Integral input-to-state stability of delay systems based on Lyapunov-Krasovskii functionals with point-wise dissipation rate

Antoine Chaillet, Pierdomenico Pepe

Integral input-to-state stability of delay systems based on Lyapunov-Krasovskii functionals with point-wise dissipation rate

Antoine Chaillet and Pierdomenico Pepe

***Abstract*— We show that a Lyapunov-Krasovskii functional whose dissipation rate involves solely the current instantaneous value of the state norm is enough to guarantee integral input-to-state stability (iISS). This result generalizes existing sufficient conditions for iISS, where the dissipation rate involves the whole Lyapunov-Krasovskii functional itself, and simplifies their applicability. Moreover, it provides a more natural bridge with the classical condition for global asymptotic stability of input-free systems. The proof strategy we employ relies on a novel characterization of global asymptotic stability, which may be of interest on its own.**

I. Introduction

The input-to-state stability (ISS) framework, standard tool in the analysis and control of finite-dimensional systems [12], [14], has more recently extended to infinite-dimensional dynamics, including time-delay systems. A handy tool to establish ISS of time-delay systems is to rely on Lyapunov-Krasovskii functionals (LKF) [10]. More precisely, just like in the finite-dimensional case, ISS is guaranteed provided that the derivative of such functional V along the system's solutions satisfies

$$\dot{V} \leq -\alpha(V) + \gamma(|u|), \tag{1}$$

where α, γ are $\mathcal{K}_\infty$ functions and u is the system's input. In this differential inequality, the dissipation rate α involves the whole LKF V. In [2], we have recently conjectured that ISS would hold also under the less stringent condition that

$$\dot{V} \leq -\alpha(|x(t)|) + \gamma(|u|), \tag{2}$$

meaning with a dissipation rate that involves only the current instantaneous value of the state norm (what we call a point-wise dissipation rate). Such a formulation would simplify the ISS analysis of time-delay systems. It would also homogenize ISS theory with the Lyapunov-Krasovskii theory of input-free systems, as global asymptotic stability of such systems is guaranteed under a point-wise dissipation rate, namely $\dot{V} \leq -\alpha(|x(t)|)$ [4]. Although this simpler characterization of ISS has been established for particular classes of systems in [2], the conjecture has not yet been proved or disproved.

In the present paper, we prove this conjecture for a weaker robustness property, namely integral input-to-state stability (iISS). This property was introduced in [13] for finite-dimensional systems. It imposes that solutions are bounded by a decaying term of the initial state plus a term depending on the energy brought by the input. iISS of time-delay systems can be established in a similar way as ISS: it was shown in [10] that if a dissipation inequality like (1) holds with a positive definite α then iISS holds. A similar result was proposed in [7] in a wider infinite-dimensional context. Here, we show that a point-wise dissipation rate like in (2) (with α merely positive definite) is enough to establish iISS.

The proof of this result requires generalizations, that may be of interest on their own, of existing finite-dimensional results to time-delay systems. More precisely, we show that if a time-delay system is globally asymptotically stable in the absence of inputs (0-GAS), then a dissipation inequality like (1) holds (with α positive definite), but with a LKF V that may not be radially unbounded. We also show that 0-GAS implies iISS provided that the system is zero-output dissipative. These two properties were originally established in [1] for finite-dimensional systems. Finally, we show that iISS holds under the even less conservative assumption that $\dot{V} \leq -\sigma(|x(t)|, \|x_t\|) + \gamma(|u|)$, where $\sigma \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$.

The outline of the paper is as follows. We start by presenting the class of systems considered here and by recalling iISS for time-delay systems and the existing LKF sufficient conditions to establish it. Then, in Section III, we state our main results. An academic example is provided in Section IV. All proofs are provided in Section V. We conclude with some perspective for future work.

Notations. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a set $I \subset \mathbb{R}$ and a measurable signal $u : I \to \mathbb{R}^m$, $\|u\| := \text{ess sup}_{t\in I}|u(t)|$. $\mathcal{U}$ denotes the set of all signals $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ that are measurable and locally essentially bounded. Given $\theta > 0$ and $n \in \mathbb{N}_{\geq 1}$, the set $C([-\theta; 0], \mathbb{R}^n)$ of all continuous functions $\phi : [-\theta; 0] \to \mathbb{R}^n$ is denoted by $\mathcal{X}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{PD}$ if is continuous and positive definite. $\alpha \in \mathcal{K}$ if $\alpha \in \mathcal{PD}$ and it is increasing. $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each $t \in \mathbb{R}_{\geq 0}$ and, for each $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is continuous, non-increasing and tends to zero as its argument tends to infinity.

II. Context and definitions

A. Time-delay systems

The paper considers nonlinear delayed systems of the form

$$\dot{x}(t) = f(x_t, u(t)), \quad \forall t \geq 0 \, a.e. \tag{3}$$

A. Chaillet is with Laboratoire des Signaux et Systèmes, CentraleSupélec, Univ. Paris Saclay, Gif sur Yvette, France and with Institut Universitaire de France antoine.chaillet@centralesupelec.fr

P. Pepe is with University of L'Aquila, Italy pierdomenico.pepe@univaq.it

$x(t) \in \mathbb{R}^n$ is the instantaneous value of the state at time $t \geq 0$. $x_t \in \mathcal{X}$ denotes its history over the time interval $[t-\theta; t]$, where θ is larger than, or equal to, the maximum delay involved in the dynamics. On its domain of existence, $x_t \in \mathcal{X}$ is defined as

$$x_t : \begin{cases} [-\theta; 0] & \to & \mathbb{R}^n \\ s & \mapsto & x(t+s). \end{cases}$$

The input u is assumed to be in the set $\mathcal{U}$. The function $f : \mathcal{X} \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be Lipschitz on any bounded subset of $\mathcal{X} \times \mathbb{R}^m \to \mathbb{R}^n$. These assumptions ensure that, given any $x_0 \in \mathcal{X}$, system (3) admits a unique and locally absolutely continuous solution on a maximal time interval $[0, b)$, $b \in (0; +\infty]$. Moreover, if $b < +\infty$, then the solution is unbounded on $[0, b)$. See [4]. We also assume that $f(0, 0) = 0$.

B. iISS

iISS of such time-delay systems was introduced in [10]. It is a natural extension the iISS property that was originally introduced in a finite-dimensional context [13].

Definition 1 (iISS, [10]) *The delay system* (3) *is integral input-to-state stable (iISS) if there exist* $\beta \in \mathcal{KL}$ *and* $\delta_1, \delta_2 \in \mathcal{K}_\infty$ *such that, given any* $x_0 \in \mathcal{X}$ *and any* $u \in \mathcal{U}$*, the corresponding solution* $x(\cdot) := x(\,\cdot\,; x_0, u)$ *satisfies*

$$|x(t)| \leq \beta(\|x_0\|, t) + \delta_1 \left(\int_0^t \delta_2(|u(\tau)|) d\tau \right), \quad \forall t \geq 0.$$

Note that iISS imposes that the system is forward complete for each $u \in \mathcal{U}$, meaning that x_t exists for all $t \geq 0$, as the above state estimate impedes any finite escape time. Moreover, just like its finite-dimensional counterpart, iISS guarantees not only that the origin of the input-free system $\dot{x}(t) = f(x_t, 0)$ is globally asymptotically stable (GAS), but also induces some robustness with respect to the disturbance u. In particular, it can be seen that, in response to any input with finite energy (as measured through the function δ_2), the state eventually converges to the origin:

$$\int_0^\infty \delta_2(|u(\tau)|) d\tau < \infty \quad \Rightarrow \quad \lim_{t \to \infty} \|x_t\| = 0.$$

C. iISS functionals

We consider two kinds of iISS Lyapunov-Krasovskii functionals: those whose dissipation rate involves the functional itself (here referred to as "strict iISS LKF") and those whose dissipation rate involves merely the present value of the state norm ("relaxed iISS LKF").

We start by recalling Driver's derivative [3]. Given any continuous function $V : \mathcal{X} \to \mathbb{R}$, its Driver's derivative along (3) is defined, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$, as

$$D^+_{(3)} V(\phi, v) := \limsup_{h \to 0^+} \frac{V(\phi^*_{h,v}) - V(\phi)}{h}, \tag{4}$$

where, for all $h \in (0; \theta)$ and all $v \in \mathbb{R}^m$, $\phi^*_{h,v} \in \mathcal{X}$ is defined as

$$\phi^*_{h,v}(s) := \begin{cases} \phi(s+h) & \text{if } s \in [-\theta; -h) \\ \phi(0) + f(\phi, v)(s+h) & \text{if } s \in [-h; 0]. \end{cases} \tag{5}$$

It was shown in [9] that, if V is Lipschitz on any bounded set of $\mathcal{X}$, then, under the assumptions made above on the vector field f, Driver's derivative of V computed at $(x_t, u(t))$ coincides almost everywhere (on the domain of existence of x_t) with the upper-right Dini derivative of $t \mapsto V(x_t)$:

$$D^+_{(3)} V(x_t, u(t)) = \limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}, \quad \forall t \geq 0 \ a.e.$$

Definition 2 (Strict/Relaxed iISS LKF) *Let* $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ *be Lipschitz on any bounded subset of* $\mathcal{X}$*. Assume there exists* $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ *such that*

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}. \tag{6}$$

Then V *is said to be a* strict iISS Lyapunov-Krasovskii functional (LKF) for (3) *if there exist* $\alpha \in \mathcal{PD}$ *and* $\gamma \in \mathcal{K}_\infty$ *such that, for all* $\phi \in \mathcal{X}$ *and all* $v \in \mathbb{R}^m$*,*

$$D^+_{(3)} V(\phi, v) \leq -\alpha(V(\phi)) + \gamma(|v|). \tag{7}$$

It is called a relaxed iISS LKF for (3) *if it satisfies* (6) *and, for all* $\phi \in \mathcal{X}$ *and all* $v \in \mathbb{R}^n$*,*

$$D^+_{(3)} V(\phi, v) \leq -\alpha(|\phi(0)|) + \gamma(|v|). \tag{8}$$

In both cases, α *and* γ *are respectively referred to as a* dissipation rate *and a* supply rate*.*

The key difference between these two iISS LKF lies in the nature of the dissipation: a strict iISS LKF dissipates in terms of the whole LKF itself, whereas a relaxed iISS LKF dissipates only in terms of the current instantaneous value of the state norm. In view of (6), any strict iISS LKF is also a relaxed iISS. Note that this distinction between relaxed and strict iISS LKF is irrelevant in a finite-dimensional context.

D. Problem statement

It is known from [10] that, if the system (3) admits a strict iISS LKF, then it is iISS. That result extended the Lyapunov sufficient condition for iISS of finite-dimensional systems. For non-delayed systems, this Lyapunov condition turns out to be also necessary for iISS[1] [1].

The sufficient condition for iISS in [10] provides a handy way to establish iISS of time-delay systems. However, in order to obtain a dissipation rate that involves the whole LKF, some mathematical tricks are often needed that complicate the analysis. More crucially, such a strict LKF is not required in the absence of inputs. To make this more precise recall that global asymptotic stability of time-delay systems is defined as follows.

Definition 3 (GAS) *The input-free system*

$$\dot{x}(t) = f(x_t) \tag{9}$$

is said to be globally asymptotically stable (GAS) *if there exists* $\beta \in \mathcal{KL}$ *such that, for all* $x_0 \in \mathcal{X}$*, its solution satisfies*

$$|x(t)| \leq \beta(\|x_0\|, t), \quad \forall t \geq 0. \tag{10}$$

[1] We are not aware of any extension of such converse results for time-delay systems, except for particular classes of systems [7].

The classical Lyapunov-Krasovskii condition, recalled in [4, Theorem 2.1, p.105], states that global asymptotic stability (GAS) of the input-free system (9) is guaranteed if there exists a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on all bounded subsets of $\mathcal{X}$, such that, for all $\phi \in \mathcal{X}$,

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|) \tag{11}$$
$$D^+_{(9)}V(\phi) \leq -\alpha(|\phi(0)|), \tag{12}$$

where $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{PD}$.

It is clear from (12) that a point-wise dissipation rate is enough to show GAS. In order to simplify the analysis and to homogenize theory with input-free systems, we here address the question whether a point-wise dissipation rate (namely, a relaxed iISS LKF) is also enough to establish iISS.

III. Main resuts

A. A relaxed condition for GAS

We start by observing that GAS can be established by a weaker requirement than (11)-(12).

Proposition 1 (GAS characterization) *The system* (9) *is GAS if and only if there exist a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on all bounded subsets of $\mathcal{X}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$, and a $\mathcal{KL}$ function σ such that, for all $\phi \in \mathcal{X}$,*

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|) \tag{13}$$
$$D^+_{(9)}V(\phi) \leq -\sigma(|\phi(0)|, \|\phi\|). \tag{14}$$

The proof lines of this result are provided in Section V-A. As we will see through an example in Section IV, this characterization may be handy for instance in situations where the employed LKF is of the form $V = \ln(1 + \tilde{V})$, where $\tilde{V}$ denotes another LKF.

B. iISS by point-wise dissipation

Our main result establishes that a point-wise dissipation rate is enough to guarantee iISS.

Theorem 1 (Relaxed iISS LKF $\Rightarrow$ iISS) *If the system* (3) *admits a relaxed iISS LKF, then it is iISS and admits a strict iISS LKF.*

As detailed in Section V-E, this result is actually a corollary of the following more general statement.

Theorem 2 (Sufficient condition for iISS) *Assume that there exist a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on each bounded subset of $\mathcal{X}$, $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, and $\sigma \in \mathcal{KL}$ such that, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$,*

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|) \tag{15}$$
$$D^+_{(3)}V(\phi, v) \leq -\sigma(|\phi(0)|, \|\phi\|) + \gamma(|v|). \tag{16}$$

Then the system (3) *is iISS and admits a strict iISS LKF.*

The proof is provided in Section V-D. It is made of two steps. The first one characterizes the internal stability of the system in terms of a semi-proper iISS LKF. The second one shows that this characterization combined with a dissipativity condition is enough to establish iISS. Each of these two steps lead to results that may be of interest on their own: we treat them in the following two sections.

C. 0-GAS characterization

Recall that the system $\dot{x}(t) = f(x_t, u(t))$ is said to be 0-GAS if the input-free system $\dot{x}(t) = f(x_t, 0)$ is GAS. The following result establishes that 0-GAS is equivalent to the existence of a semi-proper iISS LKF. The finite-dimensional counterpart of this result was originally established in [1, Proposition II.5].

Proposition 2 (0-GAS characterization) *The delay system* (3) *is 0-GAS if and only if there exist a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on each bounded subset of $\mathcal{X}$, a continuously differentiable class $\mathcal{K}$ function π satisfying $\pi'(s) > 0$ for all $s \in \mathbb{R}_{\geq 0}$, $\underline{\alpha}, \overline{\alpha}, \gamma \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{PD}$ such that*

$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}, \tag{17}$$

and the functional $W := \pi \circ V$ satisfies, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$,

$$D^+_{(3)}W(\phi, v) \leq -\alpha(\|\phi\|) + \gamma(|v|). \tag{18}$$

The proof of this result is provided in Section V-B. It is worth stressing that, in the above statement, W might not be radially unbounded (as π might not be a $\mathcal{K}_\infty$ function), which is why (18) does not readily guarantee iISS (otherwise any 0-GAS system would be iISS, which is not true even in a finite-dimensional context [1]).

It is also important to notice that the dissipation rate in (18) involves the supremum norm of the state history and that V is both upper and lower bounded by functions of $\|\phi\|$. In particular, using the terminology of e.g. [8], V is a coercive LKF. These two features are instrumental in establishing Theorem 1.

D. Link with dissipativity

It turns out that Theorems 1 and 2 are direct consequences of the result presented next, which establishes iISS based on the internal stability of the system and a dissipativity property with respect to the input.

Definition 4 (Zero-output dissipativity) *The system* (3) *is said to be* zero-output dissipative *if there exist a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on each bounded subset of $\mathcal{X}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and μ such that, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$,*

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|)$$
$$D^+_{(3)}V(\phi, v) \leq \mu(|v|).$$

This property is a natural extension of its finite-dimensional counterpart, originally introduced in [1]. It imposes that V grows at most linearly in time for any bounded input u. We underline the fact that, in the above definition, we do not impose coerciveness on V: it just needs to be lower bounded by a function of $|\phi(0)|$ rather than the whole supremum norm of the state history $\|\phi\|$. Here again, this allows to make use of standard LKF employed in stability analysis of autonomous systems.

It was shown in [1] that, for finite-dimensional systems, iISS is guaranteed by zero-output dissipativity, provided that the system is 0-GAS[2]. The following result extends this to time-delay systems.

Theorem 3 (0-GAS + dissipativity ⇒ iISS) *If the system* (3) *is 0-GAS and zero-output dissipative, then it is iISS and admits a strict iISS LKF.*

The proof is provided in Section V-C. As compared to Theorem 1, this result allows to conclude iISS for systems whose internal stability is established by other means than the construction of a Lyapunov-Krasovskii functional, such as through Krasovskii-LaSalle invariance principle arguments [4, Theorem 3.1, p. 119].

IV. Example

We provide an academic example in order to illustrate our findings. Consider the scalar system

$$\dot{x}(t) = -x(t) + c\,x(t+\theta) + x(t)u(t), \tag{19}$$

with $c \geq 0$. In the absence of an input, the system is linear:

$$\dot{x}(t) = -x(t) + c\,x(t+\theta) \tag{20}$$

and its stability can be assessed by the following LKF:

$$W(\phi) = \phi(0)^2 + c\int_{-\theta}^{0} \phi(\tau)^2 d\tau,$$

which can be bounded as in (6):

$$|\phi(0)|^2 \leq W(\phi) \leq (1 + c\,\theta)\|\phi\|^2.$$

Moreover, using the fact that $ab \leq (a^2+b^2)/2$ for all $a, b \in \mathbb{R}$, it holds that

$$\begin{aligned} D^+_{(20)}W(\phi,0) =& 2\phi(0)\big(-\phi(0) + c\,\phi(-\theta)\big) \\ &+ c\left(\phi(0)^2 - \phi(-\theta)^2\right) \\ \leq& -2(1-c)\phi(0)^2. \end{aligned}$$

If $c \in [0;1)$, 0-GAS of (19) follows from [4, Theorem 2.1, p.105]. However, this LKF is not a strict iISS LKF, as it dissipates in a point-wise manner, and therefore cannot be used to invoke existing iISS tools such as [10]. Nevertheless, it can easily be employed to construct a relaxed iISS LKF. Indeed, in the presence of an input, it holds that

$$D^+_{(19)}W(\phi,v) \leq -2(1-c)\phi(0)^2 + \phi(0)^2|v|.$$

Consequently, the functional $V := \ln(1+W)$ satisfies

$$\begin{aligned} D^+_{(19)}V(\phi,v) &= \frac{D^+_{(19)}W(\phi,v)}{1+W(\phi)} \\ &\leq -2(1-c)\frac{\phi(0)^2}{1+(1+c)\|\phi\|^2} + |v|. \end{aligned}$$

Theorem 2 then ensures that (19) is iISS and admits a strict (hence, a relaxed) iISS LKF.

V. Proofs

A. Proof of Proposition 1 (sketch)

The necessity part of this statement follows readily from converse LKF results such as [11, Lemma A.8][3]. So we now focus on showing that (13)-(14) implies GAS. First observe that stability and boundedness readily hold since $D^+_{(9)}V(\phi) \leq 0$, which ensures from (13) that, for all $x_0 \in \mathcal{X}$,

$$|x(t)| \leq \underline{\alpha}^{-1} \circ \overline{\alpha}(\|x_0\|), \quad \forall t \geq 0. \tag{21}$$

In view of Lemma 2.4 and Corollary 2.6 in [5], GAS as defined in terms of a $\mathcal{KL}$ estimate (Definition 3) follows if we manage to show that, for all $\varepsilon > 0$ and all $R \geq 0$, there exists a time $\tau \geq 0$ such that $\|x_t\| \leq \varepsilon$ for all $t \geq \tau$ provided that $\|x_0\| \leq R$ (this is the formulation of Property P3 in [5] for autonomous disturbance-free systems). This attractiveness property can be established following the lines of the proof for GAS using Lyapunov-Krasovskii approach: see [4, Theorem 2.1, p. 105]. We do not include details here due to space limitations.

B. Proof of Proposition 2

We start by showing that (17)-(18) guarantee 0-GAS. This is done by showing that V is a proper LKF for the input-free system

$$\dot{x}(t) = f(x_t, 0). \tag{22}$$

In view of (4) and (18), it holds that

$$\begin{aligned} D^+_{(22)}W(\phi) &= \limsup_{h\to 0^+} \frac{W(\phi_h^*) - W(\phi)}{h} \\ &= \limsup_{h\to 0^+} \frac{\pi\circ V(\phi_h^*) - \pi\circ V(\phi)}{h} \\ &\leq -\alpha(\|\phi\|). \end{aligned}$$

This shows in particular that $V(\phi_h^*) \leq V(\phi)$ for h small enough. Since π is continuously differentiable, the mean value theorem guarantees the existence of $c(\phi,h) \in [V(\phi_h^*); V(\phi)]$ such that

$$\pi\circ V(\phi_h^*) - \pi\circ V(\phi) = \pi'\big(c(\phi,h)\big)\big(V(\phi_h^*) - V(\phi)\big).$$

In view of (5), $\lim_{h\to 0^+} V(\phi_h^*) = V(\phi)$. Consequently, $\lim_{h\to 0^+} \pi'\big(c(\phi,h)\big) = \pi'\circ V(\phi)$. It follows that

$$D^+_{(22)}W(\phi) = \pi'\circ V(\phi)\limsup_{h\to 0^+} \frac{V(\phi_h^*) - V(\phi)}{h} \leq -\alpha(\|\phi\|).$$

Recalling that $\pi'(s) > 0$ for all $s \in \mathbb{R}_{\geq 0}$, we conclude that

$$\begin{aligned} D^+_{(22)}V(\phi) &= \limsup_{h\to 0^+} \frac{V(\phi_h^*) - V(\phi)}{h} \\ &\leq -\frac{\alpha(\|\phi\|)}{\pi'\circ V(\phi)} \\ &\leq -\frac{\alpha(|\phi(0)|)}{\pi'\circ\overline{\alpha}(\|\phi\|)} \end{aligned}$$

[2] iISS is actually equivalent to 0-GAS plus zero-output dissipativity for delay-free systems [1].

[3] [11] deals with neutral delay systems, but includes systems like (22) as special cases.

for all $\phi \in \mathcal{X}$, where we used (17). Let $\xi(s) := 1 + s + \max_{r\in[0;s]} \pi' \circ \overline{\alpha}(r)$ for all $s \in \mathbb{R}_{\geq 0}$. Then ξ is a positive continuous unbounded function satisfying $\xi(\cdot) \geq \pi' \circ \overline{\alpha}(\cdot)$. It follows that $(r, s) \mapsto \alpha(r)/\xi(s)$ is a $\mathcal{KL}$ function and

$$D^+_{(22)} V(\phi) \leq -\frac{\alpha(|\phi(0)|)}{\xi(\|\phi\|)}.$$

Unlike W, V is radially unbounded (the functions $\underline{\alpha}$ and $\overline{\alpha}$ in (17) are of class $\mathcal{K}_\infty$). GAS of (22) (hence, 0-GAS of (3)) then follows from Proposition 1.

We now proceed to establish the converse. Assume the delay system (3) is 0-GAS, meaning that the input-free system (22) is GAS. Then, it holds from [11, Lemma A.8] that there exists a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on each bounded subset of $\mathcal{X}$ such that, for all $\phi \in \mathcal{X}$,

$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|) \tag{23}$$
$$D^+_{(22)} V(\phi) \leq -\eta(\|\phi\|). \tag{24}$$

for some $\underline{\alpha}, \overline{\alpha}, \eta \in \mathcal{K}_\infty$. Proceeding as in the proof of the main result in [15], the derivative of this functional along the solutions of the perturbed system (3) reads

$$\begin{aligned} D^+_{(3)} V(\phi, v) &= \limsup_{h\to 0^+} \frac{1}{h} \left(V(\phi^*_{h,v}) - V(\phi)\right) \\ &= \limsup_{h\to 0^+} \frac{1}{h} \left(V(\phi^*_{h,0}) - V(\phi) + V(\phi^*_{h,v}) - V(\phi^*_{h,0})\right) \\ &\leq D^+_{(22)} V(\phi) + \limsup_{h\to 0^+} \frac{1}{h} \left(V(\phi^*_{h,v}) - V(\phi^*_{h,0})\right) \\ &\leq -\eta(\|\phi\|) + \limsup_{h\to 0^+} \frac{1}{h} \left|V(\phi^*_{h,v}) - V(\phi^*_{h,0})\right|. \end{aligned} \tag{25}$$

As observed in e.g. [6], since f and V are Lipschitz on bounded sets, there exist continuous nondecreasing functions $\ell_V, \ell_f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $\phi, \varphi \in \mathcal{X}$ and all $u, v \in \mathbb{R}^m$,

$$|V(\phi) - V(\varphi)| \leq \ell_V(\|\phi\| + \|\varphi\|)\|\phi - \varphi\| \tag{26}$$
$$|f(\phi, v) - f(\varphi, u)| \leq \ell_f(\|\phi\| + \|\varphi\| + |u| + |v|) \times (\|\phi - \varphi\| + |v - u|). \tag{27}$$

Note that ℓ_f and ℓ_V can be chosen greater than 1 with no loss of generality. Based on this, we have the following:

$$\left|V(\phi^*_{h,v}) - V(\phi^*_{h,0})\right| \leq \ell_V\left(\|\phi^*_{h,v}\| + \|\phi^*_{h,0}\|\right) \|\phi^*_{h,v} - \phi^*_{h,0}\|.$$

In view of (5), $\|\phi^*_{h,v}\| \leq \|\phi\| + |f(\phi, v)|h$. Consequently, from (27) and the fact that $f(0, 0) = 0$, it holds that $\|\phi^*_{h,v}\| \leq \|\phi\| + \ell_f(\|\phi\| + |v|)(\|\phi\| + |v|)h$. It follows that

$$\begin{aligned} &\left|V(\phi^*_{h,v}) - V(\phi^*_{h,0})\right| \leq \\ &\ell_V\Big(2\|\phi\| + 2\ell_f(\|\phi\| + |v|)(\|\phi\| + |v|)h\Big)\|\phi^*_{h,v} - \phi^*_{h,0}\|. \end{aligned} \tag{28}$$

In view of (5), $\phi^*_{h,v}$ and $\phi^*_{h,0}$ coincide on $[-\theta; -h)$. Hence,

$$\begin{aligned} \|\phi^*_{h,v} - \phi^*_{h,0}\| &= \sup_{s\in[-h;0]} \left|\phi^*_{h,v}(s) - \phi^*_{h,0}(s)\right| \\ &\leq \sup_{s\in[-h;0]} |f(\phi, v) - f(\phi, 0)|\,(h + s). \end{aligned}$$

Using again (27), we get that

$$\|\phi^*_{h,v} - \phi^*_{h,0}\| \leq \sup_{s\in[-h;0]} \ell_f(2\|\phi\| + |v|)|v|(h + s) \tag{29}$$
$$\leq \ell_f(2\|\phi\| + |v|)|v|h. \tag{30}$$

It then follows from (28) that

$$\limsup_{h\to 0^+} \frac{1}{h}\left|V(\phi^*_{h,v}) - V(\phi^*_{h,0})\right| \leq \ell_V(2\|\phi\|)\ell_f(2\|\phi\| + |v|)|v|.$$

Plugging this into (25), we obtain that

$$D^+_{(3)} V(\phi, v) \leq -\eta(\|\phi\|) + \ell_V(2\|\phi\|)\ell_f(2\|\phi\| + |v|)|v|.$$

Since ℓ_f is nondecreasing, it holds that $\ell_f(a+b) \leq \ell_f(2a) + \ell_f(2b)$ for all $a, b \in \mathbb{R}_{\geq 0}$. Consequently, recalling that ℓ_f was chosen not smaller than 1,

$$\begin{aligned} D^+_{(3)} V &\leq -\eta(\|\phi\|) + \ell_V(2\|\phi\|)\big(\ell_f(4\|\phi\|) + \ell_f(2|v|)\big)|v| \\ &\leq -\eta(\|\phi\|) + \ell_V(2\|\phi\|)\big(\ell_f(4\|\phi\|) + 1\big)\ell_f(2|v|)|v|. \end{aligned}$$

For future reference, we summarize these findings in the following statement[4], which is an extension of [1, Lemma IV.10] to time-delay systems.

Proposition 3 (0-GAS characterization) *The system* (3) *is 0-GAS if and only if there exist a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lispchitz on each bounded subset of $\mathcal{X}$, a nondecreasing continuous function $\ell : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and $\underline{\alpha}, \overline{\alpha}, \eta, \gamma \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$,*

$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|)$$
$$D^+_{(3)} V(\phi, v) \leq -\eta(\|\phi\|) + \ell(\|\phi\|)\gamma(|v|).$$

Now, consider the continuously differentiable class $\mathcal{K}$ function defined as

$$\pi(s) := \int_0^s \frac{dr}{1 + \ell \circ \underline{\alpha}^{-1}(r)},$$

where $\underline{\alpha} \in \mathcal{K}_\infty$ is the lower bound on V (see (23)). Note that $\pi'(s) > 0$ for all $s \in \mathbb{R}_{\geq 0}$. Letting $W := \pi \circ V$, we get

$$\begin{aligned} D^+_{(3)} W(\phi, v) &= \pi'(V(\phi))D^+_{(3)} V(\phi, v) \\ &\leq \frac{-\eta(\|\phi\|) + \ell(\|\phi\|)\gamma(|v|)}{1 + \ell \circ \underline{\alpha}^{-1}(V(\phi))} \\ &\leq -\alpha(\|\phi\|) + \gamma(|v|), \end{aligned}$$

where the functions α is defined as

$$\alpha(s) := \frac{\eta(s)}{1 + \ell \circ \underline{\alpha}^{-1} \circ \overline{\alpha}(s)}.$$

The conclusion follows since $\alpha \in \mathcal{PD}$ and $\gamma \in \mathcal{K}_\infty$.

[4]The sufficiency part of this statement is straightforward.

C. Proof of Theorem 3

The zero-output dissipativity assumption means that there exist a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on all bounded subsets of $\mathcal{X}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and μ such that, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$,

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\phi\|) \tag{31}$$
$$D^+_{(3)}V(\phi, v) \leq \mu(|v|). \tag{32}$$

Moreover, since the system is assumed to be 0-GAS, Proposition 2 guarantees that there exist $\eta \in \mathcal{PD}$, $\underline{\eta}, \overline{\eta} \in \mathcal{K}$, $\gamma \in \mathcal{K}_\infty$, and a functional $W : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on all bounded subsets of $\mathcal{X}$, such that, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$,

$$\underline{\eta}(\|\phi\|) \leq W(\phi) \leq \overline{\eta}(\phi\|) \tag{33}$$
$$D^+_{(3)}W(\phi, v) \leq -\eta(\|\phi\|) + \gamma(|v|). \tag{34}$$

Let $\tilde{V} := V + W$. Then, $\tilde{V}$ is Lipschitz on bounded sets of $\mathcal{X}$ and we get from (31) and (33) that

Wrong ! \eta is not increasing, just PD !

$$\underline{\alpha}(|\phi(0)|) \leq \tilde{V}(\phi) \leq \tilde{\overline{\alpha}}(\|\phi\|), \tag{35}$$

where $\tilde{\overline{\alpha}} := \overline{\eta} + \overline{\alpha}$. Moreover, (32) and (34) give that

$$\begin{aligned} D^+_{(3)}\tilde{V}(\phi, v) &\leq -\eta(\|\phi\|) + \gamma(|v|) + \mu(|v|) \\ &\leq -\eta \circ \tilde{\overline{\alpha}}^{-1}(\tilde{V}(\phi)) + \gamma(|v|) + \mu(|v|). \end{aligned} \tag{36}$$

Observing that $\underline{\alpha}$, $\tilde{\overline{\alpha}}$, and $\gamma + \mu$ are $\mathcal{K}_\infty$ functions and that $\eta \circ \tilde{\overline{\alpha}}^{-1} \in \mathcal{PD}$, we conclude from (35)-(36) that $\tilde{V}$ is a strict iISS LKF for (3) and iISS follows from [10, Theorem 3.7].

D. Proof of Theorem 2

By assumption, there exist a functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, Lipschitz on all bounded subsets of $\mathcal{X}$, $\underline{\alpha}, \overline{\alpha}, \gamma \in \mathcal{K}_\infty$, and $\sigma \in \mathcal{KL}$ such that, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$,

$$\begin{aligned} \underline{\alpha}(|\phi(0)|) \leq V(\phi) &\leq \overline{\alpha}(\|\phi\|) \\ D^+_{(3)}V(\phi, v) &\leq -\sigma(|\phi(0)|, \|\phi\|) + \gamma(|v|). \end{aligned} \tag{37}$$

For $v = 0$, it holds in particular that $D^+_{(3)}V(\phi, 0) \leq -\sigma(|\phi(0)|, \|\phi\|)$. In view of Proposition 1, we conclude that the input-free system $\dot{x}(t) = f(x_t, 0)$ is GAS; in other words, the system (3) is 0-GAS. Moreover, it readily holds from (37) that $D^+_{(3)}V(\phi, v) \leq \gamma(|v|)$, so the system (3) is zero-ouptut dissipative. The conclusion then follows from Theorem 3.

E. Proof of Theorem 1

Let $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be a relaxed iISS LKF, meaning that there exist $\underline{\alpha}, \overline{\alpha}, \gamma \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{PD}$ such that, for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}^m$,

$$\begin{aligned} \underline{\alpha}(|\phi(0)|) \leq V(\phi) &\leq \overline{\alpha}(\|\phi\|) \\ D^+_{(3)}V(\phi, v) &\leq -\alpha(|\phi(0)|) + \gamma(|v|). \end{aligned}$$

Let $\sigma(r, s) := \frac{\alpha(r)}{1+s}$ for all $r, s \geq 0$. Then $\sigma \in \mathcal{KL}$ and

$$D^+_{(3)}V(\phi, v) \leq -\sigma(|\phi(0)|, \|\phi\|) + \gamma(|v|).$$

We can then invoke Theorem 2 to conclude.

VI. Conclusion and Perspectives

We have shown that a point-wise dissipation rate is enough to guarantee iISS of time-delay systems. We believe that this result will simplify the iISS analysis in specific scenarios. Moreover, it makes the iISS framework more conform to the existing theory of disturbance-free systems. In establishing this result, we have in turn provided a relaxed Lyapunov-Krasovskii characterization of global asymptotic stability and extended the link between iISS and 0-GAS plus zero-output dissipativity to time-delay systems.

To the best of our knowledge, the possibility to establish ISS through a point-wise dissipation rate remains an open question. Moreover, we are not aware of any result stating iISS guarantees the existence of a (relaxed) iISS LKF. These two points would deserve further exploration.

References

[1] D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input to state stability. *IEEE Trans. Autom. Control*, 45:1082–1097, 2000.

[2] A. Chaillet, P. Pepe, P. Mason, and Y. Chitour. Is a point-wise dissipation rate enough to show ISS for time-delay systems? In *IFAC World Congress*, Toulouse, France, July 2017.

[3] R.D. Driver. Existence and stability of solutions of a delay-differential system. *Archive for Rational Mechanics and Analysis*, 10(1):401–426, 1962.

[4] J.K. Hale. Theory of functional differential equations. *Applied mathematical sciences*, pages 1–376, 1977.

[5] I. Karafyllis. Lyapunov theorems for systems described by retarded functional differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 64(3):590–617, 2006.

[6] I. Karafyllis, P. Pepe, and Z.P. Jiang. Global output stability for systems described by retarded functional differential equations: Lyapunov characterizations. *European Journal of Control*, 14(6):516–536, 2008.

[7] A. Mironchenko and H. Ito. Characterizations of integral input-to-state stability for bilinear systems in infinite dimensions. *Mathematical Control and Related Fields*, 6(3):447–466, September 2016.

[8] A. Mironchenko and F. Wirth. Global converse Lyapunov theorems for infinite-dimensional systems. In *NOLCOS*, 2016.

[9] P. Pepe. On Liapunov-Krasovskii functionals under Caratheodory conditions. *Automatica*, 43(4):701–706, 2007.

[10] P. Pepe and Z.P. Jiang. A Lyapunov–Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters*, 55(12):1006–1014, 2006.

[11] P. Pepe and I. Karafyllis. Converse Lyapunov–Krasovskii theorems for systems described by neutral functional differential equations in Hale's form. *International Journal of Control*, 86(2):232–243, February 2013.

[12] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Autom. Control*, 34(4):435–443, 1989.

[13] E.D. Sontag. Comments on integral variants of ISS. *Systems & Control Letters*, 34:93–100, 1998.

[14] E.D. Sontag. *Input to state stability: Basic concepts and results*, pages 163–220. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2008.

[15] N. Yeganefar, P. Pepe, and M. Dambrine. Input-to-state stability of time-delay systems: a Link with exponential stability. *IEEE Trans. Autom. Control*, 53(6):1526–1531, 2008.