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## Parameter Selection in Gaussian Process Interpolation: An Empirical Study of Selection Criteria\*

Sébastien J. Petit<sup>†‡</sup>, Julien Bect<sup>‡</sup>, Paul Feliot<sup>†</sup>, and Emmanuel Vazquez<sup>‡</sup>

**Abstract.** This article revisits the fundamental problem of parameter selection for Gaussian process interpolation. By choosing the mean and the covariance functions of a Gaussian process within parametric families, the user obtains a family of Bayesian procedures to perform predictions about the unknown function and must choose a member of the family that will hopefully provide good predictive performances. We base our study on the general concept of scoring rules, which provides an effective framework for building leave-one-out selection and validation criteria and a notion of extended likelihood criteria based on an idea proposed by Fasshauer et al. [*“Optimal” scaling and stable computation of meshfree kernel methods*, 2009], which makes it possible to recover standard selection criteria, such as the generalized cross-validation criterion. Under this setting, we empirically show on several test problems of the literature that the choice of an appropriate family of models is often more important than the choice of a particular selection criterion (e.g., the likelihood versus a leave-one-out selection criterion). Moreover, our numerical results show that the regularity parameter of a Matérn covariance can be selected effectively by most selection criteria.

**Key words.** Gaussian processes interpolation, model choice, parameter selection, scoring rules, likelihood, regularity

**MSC codes.** 62G08, 65D05

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**1. Introduction.** Regression and interpolation with Gaussian processes (GPs), or kriging, is a popular statistical tool for nonparametric function estimation, originating from geostatistics and time series analysis and later adopted in many other areas, such as machine learning and the design and analysis of computer experiments (see, e.g., [43, 44, 46] and references therein). It is widely used for constructing fast approximations of time-consuming computer models, with applications to calibration and validation [6, 32], engineering design [23, 31], Bayesian inference [10, 53], and the optimization of machine learning algorithms [8]—to name but a few.

A GP prior is characterized by its mean and covariance functions. They are usually chosen within parametric families (for instance, constant or linear mean functions and Matérn covariance functions), which transfers the problem of choosing the mean and covariance functions to that of selecting parameters. The selection is most often carried out by optimization of a

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criterion that measures the goodness of fit of the predictive distributions, and a variety of such criteria—the likelihood function, the leave-one-out (LOO) squared-prediction-error criterion (hereafter denoted by LOO-SPE), and others—is available from the literature. The search for arguments to guide practitioners in the choice of one particular selection criterion is the main motivation of this study.

*Remark 1.1.* In a fully Bayesian approach, one does not select a particular value for the parameters but instead averages the predictions over a *posterior* distribution on the parameters [5, 29, 44]. This approach may be preferred for more robust predictions (see, e.g., [7]) but comes at a higher computational cost. For this reason, the present article will set aside the fully Bayesian approach and concentrate on the plug-in approach, where only one parameter value is chosen to carry out predictions.

The two most popular methods for parameter selection are maximum likelihood (ML) and cross validation (CV) based on LOO criteria, which were introduced in the field of computer experiments by the seminal work of [16]. Since then, despite a fairly large number of publications dealing with ML and CV techniques for the selection of a GP model, the literature has remained in our view quite sparse about the relative merits of these methods from both theoretical and empirical perspectives.

For instance, in the framework of interpolation and infill asymptotics, where observations accumulate in some bounded domain, [55] and [56] show that some combinations of the parameters of a Matérn covariance function can be estimated consistently by ML. Again, in the case of infill asymptotics, with the additional assumption of a one-dimensional GP with an exponential covariance function, [4] show that estimation by LOO is also consistent. (Similar results exist in the expanding asymptotic framework, where observations extend over an ever-increasing horizon.)

The practical consequences of the aforementioned results are somewhat limited in our view because practitioners are primarily interested in the quality of the predictions. Knowing that the parameters of a GP can be estimated consistently is intellectually reassuring but may be considered of secondary importance. These results are indeed about the statistical model itself, but they say little about the prediction properties of the model. Besides, there does not exist at present, to our knowledge, some theoretically based evaluation of the relative performances of ML and CV selection criteria under infill asymptotics.

Turning now to empirical comparisons of selection criteria for GP interpolation, the first attempt in the domain of computer experiments can be traced back to the work of [16, 17]. The authors introduce CV and ML—which can be seen as a special kind of CV—and present some simple experiments using tensorized covariance functions, from which they conclude, “Of the various kinds of cross-validation [they] have tried, maximum likelihood seems the most reliable.” Additional experiments have been conducted by several authors, but no consensus emerges from these studies: [2, 33, 47] conclude in favor of CV, whereas [34] advocate ML. The reference textbook of Santner, Williams, and Notz [44, section 3.4.2] echoes the conclusions of [2].

These studies are limited to a rather small number of test functions and covariance functions, which may explain the discrepancy in the conclusions of those experiments. In particular, only [2] considers the popular and versatile Matérn covariance functions. Moreover,

most studies focus only on the accuracy of the posterior mean—only [47] and [2] provide results accounting for the quality of the posterior variance—whereas the full posterior predictive distribution is used in most GP-based methods (see, e.g., [12, 31]).

This article presents two main contributions. First, we improve on the results of the literature by providing an empirical ranking of selection criteria for GP interpolation, according to several metrics measuring the quality of posterior predictive distributions on a large set of test functions from the domain of computer experiments. To this end, we base our study on the general concept of scoring rules [26, 57], which provides an effective framework for building selection and validation criteria. We also introduce a notion of extended likelihood criteria, borrowing an idea from Fasshauer and coauthors [21, 22] in the literature of radial basis functions.

Second, we provide empirical evidence that the choice of an appropriate family of models is often more important—and sometimes much more important, especially when the size of the design increases—than the choice of a particular selection criterion (e.g., likelihood versus LOO-SPE). More specifically, in the case of the Matérn family, this leads us to assess and ultimately recommend the automatic selection of a suitable value of the regularity parameter against the common practice of choosing beforehand an arbitrary value of this parameter.

These recommendations are complementary to those of [46], where a particular emphasis is placed on choosing the regularity of the covariance function, and [11], where the influence of the design and the influence of the choice of the mean and covariance functions are studied empirically and ranked (in the case where the parameters are selected by ML).

The article is organized as follows. Section 2 briefly recalls the general framework of GP regression and interpolation. Section 3 reviews selection criteria for GP model parameters. After recalling the general notion of scoring rules, we present a broad variety of selection criteria from the literature. Section 4 presents experimental results on the relative performances of these criteria. Section 5 presents our conclusions and perspectives.

**2. General framework.** Let us consider the general GP approach for a scalar-valued deterministic computer code with input space  $\mathbb{X} \subseteq \mathbb{R}^d$ . The output of the computer code  $z : \mathbb{X} \rightarrow \mathbb{R}$  is modeled by a random function  $(Z(x))_{x \in \mathbb{X}}$ , which, from a Bayesian perspective, represents prior knowledge about  $z$ . If we assume that  $Z(\cdot)$  is observed on a design  $\mathbb{X}_n = \{x_1, \dots, x_n\}$  of size  $n$ , the data correspond to a sample  $\mathbf{z}_n \in \mathbb{R}^n$  of the random vector  $\mathbf{Z}_n = (Z(x_1), \dots, Z(x_n))^T$ .

The GP assumption makes it possible to easily derive posterior distributions. More precisely, it is assumed that  $Z(\cdot)$  is a GP, with (prior) mean  $\mathbb{E}(Z(x)) = \sum_{l=1}^L \beta_l \phi_l(x)$ , where the  $\beta_1, \dots, \beta_L$  are unknown regression parameters and  $\phi_1, \dots, \phi_L$  are known regression functions, and with (prior) covariance  $\text{cov}(Z(x), Z(y)) = k_\vartheta(x, y)$ , where  $\vartheta \in \Theta \subseteq \mathbb{R}^q$  is a vector of *parameters*. Throughout the article, the covariance matrix of  $\mathbf{Z}_n$  will be denoted by  $\mathbf{K}_\vartheta$ . As is often the case in applications, we assume that the prior mean of  $Z(\cdot)$  is constant (hence,  $L = 1$  with  $\phi_1 = 1$ ). Note also that this is in accordance to the recommendation in [11].

One of the most popular covariance functions for GP regression is the anisotropic stationary Matérn covariance function [35] popularized by [46],

$$(2.1) \quad k_\vartheta(x, y) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left( \sqrt{2\nu h} \right)^\nu \mathcal{K}_\nu \left( \sqrt{2\nu h} \right), \quad h = \left( \sum_{j=1}^d \frac{(x_j - y_j)^2}{\rho_j^2} \right)^{1/2},$$

**Table 1**  
Popular Matérn subfamilies.

	$\nu = \frac{1}{2}$	$\nu = \frac{3}{2}$	$\nu = \frac{5}{2}$	$\nu = +\infty$
a.k.a.	exponential			squared exponential
$k_{\vartheta}(x, y)$	$\sigma^2 e^{-h}$	$\sigma^2(1 + \sqrt{3}h)e^{-\sqrt{3}h}$	$\sigma^2(1 + \sqrt{5}h + \frac{5h^2}{3})e^{-\sqrt{5}h}$	$\sigma^2 e^{-\frac{h^2}{2}}$

where  $\Gamma$  is the gamma function,  $\mathcal{K}_{\nu}$  is the modified Bessel function of the second kind, and  $\vartheta$  denotes the vector of parameters  $\vartheta = (\sigma^2, \rho_1, \dots, \rho_d, \nu) \in \Theta = (0, \infty)^{d+2}$ . The parameter  $\sigma^2$  is the variance of  $Z(\cdot)$ , the  $\rho_i$ 's are range parameters which characterize the typical correlation length on each dimension, and  $\nu$  is a regularity parameter, whose value controls the mean-square differentiability of  $Z(\cdot)$ . Recall (see Table 1) that the Matérn covariance function with  $\nu = 1/2$  corresponds to the so-called exponential covariance function, and the limiting case  $\nu \rightarrow \infty$  can be seen as the squared exponential (also called Gaussian) covariance function.

Because  $\mathcal{K}_{\nu}$  has a closed-form expression when  $\nu - \frac{1}{2}$  is an integer and is more expensive to evaluate numerically in other cases, most implementations choose to restrict  $\nu$  to half-integer values. Moreover, a widespread practice (in applications and research papers) consists in selecting a particular value for  $\nu$  (e.g.,  $\nu = 1/2$ ,  $\nu = 3/2$ , or the limiting case  $\nu \rightarrow \infty$ ) once and for all.

Since the family of Matérn covariance functions is widely used in practice, we focus exclusively on this family in this work.

Once a GP model has been chosen, the framework of GP interpolation allows one to build predictive distributions. More precisely, denoting by  $\theta = (\beta, \vartheta) \in \mathbb{R} \times (0, +\infty)^{d+2}$  the vector of the parameters of the model, the usual predictive distribution for an unobserved  $Z(x)$  at  $x \in \mathbb{R}^d$  is the Gaussian posterior distribution  $\mathcal{N}(\mu_{\theta}(x), \sigma_{\theta}^2(x))$ , where

$$(2.2) \quad \begin{cases} \mu_{\theta}(x) &= \beta + \mathbf{k}_{\vartheta}(x)^{\top} \mathbf{K}_{\vartheta}^{-1} \mathbf{z}_n^{\circ}, \\ \sigma_{\theta}^2(x) &= k_{\vartheta}(x, x) - \mathbf{k}_{\vartheta}(x)^{\top} \mathbf{K}_{\vartheta}^{-1} \mathbf{k}_{\vartheta}(x), \end{cases}$$

with  $\mathbf{k}_{\vartheta}(x) = (k_{\vartheta}(x, x_1), \dots, k_{\vartheta}(x, x_n))^{\top}$  and  $\mathbf{z}_n^{\circ} = \mathbf{z}_n - \beta \mathbf{1}_n$  and where  $\mathbf{1}_n = (1, \dots, 1)^{\top} \in \mathbb{R}^n$ .

Using this framework, the user obtains a family of Bayesian procedures, indexed by  $\theta$ , to perform predictions about the unknown computer code at hand and must choose a member of the family that will hopefully provide good predictive performances.

### 3. Selection of a GP model from a parameterized family.

**3.1. Scoring rules.** Goodness-of-fit criteria for probabilistic predictions have been studied in the literature under the name of scoring rules by [26]. A (negatively oriented) scoring rule is a function  $S(\cdot, z) : \mathcal{P} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ , acting on a class  $\mathcal{P}$  of probability distributions on  $\mathbb{R}$ , such that  $S(P, z)$  assigns a loss for choosing a predictive distribution  $P \in \mathcal{P}$  while observing  $z \in \mathbb{R}$ . Scoring rules make it possible to quantify the quality of probabilistic predictions.

*Example 3.1* (squared prediction error [SPE]). Denoting by  $\mu$  the mean of a predictive distribution  $P$ , the SPE

$$(3.1) \quad S^{\text{SPE}}(P, z) = (z - \mu)^2$$

accounts for the deviation of  $z$  from  $\mu$ . Note that  $S^{\text{SPE}}$  ignores subsequent moments and therefore predictive uncertainties.

*Example 3.2 (negative log predictive density [NLPD]).* Denoting by  $p$  the probability density of  $P$ ,

$$(3.2) \quad S^{\text{NLPD}}(P, z) = -\log(p(z))$$

tells how likely  $z$  is according to  $P$ . It can be proved [9] that any (proper) scoring rule that only depends on  $p(z)$  can be reduced to  $S^{\text{NLPD}}$ .

*Example 3.3 (continuous ranked probability score [CRPS]).* Let  $U$  and  $U'$  be two independent random variables with distribution  $P$ . The CRPS quantifies the deviation of  $U$  from  $z$ :

$$(3.3) \quad S^{\text{CRPS}}(P, z) = \mathbb{E}(|U - z|) - \frac{1}{2}\mathbb{E}(|U - U'|).$$

Since  $S^{\text{CRPS}}(P, z) = \int (P(U \leq u) - \mathbb{1}_{z \leq u})^2 du$ , the CRPS can also be seen as a (squared) distance between the empirical cumulative distribution  $u \mapsto \mathbb{1}_{z \leq u}$  and the cumulative distribution of  $P$  [48].

Note that if absolute values in (3.3) are replaced by squared values, then  $S^{\text{SPE}}$  is recovered. The CRPS can also be extended to the so-called energy and kernel scores [26] by observing that  $(x, y) \mapsto |x - y|$  is a conditionally negative kernel.

*Example 3.4 (interval score [IS]).* The IS rule at level  $1 - \alpha$  is defined, for  $\alpha \in (0, 1)$ , by

$$(3.4) \quad S_{1-\alpha}^{\text{IS}}(P; z) = (u - l) + \frac{2}{\alpha}(l - z)\mathbb{1}_{z \leq l} + \frac{2}{\alpha}(z - u)\mathbb{1}_{z > u},$$

where  $l$  and  $u$  are the  $\alpha/2$  and  $1 - \alpha/2$  quantiles of  $P$ . The first term penalizes large intervals, while the second and third terms penalize intervals not containing  $z$ .

When the predictive distribution  $P$  is Gaussian, which is the case when  $P$  is the posterior distribution of a GP  $Z$  at a given point, the aforementioned scoring rules all have closed-form expressions. More precisely, for  $P = \mathcal{N}(\mu, \sigma^2)$ , we simply have  $S^{\text{SPE}}(P, z) = (z - \mu)^2$  and  $S^{\text{NLPD}}(P, z) = \frac{1}{2} \log 2\pi\sigma^2 + \frac{1}{2}(z - \mu)^2/\sigma^2$ .  $S_{1-\alpha}^{\text{IS}}$  can be obtained by taking the standard expressions of the  $\alpha/2$  and  $1 - \alpha/2$  quantiles of  $P$ , and it can be shown that

$$S^{\text{CRPS}}(P, z) = \sigma \left( \frac{z - \mu}{\sigma} \left( 2\Phi \left( \frac{z - \mu}{\sigma} \right) - 1 \right) + 2\phi \left( \frac{z - \mu}{\sigma} \right) - \frac{1}{\sqrt{\pi}} \right),$$

where  $\phi$  and  $\Phi$  stand, respectively, for the probability density function and the cumulative distribution function of the standard Gaussian distribution.

Note that all aforementioned scoring rules penalize large values of  $|z - \mu|$ . When  $|z - \mu| \ll 1$ , different scoring rules yield different penalties, as reported in Table 2.

**Table 2**  
Scoring rules behavior as  $|\mu - z| \ll 1$ .

	$\sigma \ll  \mu - z $	$\sigma \simeq  \mu - z $	$\sigma \gg  \mu - z $
$S^{\text{SPE}}(P, z)$	0	0	0
$S^{\text{NLPD}}(P, z)$	$+\infty$	$-\infty$	$\log(\sqrt{2\pi}\sigma)$
$S^{\text{CRPS}}(P, z)$	0	0	$\propto \sigma$
$S_{1-\alpha}^{\text{IS}}(P, z)$	0	0	$\propto \sigma$

### 3.2. Selection criteria.

**3.2.1. LOO selection criteria.** Scoring rules make it possible to build criteria for choosing the parameters of a GP. More precisely, to select  $\theta$  based on a sample  $\mathbf{z}_n = (z_1, \dots, z_n)^\top$ , one can minimize the mean score

$$(3.5) \quad J_n^S(\theta) = \frac{1}{n} \sum_{i=1}^n S(P_{\theta, -i}, z_i),$$

where  $S$  is a scoring rule and  $P_{\theta, -i}$  denotes the distribution of  $Z(x_i)$  conditional on  $Z(x_j) = z_j$ ,  $j \neq i$ .

Selection criteria written as (3.5) correspond to the well-established LOO method, which has been proposed in the domain of computer experiments by [16] and is now used in many publications (see, e.g., [43] and also [57], who formally adopt the point of view of the scoring rules but for model validation instead of parameter selection).

*Efficient computation of predictive distributions.* LOO predictive densities can be computed using fast algebraic formulas [13, 20]. More precisely, the predictive distribution  $P_{\theta, -i}$  is a normal distribution  $\mathcal{N}(\mu_{\theta, -i}, \sigma_{\theta, -i}^2)$  with

$$(3.6) \quad \mu_{\theta, -i} = z_i - \frac{(\mathbf{K}_\theta^{-1} \mathbf{z}_n^\circ)_i}{(\mathbf{K}_\theta^{-1})_{i,i}} \quad \text{and} \quad \sigma_{\theta, -i}^2 = \frac{1}{(\mathbf{K}_\theta^{-1})_{i,i}}.$$

*Remark 3.5.* Equation (3.6) provides a numerically convenient expression for  $\mu_{\theta, -i}$ , which apparently involves  $z_i$ . However,  $z_i$  appears twice in the expression, and the two occurrences cancel out when the matrix product is written explicitly.

Furthermore, [39] show that, using reverse-mode differentiation, it is possible to compute mean scores  $J_n$  and their gradients with a  $O(n^3 + dn^2)$  computational cost, which is the same computational complexity as for computing the likelihood function and its gradient (see, e.g., [43]).

*The particular case of LOO-SPE.* The LOO selection criterion

$$(3.7) \quad J_n^{\text{SPE}}(\theta) = \frac{1}{n} \sum_{i=1}^n (\mu_{\theta, -i} - z_i)^2,$$

based on the scoring rule (3.1), will be referred to as LOO-SPE. This criterion, also called prediction sum of squares (PRESS) [1, 50] or LOO squared bias [16], is well known in statistics



and machine learning and has been advocated by some authors [2, 3, 33, 44, 47] to address the case of “misspecified” covariance functions.

However, note that  $\sigma^2$  cannot be selected using  $J_n^{\text{SPE}}$ . When  $J_n^{\text{SPE}}$  is used,  $\sigma^2$  is generally chosen (see, e.g., [2, 16] and Remark 3.6) to satisfy

$$(3.8) \quad \frac{1}{n} \sum_{i=1}^n \frac{(z_i - \mu_{\theta, -i})^2}{\sigma_{\theta, -i}^2} = 1,$$

which will be referred to as Cressie’s rule for  $\sigma^2$  in reference to the claim by [14] that (3.8) should hold approximately for a good GP model.

**Other scoring rules for LOO.** The selection criteria using the NLPD scoring rule (3.2) and the CRPS scoring rule (3.3) will be referred to as the LOO-NLPD and LOO-CRPS criteria, respectively. The LOO-NLPD criterion has been called predictive deficiency in [16] and Geisser’s surrogate predictive probability in [47]. The LOO-CRPS criterion has been considered in [57] as a criterion for model validation (see also [18] for an application to model selection) and more recently in [39, 40] as a possible criterion for parameter selection as well.

**Remark 3.6.** Note that Cressie’s rule (3.8) can be derived by minimizing the LOO-NLPD criterion with respect to  $\sigma^2$ .

**Remark 3.7.** In order to limit the number of selection criteria under study, the IS rule is only used for validation in this work.

**3.2.2. ML and generalizations.** We can safely say that the most popular method for selecting  $\theta$  from data is ML estimation—and related techniques, such as restricted ML estimation. The ML estimator is obtained by maximizing the likelihood function or, equivalently, by minimizing the negative log-likelihood (NLL) selection criterion. Denoting by  $p_{\theta}(\mathbf{z}_n)$  the joint density of  $\mathbf{Z}_n$ , the NLL selection criterion may be written as

$$(3.9) \quad J_n^{\text{NLL}}(\theta) = -\log(p_{\theta}(\mathbf{z}_n)) = \frac{1}{2} \left( n \log(2\pi) + \log \det \mathbf{K}_{\vartheta} + (\mathbf{z}_n^{\circ})^{\top} \mathbf{K}_{\vartheta}^{-1} \mathbf{z}_n^{\circ} \right).$$

As pointed out by [16], the NLL criterion is closely related to the LOO-NLPD criterion through the identity

$$J_n^{\text{NLL}}(\theta) = -\log(p_{\theta}(z_1)) - \sum_{i=2}^n \log(p_{\theta}(z_i | z_1, \dots, z_{i-1})),$$

which makes appear the densities of the  $Z(x_i)$ ’s conditional on  $Z(x_j) = z_j$ ,  $j < i$  (whereas the LOO-NLPD criterion involves the densities of the  $Z(x_i)$ ’s conditional on  $Z(x_j) = z_j$ ,  $j \neq i$ ).

One can minimize (3.9) in closed form with respect to  $\sigma^2$  given other parameters. Writing  $\mathbf{K}_{\vartheta} = \sigma^2 \mathbf{R}_{\vartheta}$  and canceling  $\partial J_n^{\text{NLL}}(\theta) / \partial \sigma^2 = (n\sigma^2 - (\mathbf{z}_n^{\circ})^{\top} \mathbf{R}_{\vartheta}^{-1} \mathbf{z}_n^{\circ}) / (2\sigma^4)$  yields

$$(3.10) \quad \sigma_{\text{NLL}}^2 = \frac{1}{n} (\mathbf{z}_n^{\circ})^{\top} \mathbf{R}_{\vartheta}^{-1} \mathbf{z}_n^{\circ},$$

which will be referred to as the profiling rule for  $\sigma^2$ .



Injecting (3.10) into (3.9) yields a *profile likelihood* selection criterion that can be written as

$$(3.11) \quad J_n^{\text{PL}}(\theta) = \log \sigma_{\text{NLL}}^2 + \frac{1}{n} \log \det \mathbf{R}_\theta = \log \left( \frac{1}{n} (\mathbf{z}_n^\circ)^\top \mathbf{R}_\theta^{-1} \mathbf{z}_n^\circ \right) + \frac{1}{n} \log \det \mathbf{R}_\theta.$$

Following Fasshauer and coauthors [21, 22], we consider now a family of selection criteria that extends (3.11) Using the factorization  $\mathbf{R}_\theta = \mathbf{Q}\Lambda\mathbf{Q}^\top$ , where  $\mathbf{Q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$  is an orthogonal matrix of (orthonormal) eigenvectors and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ , notice that

$$(3.12) \quad \begin{aligned} \exp(J_n^{\text{PL}}(\theta)) &= \frac{1}{n} (\mathbf{z}_n^\circ)^\top \mathbf{R}_\theta^{-1} \mathbf{z}_n^\circ \cdot (\det \mathbf{R}_\theta)^{1/n} \\ &\propto \left( \sum_{i=1}^n (\mathbf{q}_i^\top \mathbf{z}_n^\circ)^2 / \lambda_i \right) \left( \prod_{i=1}^n \lambda_j \right)^{1/n}. \end{aligned}$$

This suggests a generalization of the likelihood criterion that we shall call Fasshauer’s Hölderized likelihood (HL), defined as

$$(3.13) \quad J_n^{\text{HL},p,q}(\theta) = \left( \sum_{i=1}^n (\mathbf{q}_i^\top \mathbf{z}_n^\circ)^2 / \lambda_i^p \right)^{1/p} \left( \frac{1}{n} \sum_{j=1}^n \lambda_j^q \right)^{1/q},$$

with  $q \in [-\infty, +\infty]$   $p \in \mathbb{R} \setminus \{0\}$  and where  $\sigma^2$  can be chosen using the rules (3.8) or (3.10) since  $J_n^{\text{HL},p,q}(\theta)$  does not depend on  $\sigma^2$ . Due to the standard property of generalized means  $(\frac{1}{n} \sum_{i=1}^n x_i^q)^{\frac{1}{q}} \xrightarrow{q \rightarrow 0} \sqrt[q]{x_1 \cdots x_n}$ , (3.12) is recovered by taking  $p = 1$  and letting  $q \rightarrow 0$ . Moreover, two other known selection criteria can be obtained for particular values of  $p$  and  $q$ , as detailed below.

**Generalized CV.** Taking  $p = 2$  and  $q = -1$  in (3.13) yields the generalized CV (GCV) criterion

$$(3.14) \quad J_n^{\text{GCV}}(\theta) = n^{-1} (J_n^{\text{HL},2,-1}(\theta))^2,$$

which was originally proposed as a rotation-invariant version of PRESS [27] for linear models. It has also been shown to be efficient for the selection of the smoothing parameter of polyharmonic splines [50] and for the selection of the degree of a spline [51].

The GCV selection criterion is a weighted SPE criterion, which can also be written as

$$(3.15) \quad J_n^{\text{GCV}}(\theta) = \frac{1}{n} \sum_{i=1}^n w_i^2(\theta) (z_i - \mu_{\theta,-i})^2, \quad w_i(\theta) = \frac{\tilde{\sigma}^2}{\sigma_{\theta,-i}^2},$$

with  $\tilde{\sigma}^2 = (\frac{1}{n} \sum_{i=1}^n \frac{1}{\sigma_{\theta,-i}^2})^{-1}$ . Notice that  $w_i(\theta)$  is lower when  $\sigma_{\theta,-i}$  is larger, which happens when points are either isolated or lying on the border/envelope of  $\mathbb{X}_n$ . Equation (3.15) shows that, similarly to the LOO criteria of section 3.2.1, the GCV criterion can be computed, along with its gradient, in  $O(n^3 + dn^2)$  time. However, since  $\tilde{\sigma}$  depends on the data, the  $J_n^{\text{GCV}}$  criterion cannot be formally derived from a scoring rule.

*Kernel alignment.* The kernel alignment (KA) selection criterion [15]

$$(3.16) \quad J_n^{\text{KA}}(\theta) = -\frac{(\mathbf{z}_n^\circ)^\top \mathbf{K}_\theta \mathbf{z}_n^\circ}{\|\mathbf{K}_\theta\|_F \|\mathbf{z}_n^\circ\|^2},$$

where  $\|\cdot\|_F$  stands for the Frobenius matrix norm, measures the (opposite of the) alignment of  $\mathbf{z}_n^\circ$  with the eigenvector of  $\mathbf{K}_\theta$  corresponding to the largest eigenvalue. This criterion is related to (3.13) with  $p = -1$  and  $q = 2$ :

$$(3.17) \quad J_n^{\text{KA}}(\theta) = -\frac{1}{\sqrt{n} \|\mathbf{z}_n^\circ\|^2 J_n^{\text{HL}, -1, 2}(\theta)}.$$

Thus, minimizing the KA criterion (3.16) is equivalent to minimizing  $J_n^{\text{HL}, -1, 2}$  when the mean  $\beta$  is known (otherwise,  $\mathbf{z}_n^\circ$  also depends on  $\theta$ ). The KA criterion is noticeably cheaper than the others, as it does not require to invert  $\mathbf{K}_\theta$  and can therefore be computed along with its gradient in  $O(dn^2)$  time.

*Remark 3.8.* The KA criterion as written above is only suitable for selecting covariance parameters and cannot be used to select both the mean  $\beta$  and the range parameters as is done in this study. Indeed, the criterion is minimized when both  $\beta$  and the range parameters  $\rho_j$  go to infinity.

*Remark 3.9.* We choose to focus in this article on two well-known selection criteria (NLL and GCV) that can be seen as special cases of (3.13), corresponding, respectively, to  $(p, q)$  equal to  $(1, 0)$ ,  $(2, -1)$ . The study of new selection criteria, obtained for other values of  $(p, q)$ , is left for future work.

**3.3. Hybrid selection criteria.** When considering several parameterized models—or, equivalently, when dealing with discrete parameters, such as half-integer values for the regularity parameter of the Matérn covariance—some authors suggest to use one selection criterion to select the parameters in each particular model and a different one to select the model itself.

For instance, in [31], the authors select the parameters of a power-exponential covariance function using the NLL selection criterion (i.e., the ML method) and then select a suitable transformation of the output of the simulator, in a finite list of possible choices, using the LOO-SPE criterion. Similarly, the NLL selection criterion is combined in [18] with a variety of model-validation criteria, including LOO-CRPS and LOO-NLPD.

In section 4, we will denote by NLL/SPE the hybrid method that selects the variance and range parameters of a Matérn covariance function using the NLL criterion and then minimizes the LOO-SPE criterion to select the regularity parameter  $\nu$  in a finite list of values.

## 4. Numerical experiments.

**4.1. Methodology.** We investigate the problem of parameter selection with an experimental approach consisting of four ingredients: (1) a set of unknown functions  $z$  to be predicted using evaluation results on a finite design  $\mathbb{X}_n = \{x_1, \dots, x_n\} \subset \mathbb{X}$ ; (2) the GP regression method that constructs predictive distributions  $P_{\theta, x}$  of  $z$  at given  $x$ 's in  $\mathbb{X}$ , indexed by parameter  $\theta$ ; (3) several selection criteria  $J_n$  to choose  $\theta$ ; and (4) several criteria to assess the quality of the  $P_{\theta, x}$ s. Details about each of these ingredients are given below (starting from the last one).

*Criteria to assess the quality of the  $P_{\theta, x}$ 's.* A natural way to construct a criterion to assess the quality of the  $P_{\theta, x}$ 's is to choose a scoring rule  $S$  and to consider the mean score on a test set  $\mathbb{X}_N^{\text{test}} = \{x_1^{\text{test}}, \dots, x_N^{\text{test}}\} \subset \mathbb{X}$  of size  $N$ :

$$(4.1) \quad R(\theta; S) = \frac{1}{N} \sum_{i=1}^N S(P_{\theta, x_i^{\text{test}}}, z(x_i^{\text{test}})).$$

*Selection criteria.* We shall consider the selection criteria  $J_n$  presented in section 3, namely, the LOO-SPE, LOO-NLPD, LOO-CRPS, NLL, GCV, and NLL/SPE selection criteria. Given a function  $f$  and a design  $\mathbb{X}_n$ , each selection criterion  $J_n$  yields a parameter  $\theta_{J_n}$ .

*Remark 4.1.* We do not include KA in our list of selection criteria since it is not suitable to select the mean  $\beta$  (cf. Remark 3.8). Note that experiments in the zero-mean case [38] suggest that the KA criterion performs poorly when compared to the other criteria considered in the article.

*Parameterized GP models.* In this work, models are implemented using a custom version of the GPy [28] Python package (see Supplementary Material (M144471\_01.pdf [local/web 1.28MB]), hereafter abbreviated as SM). We assume no observation noise, which corresponds to the interpolation setting. A constant mean function and the anisotropic Matérn covariance function (2.1) are used. The parameter, to be selected, is  $\theta = (\beta, \sigma^2, \rho_1, \dots, \rho_d, \nu)$ . The regularity parameter  $\nu$  is either set a priori to  $\nu = \chi + 1/2$ , with  $\chi \in \{0, 1, 2, 3, 4, d, 2d, \infty\}$ , or selected automatically to a value denoted by  $\hat{\nu}$ .

*Remark 4.2.* Since the covariance matrix of  $\mathbf{Z}_n$  can be nearly singular when the range parameters take large values, we define upper bounds on these values in order to avoid the use of *nugget* or *jitter* (see, e.g., [37, 42]). Details are provided in the SM (M144471\_01.pdf [local/web 1.28MB]).

*Test functions.* The test functions used in the study are described in the next section. They are grouped into collections, and we provide averaged values of mean-score metrics of the form (4.1) for each collection.

## 4.2. Test functions.

**4.2.1. Design of a low-pass filter.** Fuhrländer and Schöps [24] consider the problem of computing, using a frequency-domain PDE solver, the scattering parameters  $s_\omega$  of an electronic component called stripline low-pass filter, at several values of the excitation pulsation  $\omega$ . The geometry of the stripline filter is illustrated in Figure 1. It is parameterized using six real valued factors concatenated in a vector  $x \in \mathbb{R}^d$ ,  $d = 6$ . The objective is to satisfy the low-pass specifications  $|s_{2k\pi}(x)| \geq -1$  dB for  $0 \leq k \leq 4$  and  $|s_{2k\pi}(x)| \leq -20$  dB for  $5 \leq k \leq 7$ . Meeting such requirements is a difficult and time-consuming task.

In this article, we consider the quantities  $\text{Re}(s_{2\pi})$ ,  $\text{Re}(s_{6\pi})$ ,  $\text{Re}(s_{10\pi})$ , and  $\text{Re}(s_{14\pi})$ , where  $\text{Re}(z)$  denotes the real part of a complex number  $z$ . For three different design sizes  $n \in \{10d, 20d, 50d\}$ , we randomly sample  $M = 100$  subsets  $\mathbb{X}_n$  of size  $n$  from a database of 10000 simulation results and use the remaining  $N = 10000 - n$  points as test sets. The data are standardized so that the output values have zero mean and unit variance on the test sets. The metric (4.1) is computed and averaged on these  $M$  repetitions.

**4.2.2. Other test functions.** We supplement the above engineering problem with a collection of test functions from the literature. More precisely, we consider the Goldstein–Price func-

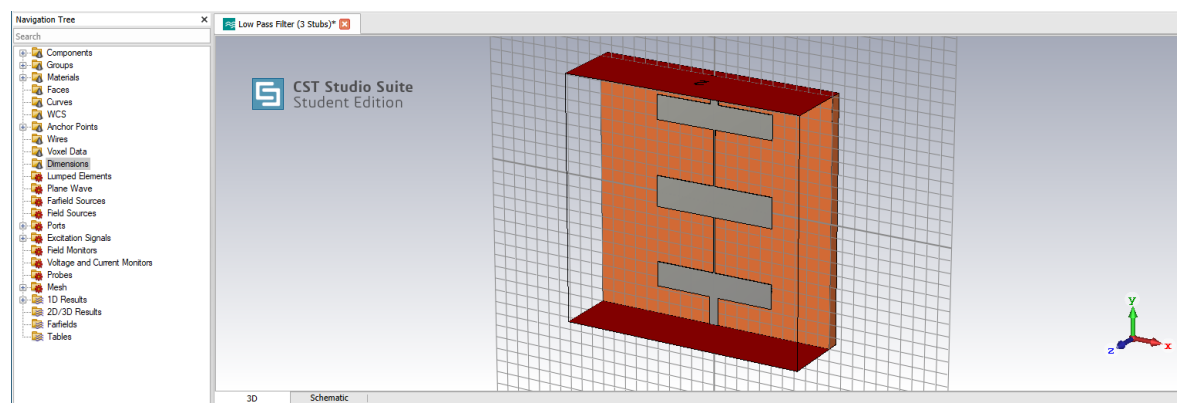


Figure 1. A low-pass filter design problem in CST Studio Suite.

Table 3

Twelve benchmark problems.

Problem	Goldstein–Price	Mystery	GKLS <sub>k=0</sub>	GKLS <sub>k=1</sub>	GKLS <sub>k=2</sub>	Rosenbrock	Borehole
$d$	{1, 2}	2	{2, 5}	{2, 5}	{2, 5}	{2, 5}	8

tion [19], a one-dimensional version of the Goldstein–Price function (see SM (M144471\_01.pdf [local/web 1.28MB]) for details), the Mystery function [33], the Borehole function [54], several collections obtained from the GKLS simulator [25], and the rotated Rosenbrock collection from the BBOB benchmark suite [30].

The GKLS simulator has a “smoothness” parameter  $k \in \{0, 1, 2\}$  controlling the presence of nondifferentiabilities on some nonlinear subspaces—the trajectories being otherwise infinitely differentiable. For both GKLS and Rosenbrock, two different values of the input dimension were considered ( $d = 2$  and  $d = 5$ ). The resulting set of 12 problems—considering that changing the value of  $k$  or  $d$  defines a new problem—is summarized in Table 3.

For each problem, we consider three design sizes  $n \in \{10d, 20d, 50d\}$ . For the GKLS and Rosenbrock collections, we used the collections of test functions provided by the authors ( $M = 100$  and 15 functions, respectively). They are evaluated on random space-filling designs  $\mathbb{X}_n$  (see below).

For the Goldstein–Price function, the one-dimensional version of the Goldstein–Price function, the Mystery function, and the Borehole function, we evaluate each test function on  $M = 100$  random space-filling designs  $\mathbb{X}_n$ .

Random space-filling designs are obtained using pseudomaximin Latin hypercube sampling obtained from 1000 random draws (see, e.g., [41]). Test sets are constructed using a Sobol’ sequence  $\mathbb{X}_N^{\text{test}}$  of size  $N = 10000$  for the evaluations of the functions. To make the results comparable, all functions are centered and normalized to unit variance on the test sets.

### 4.3. Results and findings.

**4.3.1. A close look at one of the problems.** Tables 4 and 5 provide a detailed view of the results obtained on one of the test problems—namely, the output  $\text{Re}(s_{2\pi})$  with  $n = 10d = 60$  of the low-pass filter case (see section 4.2.1).

Table 4

Averages (over the  $M = 100$  designs) of the  $R(\theta; S^{\text{SPE}})$  values for  $\text{Re}(s_{2\pi})$  with  $n = 10d = 60$ . For comparison, the rightmost column gives the optimal value  $R^*$  obtained by direct minimization of the score (4.1); see SM (M144471\_01.pdf [local/web 1.28MB]) for more details. Using  $J_n^{\text{NLL/SPE}}$  as a selection criterion, which also selects  $\nu$  (see section 3.3), we have  $R(\theta; S^{\text{SPE}}) = 2.00 \cdot 10^{-5}$ . The gray scale highlights the order of magnitude of the discrepancies.

Scoring rule: $S^{\text{SPE}}$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	GCV	$R^*$
$\nu = 1/2$	$2.56 \cdot 10^{-2}$	$2.78 \cdot 10^{-2}$	$2.43 \cdot 10^{-2}$	$2.43 \cdot 10^{-2}$	$2.35 \cdot 10^{-2}$	$2.09 \cdot 10^{-2}$
$\nu = 3/2$	$7.41 \cdot 10^{-5}$	$8.95 \cdot 10^{-5}$	$7.15 \cdot 10^{-5}$	$7.73 \cdot 10^{-5}$	$7.13 \cdot 10^{-5}$	$6.00 \cdot 10^{-5}$
$\nu = 5/2$	$1.66 \cdot 10^{-5}$	$2.30 \cdot 10^{-5}$	$1.61 \cdot 10^{-5}$	$1.73 \cdot 10^{-5}$	$1.91 \cdot 10^{-5}$	$1.02 \cdot 10^{-5}$
$\nu = 7/2$	$1.95 \cdot 10^{-5}$	$3.14 \cdot 10^{-5}$	$2.41 \cdot 10^{-5}$	$2.51 \cdot 10^{-5}$	$3.27 \cdot 10^{-5}$	$1.07 \cdot 10^{-5}$
$\nu = 9/2$	$2.58 \cdot 10^{-5}$	$4.25 \cdot 10^{-5}$	$3.26 \cdot 10^{-5}$	$3.48 \cdot 10^{-5}$	$5.25 \cdot 10^{-5}$	$1.53 \cdot 10^{-5}$
$\nu = 13/2$	$3.26 \cdot 10^{-5}$	$6.38 \cdot 10^{-5}$	$4.11 \cdot 10^{-5}$	$4.14 \cdot 10^{-5}$	$1.11 \cdot 10^{-4}$	$1.90 \cdot 10^{-5}$
$\nu = 25/2$	$4.23 \cdot 10^{-5}$	$6.79 \cdot 10^{-5}$	$5.93 \cdot 10^{-5}$	$5.57 \cdot 10^{-5}$	$1.16 \cdot 10^{-4}$	$2.30 \cdot 10^{-5}$
$\nu = \infty$	$4.87 \cdot 10^{-5}$	$7.47 \cdot 10^{-5}$	$5.79 \cdot 10^{-5}$	$5.90 \cdot 10^{-5}$	$9.08 \cdot 10^{-5}$	$2.56 \cdot 10^{-5}$
$\nu \in \{1/2, \dots, \infty\}$	$1.69 \cdot 10^{-5}$	$3.37 \cdot 10^{-5}$	$2.13 \cdot 10^{-5}$	$2.26 \cdot 10^{-5}$	$2.70 \cdot 10^{-5}$	$8.99 \cdot 10^{-6}$

Table 5

Same as Table 4 but for averages of  $R(\theta; S_{0.95}^{\text{IS}})$ . Using  $J_n^{\text{NLL/SPE}}$  gives  $R(\theta; S_{0.95}^{\text{IS}}) = 2.18 \cdot 10^{-2}$ .

Scoring rule: $S_{0.95}^{\text{IS}}$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	GCV
$\nu = 1/2$	$1.05 \cdot 10^0$	$9.86 \cdot 10^{-1}$	$9.10 \cdot 10^{-1}$	$1.03 \cdot 10^0$	$8.91 \cdot 10^{-1}$
$\nu = 3/2$	$4.25 \cdot 10^{-2}$	$4.46 \cdot 10^{-2}$	$3.86 \cdot 10^{-2}$	$4.09 \cdot 10^{-2}$	$3.82 \cdot 10^{-2}$
$\nu = 5/2$	$1.91 \cdot 10^{-2}$	$2.09 \cdot 10^{-2}$	$1.76 \cdot 10^{-2}$	$1.88 \cdot 10^{-2}$	$1.88 \cdot 10^{-2}$
$\nu = 7/2$	$2.08 \cdot 10^{-2}$	$2.50 \cdot 10^{-2}$	$2.16 \cdot 10^{-2}$	$2.40 \cdot 10^{-2}$	$2.51 \cdot 10^{-2}$
$\nu = 9/2$	$2.53 \cdot 10^{-2}$	$3.14 \cdot 10^{-2}$	$2.84 \cdot 10^{-2}$	$3.20 \cdot 10^{-2}$	$3.54 \cdot 10^{-2}$
$\nu = 13/2$	$3.04 \cdot 10^{-2}$	$4.02 \cdot 10^{-2}$	$3.55 \cdot 10^{-2}$	$3.89 \cdot 10^{-2}$	$4.59 \cdot 10^{-2}$
$\nu = 25/2$	$3.70 \cdot 10^{-2}$	$4.38 \cdot 10^{-2}$	$4.46 \cdot 10^{-2}$	$4.59 \cdot 10^{-2}$	$5.23 \cdot 10^{-2}$
$\nu = \infty$	$4.16 \cdot 10^{-2}$	$4.70 \cdot 10^{-2}$	$4.61 \cdot 10^{-2}$	$4.94 \cdot 10^{-2}$	$5.62 \cdot 10^{-2}$
$\nu \in \{1/2, \dots, \infty\}$	$1.95 \cdot 10^{-2}$	$2.82 \cdot 10^{-2}$	$2.15 \cdot 10^{-2}$	$2.43 \cdot 10^{-2}$	$2.50 \cdot 10^{-2}$

The results presented in these tables are the scores  $R(\theta; S)$ , averaged over the  $M = 100$  repetitions, where  $\theta$  is selected using different selection criteria (along columns) and the regularity of the Matérn covariance varies or is selected automatically (along rows). The scoring rule for assessing the quality of the predictions is the SPE in Table 4 and the IS at level 95% in Table 5. (A similar table, not shown here, is presented in the SM (M144471\_01.pdf [local/web 1.28MB]) for the CRPS.)

For comparison, Table 4 also provides the optimal values  $R^*$  obtained by direct minimization of the score (4.1). They can be used to assess the loss of predictive accuracy of the selected models, which are constructed using a limited number of observations, with respect to the best model that could have been obtained if the test data had also been used to select the parameters.

Tables 4 and 5 support the fact that, for this particular problem, the NLL and NLL/SPE criteria may be seen as slightly better choices for selecting  $\theta$  in terms of the SPE and the IS scores both for a prescribed regularity  $\nu$  and when  $\nu$  is selected automatically (the NLL/SPE being only available for the latter case). The other selection criteria, however, are never *very* far behind—at most a factor of approximately two for both metrics. Elements provided as SM (M144471\_01.pdf [local/web 1.28MB]) show similar findings using the CRPS validation score.

Table 6

Averages over the repetitions of the empirical coverage  $C_{0.95}(\theta)$  for the function  $s_{2\pi}$ . Using  $J_n^{\text{NLL/SPE}}$  gives  $C_{0.95}(\theta) = 0.93$ .

$C_{0.95}$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	GCV
$\nu = 1/2$	0.993	0.923	0.929	0.890	0.932
$\nu = 3/2$	0.992	0.912	0.929	0.927	0.936
$\nu = 5/2$	0.965	0.883	0.898	0.893	0.905
$\nu = 7/2$	0.927	0.842	0.838	0.823	0.837
$\nu = 9/2$	0.879	0.802	0.790	0.768	0.767
$\nu = 13/2$	0.828	0.762	0.741	0.720	0.728
$\nu = 25/2$	0.788	0.751	0.714	0.700	0.709
$\nu = \infty$	0.763	0.745	0.715	0.696	0.698
$\nu \in \{1/2, \dots, \infty\}$	0.963	0.824	0.838	0.824	0.828

Strikingly enough, for both scoring rules, the variations of the average score are much larger when the model changes (with  $\nu$ ) than when the selection criterion changes. If a Matérn covariance function with fixed regularity is used, as is often done in practice, then the best results are obtained for all criteria when  $\nu = 5/2$ . The values of  $R^*$  (Table 4) confirm that this is indeed the best fixed- $\nu$  model on this problem for the SPE score. Since this optimal value was not known beforehand, it is a relief to see (cf. the last row of each table) that comparable performances can be achieved on this problem by selecting  $\nu$  automatically.

Finally, we can also study the impact of the selection criterion and that of  $\nu$  on confidence intervals at level 95%. This is a useful complement to the results obtained using the IS for model validation. For a Gaussian predictive distribution  $P_{\theta,x}$  at  $x \in \mathbb{X}$  with mean  $\mu_\theta(x)$  and variance  $\sigma_\theta^2(x)$ , the natural confidence interval at level 95% is

$$(4.2) \quad I_\theta^{0.95}(x) = [\mu_\theta(x) - 1.96\sigma_\theta(x), \mu_\theta(x) + 1.96\sigma_\theta(x)].$$

Then, denoting by  $z : \mathbb{X} \rightarrow \mathbb{R}$  the unknown data-generating function and given a test set

$$(4.3) \quad \{x_i^{\text{test}}, i = 1, \dots, N\} \in \mathbb{X}^N$$

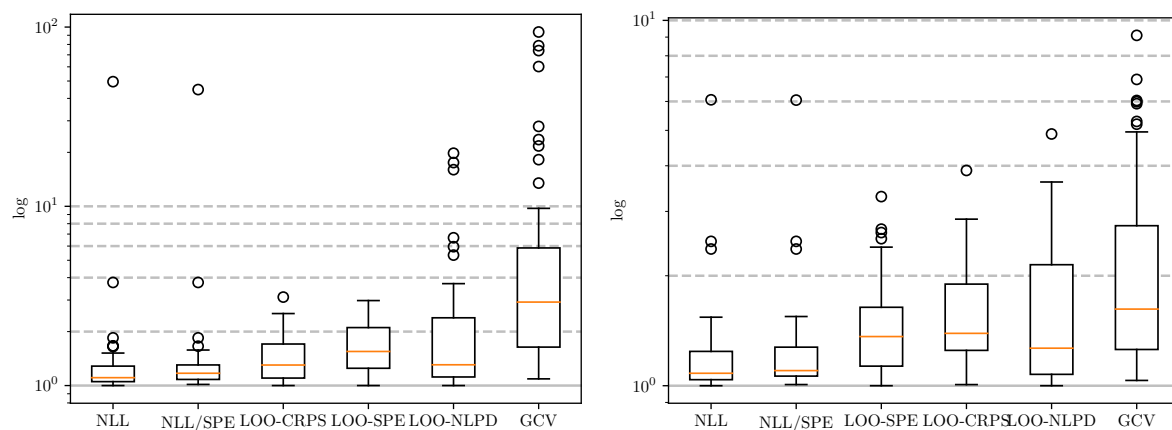
at which the values of  $z$  are known, define the *empirical coverage* as

$$(4.4) \quad C_{0.95}(\theta) = \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{z(x_i^{\text{test}}) \in I_\theta^{0.95}(x_i^{\text{test}})}.$$

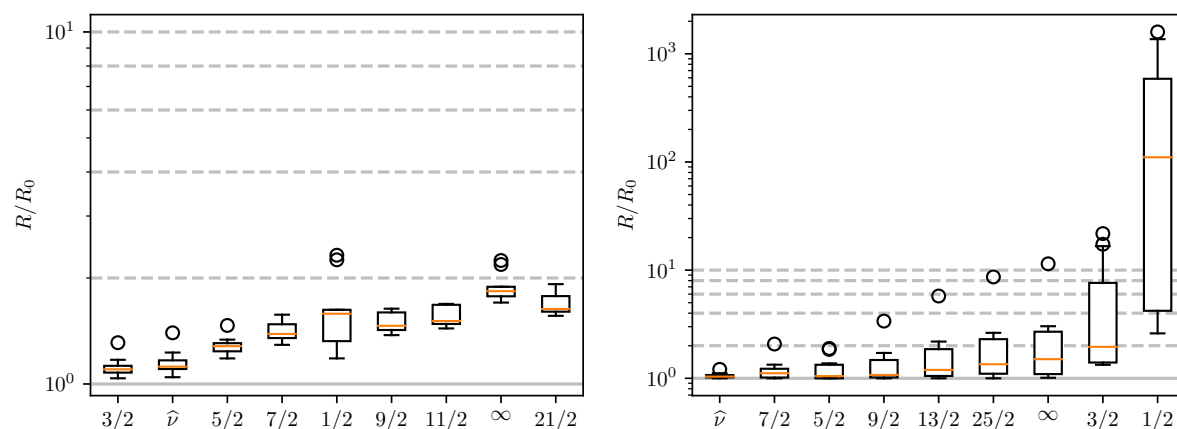
Table 6 reports the averaged values of  $C_{0.95}(\theta)$  over the  $M$  repetitions for the data-generating function  $s_{2\pi}$ . Notice that when  $\nu$  is large, empirical coverages are low, indicating that the confidence intervals are probably too small. Notice also that the NLL selection criterion tends to produce well-calibrated confidence intervals on this test function for  $\nu = 5/2$  or when  $\nu$  is selected automatically.

**4.3.2. Statistical analysis of the benchmark results.** Tables similar to Tables 4 and 5 have been produced for all the  $(4 + 12) \times 3 = 48$  combinations of problem and design size for the three scoring rules (SPE, CRPS, and IS). We present in this section some graphical summaries and statistical analyses of these results. The individual tables are provided in the SM (M144471\_01.pdf [local/web 1.28MB]).





**Figure 2.** Box plots of  $R/R_0$  for different selection criteria in the case where  $\nu$  is automatically selected. Each box plot is constructed using all problems and design sizes ( $16 \times 3 = 48$  combinations). For a given problem,  $R$  denotes the score, averaged over the repetitions, and  $R_0$  stands for the best value of  $R$  (among all models and selection criteria). The left (resp., right) panel uses the  $S^{\text{SPE}}$  (resp.,  $S_{0.95}^{\text{IS}}$ ) as a quality assessment criterion. The box plots are sorted according to their upper whisker. Gray dashed lines:  $R/R_0 = 2, 4, 6, 8, 10$ .



**Figure 3.** Box plots of  $R/R_0$  using  $J_n^{\text{NLL}}$  as a selection criterion and  $S^{\text{SPE}}$  as a quality assessment criterion for different choices of regularity. Left: all 5d GKLS problems. Right: all low-pass filter problems. Each box plot is constructed using all problems in the given class and all design sizes (left:  $3 \times 3 = 9$  combinations; right:  $4 \times 3 = 12$  combinations). For a given problem,  $R$  denotes the score, averaged over the repetitions, and  $R_0$  stands, as in Figure 2, for the best value of  $R$  (among all models and selection criteria). The box plots are sorted according to their upper whisker. Gray dashed lines:  $R/R_0 = 2, 4, 6, 8, 10$ .

**Comparison of the selection criteria.** Figure 2 compares the distributions of the average values of the SPE and IS scores for all selection criteria in the case of a Matérn covariance function with automatically selected regularity. As a preliminary observation, note that for most criteria (except GCV), the variations of the ratio  $R/R_0$  for the SPE score remain under four (second horizontal dashed line), which is to be compared to the possibly large variations due to a poor choice of covariance models (see the right plot in Figure 3 below).

A closer look at Figure 2 reveals that the rankings of criteria obtained for both scoring rules are almost identical. The ranking for the CRPS scoring rule (not shown) is the same



as the one for SPE. GCV provides the worst performance for all scoring rules, followed by LOO-NLPD, while NLL dominates the ranking (for all scoring rules as well).

*Remark 4.3.* Observe in Figure 2 that LOO-SPE is, surprisingly, much less accurate than NLL according to  $S^{\text{SPE}}$ . More generally, choosing a scoring rule  $S$  for the LOO criterion does not guarantee the highest precision according to this particular score.

*Comparison of the covariance models.* Figure 3 compares the average values of  $R(\theta; S^{\text{SPE}})$  when the NLL selection criterion is used on the set of GKLS problems, which have low regularities, and on the set of low-pass filter problems, which contain very smooth instances.

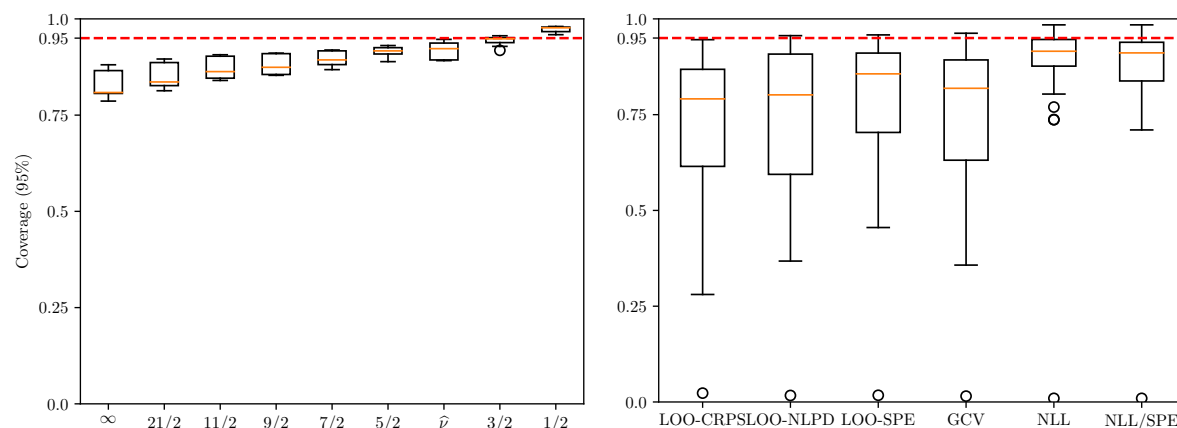
Observe first that the fixed- $\nu$  models rank differently on these two sets of problems, as expected considering the actual regularity of the underlying functions: Low values of  $\nu$  perform better on the GKLS problems and worse on the low-pass filter case. Furthermore, it appears that underestimating the regularity (on the low-pass filter case) has much more severe consequences than overestimating it (on the GKLS problems) according to the SPE score and as suggested by the theoretical results of [46], [36], [49], and [45, section 6].

Another important conclusion from Figure 3 is that very good results can be obtained by selecting the regularity parameter  $\nu$  automatically, jointly with the other parameters (using the NLL criterion in this case). On the GKLS problems, the results with selected  $\nu$  are not far from those of the best fixed- $\nu$  model under consideration ( $\nu = 3/2$ ); in the low-pass filter case, they are even better than those obtained with the best fixed- $\nu$  models ( $\nu = 5/2$  or  $7/2$ ). In other words, the regularity needs not be known in advance to achieve good performances, which is a very welcome practical result. This conclusion is also supported for NLL by the additional results provided in the SM (M144471.01.pdf [local/web 1.28MB]) for the other problems and for the three scoring rules.

Concerning the other selection criteria, the situation is more contrasted (see SM (M144471.01.pdf [local/web 1.28MB])): The automatic selection of  $\nu$  using these criteria still performs very well for smooth problems but not always, in particular with GCV, for the less regular problems of the GKLS class. This is especially true when the sample size is small ( $n = 10d$ ).

*Robustness.* LOO-SPE is commonly claimed in the literature (see, notably, [2]) to provide a certain degree of robustness with respect to model misspecification. According to this claim, LOO-SPE would be expected to somehow mitigate the loss of predictive accuracy with respect to likelihood-based approaches incurred by an ill-advised choice of covariance function. Our detailed results (see SM (M144471.01.pdf [local/web 1.28MB])) suggest that this effect indeed exists when the regularity is severely underestimated (e.g.,  $\nu = 1/2$  for the low-pass filter problems) but is actually quite small and should not be used to motivate the practice of setting  $\nu$  to an arbitrary value. A similar effect exists for LOO-CRPS, LOO-NLPD, and GCV as well. Quite surprisingly, NLL turns out to be more robust than LOO-SPE (and the other criteria) in the case of oversmoothing.

*Coverage.* To further complement the study, we analyze the results in terms of empirical coverage, as defined in section 4.3.1. Figure 4 shows the variations of the empirical coverage as a function of the covariance model on the GKLS test functions. As expected, choosing large values of  $\nu$  for these test functions tends to produce undercoverage, that is, (too) small confidence intervals or slightly overconfident predictions. Figure 4 also shows the effect of



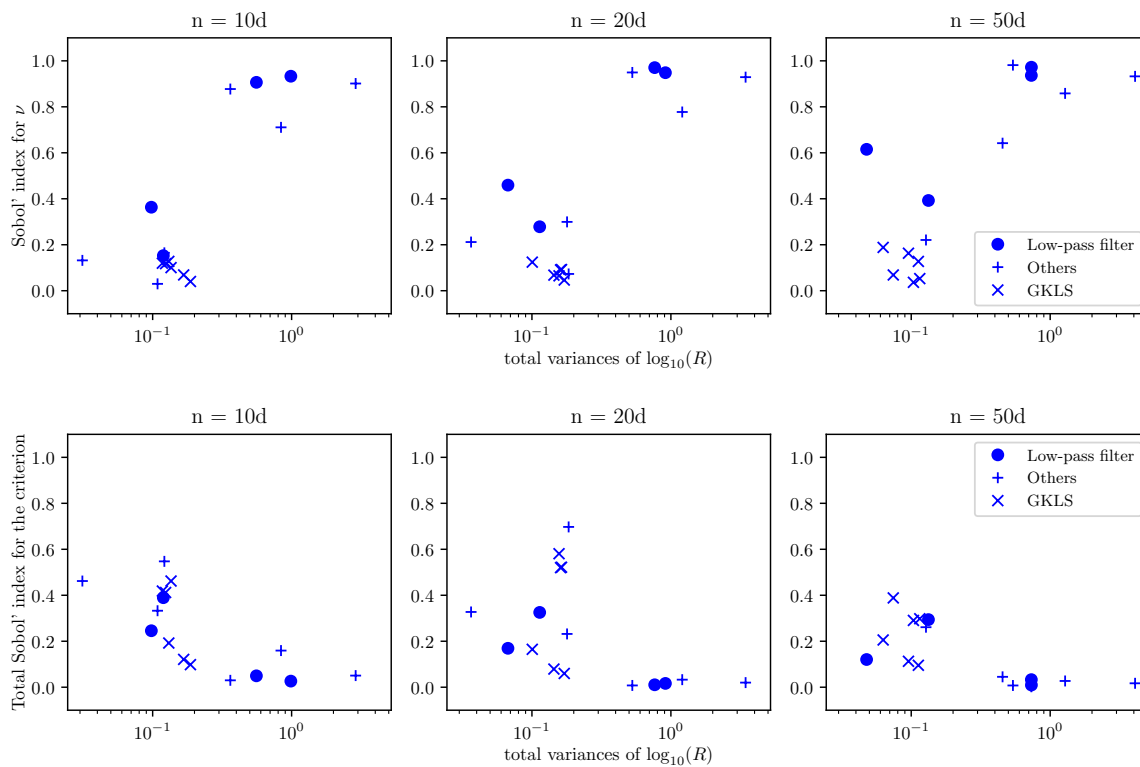
**Figure 4.** Box plots of  $C_{0.95}(\theta)$  (averaged over the repetitions for each test function). Left: for several regularities and using the NLL criterion on all 5d GKLS problems; right: for several selection criteria and for  $\nu$  automatically selected on the whole set of test functions.

the selection criterion (when  $\nu$  is automatically selected) on the whole set of test functions. As in section 4.3.1, it seems that the NLL selection criterion yields better calibrated confidence intervals.

**Sensitivity analysis.** We observed in section 4.3.1 and in the paragraphs above that the choice of the model—more specifically, of the regularity parameter of the Matérn covariance function—was more important than that of a particular selection criterion. To confirm this finding, a global sensitivity analysis of the logarithm of the score has been performed for each combination of problem and design size. The score, for a given scoring rule, depends on three discrete factors: the selection criterion, the regularity  $\nu$  of the Matérn covariance function, and the repetition index (recall that for each problem, we have a collection of  $M$  repetitions, with  $M = 100$  or  $15$ , depending on the problem). Only fixed values of  $\nu$  are considered in this analysis (no automatic selection), and therefore the NLL/SPE method, which coincides with NLL for fixed  $\nu$ , is not included as a level for the criterion factor. The factors are considered independent and uniformly distributed in the definition of the Sobol’ sensitivity indices.

We present in Figure 5, for the case of the SPE scoring rule, the first-order Sobol’ sensitivity indices with respect to  $\nu$  and the total Sobol’ sensitivity indices with respect to the selection criterion as functions of the total variance. Observe that, for the problems where the total variance is large, the total index with respect to the selection criterion is typically close to zero, which indicates a negligible impact of the selection criterion on the variations of  $R$ . On the other hand, first-order indices with respect to  $\nu$  are typically above 80% on problems with large variations of  $R$ . This indicates situations where the influence of  $\nu$  is very important.

Note that it also means that the influence of  $\nu$  dominates the variability due to the randomness of the design over the repetitions, which apparently counterbalances the conclusions of [11], where large variations of the prediction errors are reported when the design varies. However, [11] is in the case of small designs, and from a fully Bayesian point of view, the posterior distribution of  $\theta$  is very often very spread out in this case, potentially leading to poor parameter selection, as also noted in [7]. In our setting, however, our primary focus is



**Figure 5.** Sensitivity analysis of  $\log_{10}(R)$  for  $S^{\text{SPE}}$ , for each problem and design size. The top plots show the first-order Sobol' indices for  $\nu$ , i.e., the contribution of  $\nu$  to the variance of  $S^{\text{SPE}}$  measured as the ratio of the variance of the expectation of  $S^{\text{SPE}}$  when  $\nu$  is fixed to the total variance of  $S^{\text{SPE}}$ . Values close to one tell us that the variations of  $\nu$  explain almost all the variations of  $S^{\text{SPE}}$ . The bottom plots show the total Sobol' indices of the selection criterion, i.e., the variance explained by the selection criterion, taking every interaction into account. Values close to zero tell us that the variations of  $S^{\text{SPE}}$  are not explained by variations of the selection criterion. Each point represents the variations of  $\log_{10}(R)$  for one of the 16 problems from section 4.2.2, split by design size. The model explains almost all the variations for problems exhibiting significant fluctuations of  $\log_{10}(R)$  (at the right of the figure).

to study the problem of model selection, and so we have chosen to consider “informative” designs in our experiments (using maximin Latin hypercube sampling and design sizes  $n$  with  $n \geq 10d$ ).

Similar conclusions hold for the other scoring rules (results not shown; see SM (M144471.01.pdf [local/web 1.28MB])).

**5. Conclusions.** A large variety of selection criteria for GP models is available from the literature, with little theoretical or empirical guidance on how to choose the right one for applications. Our benchmark study with the Matérn family of covariance functions in the noiseless (interpolation) case indicates that the NLL selection criterion—in other words, the ML method—provides performances that are, in most situations and for all the scoring rules that were considered (SPE, CRPS, and IS at 95%), better than or comparable to those of the other criteria. Considering that all the criteria tested in the study have a similar computational complexity, this provides strong empirical support for the ML method—which is already the de facto standard in most statistical software packages implementing GP interpolation.

Another important lesson learned from our benchmark study is that the choice of the family of models, in particular of the family of covariance functions, has very often a bigger impact on performance than that of the selection criterion itself. This is especially striking when the actual function is smooth and a very irregular covariance function, such as the Matérn covariance with regularity  $1/2$ , is used to perform GP interpolation. In such a situation, NLL is actually outperformed by other criteria such as LOO-SPE or LOO-CRPS, which thus appear to be more “robust to model misspecification.” However, the small gain of performance, which is achieved by using LOO-SPE or LOO-CRPS instead of NLL in this case, is generally negligible with respect to the much larger loss induced by choosing an inappropriate covariance function in the first place.

Our final recommendation, supported by the results of the benchmark and in line with the recommendation of [46], is therefore to select the regularity of the covariance function automatically, jointly with the other parameters, using the NLL criterion. A minimal list of candidate values for the regularity parameter should typically include  $1/2$ ,  $3/2$ ,  $5/2$ ,  $7/2$ , and  $+\infty$  (the Gaussian covariance function). Should a situation arise where a default value of  $\nu$  is nevertheless needed, our recommendation would be to choose a reasonably large value, such as  $\nu = 5/2$  or  $\nu = 7/2$ , since underestimating  $\nu$  has been seen in section 4.3.2 to have much more severe consequences than overestimating it a little. Nevertheless, we stress that overestimating  $\nu$  is not only suboptimal but also likely to produce confidence intervals with undercoverage.

More generally, our numerical results support the fact that choosing carefully a suitable family of GP models is extremely important: Poor decisions at this stage are indeed likely to induce large losses in performance that no choice of selection criterion can counterbalance. We have focused in this article on the case of a constant mean with an anisotropic Matérn covariance function, with fixed regularity or not. We believe—but it should be confirmed by further numerical experiments—that this conclusion would not be altered significantly if other families of GP models with a comparable number of parameters (e.g., affine mean function, tensor-product covariance functions, or the compactly supported covariance functions proposed by [52]) were considered instead or if other noise-free settings (multivariate outputs or multifidelity) were addressed.

However, it should be kept in mind that the study focuses on cases where the number of parameters is small with respect to the number of observations (in particular, we considered GPs with an anisotropic stationary Matérn covariance function, which have  $d + 2$  parameters, and we took care of having  $n \gg d$ ). When  $n$  is small with respect to  $d$  or when the number of parameters increases, it seems to us that a fully Bayesian approach could help or that other selection criteria should be considered, with the introduction of regularization terms, for instance.

For future work, it would be of course very interesting to consider the performance of using selection criteria against a fully Bayesian approach. Another direction would be to extend this study to the case of regression, which is also used in many applications, when dealing with stochastic simulators, for instance.

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