1}_{l_i} + y^i} = \left( e^{\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i}) + y_1^i}, \dots, e^{\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i}) + y_{l_i}^i} \right)^T.$$

Let  $Y^i = \{y^i \in \mathbb{R}^{l_i} \mid y_j^i = \ln x_j^i, x^i = (x_1^i, \dots, x_{l_i}^i)^T, x^i \in X^i\}$ . The set  $Y^i$  is a compact set because it is an image of  $X^i$  under logarithmic transformation. The convexity might not be preserved under logarithmic transformation. Therefore, we consider only  $X^i$  for which  $Y^i$  is a convex set, see for instance [24]. From now onward, we consider the strategy set  $X^i$  for which  $Y^i$  is a

convex set. The reformulation of the feasible strategy set  $S_{\alpha_i}^i$  of player  $i$ ,  $i \in I$ , is given by

$$\tilde{S}_{\alpha_i}^i = \left\{ (y^i, z^i) \in Y^i \times \mathbb{R}^{K_i} \mid (y^i, z^i) \in \tilde{Q}_{\alpha_i}^i \right\}. \quad (8)$$

The convexity of reformulated feasible strategy set  $\tilde{S}_{\alpha_i}^i$  corresponding to elliptical distributions listed in Table 1 depends on the convexity of function  $\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i})$  [24]. We show that the function  $\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i})$  is convex in  $z_k^i$  for probability distributions listed in Table 1. In order to come-up with the convexity of the set  $\tilde{S}_{\alpha_i}^i$ , we consider the following assumption.

**Assumption 2** For each  $i \in I$  and  $k = 1, 2, \dots, K_i$ , all the components of  $\Sigma_k^i$  and  $\mu_k^i$  are nonnegative.

**Lemma 1** Let  $\Sigma$  be a  $k \times k$  symmetric positive definite matrix with non-negative entries and  $q(\delta) = (q_1(\delta), q_2(\delta), \dots, q_k(\delta))^T$  be a  $k \times 1$  vector of functions, where each component  $q_j(\delta)$  is a non-negative convex function of  $\delta$ . Then,  $\|\Sigma^{\frac{1}{2}}q(\delta)\|$  is a convex function of  $\delta$ .

*Proof* The proof follows from the Proposition 2.1 of [14].  $\square$

**Lemma 2** For each  $i \in I$  and  $k = 1, 2, \dots, K_i$ , let  $(\Psi_k^i)^{-1}(\cdot)$  be a quantile function of 1-dimensional spherical distribution listed in Table 1. Then, there exists an  $\bar{\alpha}_{k,i}$  such that  $\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i})$  is a convex function of  $z_k^i$  on  $[0, 1]$  for all  $\alpha_i \in [\bar{\alpha}_{k,i}, 1]$ .

*Proof* The proof is given in Appendix A.  $\square$

**Lemma 3** For each  $i \in I$ , let the convex set  $X^i$  be such that  $Y^i$  is a convex set. If Assumption 2 holds for the elliptical distributions listed in Table 1,  $\tilde{S}_{\alpha_i}^i$ ,  $i \in I$ , defined by (8) is a convex set for all  $\alpha_i \in [\bar{\alpha}_i, 1]$ ;  $\bar{\alpha}_i$  is given in Table 1.

*Proof* It is evident that  $(\mu_k^i)^T e^{y^i}$  is a convex function of  $y^i$  because the components of  $\mu_k^i$  are nonnegative. From Lemma 2,  $\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i})$ ,  $k = 1, 2, \dots, K_i$ , is a convex function of  $z_k^i$  for all  $\alpha_i \in [\bar{\alpha}_i, 1]$ . Therefore, all the components of the vector of composition functions  $e^{\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i}) \cdot \mathbb{1}_{l_i + y^i}}$  are nonnegative and convex function of  $(y^i, z_k^i)$ . The entries of  $\Sigma_k^i$  are nonnegative. Then, it follows from Lemma 1 that  $\left\| (\Sigma_k^i)^{\frac{1}{2}} e^{\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i}) \cdot \mathbb{1}_{l_i + y^i}} \right\|$  is a convex function of  $(y^i, z_k^i)$ . Therefore, the constraints

$$(\mu_k^i)^T e^{y^i} + \left\| (\Sigma_k^i)^{\frac{1}{2}} e^{\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i}) \cdot \mathbb{1}_{l_i + y^i}} \right\| \leq b_k^i, \quad \forall k = 1, 2, \dots, K_i,$$

are convex in  $(y^i, z^i)$  for all  $\alpha_i \in [\bar{\alpha}_i, 1]$ . The other constraints of  $\tilde{S}_{\alpha_i}^i$  are convex constraints. Hence,  $\tilde{S}_{\alpha_i}^i$ ,  $i \in I$ , is a convex set for all  $\alpha_i \in [\bar{\alpha}_i, 1]$ .  $\square$



Finally, we show that there exists a Nash equilibrium of a CCG for which we need the following assumption on the players' payoff functions.

**Assumption 3** For each player  $i$ ,  $i \in I$ , the payoff function  $u_i(\cdot, x^{-i})$  satisfies the non-increasing condition for every  $x^{-i} \in X^{-i}$ , i.e., for any two points  $x^i$  and  $\bar{x}^i$  such that  $x_k^i \leq \bar{x}_k^i$  for all  $k = 1, 2, \dots, l_i$ , we have  $u_i(x^i, x^{-i}) \geq u_i(\bar{x}^i, x^{-i})$ .

**Theorem 1** Consider an  $n$ -player CCG where

1. Assumptions 1 and 3 holds,
2. the row vectors of  $A^i$ ,  $i \in I$  are independent,
3. for each  $k = 1, 2, \dots, K_i$ , the row vector  $A_k^i \sim \text{Ellip}_{l_i}(\mu_k^i, \Sigma_k^i; \varphi_k^i)$  w.r.t. Table 1, where the location vector  $\mu_k^i$  and the scale matrix  $\Sigma_k^i$  satisfy Assumption 2.

Then, there exists a Nash equilibrium of a CCG for all  $\alpha \in [\bar{\alpha}_1, 1] \times [\bar{\alpha}_2, 1] \times \dots \times [\bar{\alpha}_n, 1]$ ;  $\bar{\alpha}_i$ ,  $i \in I$ , is given in Table 1.

*Proof* Let  $\alpha \in [\bar{\alpha}_1, 1] \times [\bar{\alpha}_2, 1] \times \dots \times [\bar{\alpha}_n, 1]$ . Under Assumption 1, the payoff function  $u_i(x^i, x^{-i})$ ,  $i \in I$ , is a concave function of  $x^i$  for every  $x^{-i} \in X^{-i}$ , and a continuous function of  $x$ . For each  $i \in I$ , define a composition function  $V_i = u_i \circ d_i$ , where  $d_i : \prod_{i=1}^n \mathbb{R}^{l_i} \rightarrow \prod_{i=1}^n \mathbb{R}_{++}^{l_i}$  such that

$$d_i(y^1, y^2, \dots, y^n) = (e^{y^1}, e^{y^2}, \dots, e^{y^n}).$$

Under Assumption 3,  $V_i(y^i, y^{-i})$  is a concave function of  $y^i$  for all  $y^{-i}$ . The function  $V_i(y)$  is a continuous function of  $y$ . It follows from Lemma 3 that the reformulated strategy set  $\tilde{S}_{\alpha_i}^i$  of player  $i$ ,  $i \in I$ , is a convex set. For each  $i \in I$ ,  $\tilde{S}_{\alpha_i}^i$  is also a compact set. Then, there exists a Nash equilibrium for an  $n$ -player non-cooperative game with the payoff functions  $V_i(y)$ ,  $i \in I$ , and the strategy sets  $\tilde{S}_{\alpha_i}^i$ ,  $i \in I$  [10, 11]. Therefore, there exists  $(y^{i*}, z^{i*})_{i \in I}$  such that for each  $i \in I$ ,

$$V_i(y^{i*}, y^{-i*}) \geq V_i(y^i, y^{-i*}), \forall (y^i, z^i) \in \tilde{S}_{\alpha_i}^i.$$

For  $x^* = e^{y^*}$  we have for each  $i \in I$ ,

$$u_i(x^{i*}, x^{-i*}) \geq u_i(x^i, x^{-i*}), \forall x^i \in S_{\alpha_i}^i.$$

Hence,  $x^*$  is a Nash equilibrium for a CCG for all  $\alpha \in [\bar{\alpha}_1, 1] \times [\bar{\alpha}_2, 1] \times \dots \times [\bar{\alpha}_n, 1]$ .  $\square$

A natural question is whether we can extend Theorem 1 for the entire elliptical family of distributions. The results on the existence of a Nash equilibrium are mainly based on the fixed point arguments of a certain set-valued map, which in our case depends on the convexity of function  $\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i})$ . We give an example of an 1-dimensional spherical distribution for which it is not a convex function. Due to this, the present approach cannot be used to prove the existence of Nash equilibrium for the entire class of elliptical distributions.

*Example 1* Consider an 1-dimensional spherical distribution with probability density function

$$f(y) = \begin{cases} \frac{3}{2}y^2, & y \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

The cumulative distribution function  $\Psi$  can be expressed as

$$\Psi(y) = \begin{cases} 0, & (-\infty, -1), \\ \frac{1}{2}y^3 + \frac{1}{2}, & y \in [-1, 1], \\ 1, & (1, +\infty). \end{cases}$$

Let  $h(z) = \log(\Psi)^{-1}(\alpha^z)$ . We have  $h(z) = \frac{1}{3} \log(2\alpha^z - 1)$ . For  $\alpha > \frac{1}{2}$ , the second derivative of  $h(z)$  is given by

$$h''(z) = \frac{2}{3}(\log \alpha)^2 \alpha^z \left( \frac{-1}{(2\alpha^z - 1)^2} \right).$$

It is easy to see that  $h''(z) < 0$ . This implies that  $\log(\Psi)^{-1}(\alpha^z)$  is not a convex function.

#### 4 Existence of Nash equilibrium under a finite mixture of elliptical distributions

We assume that the probability distribution of row vector  $A_k^i$ ,  $k = 1, 2, \dots, K_i$ , is a finite mixture of elliptical distributions listed in Table 1. The density function  $f_k^i(\mathbf{u})$  of  $A_k^i$  is defined by

$$f_k^i(\mathbf{u}) = \sum_{m=1}^{M_k^i} w_{k,m}^i f_{k,m}^i(\mathbf{u}), \quad (9)$$

where  $f_{k,m}^i(\mathbf{u})$  is the density function of an elliptical distribution from Table 1 with location parameter  $\mu_{k,m}^i$ , scale matrix  $\Sigma_{k,m}^i$ , and characteristic generator  $\varphi_{k,m}^i$ . In addition,  $(w_{k,m}^i)_{m=1}^{M_k^i}$  is a weight vector such that  $w_{k,m}^i \geq 0$  for all  $m = 1, 2, \dots, M_k^i$ , and  $\sum_{m=1}^{M_k^i} w_{k,m}^i = 1$ . From the definition of the mixture distribution, we have

$$\mathbb{P}\{A_k^i x^i \leq b_k^i\} = \int_{A_k^i x^i \leq b_k^i} f_k^i(A_k^i) dA_k^i = \sum_{m=1}^{M_k^i} w_{k,m}^i \int_{A_k^i x^i \leq b_k^i} f_{k,m}^i(A_k^i) dA_k^i, \quad (10)$$

In this case, the joint chance constraint (2) can be reformulated as

$$\begin{cases} \sum_{m=1}^{M_k^i} w_{k,m}^i \mathbb{P}_{\sim P_m} \{A_k^i x^i \leq b_k^i\} \geq \alpha_i^{z_k^i}, & k = 1, 2, \dots, K_i, \\ \sum_{k=1}^{K_i} z_k^i = 1, & z_k^i \geq 0, \quad k = 1, 2, \dots, K_i, \end{cases} \quad (11)$$

where  $\mathbb{P}_{\sim P_m}\{\cdot\}$  denotes the probability function of the  $m$ th probability distribution  $P_m$ . We introduce an auxiliary variables  $\eta_{k,m}^i$ ,  $m = 1, 2, \dots, M_k^i$ ,  $k = 1, 2, \dots, K_i$ , and reformulate the constraint (11) as follows

$$\begin{cases} \mathbb{P}_{\sim P_m}\{A_k^i x^i \leq b_k^i\} \geq \eta_{k,m}^i, & m = 1, 2, \dots, M_k^i, k = 1, 2, \dots, K_i, \\ \sum_{m=1}^{M_k^i} w_{k,m}^i \eta_{k,m}^i \geq \alpha_i^{z_k^i}, & k = 1, 2, \dots, K_i \\ \sum_{k=1}^{K_i} z_k^i = 1, & z_k^i \geq 0, k = 1, 2, \dots, K_i, \end{cases} \quad (12)$$

which is equivalent to

$$\begin{cases} (\mu_{k,m}^i)^T x^i + (\Psi_{k,m}^i)^{-1}(\eta_{k,m}^i) \|(\Sigma_{k,m}^i)^{\frac{1}{2}} x^i\| \leq b_k^i, & m = 1, 2, \dots, M_k^i, k = 1, 2, \dots, K_i, \\ \sum_{m=1}^{M_k^i} w_{k,m}^i \eta_{k,m}^i \geq \alpha_i^{z_k^i}, & k = 1, 2, \dots, K_i, \\ \sum_{k=1}^{K_i} z_k^i = 1, & z_k^i \geq 0, k = 1, 2, \dots, K_i, \end{cases} \quad (13)$$

where  $(\Psi_{k,m}^i)^{-1}(\cdot)$  is a quantile function of one dimensional spherical distribution. By taking  $r_{k,m}^i = (\Psi_{k,m}^i)^{-1}(\eta_{k,m}^i)$ , we have the following reformulation of joint chance constraint (2)

$$Q_{\alpha_i}^i = \begin{cases} (i) (\mu_{k,m}^i)^T x^i + r_{k,m}^i \|(\Sigma_{k,m}^i)^{\frac{1}{2}} x^i\| \leq b_k^i, & m = 1, 2, \dots, M_k^i, k = 1, 2, \dots, K_i, \\ (ii) \sum_{m=1}^{M_k^i} w_{k,m}^i \Psi_{k,m}^i(r_{k,m}^i) \geq \alpha_i^{z_k^i}, & k = 1, 2, \dots, K_i, \\ (iii) \sum_{k=1}^{K_i} z_k^i = 1, \\ (iv) z_k^i \geq 0, & k = 1, 2, \dots, K_i. \end{cases} \quad (14)$$

The reformulation (14) is not convex in the new variables  $(x^i, (r_{k,m}^i)_{k=1, m=1}^{K_i, M_k^i}, (z_k^i)_{k=1}^{K_i})$ . To apply the logarithmic transformation as in the previous case, it is necessary to have  $r_{k,m}^i > 0$  for all  $k$  and  $m$ . Next, we show that for each  $k$  and  $m$ ,  $r_{k,m}^i > 0$ , under certain conditions on weight vectors and confidence level.

**Lemma 4** For a given  $\alpha_i \in [0, 1]$  and for each  $m = 1, 2, \dots, M_k^i$  and  $k = 1, 2, \dots, K_i$  if  $w_{k,m}^i \geq \frac{1}{1-C}(1-\alpha_i)$  such that  $\sum_{m=1}^{M_k^i} w_{k,m}^i = 1$ , where  $\frac{1}{2} < C < 1$ , then  $r_{k,m}^i \geq (\Psi_{k,m}^i)^{-1}(C) > 0$ .

*Proof* For each  $k$  and  $m$ , it follows from constraint (ii) of (14) that

$$1 - w_{k,m}^i + w_{k,m}^i \Psi_{k,m}^i(r_{k,m}^i) \geq \alpha_i^{z_k^i}.$$

Therefore,

$$\Psi_{k,m}^i(r_{k,m}^i) \geq 1 - \frac{1 - \alpha_i^{z_k^i}}{w_{k,m}^i}.$$

For  $0 \leq z_k^i \leq 1$ , we have  $\alpha_i^{z_k^i} \geq \alpha_i$ . Then, under the condition on the weight vector, we have  $r_{k,m}^i \geq (\Psi_{k,m}^i)^{-1}(C) > 0$ .  $\square$

For the elliptical distributions listed in Table 1, we derive the bounds on confidence level and weights for which the condition of Lemma 4 is satisfied. Let  $w_{k,m}^i \in [\bar{w}_i, 1]$  for all  $k$  and  $m$ , where  $\bar{w}_i$  is defined as in the third column of Table 2. Such a choice of  $w_{k,m}^i$  does not need to form a weight vector for an arbitrary choice of  $\alpha_i$ . The condition  $0 \leq w_{k,m}^i \leq 1$  gives  $\alpha_i \geq 1 - \frac{1}{\beta_{k,m}^i}$ , and the condition  $\sum_{m=1}^{M_k^i} w_{k,m}^i = 1$  leads to  $\alpha_i \geq 1 - \frac{1}{\sum_{m=1}^{M_k^i} \beta_{k,m}^i}$ . The uniform lower bound  $\bar{\alpha}_i$  on  $\alpha_i$  can be obtained by taking into account all the constraints and it is given in the last column of Table 2. Therefore, for  $\alpha_i \in [\bar{\alpha}_i, 1]$  and  $w_{k,m}^i \in [\bar{w}_i, 1]$  for all  $k$  and  $m$ , it is easy to see that the condition of Lemma 4 is satisfied. Therefore,  $r_{k,m}^i > 0$  for all  $k$  and  $m$ .

Table 2: Bounds for confidence level and weights used in the mixture distribution

Probability distribution	$\beta_{k,m}^i$	$\bar{w}_i$	$\bar{\alpha}_i$
normal	$\frac{1}{1 - \Psi_{k,m}^i(1)}$	$\max_{\substack{k=1,2,\dots,K_i, \\ m=1,2,\dots,M_k^i}} \{\beta_{k,m}^i(1 - \alpha_i)\}$	$\max_{k=1,2,\dots,K_i} \left\{ 1 - \frac{1}{\sum_{m=1}^{M_k^i} \beta_{k,m}^i} \right\}$
$t$	$\frac{1}{1 - \Psi_{k,m}^i(1)}$		
Cauchy	$\frac{1}{1 - \Psi_{k,m}^i(1)}$		
Laplace	$\frac{1}{1 - \Psi_{k,m}^i(1)}$		
Kotz type	$\frac{1}{1 - \Psi_{k,m}^i\left(\left(\frac{2N-1}{2sr}\right)^{\frac{1}{2s}}\right)}$		
Pearson Type VII	$\frac{1}{1 - \Psi_{k,m}^i\left(\sqrt{\frac{m}{2N-1}}\right)}$		

Now, we reformulate (14) by using the change of variables under logarithmic transformation. We transform the vector  $x^i \in X^i$  into a vector  $y^i \in \mathbb{R}^{l_i}$ , where  $y_j^i = \ln x_j^i$ ,  $j = 1, 2, \dots, l_i$ , and  $\tau_{k,m}^i = \ln r_{k,m}^i$ ,  $m = 1, 2, \dots, M_k^i$ ,  $k =$

$1, 2, \dots, K_i$ . We have the following reformulation of (14)

$$\tilde{Q}_{\alpha_i}^i = \begin{cases} (i) \ (\mu_{k,m}^i)^T e^{y^i} + \|(\Sigma_{k,m}^i)^{\frac{1}{2}} e^{y^i + \tau_{k,m}^i \cdot \mathbb{1}_{l_i}}\| \leq b_k^i, \\ \qquad \qquad \qquad m = 1, 2, \dots, M_k^i, \ k = 1, 2, \dots, K_i, \\ (ii) \ \sum_{m=1}^{M_k^i} w_{k,m}^i \Psi_{k,m}^i(e^{\tau_{k,m}^i}) \geq \alpha_i^{z_k^i}, \ k = 1, 2, \dots, K_i, \\ (iii) \ \sum_{k=1}^{K_i} z_k^i = 1, \\ (iv) \ z_k^i \geq 0, \ k = 1, 2, \dots, K_i, \end{cases} \quad (15)$$

The reformulation of the feasible strategy set  $S^i(\alpha_i)$  of player  $i$ ,  $i \in I$ , is given by

$$\tilde{S}_{\alpha_i}^i = \left\{ (y^i, z^i, \tau_1^i, \dots, \tau_{K_i}^i) \in Y^i \times \mathbb{R}_+^{K_i} \times \mathbb{R}^{M_1^i} \times \dots \times \mathbb{R}^{M_{K_i}^i} \mid (y^i, z^i, \tau_1^i, \dots, \tau_{K_i}^i) \in \tilde{Q}_{\alpha_i}^i \right\}. \quad (16)$$

It is easy to see that  $\tilde{S}_{\alpha_i}^i$  is a compact set. As similar to Assumption 2, we have the following assumption on the location and scale parameters of the elliptical distributions used in the mixture.

**Assumption 4** For each  $i \in I$ ,  $m = 1, 2, \dots, M_k^i$ ,  $k = 1, 2, \dots, K_i$ , all the components of  $\Sigma_{k,m}^i$  and  $\mu_{k,m}^i$  are nonnegative.

To prove  $\tilde{S}_{\alpha_i}^i$  a convex set for a given  $\alpha_i$ , it is enough to show that the constraints (i) and (ii) of (15) are convex.

**Lemma 5** For each  $i \in I$ , suppose  $\alpha_i \in [\bar{\alpha}_i, 1]$  and  $w_{k,m}^i \in [\bar{w}_i, 1]$  for all  $k = 1, 2, \dots, K_i$  and  $m = 1, 2, \dots, M_k^i$ , where  $\bar{\alpha}_i$  and  $\bar{w}_i$  are given as in Table 2. If  $\Psi_{k,m}^i(\cdot)$  is one dimensional distribution function of spherical distributions listed in Table 2,  $\Psi_{k,m}^i(e^{\tau_{k,m}^i})$  is a concave function of  $\tau_{k,m}^i$ .

*Proof* The proof is given in Appendix B.  $\square$

**Lemma 6** For each  $i \in I$ , let the convex set  $X^i$  be such that  $Y^i$  is a convex set. Suppose that the probability distributions used in the mixture are from the list given in Table 2. Then, under Assumption 4, the strategy set  $\tilde{S}_{\alpha_i}^i$ ,  $i \in I$ , defined by (16) is a convex set for all  $\alpha_i \in [\bar{\alpha}_i, 1]$  and  $w_{k,m}^i \in [\bar{w}_i, 1]$ ,  $m = 1, 2, \dots, M_k^i$ ,  $k = 1, 2, \dots, K_i$ ;  $\bar{\alpha}_i$  and  $\bar{w}_i$  are given in Table 2.

*Proof* As a nonnegative linear combination of concave function is a concave function, it follows from Lemma 5 that  $\sum_{m=1}^{M_k^i} w_{k,m}^i \Psi_{k,m}^i(e^{\tau_{k,m}^i}) - \alpha_i^{z_k^i}$  is a concave function of  $(\tau_{k,m}^i)_{m=1}^{M_k^i}, z_k^i$ . Hence, for each  $k = 1, 2, \dots, K_i$ , the constraint (ii) of (15) is a convex constraint. Under Assumption 4, it follows

from Lemma 1 that  $\|(\Sigma_{k,m}^i)^{\frac{1}{2}} e^{y^i + \tau_{k,m}^i \cdot \mathbb{1}_{l_i}}\|$  is a convex function of  $(y^i, \tau_{k,m}^i)$  for each  $k$  and each  $m$ . For each  $k$  and each  $m$ ,  $(\mu_{k,m}^i)^T e^{y^i}$  is a convex function of  $y^i$  because  $\mu_{k,m}^i \geq 0$ . Therefore, for each  $k$  and each  $m$  the constraint (i) of (15) is a convex constraint. It is easy to see that the other constraints of  $\tilde{S}_{\alpha_i}^i$  are convex constraints. Therefore,  $\tilde{S}_{\alpha_i}^i$  is a convex set.  $\square$

Finally, thanks to the convex reformulation of feasible strategy set for each player, we show that there exists a Nash equilibrium.

**Theorem 2** Consider an  $n$ -player CCG where

1. Assumptions 1 and 3 holds,
2. the row vectors of  $A^i$ ,  $i \in I$  are independent,
3. for each  $k = 1, 2, \dots, K_i$ , the row vector  $A_k^i$ ,  $i \in I$ , follows a finite mixture of multivariate elliptical distributions listed in the Table 2, where the location vector  $\mu_{k,m}^i$  and the scale matrix  $\Sigma_{k,m}^i$  of the  $m$ th elliptical distribution satisfy Assumption 4.
4. for each  $i \in I$ , the weights  $w_{k,m}^i \in [\bar{w}_i, 1]$ ,  $k = 1, 2, \dots, K_i$ ,  $m = 1, 2, \dots, M_k^i$ , where  $\bar{w}_i$  is given in Table 2.

Then, there exists a Nash equilibrium of the CCG for all  $\alpha \in [\bar{\alpha}_1, 1] \times [\bar{\alpha}_2, 1] \times \dots \times [\bar{\alpha}_n, 1]$ ;  $\bar{\alpha}_i$ ,  $i \in I$ , is given in Table 2.

*Proof* It follows from Lemma 6 that the reformulated strategy set  $\tilde{S}_{\alpha_i}^i$ ,  $i \in I$ , is convex for all  $\alpha \in [\bar{\alpha}_i, 1]$ . Then, the proof follows from the similar arguments used in Theorem 1.  $\square$

## 5 Competition in financial market

We consider the case of two investment firms which compete on the same market. Each firm invests in the same set of portfolios consisting of different assets. Let  $J$  be the set of portfolios and  $\mathcal{A}_k$  be the set of assets in the portfolio  $k$ . We assume that the portfolios are disjoint, i.e., for any  $k, l \in J$ ,  $\mathcal{A}_k$  and  $\mathcal{A}_l$  are disjoint sets. Each firm decides to invest its money in different portfolios whose returns are defined by random vectors. Let  $x_k^i = (x_{kj}^i)_{j \in \mathcal{A}_k}$  be the investment vector of firm  $i$ ,  $i = 1, 2$ , for  $k$ th portfolio;  $x_{kj}^i$  represents the amount of money firm  $i$ , invested in the  $j$ th asset of  $k$ th portfolio. Let  $x^i = (x_k^i)_{k \in J}$  denotes an investment vector of firm  $i$ . The set of investments  $X^i$  of firm  $i$  is defined as

$$X^i = \left\{ x^i \mid \sum_{j \in \mathcal{A}_k} x_{kj}^i \leq W_k^i, x_{kj}^i \geq \varepsilon_i, \forall k \in J, j \in \mathcal{A}_k \right\},$$

where  $\varepsilon_i > 0$  is the minimum amount invested in any asset by firm  $i$  and  $W_k^i$  denotes the maximum amount that can be invested by firm  $i$  in portfolio  $k$ . The vector  $x = (x^1, x^2) \in X^1 \times X^2$  represents an investment profile of

both the firms. Let  $R_k^i = (R_{kj}^i)_{j \in \mathcal{A}_k}$  be a random return vector of firm  $i$  from portfolio  $k$ . For a given investment vector  $x_k^i$ , the total return received by firm  $i$  from portfolio  $k$  is defined as  $(x_k^i)^T R_k^i$  and  $(-x_k^i)^T R_k^i$  represents random loss incurred by firm  $i$  (negative return) from portfolio  $k$ . Let  $d_k^i$  be the maximal allowable loss level of firm  $i$  from portfolio  $k$ . Firm  $i$  wants to keep its random losses below the maximal allowable level with at least  $\alpha_i$  probability, i.e.,

$$P \{ (-x_k^i)^T R_k^i \leq d_k^i, k \in J \} \geq \alpha_i, \forall i = 1, 2.$$

Therefore, the set of feasible investments of firm  $i$  is given by

$$S_{\alpha_i}^i = \left\{ x^i \in X^i \mid P \{ (-x_k^i)^T R_k^i \leq d_k^i, k \in J \} \geq \alpha_i \right\}. \quad (17)$$

We consider the case where each firm incurs transaction costs. The transaction costs can be used to model a number of costs involved in a financial market, e.g., brokerage fees, bid-ask spread and market impact [19, 21]. Usually the transaction costs are higher for popular assets due to the participation of many firms and it is directly proportional to the money invested in it. Therefore, the transaction cost incurred by a firm also depends on the investment of other firms. In this paper, we consider the quadratic transaction costs, i.e., for a given investment profile  $(x_k^1, x_k^2)$  for portfolio  $k$ , the firm 1 incurs cost  $c_k^1 (x_k^1)^T (x_k^1 + x_k^2)$  and firm 2 incurs cost  $c_k^2 (x_k^2)^T (x_k^1 + x_k^2)$  to manage portfolio  $k$ . Then, for a given investment profile  $x = (x^1, x^2)$ , the payoff function of firm  $i$ ,  $i = 1, 2$ , is given by

$$u_i(x^1, x^2) = \sum_{k \in J} (\mathbb{E}[R_k^i])^T x_k^i - \sum_{k \in J} c_k^i (x_k^i)^T (x_k^1 + x_k^2).$$

Under the logarithmic transformation, the payoff functions of firm 1 and firm 2, are given by

$$\begin{aligned} V_1(y^1, y^2) &= \sum_{k \in J} (\mathbb{E}[R_k^1] - c_k^1 e^{y_k^2})^T e^{y_k^1} - \sum_{k \in J} c_k^1 \sum_{j \in \mathcal{A}_k} e^{2y_{kj}^1}, \\ V_2(y^1, y^2) &= \sum_{k \in J} (\mathbb{E}[R_k^2] - c_k^2 e^{y_k^1})^T e^{y_k^2} - \sum_{k \in J} c_k^2 \sum_{j \in \mathcal{A}_k} e^{2y_{kj}^2}, \end{aligned}$$

where  $e^{y_k^i} = (e^{y_{kj}^i})_{j \in \mathcal{A}_k}$ ,  $i = 1, 2$ .

For a fixed  $y^2$ , the function  $V_1(y^1, y^2)$  is a concave function of  $y^1$  if its Hessian matrix is negative semidefinite. Therefore,  $V_1(y^1, y^2)$  is a concave function of  $y^1$  for a given  $y^2$  if

$$4c_k^1 e^{y_{kj}^1} + c_k^1 e^{y_{kj}^2} \geq \mathbb{E}[R_{kj}^1], j \in \mathcal{A}_k, k \in J. \quad (18)$$

Similarly,  $V_2(y^1, y^2)$  is a concave function of  $y^2$  for a given  $y^1$  if

$$4c_k^2 e^{y_{kj}^2} + c_k^2 e^{y_{kj}^1} \geq \mathbb{E}[R_{kj}^2], j \in \mathcal{A}_k, k \in J. \quad (19)$$

The conditions (18) and (19) are satisfied for suitable choice of the minimum invested amount  $\varepsilon_i$  of firm  $i$ . The mixture of elliptical distributions are often

used in the literature to model the random return vectors in financial market [3, 13, 17]. We assume that the random return vectors  $R_k^i$ ,  $k \in J$ , are independent and follow a mixture of elliptical distributions. Then, under logarithmic transformation the convex reformulation  $\tilde{S}_{\alpha_i}^i$  of (17) is given by (16), where

$$Y^i = \left\{ y^i \mid \sum_{j \in \mathcal{A}_k} e^{y_{kj}^i} \leq W_k^i, y_{kj}^i \geq \ln(\varepsilon_i), \forall k \in J, j \in \mathcal{A}_k \right\}.$$

If the conditions (18) and (19) are satisfied, it follows from Theorem 2 that there exists a Nash equilibrium of the game.

### 5.1 Best response algorithm

In this section, we present a best response algorithm which is used to compute the Nash equilibria of the above game problem. For a fixed  $(y^2, z^2, (\tau_k^2)_{k \in J}) \in \tilde{S}_{\alpha_2}^2$ , player 1 solves the following convex optimization problem

$$\begin{aligned} [P_1] \quad & \max_{y^1, z^1, (\tau_k^1)_{k \in J}} V_1(y^1, y^2) \\ \text{s.t.} \quad & (y^1, z^1, (\tau_k^1)_{k \in J}) \in \tilde{S}_{\alpha_1}^1. \end{aligned}$$

The set of optimal solution of  $[P_1]$ , which is also called the best response set of player 1, is given by

$$BR_1(y^2) = \{(\bar{y}^1, \bar{z}^1, (\bar{\tau}_k^1)_{k \in J}) \mid V_1(\bar{y}^1, y^2) \geq V_1(y^1, y^2), \forall (y^1, z^1, (\tau_k^1)_{k \in J}) \in \tilde{S}_{\alpha_1}^1\}.$$

Similarly, for a fixed  $(y^1, z^1, (\tau_k^1)_{k \in J}) \in \tilde{S}_{\alpha_1}^1$ , player 2 solves the following convex optimization problem

$$\begin{aligned} [P_2] \quad & \max_{y^2, z^2, (\tau_k^2)_{k \in J}} V_2(y^1, y^2) \\ \text{s.t.} \quad & (y^2, z^2, (\tau_k^2)_{k \in J}) \in \tilde{S}_{\alpha_2}^2. \end{aligned}$$

The best response set of player 2, is given by

$$BR_2(y^1) = \{(\bar{y}^2, \bar{z}^2, (\bar{\tau}_k^2)_{k \in J}) \mid V_2(y^1, \bar{y}^2) \geq V_2(y^1, y^2), \forall (y^2, z^2, (\tau_k^2)_{k \in J}) \in \tilde{S}_{\alpha_2}^2\}.$$

It is clear that, if  $(y^{1*}, z^{1*}, (\tau_k^{1*})_{k \in J}) \in BR_1(y^{2*})$  and  $(y^{2*}, z^{2*}, (\tau_k^{2*})_{k \in J}) \in BR_2(y^{1*})$ ,  $(x^{1*}, x^{2*}) = (e^{y^{1*}}, e^{y^{2*}})$  is a Nash equilibrium of the game. For computational purpose, we use the best response algorithm as outlined below:

**Algorithm 1 (Best response algorithm)**    **Step-1** *Select initial feasible point  $(y^{2(0)}, z^{2(0)}, (\tau_k^{2(0)})_{k \in J}) \in \tilde{S}_{\alpha_2}^2$  for player 2. Set  $n := 0$ .*  
**Step-2** *Solve convex optimization problem  $[P_1]$  and find a point  $(y^{1(n)}, z^{1(n)}, (\tau_k^{1(n)})_{k \in J}) \in BR_1(y^{2(n)})$ .*



**Step-3** If  $(y^{2(n)}, z^{2(n)}, (\tau_k^{2(n)})_{k \in J}) \in BR_1(y^{1(n)})$ , then set  $(x^{1*}, x^{2*}) = (e^{y^{1(n)}}, e^{y^{2(n)}})$  and stop. Otherwise, solve convex optimization problem  $[P_2]$  and find a point  $(y^{2(n)}, z^{2(n)}, (\tau_k^{2(n)})_{k \in J}) \in BR_1(y^{1(n)})$ , set  $n = n + 1$  and go to step 2.

If the Algorithm 1 stops,  $(x^{1*}, x^{2*})$  is a Nash equilibrium of the game. The proof that Algorithm 1 never cycles is still an open problem.

## 5.2 Case Study

*Example 2* We consider two investment firms with two portfolios where each portfolio consists of three assets, i.e.,  $J = \{1, 2\}$  and  $\mathcal{A}_k = \{1, 2, 3\}, k \in J$ . We take the confidence level values of firm 1 and firm 2 as 0.95, i.e.,  $\alpha_1 = \alpha_2 = 0.95$ . The probability distribution of the return vector  $R_k^i$  is assumed to be an equiprobable mixture of two normal distributions  $\mathcal{N}(\mu_{k,1}^i, \Sigma_{k,1}^i)$  and  $\mathcal{N}(\mu_{k,2}^i, \Sigma_{k,2}^i)$ , where  $\mu_{k,m}^i$  is generated uniformly on  $[0.1, 0.3]^3$ , and  $\Sigma_{k,m}^i$  is generated randomly with each element drawn from  $[0, 0.1]$  uniformly. The randomly generated mean vectors and covariance matrices are summarized as follows:

$$\mu_{1,1}^1 = \begin{pmatrix} 0.1131 \\ 0.1995 \\ 0.2909 \end{pmatrix}, \mu_{1,2}^1 = \begin{pmatrix} 0.2991 \\ 0.1587 \\ 0.1081 \end{pmatrix}, \mu_{2,1}^1 = \begin{pmatrix} 0.1092 \\ 0.2006 \\ 0.2921 \end{pmatrix}, \mu_{2,2}^1 = \begin{pmatrix} 0.2967 \\ 0.2022 \\ 0.1045 \end{pmatrix}.$$

$$\mu_{1,1}^2 = \begin{pmatrix} 0.1103 \\ 0.2022 \\ 0.2931 \end{pmatrix}, \mu_{1,2}^2 = \begin{pmatrix} 0.2896 \\ 0.2011 \\ 0.1052 \end{pmatrix}, \mu_{2,1}^2 = \begin{pmatrix} 0.1048 \\ 0.1988 \\ 0.2957 \end{pmatrix}, \mu_{2,2}^2 = \begin{pmatrix} 0.2997 \\ 0.2053 \\ 0.1052 \end{pmatrix}.$$

$$\Sigma_{1,1}^1 = \begin{pmatrix} 0.1 & 0.0927 & 0.0202 \\ 0.0927 & 0.1 & 0.0187 \\ 0.0202 & 0.0187 & 0.1 \end{pmatrix}, \Sigma_{1,2}^1 = \begin{pmatrix} 0.1 & 0.0379 & 0.0520 \\ 0.0379 & 0.1 & 0.0473 \\ 0.052 & 0.0473 & 0.1 \end{pmatrix},$$

$$\Sigma_{2,1}^1 = \begin{pmatrix} 0.1 & 0.0589 & 0.0382 \\ 0.0589 & 0.1 & 0.0554 \\ 0.0382 & 0.0554 & 0.1 \end{pmatrix}, \Sigma_{2,2}^1 = \begin{pmatrix} 0.1 & 0.0072 & 0.0250 \\ 0.0072 & 0.1 & 0.0476 \\ 0.025 & 0.0476 & 0.1 \end{pmatrix}.$$

$$\Sigma_{1,1}^2 = \begin{pmatrix} 0.1 & 0.0612 & 0.0049 \\ 0.0612 & 0.1 & 0.0732 \\ 0.0049 & 0.0732 & 0.1 \end{pmatrix}, \Sigma_{1,2}^2 = \begin{pmatrix} 0.1 & 0.031 & 0.0421 \\ 0.031 & 0.1 & 0.0404 \\ 0.0421 & 0.0404 & 0.1 \end{pmatrix},$$

$$\Sigma_{2,1}^2 = \begin{pmatrix} 0.1 & 0.0469 & 0.0245 \\ 0.0469 & 0.1 & 0.0167 \\ 0.0245 & 0.0167 & 0.1 \end{pmatrix}, \Sigma_{2,2}^2 = \begin{pmatrix} 0.1 & 0.023 & 0.0522 \\ 0.023 & 0.1 & 0.0384 \\ 0.0522 & 0.0384 & 0.1 \end{pmatrix}.$$

The values of other parameters are given by  $\varepsilon_1 = 0.3, \varepsilon_2 = 0.25, d_k^1 = 0.75, d_k^2 = 0.6, W_k^1 = W_k^2 = 1, k \in J$  and  $c_k^1 = c_k^2 = 0.2, k \in J$ .

We compute the Nash equilibrium of the game using the Algorithm 1. We run the numerical experiments on an Intel Core i7-8550U CPU 1.80 GHz 2.00 GHz with 16.0 GB RAM machine. To solve the best response convex optimization problem of both firms, we use CVX package in MATLAB. For the above data the algorithm converges to a Nash equilibrium point  $(x^{1*}, x^{2*})$  given by

$$\begin{aligned} x^{1*} &= ((0.3478, 0.3, 0.3249), (0.3431, 0.3319, 0.3250)), \\ x^{2*} &= ((0.3385, 0.3067, 0.3390), (0.3333, 0.3314, 0.3353)). \end{aligned}$$

Figure 1 shows that the best response algorithm converges to a Nash equi-

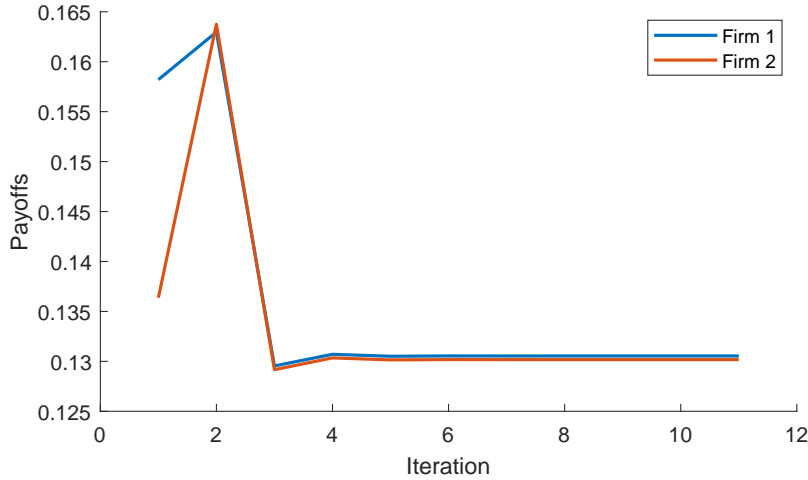


Fig. 1: Convergence of Nash equilibrium payoff

librium payoffs of both the firms after few iterations. The total CPU time to compute Nash equilibrium is 23.12 seconds.

We also perform numerical experiments for a relatively large size model with 10 sets of portfolios and 20 assets in each portfolio. The distribution of  $R_k^i$  is considered as an equiprobable mixture of two normal distributions whose mean vectors and covariance matrices are generated similar to the Example 2. The parameters  $\varepsilon_1$  and  $\varepsilon_2$  are generated uniformly on the interval  $[0.2, 0.3]$ , and  $W_k^1 = W_k^2 = 10$ ,  $c_k^1 = c_k^2 = 0.2$  for all  $k \in J$ , and  $d_k^1, d_k^2, k \in J$ , are drawn randomly from the interval  $[5, 8]$ , and  $\alpha_1 = \alpha_2 = 0.95$ . We consider 20 different instances of this model, and for each instance, the Algorithm 1 converges to a Nash equilibrium. The average time to compute a Nash equilibrium is 805.31 seconds. The Figure 2 represents the variation in the time for different instances.

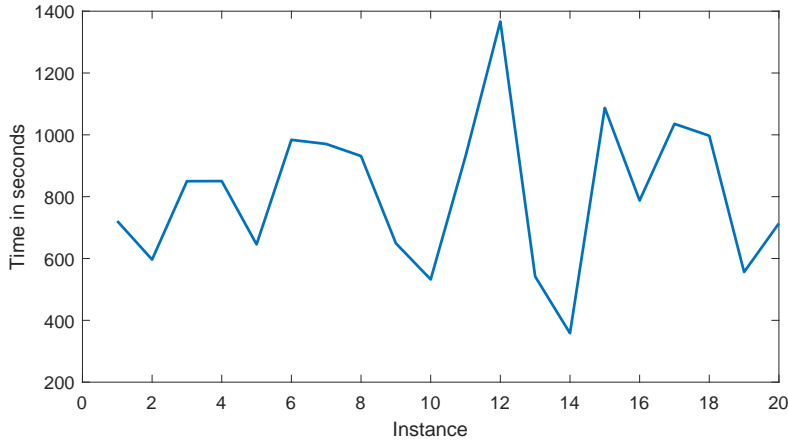


Fig. 2: Time vs Instances

## 6 Conclusion

We consider an  $n$ -player non-cooperative game under joint chance constraints. The random constraint vectors are independent. We identify a list of prominent probability distributions from the class of elliptical distributions. We consider the case when the probability distribution of each random constraint vector is among the identified elliptical distributions as well as the case when it is a finite mixture of the identified elliptical distributions. We derive the bounds for the players' confidence levels and the weights used in the mixture distribution and propose a new convex reformulation of the joint chance constraint when the players' confidence levels and the weights used in the mixture distributions are within the bounds. Under mild conditions, we show that there exists a Nash equilibrium of the CCG. As an application of these games, we study a competition between two investment firms from the same market. The numerical experiments are performed on randomly generated instances of different sizes by using a best response algorithm.

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## Appendix A Proof of Lemma 2

*Proof* Let  $h_1 : [0, 1] \rightarrow [\bar{\alpha}_{k,i}, 1]$  such that  $h_1(z_k^i) = \alpha_i^{z_k^i}$ , and  $h_2 : [\bar{\alpha}_{k,i}, 1] \rightarrow \mathbb{R}$  such that  $h_2(p) = \log(\Psi_k^i)^{-1}(p)$  be two functions. Then, the function composition  $(h_2 \circ h_1)(z_k^i) =$

$\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i})$ . It is easy to see that  $h_1(z_k^i)$  is a convex function of  $z_k^i$  and  $h_2(p)$  is a nondecreasing function. Then, from Lemma 3.11 of [24], it suffices to show that  $h_2(p)$  is a convex function of  $p$  in order to show  $\log(\Psi_k^i)^{-1}(\alpha_i^{z_k^i})$  is a convex function. The second order derivative of  $h_2(p)$  is given by

$$-\frac{ug_k^i(u) + g_k^i(u)}{c^2u^2(g_k^i(u))^3},$$

where  $u = (\Psi_k^i)^{-1}(p)$ ,  $g_k^i(u)$  is the radial density function corresponding to distribution function  $\Psi_k^i(u)$ ,  $g_k^i(u)$  is the derivative of  $g_k^i(u)$ . Therefore, the function  $h_2(p)$  is convex if

$$(\Psi_k^i)^{-1}(p)g_k^i((\Psi_k^i)^{-1}(p)) + g_k^i((\Psi_k^i)^{-1}(p)) \leq 0. \quad (20)$$

The verification of (20) for each distribution listed in Table 1 is as follows:

**Normal Distribution:** Using the radial density function of normal distribution given in Table 1, the condition (20) can be written as

$$e^{-\frac{u^2}{2}}(1-u)(1+u) \leq 0, \quad (21)$$

where  $u = (\Psi_k^i)^{-1}(p)$ . From Table 1, the value of  $\bar{\alpha}_{k,i}$  associated with normal distribution is  $\Psi_k^i(1)$ . Since, the function  $g_2(p)$  is defined on  $[\bar{\alpha}_{k,i}, 1]$ ,  $p \geq \bar{\alpha}_{k,i} = \Psi_k^i(1)$ . Hence, the condition (21) is satisfied.

**$t$  distribution:** Using the radial density function of  $t$  distribution given in Table 1, the condition (20) can be written as

$$\left(1 + \frac{u^2}{\nu}\right)^{-\frac{1+\nu}{2}-1} (1+u)(1-u) \leq 0, \quad (22)$$

where  $u = (\Psi_k^i)^{-1}(p)$ . From Table 1, the value of  $\bar{\alpha}_{k,i}$  associated with  $t$  distribution is  $\Psi_k^i(1)$ . From the similar arguments used in case of normal distribution, we have  $p \geq \Psi_k^i(1)$ . Hence, the condition (22) is satisfied.

**Cauchy distribution:** The Cauchy distribution is a special case of  $t$  distribution when  $\nu = 1$ . Therefore, the condition (20) holds in this case using the similar arguments used in case of  $t$  distribution.

**Laplace distribution:** Using the radial density function of Laplace distribution given in Table 1, the condition (20) can be written as

$$e^{-u}(1-u) \leq 0, \quad (23)$$

where  $u = (\Psi_k^i)^{-1}(p)$ . From Table 1, the value of  $\bar{\alpha}_{k,i}$  associated with Laplace distribution is  $\Psi_k^i(1)$  which gives  $p \geq \Psi_k^i(1)$ . Hence, the condition (23) is satisfied.

**Kotz type distribution:** Using the radial density function of Kotz type distribution given in Table 1, the condition (20) can be written as

$$u^{2(N-1)}e^{-ru^{2s}}(2N-1-2rsu^{2s}) \leq 0, \quad (24)$$

where  $u = (\Psi_k^i)^{-1}(p)$ . From Table 1, the value of  $\bar{\alpha}_{k,i}$  associated with Kotz type distribution is  $\Psi_k^i\left(\left(\frac{2N-1}{2sr}\right)^{\frac{1}{2s}}\right)$  which gives  $p \geq \Psi_k^i\left(\left(\frac{2N-1}{2sr}\right)^{\frac{1}{2s}}\right)$ . Hence, the condition

(24) is satisfied.

**Pearson type VII distribution:** Using the radial density function of Pearson Type VII distribution given in Table 1, the condition (20) can be written as

$$\left(1 + \frac{u^2}{m}\right)^{-N-1} \left(1 - \frac{2N-1}{m}u^2\right) \leq 0, \quad (25)$$

where  $u = (\Psi_k^i)^{-1}(p)$ . From Table 1, the value of  $\bar{\alpha}_{k,i}$  associated with Pearson type VII distribution is  $\Psi_k^i\left(\sqrt{\frac{m}{2N-1}}\right)$  which gives  $p \geq \Psi_k^i\left(\sqrt{\frac{m}{2N-1}}\right)$ . Hence, the condition (25) is satisfied.  $\square$

## Appendix B Proof of Lemma 5

*Proof* The second order derivative of  $\Psi_{k,m}^i(e^{\tau_{k,m}^i})$  is

$$\Psi_{k,m}^{\prime\prime i}(e^{\tau_{k,m}^i})(e^{2\tau_{k,m}^i}) + \Psi_{k,m}^{\prime i}(e^{\tau_{k,m}^i})(e^{\tau_{k,m}^i}),$$

where  $\Psi_{k,m}^{\prime i}(\cdot)$  and  $\Psi_{k,m}^{\prime\prime i}(\cdot)$  are the first and second order derivative functions of  $\Psi_{k,m}^i(\cdot)$ , respectively. Let  $g_{k,m}^i(\cdot)$  be a radial density associated with the distribution function  $\Psi_{k,m}^i(\cdot)$ .

Then, the concavity of  $\Psi_{k,m}^i(e^{\tau_{k,m}^i})$  is equivalent to

$$g_{k,m}^{\prime i}(e^{\tau_{k,m}^i})(e^{\tau_{k,m}^i}) + g_{k,m}^i(e^{\tau_{k,m}^i}) \leq 0. \quad (26)$$

The verification of (26) for each distribution listed in Table 2 is as follows:

**Normal distribution:** The radial density function of normal distribution is  $g_{k,m}^i(u) = e^{-\frac{u^2}{2}}$  and its first derivative is  $g_{k,m}^{\prime i}(u) = -ue^{-\frac{u^2}{2}}$ . The condition (26) can be written as

$$e^{-\frac{u^2}{2}}(1-u)(1+u) \leq 0, \quad (27)$$

where  $u = e^{\tau_{k,m}^i}$ . From Table 2, the weight  $w_{k,m}^i$  associated with normal distribution is such that  $w_{k,m}^i \geq \frac{1}{1-\Psi_{k,m}^i(1)}(1-\alpha_i)$ . Then, from Lemma 4,  $r_{k,m}^i \geq 1$  which in turn implies

that  $e^{\tau_{k,m}^i} \geq 1$ . Hence, the condition (27) is satisfied.

**$t$  distribution:** Using the radial density function of  $t$  distribution given in Table 1, the condition (26) can be written as

$$\left(1 + \frac{u^2}{\nu}\right)^{-\frac{1+\nu}{2}-1} (1+u)(1-u) \leq 0, \quad (28)$$

where  $u = e^{\tau_{k,m}^i}$ . As similar to the case of normal distribution, the weight  $w_{k,m}^i$  associated with  $t$  distribution is such that  $w_{k,m}^i \geq \frac{1}{1-\Psi_{k,m}^i(1)}(1-\alpha_i)$ . From the similar arguments used in the case of normal distribution, the condition (28) is satisfied.

**Cauchy distribution:** The Cauchy distribution is a special case of  $t$  distribution when  $\nu = 1$ . Therefore, the condition (26) holds in this case using the similar arguments used in the case of  $t$  distribution.

**Laplace distribution:** Using the radial density function of Laplace distribution given in Table 1, the condition (26) can be written as

$$e^{-u}(1-u) \leq 0, \quad (29)$$

where  $u = e^{\tau_{k,m}^i}$ . From Table 2, the weight  $w_{k,m}^i$  associated with Laplace distribution is such that  $w_{k,m}^i \geq \frac{1}{1-\psi_{k,m}^i(1)}(1-\alpha_i)$ . Then, from Lemma 4,  $r_{k,m}^i \geq 1$  which in turn implies that  $e^{\tau_{k,m}^i} \geq 1$ . Hence, the condition (29) is satisfied.

**Kotz type distribution:** Using the radial density function of Kotz type distribution given in Table 1, the condition (26) can be written as

$$u^{2(N-1)}e^{-ru^{2s}}(2N-1-2rsu^{2s}) \leq 0, \quad (30)$$

where  $u = e^{\tau_{k,m}^i}$ . From Table 2, the weight  $w_{k,m}^i$  associated with Kotz type distribution is such that  $w_{k,m}^i \geq \frac{1}{1-\psi_{k,m}^i\left(\frac{2N-1}{2sr}\right)^{\frac{1}{2s}}}(1-\alpha_i)$ . Then, from Lemma 4,  $r_{k,m}^i \geq \left(\frac{2N-1}{2sr}\right)^{\frac{1}{2s}}$  which in turn implies that  $e^{\tau_{k,m}^i} \geq \left(\frac{2N-1}{2sr}\right)^{\frac{1}{2s}}$ . Hence, the condition (30) is satisfied.

**Pearson Type VII distribution:** Using the radial density function of Pearson Type VII distribution given in Table 1, the condition (26) can be written as

$$\left(1 + \frac{u^2}{m}\right)^{-N-1} \left(1 - \frac{2N-1}{m}u^2\right) \leq 0, \quad (31)$$

where  $u = e^{\tau_{k,m}^i}$ . From Table 2, the weight  $w_{k,m}^i$  associated with Pearson Type VII is such that  $w_{k,m}^i \geq \frac{1}{1-\psi_{k,m}^i\left(\sqrt{\frac{m}{2N-1}}\right)}(1-\alpha_i)$ . Then, from Lemma 4,  $r_{k,m}^i \geq \sqrt{\frac{m}{2N-1}}$  which in turn implies that  $e^{\tau_{k,m}^i} \geq \sqrt{\frac{m}{2N-1}}$ . Hence, the condition (31) is satisfied.  $\square$

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