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# Gaussian process interpolation: the choice of the family of models is more important than that of the selection criterion

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## Abstract

This article revisits the fundamental problem of parameter selection for Gaussian process interpolation. By choosing the mean and the covariance functions of a Gaussian process within parametric families, the user obtains a family of Bayesian procedures to perform predictions about the unknown function, and must choose a member of the family that will hopefully provide good predictive performances. We base our study on the general concept of scoring rules, which provides an effective framework for building leave-one-out selection and validation criteria, and a notion of extended likelihood criteria based on an idea proposed by Fasshauer and co-authors in 2009, which makes it possible to recover standard selection criteria such as, for instance, the generalized cross-validation criterion. Under this setting, we empirically show on several test problems of the literature that the choice of an appropriate family of models is often more important than the choice of a particular selection criterion (e.g., the likelihood versus a leave-one-out selection criterion). Moreover, our numerical results show that the regularity parameter of a Matérn covariance can be selected effectively by most selection criteria.

*Keywords:* Gaussian processes interpolation, Model choice; Parameter selection; Scoring rules; Likelihood; Regularity

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# 1 Introduction

Regression and interpolation with Gaussian processes, or kriging, is a popular statistical tool for non-parametric function estimation, originating from geostatistics and time series analysis, and later adopted in many other areas such as machine learning and the design and analysis of computer experiments (see, e.g., [Stein, 1999](#); [Santner et al., 2003](#); [Rasmussen and Williams, 2006](#), and references therein). It is widely used for constructing fast approximations of time-consuming computer models, with applications to calibration and validation ([Kennedy and O’Hagan, 2001](#); [Bayarri et al., 2007](#)), engineering design ([Jones et al., 1998](#); [Forrester et al., 2008](#)), Bayesian inference ([Calderhead et al., 2009](#); [Wilkinson, 2014](#)), and the optimization of machine learning algorithms ([Bergstra et al., 2011](#))—to name but a few.

A Gaussian process (GP) prior is characterized by its mean and covariance functions. They are usually chosen within parametric families (for instance, constant or linear mean functions, and Matérn covariance functions), which transfers the problem of choosing the mean and covariance functions to that of selecting parameters. The selection is most often carried out by optimization of a criterion that measures the goodness of fit of the predictive distributions, and a variety of such criteria—the likelihood function, the leave-one-out (LOO) squared-prediction-error criterion (hereafter denoted by LOO-SPE), and others—is available from the literature. The search for arguments to guide practitioners in the choice of one particular criterion is the main motivation of this study.

As a necessary parenthesis, note that the fully Bayesian statistician does not select a particular value for the parameters, but chooses instead to average the predictions over a *posterior* distribution on the parameters ([Handcock and Stein, 1993](#)). This approach may be preferred for more robust predictions (see, e.g., [Benassi et al., 2011](#)), but comes at a higher computational cost. For this reason, the present article will set aside the fully Bayesian approach and concentrate on the plugin approach, where only one parameter value is chosen to carry out predictions.

The two most popular methods for parameter selection are maximum likelihood (ML) and cross-validation (CV) based on LOO criteria, which were introduced in the field of computer experiments by the seminal work of [Currin et al. \(1988\)](#). Since then, despite a fairly large number of publications dealing with ML and CV techniques for the selection of a GP model, the literature has remained in our view quite sparse about the relative merits of these methods, from both theoretical and empirical perspectives.

For instance, in the framework of interpolation and infill asymptotics, where observations accumulate in some bounded domain, [Ying \(1991\)](#) and [Zhang \(2004\)](#) show that some combinations

of the parameters of a Matérn covariance function can be estimated consistently by ML. Again in the case of infill asymptotics, with the additional assumption of a one-dimensional GP with an exponential covariance function, [Bachoc et al. \(2016\)](#) show that estimation by LOO is also consistent. (Similar results exist in the expanding asymptotic framework, where observations extend over an ever-increasing horizon.)

The practical consequences of the aforementioned results are somewhat limited because practitioners are primarily interested in the quality of the predictions. Knowing that the parameters of a GP can be estimated consistently is intellectually reassuring, but may be considered of secondary importance. These results are indeed about the statistical model itself but they say little about the prediction properties of the model. Besides, there does not exist at present, to our knowledge, some theoretically-based evaluation of the relative performances of ML and CV selection criteria under infill asymptotics.

Turning now to empirical comparisons of selection criteria for GP interpolation, the first attempt in the domain of computer experiments can be traced back to the work of [Currin et al. \(1988, 1991\)](#). The authors introduce CV and ML—which can be seen as a special kind of cross-validation—and present some simple experiments using tensorized covariance functions, from which they conclude that, “Of the various kinds of cross-validation [they] have tried, maximum likelihood seems the most reliable”. Additional experiments have been conducted by several authors, but no consensus emerges from these studies: [Martin and Simpson \(2003\)](#); [Bachoc \(2013\)](#); [Sundararajan and Keerthi \(2001\)](#) conclude in favor of CV whereas [Martin and Simpson \(2005\)](#) advocate ML.

These studies are limited to a rather small number of test functions and covariance functions, which may explain the discrepancy in the conclusions of those experiments. In particular, only [Bachoc \(2013\)](#) considers the popular and versatile Matérn covariance functions. Moreover, most studies focus only on the accuracy of the posterior mean—only [Sundararajan and Keerthi \(2001\)](#) and [Bachoc \(2013\)](#) provide results accounting for the quality of the posterior variance—whereas the full posterior predictive distribution is used in most GP-based methods (see, e.g., [Jones et al., 1998](#); [Chevalier et al., 2014](#)).

This article presents two main contributions. First, we improve upon the results of the literature by providing an empirical ranking of selection criteria for GP interpolation, according to several metrics measuring the quality of posterior predictive distributions on a large set of test functions from the domain of computer experiments. To this end, we base our study on the general concept of scoring rules ([Gneiting and Raftery, 2007](#); [Zhang and Wang, 2010](#)), which provides an effective framework for building selection and validation criteria. We also intro-

duce a notion of extended likelihood criteria, borrowing an idea from Fasshauer and co-authors (Fasshauer et al., 2009; Fasshauer, 2011) in the literature of radial basis functions.

Second, we provide empirical evidence that the choice of an appropriate family of models is often more important—and sometimes much more important, especially when the size of the design increases—than the choice of a particular selection criterion (e.g., likelihood versus LOO-SPE). More specifically, in the case of the Matérn family, this leads us to assess, and ultimately recommend, the automatic selection of a suitable value of the regularity parameter, against the common practice of choosing beforehand an arbitrary value of this parameter.

The article is organized as follows. Section 2 briefly recalls the general framework of GP regression and interpolation. Section 3 deals with selection criteria. After recalling the general notion of scoring rules, we derive a broad variety of standard and new selection criteria. Section 4 presents experimental results on the relative performances of these criteria. Section 5 presents our conclusions and perspectives.

## 2 General framework

Let us consider the general GP approach for modeling the scalar output of a deterministic, but unknown, computer code, with input space  $\mathbb{X} \subseteq \mathbb{R}^d$ . Gaussian process modeling is a classical Bayesian point of view, under which uncertainty about the computer code is expressed by means of probabilities. More precisely, the output of the computer code  $z : \mathbb{X} \rightarrow \mathbb{R}$  is modeled by a random function  $\{Z(x), x \in \mathbb{X}\}$ , which represents user’s prior knowledge about  $z$ . If we assume that  $Z(\cdot)$  is observed on a design  $\mathbb{X}_n = \{x_1, \dots, x_n\}$  of size  $n$ , the data corresponds to a sample of the random vector  $\mathbf{Z}_n = (Z(x_1), \dots, Z(x_n))^T$ .

The GP assumption makes it possible to easily derive posterior distributions. More precisely, it is assumed that  $Z(\cdot)$  is a Gaussian process, with (prior) mean  $E(Z(x)) = \sum_{l=1}^L \beta_l \phi_l(x)$ , where the  $\beta_1, \dots, \beta_L$  are unknown regression parameters and  $\phi_1, \dots, \phi_L$  are known regression functions, and with (prior) covariance  $\text{cov}(Z(x), Z(y)) = k_\theta(x, y)$ , where  $\theta \in \Theta \subseteq \mathbb{R}^q$  is a vector of *parameters*. Throughout the article, the covariance matrix of  $\mathbf{Z}_n$  will be denoted by  $\mathbf{K}_\theta$ . We assume for simplicity that the prior mean of  $Z(\cdot)$  is zero (hence,  $L = 0$ ), which is a common (and reasonable) practice if the user has taken care to center the data.

One of the most popular covariance functions for GP regression is the anisotropic stationary Matérn covariance function (Matérn, 1986) popularized by Stein (1999):

$$k_\theta(x, y) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left( \sqrt{2\nu h} \right)^\nu \mathcal{K}_\nu \left( \sqrt{2\nu h} \right), \quad h = \left( \sum_{j=1}^d \frac{(x_j - y_j)^2}{\rho_j^2} \right)^{1/2}, \quad (1)$$

Table 1: Popular Matérn subfamilies

	$\nu = \frac{1}{2}$	$\nu = \frac{3}{2}$	$\nu = \frac{5}{2}$	$\nu = +\infty$
a.k.a.	exponential			squared exponential
$k_\theta(x, y)$	$\sigma^2 e^{-h}$	$\sigma^2 (1 + \sqrt{3}h) e^{-\sqrt{3}h}$	$\sigma^2 (1 + \sqrt{5}h + \frac{5h^2}{3}) e^{-\sqrt{5}h}$	$\sigma^2 e^{-\frac{h^2}{2}}$

where  $\Gamma$  is the Gamma function,  $\mathcal{K}_\nu$  is the modified Bessel function of the second kind, and  $\theta$  denotes the vector of parameters  $\theta = (\sigma^2, \rho_1, \dots, \rho_d, \nu) \in \Theta = ]0, \infty[^{d+2}$ . The parameter  $\sigma^2$  is the variance of  $Z(\cdot)$ , the  $\rho_i$ s are range parameters which characterize the typical correlation length on each dimension, and  $\nu$  is a regularity parameter, whose value controls the mean-square differentiability of  $Z(\cdot)$ . Recall (see Table 1) that the Matérn covariance function with  $\nu = 1/2$  corresponds to the so-called exponential covariance function, and the limiting case  $\nu \rightarrow \infty$  can be seen as the “squared exponential” (also called Gaussian) covariance function.

Because  $\mathcal{K}_\nu$  can be expressed in simple closed-form when  $\nu = \chi + \frac{1}{2}$  and  $\chi$  is an integer, and it is expensive to obtain numerical approximations of  $\mathcal{K}_\nu$  in other cases, most implementations choose to restrict  $\nu$  to half-integer values. Moreover, a widespread practice (in applications and research papers) consists in selecting a particular value for  $\nu$  (e.g.,  $\nu = 1/2$ ,  $\nu = 3/2 \dots$  or the limiting case  $\nu \rightarrow \infty$ ), once and for all.

Since the family of Matérn covariance functions is widely used in practice, we focus exclusively on this family in this work. We believe that the conclusions of the present study would not be altered significantly if other families of covariance functions (e.g., the compactly supported covariance functions proposed by [Wendland \(1995\)](#)) were considered.

Once a GP model has been chosen, the framework of Gaussian process regression allows one to build a predictive distribution  $\mathcal{N}(\mu_\theta(x), \sigma_\theta^2(x))$  for an unobserved  $Z(x)$  at  $x \in \mathbb{R}^d$ , where

$$\begin{cases} \mu_\theta(x) &= \mathbf{k}_\theta^*(x)^\top \mathbf{K}_\theta^{-1} \mathbf{Z}_n, \\ \sigma_\theta^2(x) &= k_\theta(x, x) - \mathbf{k}_\theta^*(x)^\top \mathbf{K}_\theta^{-1} \mathbf{k}_\theta^*(x) \end{cases} \quad (2)$$

with  $\mathbf{k}_\theta^*(x) = (k_\theta(x, x_1), \dots, k_\theta(x, x_n))^\top$ . More generally, predictive distributions can be built for a larger range of quantities of interest such as joint observations, derivatives, integrals or excursions of  $Z$  above a given threshold.

Using this framework, the user obtains a family of Bayesian procedures, indexed by  $\theta$ , to perform predictions about the unknown computer code at hand, and must choose a member of the family that will hopefully provide good predictive performances.

### 3 Selection of a GP model from a parameterized family

#### 3.1 Scoring rules

Goodness-of-fit criteria for probabilistic predictions have been studied in the literature under the name of scoring rules by [Gneiting and Raftery \(2007\)](#). A (negatively oriented) scoring rule is a function  $S(\cdot; z) : \mathcal{P} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ , acting on a class  $\mathcal{P}$  of probability distributions on  $\mathbb{R}$ , such that  $S(P; z)$  assigns a loss for choosing a predictive distribution  $P \in \mathcal{P}$ , while observing  $z \in \mathbb{R}$ . Scoring rules make it possible to quantify the quality of probabilistic predictions.

**Example 1 (squared prediction error).** Denoting by  $\mu$  the mean of a predictive distribution  $P$ , the squared prediction error

$$S^{\text{SPE}}(P; z) = (z - \mu)^2 \quad (3)$$

accounts for the deviation of  $z$  from  $\mu$ . Note that  $S^{\text{SPE}}$  ignores subsequent moments and therefore predictive uncertainties.

**Example 2 (negative log predictive density).** Denoting by  $p$  the probability density of  $P$ ,

$$S^{\text{NLPD}}(P; z) = -\log(p(z)) \quad (4)$$

tells how likely  $z$  is according to  $P$ . [Bernardo \(1979\)](#) shows that any (proper) scoring rule that depends on  $p(z)$  (and only on  $p(z)$ ) can be reduced to  $S^{\text{NLPD}}$ .

**Example 3 (continuous ranked probability score).** Let  $U$  and  $U'$  be two independent random variables with distribution  $P$ . The CRPS quantifies the deviation of  $U$  from  $z$ :

$$S^{\text{CRPS}}(P; z) = \mathbb{E}(|U - z|) - \frac{1}{2}\mathbb{E}(|U - U'|). \quad (5)$$

Since it is possible to show that  $S^{\text{CRPS}}(P; z) = \int (P(U \leq u) - \mathbb{1}_{z \leq u})^2 du$ , the CRPS can also be seen as a (squared) distance between the empirical cumulative distribution  $u \mapsto \mathbb{1}_{z \leq u}$  and the cumulative distribution of  $P$ .

Note that if absolute values in (5) are replaced by squared values, then  $S^{\text{SPE}}$  is recovered. The CRPS can also be extended to the so-called energy and kernel scores ([Gneiting and Raftery, 2007](#)) by observing that  $(x, y) \mapsto |x - y|$  is a conditionally negative kernel.

**Example 4 (interval score).** The interval scoring rule at level  $1 - \alpha$  is defined, for  $\alpha \in ]0, 1[$ , by

$$S_{1-\alpha}^{\text{IS}}(P; z) = (u - l) + \frac{2}{\alpha}(l - z)\mathbb{1}_{z \leq l} + \frac{2}{\alpha}(z - u)\mathbb{1}_{z > u} \quad (6)$$

where  $l$  and  $u$  are the  $\alpha/2$  and  $1 - \alpha/2$  quantiles of  $P$ . The first term penalizes large intervals, while the second and third terms penalize intervals not containing  $z$ .



Table 2: Scoring rules behavior as  $|\mu - z| \ll 1$ .

	$\sigma \ll  \mu - z $	$\sigma \simeq  \mu - z $	$\sigma \gg  \mu - z $
$S^{\text{SPE}}(P; z)$	0	0	0
$S^{\text{NLPD}}(P; z)$	$+\infty$	$-\infty$	$\log(2\pi\sigma)$
$S^{\text{CRPS}}(P; z)$	0	0	$\propto \sigma$
$S_{1-\alpha}^{\text{IS}}(P; z)$	0	0	$\propto \sigma$

When the predictive distribution  $P$  is Gaussian, which is the case when  $P$  is the posterior distribution of a GP  $Z$  at a given point, the aforementioned scoring rules all have closed-form expressions. More precisely, for  $P = \mathcal{N}(\mu, \sigma^2)$ , we simply have  $S^{\text{SPE}}(P; z) = (z - \mu)^2$  and  $S^{\text{NLPD}}(P; z) = \frac{1}{2} \log 2\pi\sigma^2 + \frac{1}{2}(z - \mu)^2/\sigma^2$ .  $S_{1-\alpha}^{\text{IS}}$  can be obtained by taking the standard expressions of the  $\alpha/2$  and  $1 - \alpha/2$  quantiles of  $P$ , and it can be shown that

$$S^{\text{CRPS}}(P; z) = \sigma \left( \frac{z - \mu}{\sigma} \left( 2\Phi\left(\frac{z - \mu}{\sigma}\right) - 1 \right) + 2\phi\left(\frac{z - \mu}{\sigma}\right) - \frac{1}{\sqrt{\pi}} \right),$$

where  $\phi$  and  $\Phi$  stand respectively for the probability density function and the cumulative distribution function of the standard Gaussian distribution.

Note that all aforementioned scoring rules penalize large values of  $|z - \mu|$ . When  $|z - \mu| \ll 1$  different scoring rules yield different penalties, as reported in Table 2.

## 3.2 Selection criteria

### 3.2.1 Leave-one-out selection criteria

Scoring rules make it possible to build criteria for choosing the parameters of a GP. More precisely, to select  $\theta$  based on a sample  $Z_1, \dots, Z_n$ , one can minimize the mean score

$$J_n^S(\theta) = \frac{1}{n} \sum_{i=1}^n S(P_{\theta, -i}; Z_i), \quad (7)$$

where  $S$  is a scoring rule and  $P_{\theta, -i}$  denotes the distribution of  $Z_i$  conditional on the  $Z_j$ s, for  $1 \leq j \leq n$ ,  $j \neq i$ , indexed by  $\theta$ .

Selection criteria written as (7) correspond to the well-established leave-one-out (LOO) method, which has been proposed in the domain of computer experiments by [Currin et al. \(1988\)](#) under the name of predictive deficiency, and is now used in many publications (see, e.g., [Rasmussen and Williams \(2006\)](#), and also [Zhang and Wang \(2010\)](#), who formally adopt the point of view of the scoring rules, but for model validation instead of parameter selection).

**Efficient computation of predictive distributions.** Leave-one-out predictive densities can be computed using fast algebraic formulas (Craven and Wahba, 1979; Dubrule, 1983). More precisely, the predictive distribution  $P_{\theta,-i}$  is a normal distribution  $\mathcal{N}(\mu_{\theta,-i}, \sigma_{\theta,-i}^2)$  with

$$\mu_{\theta,-i} = Z(x_i) - \frac{(\mathbf{K}_{\theta}^{-1} \mathbf{Z}_n)_i}{\mathbf{K}_{\theta,i,i}^{-1}} \quad \text{and} \quad \sigma_{\theta,-i}^2 = \frac{1}{\mathbf{K}_{\theta,i,i}^{-1}}. \quad (8)$$

Furthermore, Petit et al. (2020) show that, using reverse-mode differentiation, it is possible to compute mean scores  $J_n$  and their gradients with a  $O(n^3 + dn^2)$  computational cost, which is the same computational complexity as for computing the likelihood function and its gradient (see, e.g., Rasmussen and Williams (2006)).

**The particular case of LOO-SPE.** The LOO selection criterion

$$J_n^{\text{SPE}}(\theta) = \frac{1}{n} \sum_{i=1}^n (\mu_{\theta,-i} - Z(x_i))^2, \quad (9)$$

based on the scoring rule (3) will be referred to as LOO-SPE. This criterion, also known as PRESS (Allen, 1974; Wahba, 1990), is well known in statistics and machine learning, and has been advocated by Sundararajan and Keerthi (2001), Martin and Simpson (2003), Bachoc (2013, 2018), Santner et al. (2003) to address the case of “misspecified” covariance functions.

However, note that  $\sigma^2$  cannot be selected using  $J_n^{\text{SPE}}$ . When  $J_n^{\text{SPE}}$  is used,  $\sigma^2$  is generally chosen (see, e.g., Currin et al., 1988; Bachoc, 2013, and Remark 1) to satisfy

$$\frac{1}{n} \sum_{i=1}^n \frac{(Z(x_i) - \mu_{\theta,-i})^2}{\sigma_{\theta,-i}^2} = 1, \quad (10)$$

which will be referred to as Cressie’s rule for  $\sigma^2$ , in reference to the claim by Cressie (1993) that (10) should hold approximately for a good GP model.

**Other scoring rules for LOO.** The selection criteria using the NLPD scoring rule (4) and the CRPS scoring rule (5) will be referred to as the LOO-NLPD and the LOO-CRPS respectively. The LOO-NLPD criterion has already been proposed by Currin et al. (1988); Sundararajan and Keerthi (2001) and the LOO-CRPS has been proposed as a validation criterion by Zhang and Wang (2010). To the best of our knowledge, using LOO-CRPS as a selection criterion is new.

*Remark 1.* Minimization of the LOO-NLPD criterion with respect to  $\sigma^2$  yields Cressie’s rule (10).

*Remark 2.* In order to limit the number of selection criteria under study, the interval scoring rule is only used for validation in this work.

### 3.2.2 Maximum likelihood and generalizations

We can safely say that the most popular method for selecting  $\theta$  from data is maximum likelihood estimation (and related techniques, such as restricted maximum likelihood estimation). With the conventional abuse of notations, denoting by  $p_\theta(\mathbf{Z}_n)$  the joint density of  $\mathbf{Z}_n$ , the negative log-likelihood (NLL) selection criterion may be written as

$$J_n^{\text{NLL}}(\theta) = -\log(p_\theta(\mathbf{Z}_n)) = \frac{1}{2} \left( n \log(2\pi) + \log \det \mathbf{K}_\theta + \mathbf{Z}_n^\top \mathbf{K}_\theta^{-1} \mathbf{Z}_n \right). \quad (11)$$

As pointed out by [Currin et al. \(1988\)](#), the NLL criterion is closely related to the LOO-NLPD criterion, through the identity

$$J_n^{\text{NLL}}(\theta) = -\log(p_\theta(Z(x_1))) - \sum_{i=2}^n \log(p_\theta(Z(x_i) | Z(x_1), \dots, Z(x_{i-1}))),$$

where the predictive distributions of the  $Z(x_i)$ s given the  $Z(x_1), \dots, Z(x_{i-1})$  explicitly appear.

One can minimize (11) in closed-form with respect to  $\sigma^2$ , given other parameters. Writing  $\mathbf{K}_\theta = \sigma^2 \mathbf{R}_\theta$  and canceling  $\partial J_n^{\text{NLL}}(\theta) / \partial \sigma^2 = (n\sigma^2 - \mathbf{Z}_n^\top \mathbf{R}_\theta^{-1} \mathbf{Z}_n) / (2\sigma^2)$  yields

$$\sigma_{\text{NLL}}^2 = \frac{1}{n} \mathbf{Z}_n^\top \mathbf{R}_\theta^{-1} \mathbf{Z}_n, \quad (12)$$

which will be referred to as the profiling rule for  $\sigma^2$ .

Injecting (12) into (11) yields a *profiled likelihood* selection criterion, that can be written as

$$J_n^{\text{PL}}(\theta) = \log \sigma_{\text{NLL}}^2 + \frac{1}{n} \log \det \mathbf{R}_\theta = \log \left( \frac{1}{n} \mathbf{Z}_n^\top \mathbf{R}_\theta^{-1} \mathbf{Z}_n \right) + \frac{1}{n} \log \det \mathbf{R}_\theta. \quad (13)$$

Following [Fasshauer and co-authors \(Fasshauer et al., 2009; Fasshauer, 2011\)](#), we now suggest a new family of selection criteria extending (11). Using the factorization  $\mathbf{R}_\theta = \mathbf{Q}\Lambda\mathbf{Q}^\top$ , where  $\mathbf{Q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$  is an orthogonal matrix of (orthonormal) eigenvectors and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ , notice that

$$\exp(J_n^{\text{PL}}(\theta)) = \frac{1}{n} \mathbf{Z}_n^\top \mathbf{R}_\theta^{-1} \mathbf{Z}_n \cdot (\det \mathbf{R}_\theta)^{1/n} \propto \left( \sum_{i=1}^n (\mathbf{q}_i^\top \mathbf{Z}_n)^2 / \lambda_i \right) \left( \prod_{i=1}^n \lambda_j \right)^{1/n}. \quad (14)$$

This suggests a generalization of the likelihood criterion that we shall call Fasshauer's Hölderized likelihood (HL), defined as

$$J_n^{\text{HL}, p, q}(\theta) = \left( \sum_{i=1}^n (\mathbf{q}_i^\top \mathbf{Z}_n)^2 / \lambda_i^p \right)^{1/p} \left( \frac{1}{n} \sum_{j=1}^n \lambda_j^q \right)^{1/q}, \quad (15)$$

with  $q \in [-\infty, +\infty]$ , and  $p \in \mathbb{R} \setminus \{0\}$ , and where  $\sigma^2$  can be chosen using the rules (10) or (12), since  $J_n^{\text{HL}, p, q}(\theta)$  does not depend on  $\sigma^2$ . Owing to the standard property of generalized means

$(\frac{1}{n} \sum_{i=1}^n x_i^q)^{\frac{1}{q}} \xrightarrow{q \rightarrow 0} \sqrt[n]{x_1 \cdots x_n}$ , (14) is recovered by taking  $p = 1$  and letting  $q \rightarrow 0$ . Moreover, two other known selection criteria can be obtained for particular values of  $p$  and  $q$ , as detailed below.

**Generalized cross-validation** – Taking  $p = 2$  and  $q = -1$  in (15) yields the generalized cross-validation (GCV) criterion

$$J_n^{\text{GCV}}(\boldsymbol{\theta}) = n^{-1} (J_n^{\text{HL}, 2, -1}(\boldsymbol{\theta}))^2,$$

which was originally proposed as a rotation-invariant version of PRESS (Golub et al., 1979) for linear models. It has also been shown to be efficient for the selection of the smoothing parameter of polyharmonic splines by Wahba (1990), and for the selection of the degree of a spline by Wahba and Wendelberger (1980).

The GCV selection criterion is a weighted SPE criterion, which can also be written as

$$J_n^{\text{GCV}}(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^n w_i^2(\boldsymbol{\theta}) (Z(x_i) - \mu_{\boldsymbol{\theta}, -i})^2, \quad w_i(\boldsymbol{\theta}) = \frac{\tilde{\sigma}^2}{\sigma_{\boldsymbol{\theta}, -i}^2}, \quad (16)$$

with  $\tilde{\sigma}^2 = (\frac{1}{n} \sum_{i=1}^n \frac{1}{\sigma_{\boldsymbol{\theta}, -i}^2})^{-1}$ . Notice that  $w_i(\boldsymbol{\theta})$  is lower when  $\sigma_{\boldsymbol{\theta}, -i}$  is larger, which happens when points are either isolated or lying on the border / envelope of  $\mathbb{X}_n$ . However, since  $\tilde{\sigma}$  depends on the data, the  $J_n^{\text{GCV}}$  criterion cannot be formally derived from a scoring rule.

**Kernel alignment**— The kernel alignment selection criterion is defined as

$$J_n^{\text{KA}}(\boldsymbol{\theta}) = - \frac{\mathbf{Z}_n^\top \mathbf{K}_\boldsymbol{\theta} \mathbf{Z}_n}{\|\mathbf{K}_\boldsymbol{\theta}\|_F \|\mathbf{Z}_n\|^2}, \quad (17)$$

where  $\|\cdot\|_F$  stands for the Frobenius matrix norm. This criterion can be derived from (15) by taking  $p = -1$  and  $q = 2$ :

$$J_n^{\text{KA}}(\boldsymbol{\theta}) = - \frac{1}{\sqrt{n} \|\mathbf{Z}_n\|^2 J_n^{\text{HL}, -1, 2}(\boldsymbol{\theta})}.$$

It was originally proposed in the literature of machine learning by Cristianini et al. (2001) to maximize the alignment of  $\mathbf{Z}_n$  with the eigenvector of  $\mathbf{K}_\boldsymbol{\theta}$  corresponding to the largest eigenvalue.

### 3.3 Hybrid selection criteria

When considering several parameterized models, or when dealing with discrete parameters, such as half-integer values for the regularity parameter of the Matérn covariance, we may want to use one selection criterion to select one set of parameters and a different one for other parameters. For instance, Jones et al. (1998) select some parameters of a covariance function using the

NLL selection criterion, and then study the effect of several transformations of the output of the simulator using LOO-SPE.

When using Matérn covariance functions, one may use the NLL criterion (i.e., the ML method) to select  $\sigma^2$  and  $\rho_1, \dots, \rho_d$ , and then LOO-SPE to select the regularity parameter  $\nu$ . We shall call this particular selection procedure NLL/SPE in the next section.

## 4 Numerical experiments

### 4.1 Methodology

We investigate the problem of parameter selection with an experimental approach consisting of four ingredients: 1) a set of unknown functions  $f$  to be predicted using evaluation results on a finite design  $\mathbb{X}_n = \{x_1, \dots, x_n\} \subset \mathbb{X}$ ; 2) the GP regression method that constructs predictive distributions  $P_{\theta, x}$  of  $f$  at given  $x$ s in  $\mathbb{X}$ , indexed by parameter  $\theta$ ; 3) several selection criteria  $J_n$  to choose  $\theta$ ; 4) several criteria to assess the quality of the  $P_{\theta, x}$ s. Details about each of these ingredients are given below (starting from the last one).

**Criteria to assess the quality of the  $P_{\theta, x}$ s.** A natural way to construct a criterion to assess the quality of the  $P_{\theta, x}$ s is to choose a scoring rule  $S$  and to consider the mean score on a test set  $\mathbb{X}_N^{\text{test}} = \{x_1^{\text{test}}, \dots, x_N^{\text{test}}\} \subset \mathbb{X}$  of size  $N$ :

$$R(\theta; S) = \frac{1}{N} \sum_{i=1}^N S(P_{\theta, x_i^{\text{test}}}; f(x_i^{\text{test}})). \quad (18)$$

**Selection criteria.** We shall consider the selection criteria  $J_n$  presented in Section 3, namely, the LOO-SPE, LOO-NLPD, LOO-CRPS, NLL, GCV, KA and NLL/SPE selection criteria. Given a function  $f$  and a design  $\mathbb{X}_n$ , each selection criterion  $J_n$  yields a parameter  $\theta_{J_n}$ .

**Parameterized GP models.** In this work, models are implemented using a custom version of the `GPpy` (2019) Python package (see Supplementary Material, hereafter abbreviated as SM). We assume no observation noise, which corresponds to the interpolation setting. All functions will be centered beforehand to have zero-mean on  $\mathbb{X}_N^{\text{test}}$ , and we will consider zero-mean GPs only. The anisotropic Matérn covariance function (1) is used, with parameter  $\theta = (\sigma^2, \rho_1, \dots, \rho_d, \nu)$ , and the regularity parameter  $\nu$  is either set a priori to  $\nu = \chi + 1/2$ , with  $\chi \in \{0, 1, 2, 3, 4, d, 2d, \infty\}$ , or selected automatically. The latter case will be denoted by  $\hat{\nu}$ .

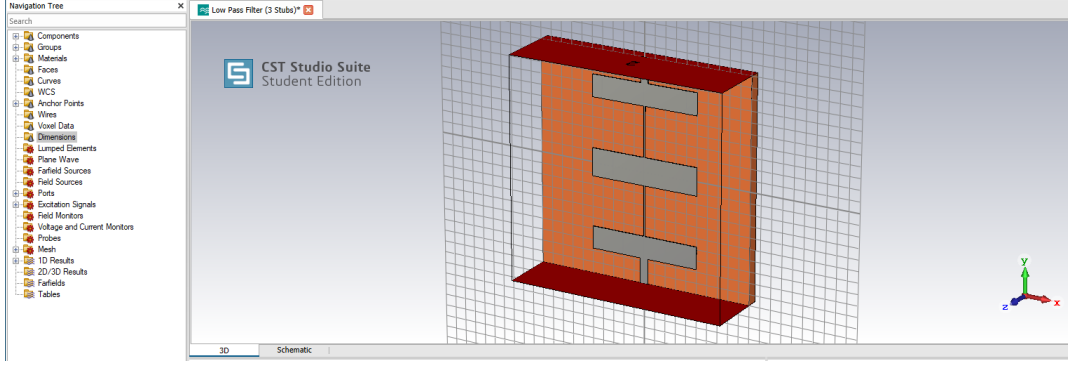


Figure 1: A low-pass filter design problem in CST Studio Suite®.

*Remark 3.* Since the covariance matrix of  $\mathbf{Z}_n$  can be nearly singular when the range parameters take large values, we define upper bounds on these values in order to avoid the use of *nugget* or *jitter* (see, e.g., Ranjan et al., 2011; Peng and Wu, 2014). Details are provided in the SM.

**Test functions.** The test functions used in the study are described in the next section. They are grouped into collections, and we provide averaged values of mean-score metrics of the form (18) for each collection.

## 4.2 Test functions

### 4.2.1 Design of a low-pass filter

Fuhrländer and Schöps (2020) consider the problem of computing, using a frequency-domain PDE solver, the scattering parameters  $S_\omega$  of an electronic component called stripline low-pass filter, at several values of the excitation pulsation  $\omega$ . The geometry of the stripline filter is illustrated on Figure 1. It is parameterized using six real valued factors concatenated in a vector  $x \in \mathbb{R}^d$ ,  $d = 6$ . The objective is to satisfy the low-pass specifications  $|S_{2k\pi}(x)| \geq -1\text{dB}$  for  $0 \leq k \leq 4$  and  $|S_{2k\pi}(x)| \leq -20\text{dB}$  for  $5 \leq k \leq 7$ . Meeting such requirements is a difficult and time-consuming task.

In this article we consider the quantities  $\text{Re}(S_{2\pi})$ ,  $\text{Re}(S_{6\pi})$ ,  $\text{Re}(S_{10\pi})$  and  $\text{Re}(S_{14\pi})$ . We randomly sample  $M = 100$  designs  $\mathbb{X}_n$  of size  $n = 300$  from a database of 10000 simulation results, and use the remaining  $N = 10000 - n = 9700$  points as test sets. The metric (18) is computed and averaged on these  $M$  test sets.

Table 3: Twelve benchmark problems

Problem	Goldstein-Price	Mystery	GKLS <sub>k=0</sub>	GKLS <sub>k=1</sub>	GKLS <sub>k=2</sub>	Rosenbrock	Borehole
$d$	{1, 2}	2	{2, 5}	{2, 5}	{2, 5}	{2, 5}	8

### 4.2.2 Other test functions

We supplement the above engineering problem with a collection of test functions from the literature. More precisely, we consider the Goldstein-Price function (Dixon and Szegö, 1978), a one-dimensional version of the Goldstein-Price function (see SM for details), the Mystery function (Martin and Simpson, 2003), the Borehole function (Worley, 1987), several collections obtained from the GKLS simulator (Gaviano et al., 2003), and the rotated Rosenbrock collection from the BBOB benchmark suite (Hansen et al., 2009).

The GKLS simulator has a “smoothness” parameter  $k \in \{0, 1, 2\}$  controlling the presence of non-differentiabilities on some nonlinear subspaces—the trajectories being otherwise infinitely differentiable. For both GKLS and Rosenbrock, two different values of the input dimension were considered ( $d = 2$  and  $d = 5$ ). The resulting set of twelve problems—considering that changing the value of  $k$  or  $d$  defines a new problem—is summarized in Table 3.

For each problem, we consider three design sizes  $n \in \{10d, 20d, 50d\}$ . For the GKLS and Rosenbrock collections, we directly used the collections of test functions provided by the authors ( $M = 100$  and 15 functions, respectively). For a given dimension, they are all evaluated on the same space-filling designs  $\mathbb{X}_n$ . For each of the remaining problems, we used a single test function, evaluated on  $M = 100$  random space-filling designs  $\mathbb{X}_n$ , thereby creating collections of 100 data sets.

A Sobol’ sequence  $\mathbb{X}_N^{\text{test}}$  of size  $N = 10000$  is used as test set and the functions are centered and normalized to unit variance on these test sets.

## 4.3 Results and findings

### 4.3.1 A close look at one of the problems

Tables 4 and 5 provide a detailed view of the results obtained on one of the test problems—namely, the output  $f = \text{Re}(S_{6\pi})$  with  $n = 10d = 60$  of the low-pass filter case (see Section 4.2.1).

The results presented in these tables are the scores  $R(\theta; S)$ , averaged over the  $M = 100$  random instances of the problem, where  $\theta$  is selected using different selection criteria (along columns), and the regularity of the Matérn covariance varies or is selected automatically (along rows).

Scoring rule: $S^{\text{SPE}}$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV	$R^*$
$\nu = 1/2$	$4.94 \cdot 10^{-2}$	$5.11 \cdot 10^{-2}$	$4.84 \cdot 10^{-2}$	$4.84 \cdot 10^{-2}$	$4.29 \cdot 10^{-1}$	$4.73 \cdot 10^{-2}$	$4.44 \cdot 10^{-2}$
$\nu = 3/2$	$3.85 \cdot 10^{-3}$	$4.24 \cdot 10^{-3}$	$3.52 \cdot 10^{-3}$	$3.59 \cdot 10^{-3}$	$3.82 \cdot 10^{-1}$	$3.45 \cdot 10^{-3}$	$2.97 \cdot 10^{-3}$
$\nu = 5/2$	$4.02 \cdot 10^{-4}$	$5.11 \cdot 10^{-4}$	$4.18 \cdot 10^{-4}$	$4.31 \cdot 10^{-4}$	$3.71 \cdot 10^{-1}$	$4.93 \cdot 10^{-4}$	$3.21 \cdot 10^{-4}$
$\nu = 7/2$	$2.88 \cdot 10^{-4}$	$4.33 \cdot 10^{-4}$	$3.73 \cdot 10^{-4}$	$3.86 \cdot 10^{-4}$	$3.54 \cdot 10^{-1}$	$4.75 \cdot 10^{-4}$	$2.26 \cdot 10^{-4}$
$\nu = 9/2$	$2.96 \cdot 10^{-4}$	$4.39 \cdot 10^{-4}$	$4.22 \cdot 10^{-4}$	$3.95 \cdot 10^{-4}$	$3.42 \cdot 10^{-1}$	$5.44 \cdot 10^{-4}$	$2.26 \cdot 10^{-4}$
$\nu = 13/2$	$3.15 \cdot 10^{-4}$	$4.80 \cdot 10^{-4}$	$4.48 \cdot 10^{-4}$	$4.25 \cdot 10^{-4}$	$3.29 \cdot 10^{-1}$	$6.43 \cdot 10^{-4}$	$2.32 \cdot 10^{-4}$
$\nu = 25/2$	$3.46 \cdot 10^{-4}$	$5.29 \cdot 10^{-4}$	$4.92 \cdot 10^{-4}$	$4.61 \cdot 10^{-4}$	$3.14 \cdot 10^{-1}$	$7.43 \cdot 10^{-4}$	$2.45 \cdot 10^{-4}$
$\nu = \infty$	$3.80 \cdot 10^{-4}$	$6.34 \cdot 10^{-4}$	$5.38 \cdot 10^{-4}$	$5.38 \cdot 10^{-4}$	$2.94 \cdot 10^{-1}$	$8.75 \cdot 10^{-4}$	$2.60 \cdot 10^{-4}$
$\nu \in \{1/2, \dots, \infty\}$	$3.07 \cdot 10^{-4}$	$4.90 \cdot 10^{-4}$	$4.64 \cdot 10^{-4}$	$4.31 \cdot 10^{-4}$	$2.98 \cdot 10^{-1}$	$6.89 \cdot 10^{-4}$	$2.13 \cdot 10^{-4}$

Table 4: Averages (over the  $M = 100$  designs) of the  $R(\theta; S^{\text{SPE}})$  values for  $\text{Re}(S_{6\pi})$  with  $n = 10d = 60$ . Optimal  $R^*$  values are given in the rightmost column for comparison. For comparison also,  $R(\theta; S^{\text{SPE}}) = 3.26 \cdot 10^{-4}$  for the  $J_n^{\text{NLL/SPE}}$  selection criterion, which also selects  $\nu$  (see Section 3.3). The gray scale highlights the order of magnitude of the discrepancies.

Scoring rule: $S_{0.95}^{\text{IS}}$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$\nu = 1/2$	$1.44 \cdot 10^0$	$1.38 \cdot 10^0$	$1.34 \cdot 10^0$	$1.48 \cdot 10^0$	$3.69 \cdot 10^0$	$1.32 \cdot 10^0$
$\nu = 3/2$	$2.80 \cdot 10^{-1}$	$3.05 \cdot 10^{-1}$	$2.75 \cdot 10^{-1}$	$2.88 \cdot 10^{-1}$	$3.43 \cdot 10^0$	$2.69 \cdot 10^{-1}$
$\nu = 5/2$	$9.30 \cdot 10^{-2}$	$9.42 \cdot 10^{-2}$	$8.55 \cdot 10^{-2}$	$9.11 \cdot 10^{-2}$	$3.36 \cdot 10^0$	$9.18 \cdot 10^{-2}$
$\nu = 7/2$	$6.82 \cdot 10^{-2}$	$8.82 \cdot 10^{-2}$	$8.42 \cdot 10^{-2}$	$9.10 \cdot 10^{-2}$	$3.23 \cdot 10^0$	$9.17 \cdot 10^{-2}$
$\nu = 9/2$	$6.50 \cdot 10^{-2}$	$9.08 \cdot 10^{-2}$	$9.30 \cdot 10^{-2}$	$9.53 \cdot 10^{-2}$	$3.14 \cdot 10^0$	$1.00 \cdot 10^{-1}$
$\nu = 13/2$	$6.48 \cdot 10^{-2}$	$9.95 \cdot 10^{-2}$	$9.99 \cdot 10^{-2}$	$1.03 \cdot 10^{-1}$	$3.02 \cdot 10^0$	$1.16 \cdot 10^{-1}$
$\nu = 25/2$	$6.77 \cdot 10^{-2}$	$1.10 \cdot 10^{-1}$	$1.08 \cdot 10^{-1}$	$1.12 \cdot 10^{-1}$	$2.90 \cdot 10^0$	$1.29 \cdot 10^{-1}$
$\nu = \infty$	$7.23 \cdot 10^{-2}$	$1.23 \cdot 10^{-1}$	$1.16 \cdot 10^{-1}$	$1.23 \cdot 10^{-1}$	$2.74 \cdot 10^0$	$1.44 \cdot 10^{-1}$
$\nu \in \{1/2, \dots, \infty\}$	$6.70 \cdot 10^{-2}$	$1.04 \cdot 10^{-1}$	$1.06 \cdot 10^{-1}$	$1.08 \cdot 10^{-1}$	$2.78 \cdot 10^0$	$1.23 \cdot 10^{-1}$

Table 5: Same as Table 4 but for averages of  $R(\theta; S_{0.95}^{\text{IS}})$ . Using  $J_n^{\text{NLL/SPE}}$  gives  $R(\theta; S_{0.95}^{\text{IS}}) = 7.06 \cdot 10^{-2}$ .



The scoring rule for assessing the quality of the predictions is the SPE in Table 4 and the IS at level 95% in Table 5. (A similar table, not shown here, is presented in the SM for the CRPS.)

For comparison, Table 4 also provides the optimal values  $R^*$  obtained by direct minimization of the score (18). They can be used to assess the loss of predictive accuracy of the selected models, which are constructed using a limited number of observations, with respect to the best model that could have been obtained if the test data had also been used to select the parameters.

Table 4 and Table 5 support the fact that, for this particular problem, the NLL and NLL/SPE criteria are the best choices for selecting  $\theta$  in terms of the SPE and the IS scores, both for a prescribed regularity  $\nu$  and when  $\nu$  is selected automatically (the NLL/SPE being only available for the latter case). Except for the KA criterion, however, the other selection criteria are never very far behind. Elements provided as SM show similar findings using the CRPS validation score.

Strikingly enough, for both scoring rules, the variations of the average score are much larger when the model changes than when the selection criterion changes. If a Matérn covariance function with fixed regularity is used, as is often done in practice, then the best results are obtained for all criteria (except KA) when  $\nu$  takes the values  $7/2$ ,  $9/2$  and  $13/2$ . The values of  $R^*$  (Table 4) confirm that these are indeed the best fixed- $\nu$  models on this problem for the SPE score. Since these optimal values were not known beforehand, it is a relief to see (cf. last row of each table) that comparable performances can be achieved on this problem by selecting  $\nu$  automatically.

### 4.3.2 Statistical analysis of the benchmark results

Tables similar to Tables 4 and 5 have been produced for all the  $(4 + 12) \times 3 = 48$  test problems presented in Section 4.2, for the three scoring rules (SPE, CRPS and IS). We present in this section some graphical summaries and statistical analyses of these results. The individual tables for each problem are provided in the SM.

*Remark 4.* The poor performance of the KA criterion, already observed in Tables 4 and 5, is confirmed by the results (not shown) on all the other problems. We conclude that this selection criterion should not be used in practice, and exclude it from the analyses of this section in order to refine the comparison between the remaining ones.

**Sensitivity analysis.** We observed in Section 4.3.1 that the choice of the model—more specifically, of the regularity parameter of the Matérn covariance function—was more important than that of a particular selection criterion (excluding KA of course). To confirm this finding, a global sensitivity analysis of the logarithm of the average score, where the average is taken over the  $M = 100$  instances of each problem, has been performed on each problem. The average score,

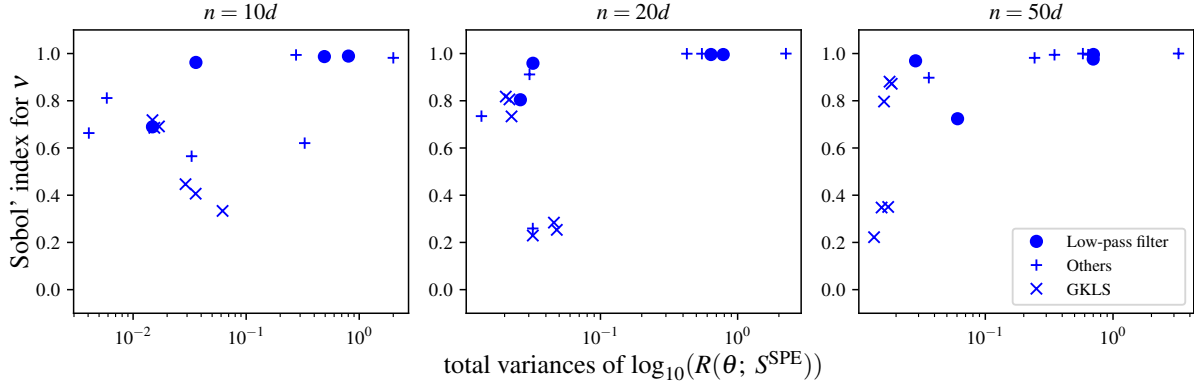


Figure 2: Parts of total variances of  $\log_{10}(R)$ s explained by  $\nu$  for  $S^{\text{SPE}}$  using a one-factor ANOVA. Each point represents the variations of  $\log_{10}(R)$  for one of the 16 problems from Section 4.2.2, split by design size, with KA and GCV excluded. The model explains almost all the variations for problems that exhibit significant fluctuations of  $\log_{10}(R)$  (at the right of the figure).

for a given scoring rule, depends on two discrete factors: the selection criterion and the regularity  $\nu$  of the Matérn covariance function. We present on Figure 2, for the SPE scoring rule, the Sobol' sensitivity index for the latter factor as a function of the total variance. Observe that, for the problems where the total variance is large, the Sobol index is typically very close to one, which indicates that the variability is indeed mainly explained by the choice of model. Similar conclusions hold for the other scoring rules (results not shown, see SM).

**Comparison of the covariance models.** Figure 3 compares the average values of  $R(\theta; S^{\text{SPE}})$  when ML is used on the set of GKLS problems, which have low regularities, and on the set of low-pass filter problems, which contains very smooth instances.

Observe first that the fixed- $\nu$  models rank differently on these two sets of problems, as expected considering the actual regularity of the underlying functions: low values of  $\nu$  perform better on the GKLS problems and worse on the low-pass filter case. Furthermore, it appears that underestimating the regularity (on the low-pass filter case) has much more severe consequences than overestimating it (on the GKLS problems) according to the SPE score, as suggested by the theoretical results of Stein (1999, Chapter 3) and Tuo and Wang (2020).

Another important conclusion from Figure 3 is that very good results can be obtained by selecting the regularity parameter  $\nu$  automatically, jointly with the other parameters (using the NLL criterion in this case). On the GKLS problems, the results with selected  $\nu$  are not far from

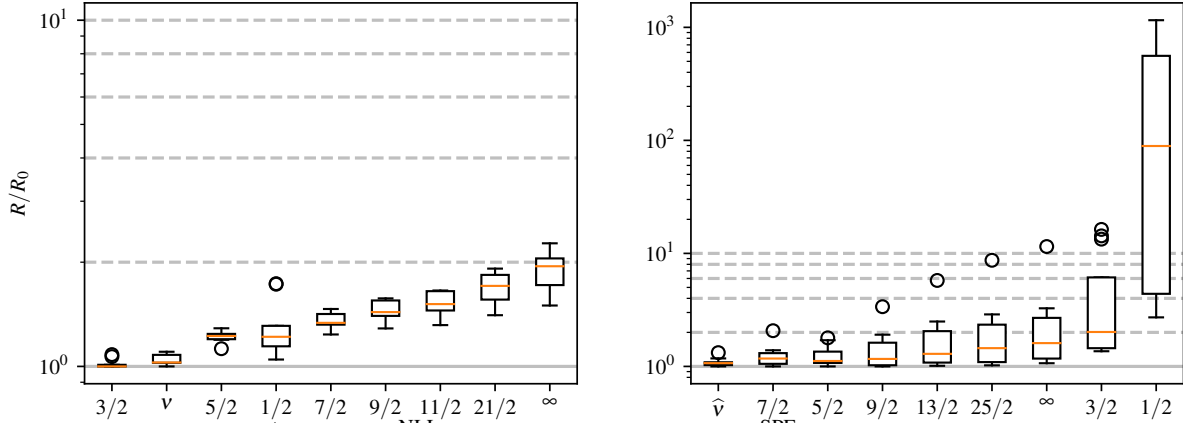


Figure 3: Box plots of  $R/R_0$  using  $J_n^{\text{NLL}}$  as selection criterion and  $S^{\text{SPE}}$  as quality assessment criterion, for different choices of regularity. Here,  $R_0$  stands for the best value of  $R$  on each problem (among all models). Left: All 5d GKLS problems. Right: All low-pass filter problems. The box plots are sorted according to their upper whisker. Grey dashed lines:  $R/R_0 = 2, 4, 6, 8, 10$ .

those of the best fixed- $\nu$  model under consideration ( $\nu = 3/2$ ); in the low-pass filter case, they are even better than those obtained with the best fixed- $\nu$  models ( $\nu = 5/2$  or  $7/2$ ). In other words, the regularity needs not be known in advance to achieve good performances, which is a very welcome practical result. This conclusion is also supported, for NLL, by the additional results provided in the SM for the other problems and for the three scoring rules.

Concerning the other selection criteria the situation is more contrasted (see SM): the automatic selection of  $\nu$  using these criteria still performs very well for smooth problems, but not always, in particular with GCV, for the less regular problems of the GKLS class. This is especially true when the sample size is small ( $n = 10d$ ).

**Comparison of the selection criteria.** Figure 4 compares the distributions of the average values of the SPE and IS scores for all selection criteria (except KA) on all test instances, in the case of a Matérn covariance function with automatically selected regularity. As a preliminary observation, note that for most cases the ratio  $R/R_0$  remains under two (first horizontal dashed line), which confirms that the differences between selection criteria are much milder than those between covariance models (recall Figure 3).

A closer look at Figure 4 reveals that the rankings of criteria obtained for both scoring rules are almost identical. The ranking for the CRPS scoring rule (not shown) is the same as the one for SPE. GCV provides the worst performance for all scoring rules, followed by LOO-NLPD, while NLL dominates the ranking (for all scoring rules as well).

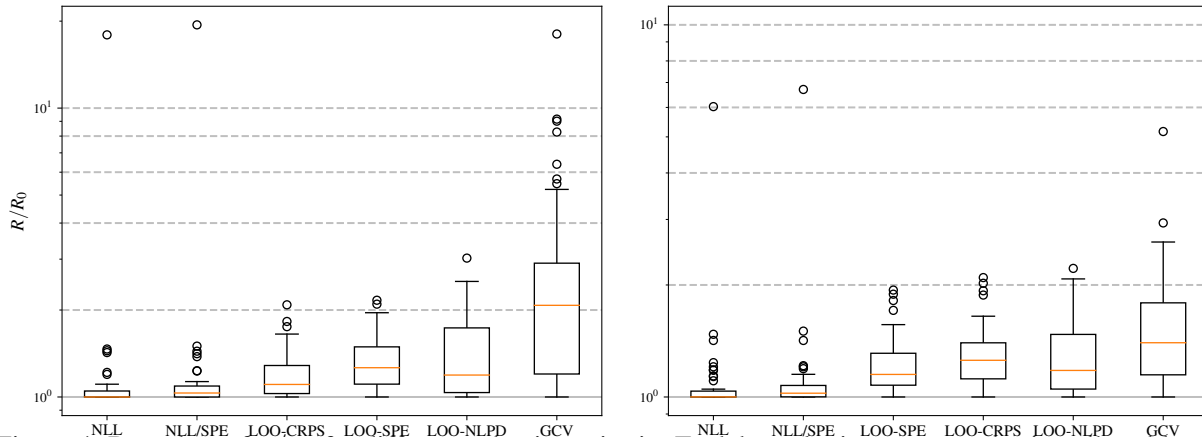


Figure 4: Box plots of  $R/R_0$  for different selection criteria. Each box plot is constructed using all problems, with automatically selected  $\nu$ , and  $R_0$  stands for the best value of  $R$  on each problem (among all selection criteria). The left (resp. right) panel uses the  $S^{\text{SPE}}$  (resp.  $S_{0.95}^{\text{IS}}$ ) as quality assessment criterion. Sorting of box plots and horizontal lines: as in Figure 3.

*Remark 5.* Observe on Figure 4 that LOO-SPE is surprisingly significantly less accurate than NLL according to  $S^{\text{SPE}}$ . More generally, choosing a scoring rule  $S$  for the LOO criterion does not guarantee the highest precision according to this particular score.

**Robustness?** LOO-SPE is commonly claimed in the literature (see, notably, [Bachoc, 2013](#)) to provide a certain degree of robustness with respect to model misspecification. According to this claim, LOO-SPE would be expected to somehow mitigate the loss of predictive accuracy with respect to likelihood-based approaches incurred by an ill-advised choice of covariance function. Our detailed results (see SM) suggest that this effect indeed exists when the regularity is severely under-estimated (e.g.,  $\nu = 1/2$  for the low-pass filter problems), but is actually quite small, and should not be used to motivate the practice of setting  $\nu$  to an arbitrary value. A similar effect exists for LOO-CRPS, LOO-NLPD and GCV as well. Quite surprisingly, NLL turns out to be more robust than LOO-SPE (and the other criteria) in the case of over-smoothing.

## 5 Conclusions

A large variety of selection criteria for Gaussian process models is available from the literature, with little theoretical or empirical guidance on how to choose the right one for applications. Our benchmark study with the Matérn family of covariance functions in the noiseless (interpolation) case indicates that the NLL selection criterion—in other words, the ML method—provides per-

performances that are, in most situations and for all the scoring rules that were considered (SPE, CRPS and IS at 95%), better than or comparable to those of the other criteria. This conclusion provides a strong empirical support to the fact that the ML method (or its “restricted” variant ReML), which is already the *de facto* standard in most statistical software packages implementing Gaussian process interpolation (a.k.a. kriging), should be used in practice.

Another important lesson learned from our benchmark study is that the choice of the family of models, and in particular of the family of covariance functions, has very often a bigger impact on performance than that of the selection criterion itself. This is especially striking when the actual function is smooth, and very irregular covariance function such as the Matérn covariance with regularity  $1/2$  is used to perform Gaussian process interpolation. In such a situation, NLL is actually outperformed by other criteria such as LOO-SPE or LOO-CRPS, which thus appear to be more “robust to model misspecification”. However, the small gain of performance, which is achieved by using LOO-SPE or LOO-CRPS instead of NLL in this case, is generally negligible with respect to the much larger loss induced by choosing an inappropriate covariance function in the first place.

Our final recommendation, supported by the results of the benchmark, is therefore to select, if possible, the regularity of the covariance function automatically, jointly with the other parameters, using the NLL criterion. A minimal list of candidate values for the regularity parameter should typically include  $1/2$ ,  $3/2$ ,  $5/2$ ,  $7/2$  and  $+\infty$  (the Gaussian covariance function). Should a situation arise where a default value of  $\nu$  is nevertheless needed, our recommendation would be to choose a reasonably large value such as  $\nu = 7/2$ , since under-smoothing has been seen to have much more severe consequences than over-smoothing. More generally, our numerical results support the fact that choosing a model carefully is important, and probably so not only in the class of Matérn covariance functions.

However, it should be kept in mind that the study focuses on cases where the number of parameters is small with respect to the number of observations (in particular, we considered zero-mean GPs with an anisotropic stationary Matérn covariance function, which have  $d + 2$  parameters, and we took care of having  $n \gg d$ ). When  $d$  is large, or when the number of parameters increases, it seems to us that other selection criteria should be considered, and that the introduction of regularization terms would be required.

For future work, it would be very interesting to consider the performance of using selection criteria against a fully Bayesian approach. Another direction would be to extend this study to the case of regression, which is also used in many applications, when dealing with stochastic simulators, for instance.

**Acknowledgments.** The authors thank Mona Fuhrländer and Sebastian Schöps for kindly providing the data for the “Design of a low-pass filter” example.

## SUPPLEMENTARY MATERIAL

**Results:** Raw benchmark results, and some Python code to analyze them. (.zip file)

**Supplementary Material:** Additional information about the experiments. Detailed analysis of benchmark results: tables and figures. (*See below.*)

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## SM-1 Additional information about the experiments

### SM-1.1 Some remarkable features of two problems

#### SM-1.1.1 GKLS setup

We chose the following parameters for the collections of functions taken from the GKLS simulator:

- $[0, 1]^d$  as inputs domain,
- 10 local minima,
- the default value of the global minimum value was set to  $-1.0$ ,
- the distance from the global minimizer to the vertex of the paraboloid was set to  $1/3$ ,
- and the radius of the attraction region of the global minimizer was set to  $1/6$ .

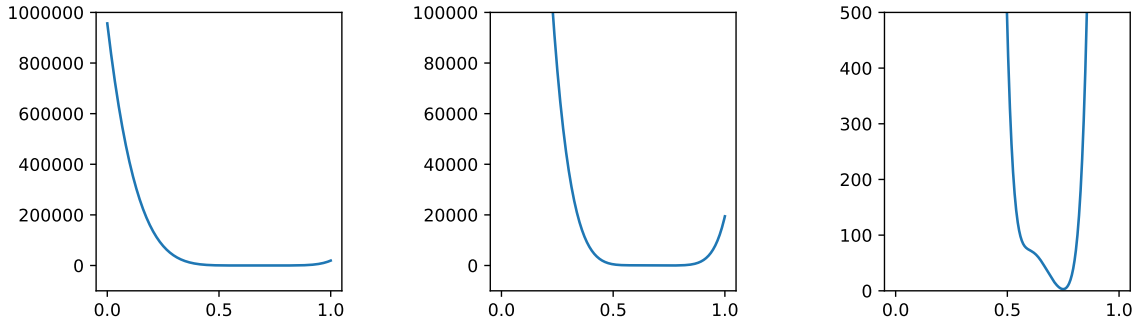


Figure SM1: Left: Goldstein-Price (1D) with two zooms on the y-axis (middle and right).

### SM-1.1.2 Goldstein-Price (1D)

The Goldstein-Price function (Dixon and Szegö, 1978) is a function that is defined on  $[-2, 2] \times [-2, 2]$ . It is relatively well-behaved close to the global minimizer and at the center of the domain, but has large variations on the edges. For illustration purpose, we consider a section along an axis going from the corner with highest values and passing through the local minimizer. The resulting function is defined by

$$f(t) = g(t(2/3, -2)^T + (1-t)(-2, 2)^T), t \in [0, 1]$$

where  $g$  is the Goldstein-Price function. This one-dimensional version of the problem is represented on Figure SM1.

### SM-1.2 Custom version of GPpy

In our experiments, we use the GPpy Python package (GPpy, 2019) with a few modifications.

- We use the levers identified by Basak et al. (2021) to facilitate the optimization of the criteria (especially the “log” parameterization),
- we added the general expression for a Matérn covariance function with regularity  $\nu = \chi + 1/2$  (see, e.g. Rasmussen and Williams, 2006, Section 4.2.1),
- we deactivated all kinds of *jitter* or *nugget* (see next subsection).

### SM-1.3 Numerics

Some criteria sometimes select very large  $\sigma^2$  and  $\rho_i$ , as reported by Martin and Simpson (2005) and Bachoc (2013). Unfortunately, most kriging implementations suffer from large  $\rho_i$ s, because

this entails nearly singular  $\mathbf{K}_\theta$ . A common way to circumvent this difficulty is to add a small quantity called *jitter* or *nugget* to the diagonal of  $\mathbf{K}_\theta$  (Ranjan et al., 2011; Peng and Wu, 2014). Note that the effect of this parameter on the model’s predictions and on the shape of the likelihood can be nonintuitive, especially for high  $\rho_i$ s as showed by Andrianakis and Challenor (2012).

For the sake of simplicity, we chose to circumvent this difficulty by setting bounds for the range parameters. We describe those bounds here assuming that the input domain is  $[0, 1]^d$ . Given  $\mathbf{v}$  and a design, a bound  $L$  is taken to be the largest  $\rho$  such that the condition number of  $\mathbf{K}_{(1,\rho,\dots,\rho,\mathbf{v})}$  stays below some threshold  $\kappa$  that must be small compared to the invert of the double precision  $10^{-16}$ . The search space for  $(\rho_1, \dots, \rho_d)$  is then restricted to  $(0, L]^d$ .

The  $L$ -values used in this work are provided as a function<sup>1</sup> of  $\mathbf{v}$ ,  $n$ ,  $d$  and a datasets type—with *unif* referring to a “space-filling” and *gaussian* referring to a sample from the low-pass filter data—in the file *rho\_bounds.csv*. The bounds used for the problems must then be computed taking the scales of the domains into account.

Unfortunately, there are situations where keeping a suitable condition number without further restrictions is impossible. This happens for high  $\mathbf{v}$ , high  $n$  and small  $d^2$ . To prevent this issue, given a design, we also restrict  $\mathbf{v}$  to have  $L \geq 0.35\sqrt{d}$ .

In our experiments, we chose  $\kappa = 10^{11}$  for all problems but for the low-pass filter cases with  $n = 300$ , where taking  $\kappa = 10^{12}$  allows to include the Gaussian covariance function.

Finally, predictions on very dense validation grids can suffer from numerical instabilities that may yield numerically negative values when predicting the kriging variance at a location that is very near an element of the initial design. We choose to truncate predicted variances to be greater than  $10^{-15}$  in our experiments.

## SM-1.4 Optimal values $R^*$

Here we present how we obtain optimal values  $R^*$  for the goodness-of-fit criteria  $R(\theta; S)$ . Given a function  $f$ , a design  $(x_i, z_i)_{1 \leq i \leq n}$  and a test set  $(x_j^t, z_j^t)_{1 \leq j \leq n_t}$ , these are obtained by direct minimization of

$$R(\theta; S) = \frac{1}{n_t} \sum_{j=1}^{n_t} S(P_{\theta, x_j^t}; f(x_j^t)) \quad (\text{SM1})$$

To obtain the gradient of (SM1) with respect to  $\theta$ , in the spirit of (Petit et al., 2020), we

---

<sup>1</sup>We therefore use common bounds for several similar datasets since we observed the fact that the variations were negligible.

<sup>2</sup>Note that, in such cases, kriging without any nugget is either numerically impossible or lacking of numerical stability.

decompose the evaluation of (SM1) in

$$\left\{ \begin{array}{l} \Gamma : \boldsymbol{\theta} \in \Theta \mapsto \mathbf{G} \in \mathbb{R}^{n^2+n \times n_t+n_t}, \\ \rho : \mathbf{G} \in \mathbb{R}^{n^2+n \times n_t+n_t} \mapsto (\boldsymbol{\mu}, \boldsymbol{\sigma}^2) \in \mathbb{R}^{2n_t}, \\ \varphi : (\boldsymbol{\mu}, \boldsymbol{\sigma}^2) \in \mathbb{R}^{2n_t} \mapsto R \in \mathbb{R}, \end{array} \right.$$

where

$$\left\{ \begin{array}{l} \mathbf{G} = (\mathbf{K}_\theta, \mathbf{K}_\theta^*, \mathbf{k}_\theta^{*,*}), \quad w = (\boldsymbol{\mu}, \boldsymbol{\sigma}^2), \\ \mathbf{K}_\theta^* = (k_\theta(x_i, x_j^t))_{1 \leq i \leq n, 1 \leq j \leq n_t}, \quad \mathbf{k}_\theta^{*,*} = (k_\theta(x_j^t, x_j^t))_{1 \leq j \leq n_t}, \\ \boldsymbol{\mu} = (\mu_\theta(x_1^t), \dots, \mu_\theta(x_{n_t}^t))^T, \quad \boldsymbol{\sigma}^2 = (\sigma_\theta^2(x_1^t), \dots, \sigma_\theta^2(x_{n_t}^t))^T, \end{array} \right.$$

in such a way that  $R(\boldsymbol{\theta}; S) = (\varphi \circ \rho \circ \Gamma)(\boldsymbol{\theta})$ .

Using elements from (Petit et al., 2020), the computation of the gradient of (SM1) boils down to the implementation of the adjoint operators of the differentials of  $\rho$ :

$$\mathcal{L}_\rho^* : (\mathbf{G}, \delta_w) \mapsto J_{\rho, \mathbf{G}}^T \delta_w.$$

where  $J_{\rho, \mathbf{G}}$  is the Jacobian matrix of  $\rho$  at  $\mathbf{G}$ . A generic implementation of those operators is given by Algorithm 1. The overall computation of this gradient has complexity  $\mathcal{O}(n^3 + dn^2 + n_t n^2)$  for a Matérn covariance function. It can be used to optimize (SM1) with respect to  $\boldsymbol{\theta}$  with a quasi-Newton type algorithm.

## SM-2 Detailed results

### SM-2.1 Conventions

We state here more precisely the conventions used for the tables of section SM-2.2. For each problem and design size, the first table represents the selection criteria considered in this work, without NLL/SPE. In this table, the gray scale indicates how far a cell is from the best value. More precisely, the combination of model and criterion that reaches the best score  $R_0$  is colored in white and we use a log-uniform scale of gray from  $R_0$  to the values that are greater than  $10R_0$ .

For comparison, NLL/SPE values are provided in another table along with results for a linear model and  $k$ -nearest neighbors models ( $k$ NN) for  $S^{\text{SPE}}$ .

Finally, optimal values  $R^*$  are provided for  $S^{\text{SPE}}$  in a third table. For all three tables, estimated standard deviations for the averages are provided in parentheses.

---

**Algorithm 1:** Computing  $\delta_G = J_{\rho, G}^T \delta_w$ , from  $G$  and  $\delta_w$ . Inputs at first step refer to what has already been computed for evaluating  $\mu$  and  $\sigma^2$  efficiently. For vectors or matrices  $a$  and  $b$ ,  $a \circ b$  denotes the Hadamard element-wise multiplication. Moreover,  $\mathbf{1}_n = (1, \dots, 1)^T \in \mathbb{R}^n$ .

---

**Input:**

$$G = (K_\theta, K_\theta^*, k_\theta^{*,*}), z = (z_1, \dots, z_n), B = K_\theta^{-1}, \Gamma = BK_\theta^*, \delta_w = (\delta_\mu, \delta_{\sigma^2})$$

**Output:**  $\delta_K = J_{\rho, K}^T \delta_w$

- 1  $\Delta = \mathbf{1}_n \sigma^{2T}$
  - 2  $\delta_\Gamma = -K_\theta^* \circ \Delta$
  - 3  $\delta_{K_\theta^*} = (Bz) \delta_\mu^T - \Gamma \circ \Delta + B^T \delta_\Gamma$
  - 4  $\delta_{k_\theta^{*,*}} = \delta_{\sigma^2}$
  - 5  $\delta_B = (K_\theta^* \delta_\mu) z^T + \delta_\Gamma K_\theta^{*T}$
  - 6  $\delta_{K_\theta} = -B^T \delta_B B^T$
- 

## SM-2.2 Results

### SM-2.2.1 Goldstein-Price (1D), $n = 10, d = 1$

Averages of  $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.45 \cdot 10^{-2} (3.12 \cdot 10^{-3})$	$2.04 \cdot 10^{-2} (2.42 \cdot 10^{-3})$	$2.04 \cdot 10^{-2} (2.42 \cdot 10^{-3})$	$2.04 \cdot 10^{-2} (2.42 \cdot 10^{-3})$	$5.51 \cdot 10^{-2} (6.46 \cdot 10^{-3})$	$2.04 \cdot 10^{-2} (2.42 \cdot 10^{-3})$
$v = 3/2$	$4.57 \cdot 10^{-3} (5.67 \cdot 10^{-4})$	$2.29 \cdot 10^{-3} (2.79 \cdot 10^{-4})$	$2.29 \cdot 10^{-3} (2.79 \cdot 10^{-4})$	$2.29 \cdot 10^{-3} (2.79 \cdot 10^{-4})$	$3.49 \cdot 10^{-2} (4.54 \cdot 10^{-3})$	$2.29 \cdot 10^{-3} (2.79 \cdot 10^{-4})$
$v = 5/2$	$9.50 \cdot 10^{-4} (1.41 \cdot 10^{-4})$	$2.41 \cdot 10^{-4} (3.49 \cdot 10^{-5})$	$2.41 \cdot 10^{-4} (3.49 \cdot 10^{-5})$	$2.41 \cdot 10^{-4} (3.49 \cdot 10^{-5})$	$3.46 \cdot 10^{-2} (3.98 \cdot 10^{-3})$	$2.41 \cdot 10^{-4} (3.49 \cdot 10^{-5})$
$v = 7/2$	$2.09 \cdot 10^{-4} (3.63 \cdot 10^{-5})$	$3.75 \cdot 10^{-5} (5.45 \cdot 10^{-6})$	$3.75 \cdot 10^{-5} (5.45 \cdot 10^{-6})$	$3.75 \cdot 10^{-5} (5.45 \cdot 10^{-6})$	$3.48 \cdot 10^{-2} (3.72 \cdot 10^{-3})$	$3.75 \cdot 10^{-5} (5.45 \cdot 10^{-6})$
$v = 9/2$	$4.70 \cdot 10^{-5} (8.82 \cdot 10^{-6})$	$1.19 \cdot 10^{-5} (1.52 \cdot 10^{-6})$	$1.19 \cdot 10^{-5} (1.52 \cdot 10^{-6})$	$1.19 \cdot 10^{-5} (1.52 \cdot 10^{-6})$	$3.50 \cdot 10^{-2} (3.58 \cdot 10^{-3})$	$1.19 \cdot 10^{-5} (1.52 \cdot 10^{-6})$
$v = \infty$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$	$3.59 \cdot 10^{-2} (3.00 \cdot 10^{-3})$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$
$v \in \{1/2, \dots, \infty\}$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$	$3.90 \cdot 10^{-2} (4.21 \cdot 10^{-3})$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.91 \cdot 10^{-2} (2.50 \cdot 10^{-3})$
$v = 3/2$	$2.29 \cdot 10^{-3} (2.79 \cdot 10^{-4})$
$v = 5/2$	$2.41 \cdot 10^{-4} (3.49 \cdot 10^{-5})$
$v = 7/2$	$3.75 \cdot 10^{-5} (5.45 \cdot 10^{-6})$
$v = 9/2$	$1.19 \cdot 10^{-5} (1.52 \cdot 10^{-6})$
$v = \infty$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$
$v \in \{1/2, \dots, \infty\}$	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.22 \cdot 10^{-6} (1.69 \cdot 10^{-7})$
linear	$8.81 \cdot 10^{-1} (1.98 \cdot 10^{-5})$
1NN	$8.46 \cdot 10^{-2} (2.42 \cdot 10^{-3})$
2NN	$1.42 \cdot 10^{-1} (4.75 \cdot 10^{-3})$

Averages of  $R(\theta; S^{\text{CRPS}})$



$R(\theta; S^{CRPS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$9.93 \cdot 10^{-2} (7.44 \cdot 10^{-4})$	$8.93 \cdot 10^{-2} (7.00 \cdot 10^{-4})$	$8.93 \cdot 10^{-2} (7.00 \cdot 10^{-4})$	$5.55 \cdot 10^{-2} (1.16 \cdot 10^{-3})$	$1.64 \cdot 10^{-1} (1.40 \cdot 10^{-3})$	$8.93 \cdot 10^{-2} (7.00 \cdot 10^{-4})$
$v = 3/2$	$2.49 \cdot 10^{-2} (5.37 \cdot 10^{-4})$	$1.78 \cdot 10^{-2} (3.61 \cdot 10^{-4})$	$1.78 \cdot 10^{-2} (3.61 \cdot 10^{-4})$	$1.72 \cdot 10^{-2} (4.64 \cdot 10^{-4})$	$1.08 \cdot 10^{-1} (1.43 \cdot 10^{-3})$	$1.78 \cdot 10^{-2} (3.61 \cdot 10^{-4})$
$v = 5/2$	$9.18 \cdot 10^{-3} (2.96 \cdot 10^{-4})$	$4.58 \cdot 10^{-3} (1.35 \cdot 10^{-4})$	$4.58 \cdot 10^{-3} (1.35 \cdot 10^{-4})$	$4.80 \cdot 10^{-3} (1.28 \cdot 10^{-4})$	$9.70 \cdot 10^{-2} (1.47 \cdot 10^{-3})$	$4.58 \cdot 10^{-3} (1.35 \cdot 10^{-4})$
$v = 7/2$	$3.77 \cdot 10^{-3} (1.55 \cdot 10^{-4})$	$1.59 \cdot 10^{-3} (5.31 \cdot 10^{-5})$	$1.59 \cdot 10^{-3} (5.32 \cdot 10^{-5})$	$1.60 \cdot 10^{-3} (4.83 \cdot 10^{-5})$	$9.28 \cdot 10^{-2} (1.50 \cdot 10^{-3})$	$1.59 \cdot 10^{-3} (5.31 \cdot 10^{-5})$
$v = 9/2$	$1.62 \cdot 10^{-3} (7.74 \cdot 10^{-5})$	$8.56 \cdot 10^{-4} (2.77 \cdot 10^{-5})$	$8.56 \cdot 10^{-4} (2.77 \cdot 10^{-5})$	$8.12 \cdot 10^{-4} (2.53 \cdot 10^{-5})$	$9.06 \cdot 10^{-2} (1.53 \cdot 10^{-3})$	$8.56 \cdot 10^{-4} (2.77 \cdot 10^{-5})$
$v = \infty$	$2.94 \cdot 10^{-4} (1.01 \cdot 10^{-5})$	$2.91 \cdot 10^{-4} (9.78 \cdot 10^{-6})$	$2.91 \cdot 10^{-4} (9.78 \cdot 10^{-6})$	$2.83 \cdot 10^{-4} (9.82 \cdot 10^{-6})$	$8.35 \cdot 10^{-2} (1.71 \cdot 10^{-3})$	$2.91 \cdot 10^{-4} (9.78 \cdot 10^{-6})$
$v \in \{1/2, \dots, \infty\}$	$2.94 \cdot 10^{-4} (1.01 \cdot 10^{-5})$	$2.91 \cdot 10^{-4} (9.78 \cdot 10^{-6})$	$2.91 \cdot 10^{-4} (9.78 \cdot 10^{-6})$	$2.84 \cdot 10^{-4} (1.04 \cdot 10^{-5})$	$1.21 \cdot 10^{-1} (4.59 \cdot 10^{-3})$	$2.91 \cdot 10^{-4} (9.78 \cdot 10^{-6})$

$R(\theta; S^{CRPS})$	
NLL/SPE	$2.94 \cdot 10^{-4} (1.01 \cdot 10^{-5})$

## Averages of $R(\theta; S_{0.95}^{IS})$

$R(\theta; S_{0.95}^{IS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.56 \cdot 10^0 (2.36 \cdot 10^{-2})$	$1.38 \cdot 10^0 (1.99 \cdot 10^{-2})$	$1.38 \cdot 10^0 (1.99 \cdot 10^{-2})$	$9.35 \cdot 10^{-1} (4.97 \cdot 10^{-2})$	$2.67 \cdot 10^0 (3.53 \cdot 10^{-2})$	$1.38 \cdot 10^0 (1.99 \cdot 10^{-2})$
$v = 3/2$	$3.35 \cdot 10^{-1} (1.46 \cdot 10^{-2})$	$2.26 \cdot 10^{-1} (8.24 \cdot 10^{-3})$	$2.26 \cdot 10^{-1} (8.24 \cdot 10^{-3})$	$2.51 \cdot 10^{-1} (1.23 \cdot 10^{-2})$	$1.74 \cdot 10^0 (3.70 \cdot 10^{-2})$	$2.26 \cdot 10^{-1} (8.24 \cdot 10^{-3})$
$v = 5/2$	$1.26 \cdot 10^{-1} (7.53 \cdot 10^{-3})$	$4.78 \cdot 10^{-2} (2.34 \cdot 10^{-3})$	$4.78 \cdot 10^{-2} (2.34 \cdot 10^{-3})$	$5.12 \cdot 10^{-2} (1.70 \cdot 10^{-3})$	$1.52 \cdot 10^0 (3.84 \cdot 10^{-2})$	$4.78 \cdot 10^{-2} (2.34 \cdot 10^{-3})$
$v = 7/2$	$5.04 \cdot 10^{-2} (3.60 \cdot 10^{-3})$	$1.22 \cdot 10^{-2} (6.30 \cdot 10^{-4})$	$1.22 \cdot 10^{-2} (6.30 \cdot 10^{-4})$	$1.39 \cdot 10^{-2} (3.46 \cdot 10^{-4})$	$1.42 \cdot 10^0 (3.93 \cdot 10^{-2})$	$1.22 \cdot 10^{-2} (6.30 \cdot 10^{-4})$
$v = 9/2$	$1.87 \cdot 10^{-2} (1.61 \cdot 10^{-3})$	$4.93 \cdot 10^{-3} (2.63 \cdot 10^{-4})$	$4.93 \cdot 10^{-3} (2.63 \cdot 10^{-4})$	$5.75 \cdot 10^{-3} (1.51 \cdot 10^{-4})$	$1.37 \cdot 10^0 (3.99 \cdot 10^{-2})$	$4.93 \cdot 10^{-3} (2.63 \cdot 10^{-4})$
$v = \infty$	$8.11 \cdot 10^{-3} (2.72 \cdot 10^{-4})$	$7.71 \cdot 10^{-3} (2.64 \cdot 10^{-4})$	$7.71 \cdot 10^{-3} (2.64 \cdot 10^{-4})$	$6.58 \cdot 10^{-3} (3.41 \cdot 10^{-4})$	$1.20 \cdot 10^0 (4.26 \cdot 10^{-2})$	$7.71 \cdot 10^{-3} (2.64 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$8.11 \cdot 10^{-3} (2.72 \cdot 10^{-4})$	$7.71 \cdot 10^{-3} (2.64 \cdot 10^{-4})$	$7.71 \cdot 10^{-3} (2.64 \cdot 10^{-4})$	$6.73 \cdot 10^{-3} (4.03 \cdot 10^{-4})$	$1.91 \cdot 10^0 (8.21 \cdot 10^{-2})$	$7.71 \cdot 10^{-3} (2.64 \cdot 10^{-4})$

$R(\theta; S_{0.95}^{IS})$	
NLL/SPE	$8.11 \cdot 10^{-3} (2.72 \cdot 10^{-4})$

## SM-2.2.2 Goldstein-Price (1D), $n = 20, d = 1$

### Averages of $R(\theta; S^{SPE})$

$R(\theta; S^{SPE})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$6.33 \cdot 10^{-3} (8.62 \cdot 10^{-4})$	$5.87 \cdot 10^{-3} (7.94 \cdot 10^{-4})$	$5.87 \cdot 10^{-3} (7.94 \cdot 10^{-4})$	$5.87 \cdot 10^{-3} (7.94 \cdot 10^{-4})$	$2.05 \cdot 10^{-2} (2.73 \cdot 10^{-3})$	$5.87 \cdot 10^{-3} (7.94 \cdot 10^{-4})$
$v = 3/2$	$2.55 \cdot 10^{-4} (3.37 \cdot 10^{-5})$	$1.97 \cdot 10^{-4} (2.61 \cdot 10^{-5})$	$1.97 \cdot 10^{-4} (2.61 \cdot 10^{-5})$	$1.97 \cdot 10^{-4} (2.61 \cdot 10^{-5})$	$9.47 \cdot 10^{-3} (1.21 \cdot 10^{-3})$	$1.97 \cdot 10^{-4} (2.61 \cdot 10^{-5})$
$v = 5/2$	$8.88 \cdot 10^{-6} (1.26 \cdot 10^{-6})$	$8.19 \cdot 10^{-6} (1.15 \cdot 10^{-6})$	$8.19 \cdot 10^{-6} (1.15 \cdot 10^{-6})$	$8.19 \cdot 10^{-6} (1.15 \cdot 10^{-6})$	$6.44 \cdot 10^{-3} (7.91 \cdot 10^{-4})$	$8.19 \cdot 10^{-6} (1.15 \cdot 10^{-6})$
$v = 7/2$	$1.18 \cdot 10^{-6} (1.67 \cdot 10^{-7})$	$1.18 \cdot 10^{-6} (1.67 \cdot 10^{-7})$	$1.18 \cdot 10^{-6} (1.67 \cdot 10^{-7})$	$1.18 \cdot 10^{-6} (1.67 \cdot 10^{-7})$	$4.80 \cdot 10^{-3} (5.95 \cdot 10^{-4})$	$1.18 \cdot 10^{-6} (1.67 \cdot 10^{-7})$
$v = 9/2$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$	$3.82 \cdot 10^{-3} (4.82 \cdot 10^{-4})$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$
$v \in \{1/2, \dots, 9/2\}$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$	$7.20 \cdot 10^{-3} (1.05 \cdot 10^{-3})$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$

$R^*(S^{SPE})$	
$v = 1/2$	$5.77 \cdot 10^{-3} (8.00 \cdot 10^{-4})$
$v = 3/2$	$1.97 \cdot 10^{-4} (2.61 \cdot 10^{-5})$
$v = 5/2$	$8.19 \cdot 10^{-6} (1.15 \cdot 10^{-6})$
$v = 7/2$	$1.18 \cdot 10^{-6} (1.67 \cdot 10^{-7})$
$v = 9/2$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$
$v \in \{1/2, \dots, 9/2\}$	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$

$R(\theta; S^{SPE})$	
NLL/SPE	$5.40 \cdot 10^{-7} (7.88 \cdot 10^{-8})$
linear	$8.81 \cdot 10^{-1} (4.82 \cdot 10^{-6})$
1NN	$2.64 \cdot 10^{-2} (6.90 \cdot 10^{-4})$
2NN	$4.38 \cdot 10^{-2} (1.58 \cdot 10^{-3})$

## Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.81 \cdot 10^{-2}$ ( $3.63 \cdot 10^{-4}$ )	$3.85 \cdot 10^{-2}$ ( $4.43 \cdot 10^{-4}$ )	$3.85 \cdot 10^{-2}$ ( $4.43 \cdot 10^{-4}$ )	$1.80 \cdot 10^{-2}$ ( $7.18 \cdot 10^{-4}$ )	$7.70 \cdot 10^{-2}$ ( $6.89 \cdot 10^{-4}$ )	$3.85 \cdot 10^{-2}$ ( $4.43 \cdot 10^{-4}$ )
$v = 3/2$	$4.85 \cdot 10^{-3}$ ( $1.09 \cdot 10^{-4}$ )	$3.92 \cdot 10^{-3}$ ( $1.04 \cdot 10^{-4}$ )	$3.92 \cdot 10^{-3}$ ( $1.04 \cdot 10^{-4}$ )	$3.09 \cdot 10^{-3}$ ( $1.32 \cdot 10^{-4}$ )	$3.35 \cdot 10^{-2}$ ( $6.07 \cdot 10^{-4}$ )	$3.92 \cdot 10^{-3}$ ( $1.04 \cdot 10^{-4}$ )
$v = 5/2$	$6.44 \cdot 10^{-4}$ ( $2.19 \cdot 10^{-5}$ )	$6.07 \cdot 10^{-4}$ ( $2.20 \cdot 10^{-5}$ )	$6.07 \cdot 10^{-4}$ ( $2.20 \cdot 10^{-5}$ )	$6.45 \cdot 10^{-4}$ ( $2.23 \cdot 10^{-5}$ )	$2.31 \cdot 10^{-2}$ ( $5.33 \cdot 10^{-4}$ )	$6.07 \cdot 10^{-4}$ ( $2.20 \cdot 10^{-5}$ )
$v = 7/2$	$1.98 \cdot 10^{-4}$ ( $7.86 \cdot 10^{-6}$ )	$2.00 \cdot 10^{-4}$ ( $8.20 \cdot 10^{-6}$ )	$2.00 \cdot 10^{-4}$ ( $8.20 \cdot 10^{-6}$ )	$2.05 \cdot 10^{-4}$ ( $7.21 \cdot 10^{-6}$ )	$1.81 \cdot 10^{-2}$ ( $4.88 \cdot 10^{-4}$ )	$2.00 \cdot 10^{-4}$ ( $8.20 \cdot 10^{-6}$ )
$v = 9/2$	$1.22 \cdot 10^{-4}$ ( $5.38 \cdot 10^{-6}$ )	$1.23 \cdot 10^{-4}$ ( $5.48 \cdot 10^{-6}$ )	$1.23 \cdot 10^{-4}$ ( $5.48 \cdot 10^{-6}$ )	$1.20 \cdot 10^{-4}$ ( $4.77 \cdot 10^{-6}$ )	$1.52 \cdot 10^{-2}$ ( $4.57 \cdot 10^{-4}$ )	$1.23 \cdot 10^{-4}$ ( $5.48 \cdot 10^{-6}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.22 \cdot 10^{-4}$ ( $5.38 \cdot 10^{-6}$ )	$1.23 \cdot 10^{-4}$ ( $5.48 \cdot 10^{-6}$ )	$1.23 \cdot 10^{-4}$ ( $5.48 \cdot 10^{-6}$ )	$1.20 \cdot 10^{-4}$ ( $4.77 \cdot 10^{-6}$ )	$4.24 \cdot 10^{-2}$ ( $2.65 \cdot 10^{-3}$ )	$1.23 \cdot 10^{-4}$ ( $5.48 \cdot 10^{-6}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.22 \cdot 10^{-4}$ ( $5.38 \cdot 10^{-6}$ )

## Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$7.83 \cdot 10^{-1}$ ( $9.74 \cdot 10^{-3}$ )	$6.54 \cdot 10^{-1}$ ( $1.44 \cdot 10^{-2}$ )	$6.54 \cdot 10^{-1}$ ( $1.44 \cdot 10^{-2}$ )	$4.04 \cdot 10^{-1}$ ( $2.99 \cdot 10^{-2}$ )	$1.35 \cdot 10^0$ ( $2.60 \cdot 10^{-2}$ )	$6.54 \cdot 10^{-1}$ ( $1.44 \cdot 10^{-2}$ )
$v = 3/2$	$6.95 \cdot 10^{-2}$ ( $2.42 \cdot 10^{-3}$ )	$6.18 \cdot 10^{-2}$ ( $2.83 \cdot 10^{-3}$ )	$6.18 \cdot 10^{-2}$ ( $2.83 \cdot 10^{-3}$ )	$8.20 \cdot 10^{-2}$ ( $4.68 \cdot 10^{-3}$ )	$5.59 \cdot 10^{-1}$ ( $2.23 \cdot 10^{-2}$ )	$6.18 \cdot 10^{-2}$ ( $2.83 \cdot 10^{-3}$ )
$v = 5/2$	$7.87 \cdot 10^{-3}$ ( $3.88 \cdot 10^{-4}$ )	$8.98 \cdot 10^{-3}$ ( $5.21 \cdot 10^{-4}$ )	$8.98 \cdot 10^{-3}$ ( $5.21 \cdot 10^{-4}$ )	$8.88 \cdot 10^{-3}$ ( $5.13 \cdot 10^{-4}$ )	$3.85 \cdot 10^{-1}$ ( $1.85 \cdot 10^{-2}$ )	$8.98 \cdot 10^{-3}$ ( $5.21 \cdot 10^{-4}$ )
$v = 7/2$	$2.16 \cdot 10^{-3}$ ( $1.29 \cdot 10^{-4}$ )	$2.69 \cdot 10^{-3}$ ( $1.73 \cdot 10^{-4}$ )	$2.69 \cdot 10^{-3}$ ( $1.73 \cdot 10^{-4}$ )	$2.02 \cdot 10^{-3}$ ( $6.81 \cdot 10^{-5}$ )	$3.09 \cdot 10^{-1}$ ( $1.56 \cdot 10^{-2}$ )	$2.69 \cdot 10^{-3}$ ( $1.73 \cdot 10^{-4}$ )
$v = 9/2$	$1.49 \cdot 10^{-3}$ ( $9.76 \cdot 10^{-5}$ )	$1.64 \cdot 10^{-3}$ ( $1.11 \cdot 10^{-4}$ )	$1.64 \cdot 10^{-3}$ ( $1.11 \cdot 10^{-4}$ )	$1.05 \cdot 10^{-3}$ ( $3.82 \cdot 10^{-5}$ )	$2.63 \cdot 10^{-1}$ ( $1.37 \cdot 10^{-2}$ )	$1.64 \cdot 10^{-3}$ ( $1.11 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.49 \cdot 10^{-3}$ ( $9.76 \cdot 10^{-5}$ )	$1.64 \cdot 10^{-3}$ ( $1.11 \cdot 10^{-4}$ )	$1.64 \cdot 10^{-3}$ ( $1.11 \cdot 10^{-4}$ )	$1.05 \cdot 10^{-3}$ ( $3.82 \cdot 10^{-5}$ )	$7.30 \cdot 10^{-1}$ ( $4.43 \cdot 10^{-2}$ )	$1.64 \cdot 10^{-3}$ ( $1.11 \cdot 10^{-4}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.49 \cdot 10^{-3}$ ( $9.76 \cdot 10^{-5}$ )

## SM-2.2.3 Goldstein-Price (1D), $n = 50, d = 1$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$6.88 \cdot 10^{-4}$ ( $8.27 \cdot 10^{-5}$ )	$6.70 \cdot 10^{-4}$ ( $8.05 \cdot 10^{-5}$ )	$6.70 \cdot 10^{-4}$ ( $8.05 \cdot 10^{-5}$ )	$6.70 \cdot 10^{-4}$ ( $8.05 \cdot 10^{-5}$ )	$3.09 \cdot 10^{-3}$ ( $3.72 \cdot 10^{-4}$ )	$6.70 \cdot 10^{-4}$ ( $8.05 \cdot 10^{-5}$ )
$v = 3/2$	$3.71 \cdot 10^{-6}$ ( $4.01 \cdot 10^{-7}$ )	$3.55 \cdot 10^{-6}$ ( $3.84 \cdot 10^{-7}$ )	$3.55 \cdot 10^{-6}$ ( $3.84 \cdot 10^{-7}$ )	$3.55 \cdot 10^{-6}$ ( $3.84 \cdot 10^{-7}$ )	$4.03 \cdot 10^{-4}$ ( $4.63 \cdot 10^{-5}$ )	$3.55 \cdot 10^{-6}$ ( $3.84 \cdot 10^{-7}$ )
$v = 5/2$	$6.29 \cdot 10^{-8}$ ( $6.96 \cdot 10^{-9}$ )	$6.29 \cdot 10^{-8}$ ( $6.96 \cdot 10^{-9}$ )	$6.29 \cdot 10^{-8}$ ( $6.96 \cdot 10^{-9}$ )	$6.29 \cdot 10^{-8}$ ( $6.96 \cdot 10^{-9}$ )	$9.27 \cdot 10^{-5}$ ( $1.08 \cdot 10^{-5}$ )	$6.29 \cdot 10^{-8}$ ( $6.96 \cdot 10^{-9}$ )
$v = 7/2$	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )	$2.77 \cdot 10^{-5}$ ( $3.38 \cdot 10^{-6}$ )	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )
$v \in \{1/2, \dots, 7/2\}$	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )	$2.32 \cdot 10^{-4}$ ( $4.34 \cdot 10^{-5}$ )	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$6.68 \cdot 10^{-4}$ ( $8.07 \cdot 10^{-5}$ )
$v = 3/2$	$3.55 \cdot 10^{-6}$ ( $3.84 \cdot 10^{-7}$ )
$v = 5/2$	$6.30 \cdot 10^{-8}$ ( $6.95 \cdot 10^{-9}$ )
$v = 7/2$	$1.52 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )
$v \in \{1/2, \dots, 7/2\}$	$1.52 \cdot 10^{-8}$ ( $1.77 \cdot 10^{-9}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.51 \cdot 10^{-8}$ ( $1.78 \cdot 10^{-9}$ )
linear	$8.81 \cdot 10^{-1}$ ( $3.44 \cdot 10^{-7}$ )
1NN	$4.48 \cdot 10^{-3}$ ( $6.96 \cdot 10^{-5}$ )
2NN	$6.34 \cdot 10^{-3}$ ( $1.36 \cdot 10^{-4}$ )

## Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.91 \cdot 10^{-2}$ ( $7.96 \cdot 10^{-5}$ )	$1.09 \cdot 10^{-2}$ ( $1.13 \cdot 10^{-4}$ )	$1.09 \cdot 10^{-2}$ ( $1.13 \cdot 10^{-4}$ )	$3.80 \cdot 10^{-3}$ ( $1.77 \cdot 10^{-4}$ )	$2.37 \cdot 10^{-2}$ ( $2.46 \cdot 10^{-4}$ )	$1.09 \cdot 10^{-2}$ ( $1.13 \cdot 10^{-4}$ )
$v = 3/2$	$6.32 \cdot 10^{-4}$ ( $7.24 \cdot 10^{-6}$ )	$4.26 \cdot 10^{-4}$ ( $8.45 \cdot 10^{-6}$ )	$4.26 \cdot 10^{-4}$ ( $8.45 \cdot 10^{-6}$ )	$2.55 \cdot 10^{-4}$ ( $1.01 \cdot 10^{-5}$ )	$4.85 \cdot 10^{-3}$ ( $9.52 \cdot 10^{-5}$ )	$4.26 \cdot 10^{-4}$ ( $8.45 \cdot 10^{-6}$ )
$v = 5/2$	$4.89 \cdot 10^{-5}$ ( $1.00 \cdot 10^{-6}$ )	$4.08 \cdot 10^{-5}$ ( $1.14 \cdot 10^{-6}$ )	$4.08 \cdot 10^{-5}$ ( $1.14 \cdot 10^{-6}$ )	$3.91 \cdot 10^{-5}$ ( $1.30 \cdot 10^{-6}$ )	$1.73 \cdot 10^{-3}$ ( $4.70 \cdot 10^{-5}$ )	$4.08 \cdot 10^{-5}$ ( $1.14 \cdot 10^{-6}$ )
$v = 7/2$	$1.69 \cdot 10^{-5}$ ( $5.12 \cdot 10^{-7}$ )	$1.61 \cdot 10^{-5}$ ( $5.58 \cdot 10^{-7}$ )	$1.61 \cdot 10^{-5}$ ( $5.58 \cdot 10^{-7}$ )	$1.73 \cdot 10^{-5}$ ( $5.89 \cdot 10^{-7}$ )	$7.70 \cdot 10^{-4}$ ( $2.61 \cdot 10^{-5}$ )	$1.61 \cdot 10^{-5}$ ( $5.58 \cdot 10^{-7}$ )
$v \in \{1/2, \dots, 7/2\}$	$1.69 \cdot 10^{-5}$ ( $5.12 \cdot 10^{-7}$ )	$1.61 \cdot 10^{-5}$ ( $5.58 \cdot 10^{-7}$ )	$1.61 \cdot 10^{-5}$ ( $5.58 \cdot 10^{-7}$ )	$1.73 \cdot 10^{-5}$ ( $5.89 \cdot 10^{-7}$ )	$2.25 \cdot 10^{-3}$ ( $2.09 \cdot 10^{-4}$ )	$1.61 \cdot 10^{-5}$ ( $5.58 \cdot 10^{-7}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.69 \cdot 10^{-5}$ ( $5.12 \cdot 10^{-7}$ )

## Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.16 \cdot 10^{-1}$ ( $1.79 \cdot 10^{-3}$ )	$2.01 \cdot 10^{-1}$ ( $3.82 \cdot 10^{-3}$ )	$2.01 \cdot 10^{-1}$ ( $3.82 \cdot 10^{-3}$ )	$9.84 \cdot 10^{-2}$ ( $7.14 \cdot 10^{-3}$ )	$4.43 \cdot 10^{-1}$ ( $8.46 \cdot 10^{-3}$ )	$2.01 \cdot 10^{-1}$ ( $3.82 \cdot 10^{-3}$ )
$v = 3/2$	$9.59 \cdot 10^{-3}$ ( $1.41 \cdot 10^{-4}$ )	$8.26 \cdot 10^{-3}$ ( $2.74 \cdot 10^{-4}$ )	$8.26 \cdot 10^{-3}$ ( $2.74 \cdot 10^{-4}$ )	$9.90 \cdot 10^{-3}$ ( $4.05 \cdot 10^{-4}$ )	$9.36 \cdot 10^{-2}$ ( $3.38 \cdot 10^{-3}$ )	$8.26 \cdot 10^{-3}$ ( $2.74 \cdot 10^{-4}$ )
$v = 5/2$	$6.77 \cdot 10^{-4}$ ( $1.73 \cdot 10^{-5}$ )	$8.29 \cdot 10^{-4}$ ( $3.43 \cdot 10^{-5}$ )	$8.29 \cdot 10^{-4}$ ( $3.43 \cdot 10^{-5}$ )	$9.48 \cdot 10^{-4}$ ( $5.04 \cdot 10^{-5}$ )	$3.64 \cdot 10^{-2}$ ( $1.53 \cdot 10^{-3}$ )	$8.29 \cdot 10^{-4}$ ( $3.43 \cdot 10^{-5}$ )
$v = 7/2$	$2.52 \cdot 10^{-4}$ ( $1.01 \cdot 10^{-5}$ )	$3.49 \cdot 10^{-4}$ ( $1.63 \cdot 10^{-5}$ )	$3.49 \cdot 10^{-4}$ ( $1.63 \cdot 10^{-5}$ )	$3.26 \cdot 10^{-4}$ ( $1.96 \cdot 10^{-5}$ )	$1.72 \cdot 10^{-2}$ ( $8.14 \cdot 10^{-4}$ )	$3.49 \cdot 10^{-4}$ ( $1.63 \cdot 10^{-5}$ )
$v \in \{1/2, \dots, 7/2\}$	$2.52 \cdot 10^{-4}$ ( $1.01 \cdot 10^{-5}$ )	$3.49 \cdot 10^{-4}$ ( $1.63 \cdot 10^{-5}$ )	$3.49 \cdot 10^{-4}$ ( $1.63 \cdot 10^{-5}$ )	$3.26 \cdot 10^{-4}$ ( $1.96 \cdot 10^{-5}$ )	$4.91 \cdot 10^{-2}$ ( $4.94 \cdot 10^{-3}$ )	$3.49 \cdot 10^{-4}$ ( $1.63 \cdot 10^{-5}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$2.52 \cdot 10^{-4}$ ( $1.01 \cdot 10^{-5}$ )

## SM-2.2.4 Goldstein-Price, $n = 20, d = 2$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.61 \cdot 10^{-1}$ ( $2.29 \cdot 10^{-2}$ )	$3.17 \cdot 10^{-1}$ ( $1.86 \cdot 10^{-2}$ )	$3.46 \cdot 10^{-1}$ ( $2.25 \cdot 10^{-2}$ )	$3.26 \cdot 10^{-1}$ ( $1.87 \cdot 10^{-2}$ )	$4.56 \cdot 10^{-1}$ ( $1.92 \cdot 10^{-2}$ )	$3.63 \cdot 10^{-1}$ ( $2.36 \cdot 10^{-2}$ )
$v = 3/2$	$3.04 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )	$3.00 \cdot 10^{-1}$ ( $2.34 \cdot 10^{-2}$ )	$2.96 \cdot 10^{-1}$ ( $2.54 \cdot 10^{-2}$ )	$2.71 \cdot 10^{-1}$ ( $1.80 \cdot 10^{-2}$ )	$4.05 \cdot 10^{-1}$ ( $2.05 \cdot 10^{-2}$ )	$3.14 \cdot 10^{-1}$ ( $3.08 \cdot 10^{-2}$ )
$v = 5/2$	$2.99 \cdot 10^{-1}$ ( $2.10 \cdot 10^{-2}$ )	$2.97 \cdot 10^{-1}$ ( $2.41 \cdot 10^{-2}$ )	$2.65 \cdot 10^{-1}$ ( $2.27 \cdot 10^{-2}$ )	$2.68 \cdot 10^{-1}$ ( $1.75 \cdot 10^{-2}$ )	$3.92 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )	$2.83 \cdot 10^{-1}$ ( $3.64 \cdot 10^{-2}$ )
$v = 7/2$	$2.97 \cdot 10^{-1}$ ( $2.10 \cdot 10^{-2}$ )	$2.95 \cdot 10^{-1}$ ( $2.28 \cdot 10^{-2}$ )	$2.40 \cdot 10^{-1}$ ( $1.67 \cdot 10^{-2}$ )	$2.64 \cdot 10^{-1}$ ( $1.77 \cdot 10^{-2}$ )	$3.89 \cdot 10^{-1}$ ( $2.13 \cdot 10^{-2}$ )	$2.36 \cdot 10^{-1}$ ( $1.97 \cdot 10^{-2}$ )
$v = 9/2$	$2.79 \cdot 10^{-1}$ ( $1.91 \cdot 10^{-2}$ )	$2.87 \cdot 10^{-1}$ ( $2.19 \cdot 10^{-2}$ )	$2.36 \cdot 10^{-1}$ ( $1.73 \cdot 10^{-2}$ )	$2.63 \cdot 10^{-1}$ ( $1.80 \cdot 10^{-2}$ )	$3.86 \cdot 10^{-1}$ ( $2.16 \cdot 10^{-2}$ )	$2.14 \cdot 10^{-1}$ ( $1.67 \cdot 10^{-2}$ )
$v = \infty$	$2.81 \cdot 10^{-1}$ ( $1.89 \cdot 10^{-2}$ )	$2.33 \cdot 10^{-1}$ ( $1.78 \cdot 10^{-2}$ )	$2.32 \cdot 10^{-1}$ ( $1.85 \cdot 10^{-2}$ )	$2.44 \cdot 10^{-1}$ ( $1.85 \cdot 10^{-2}$ )	$3.73 \cdot 10^{-1}$ ( $2.10 \cdot 10^{-2}$ )	$1.94 \cdot 10^{-1}$ ( $1.62 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, \infty\}$	$2.97 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )	$2.60 \cdot 10^{-1}$ ( $2.31 \cdot 10^{-2}$ )	$2.24 \cdot 10^{-1}$ ( $1.76 \cdot 10^{-2}$ )	$2.49 \cdot 10^{-1}$ ( $1.92 \cdot 10^{-2}$ )	$3.97 \cdot 10^{-1}$ ( $2.09 \cdot 10^{-2}$ )	$2.05 \cdot 10^{-1}$ ( $1.72 \cdot 10^{-2}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.53 \cdot 10^{-1}$ ( $1.80 \cdot 10^{-2}$ )
$v = 3/2$	$1.44 \cdot 10^{-1}$ ( $1.12 \cdot 10^{-2}$ )
$v = 5/2$	$1.16 \cdot 10^{-1}$ ( $9.00 \cdot 10^{-3}$ )
$v = 7/2$	$1.06 \cdot 10^{-1}$ ( $8.51 \cdot 10^{-3}$ )
$v = 9/2$	$9.93 \cdot 10^{-2}$ ( $8.36 \cdot 10^{-3}$ )
$v = \infty$	$8.08 \cdot 10^{-2}$ ( $5.31 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, \infty\}$	$7.65 \cdot 10^{-2}$ ( $5.26 \cdot 10^{-3}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$2.90 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )
linear	$8.46 \cdot 10^{-1}$ ( $8.65 \cdot 10^{-3}$ )
1NN	$4.49 \cdot 10^{-1}$ ( $1.61 \cdot 10^{-2}$ )
2NN	$4.46 \cdot 10^{-1}$ ( $1.65 \cdot 10^{-2}$ )

## Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.34 \cdot 10^{-1} (5.07 \cdot 10^{-3})$	$2.24 \cdot 10^{-1} (3.59 \cdot 10^{-3})$	$2.28 \cdot 10^{-1} (5.27 \cdot 10^{-3})$	$1.89 \cdot 10^{-1} (4.38 \cdot 10^{-3})$	$2.84 \cdot 10^{-1} (5.19 \cdot 10^{-3})$	$2.35 \cdot 10^{-1} (6.04 \cdot 10^{-3})$
$v = 3/2$	$1.92 \cdot 10^{-1} (5.28 \cdot 10^{-3})$	$1.94 \cdot 10^{-1} (6.36 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (7.26 \cdot 10^{-3})$	$1.80 \cdot 10^{-1} (5.34 \cdot 10^{-3})$	$2.54 \cdot 10^{-1} (6.75 \cdot 10^{-3})$	$1.99 \cdot 10^{-1} (8.68 \cdot 10^{-3})$
$v = 5/2$	$1.89 \cdot 10^{-1} (5.54 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (6.61 \cdot 10^{-3})$	$1.83 \cdot 10^{-1} (5.80 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (5.25 \cdot 10^{-3})$	$2.46 \cdot 10^{-1} (7.21 \cdot 10^{-3})$	$1.90 \cdot 10^{-1} (9.63 \cdot 10^{-3})$
$v = 7/2$	$1.89 \cdot 10^{-1} (5.46 \cdot 10^{-3})$	$1.94 \cdot 10^{-1} (6.55 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (4.88 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (5.29 \cdot 10^{-3})$	$2.43 \cdot 10^{-1} (7.41 \cdot 10^{-3})$	$1.74 \cdot 10^{-1} (6.38 \cdot 10^{-3})$
$v = 9/2$	$1.84 \cdot 10^{-1} (4.50 \cdot 10^{-3})$	$1.92 \cdot 10^{-1} (6.56 \cdot 10^{-3})$	$1.75 \cdot 10^{-1} (5.10 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (5.17 \cdot 10^{-3})$	$2.41 \cdot 10^{-1} (7.51 \cdot 10^{-3})$	$1.68 \cdot 10^{-1} (5.48 \cdot 10^{-3})$
$v = \infty$	$1.89 \cdot 10^{-1} (4.37 \cdot 10^{-3})$	$1.73 \cdot 10^{-1} (5.27 \cdot 10^{-3})$	$1.76 \cdot 10^{-1} (5.65 \cdot 10^{-3})$	$1.78 \cdot 10^{-1} (5.52 \cdot 10^{-3})$	$2.34 \cdot 10^{-1} (7.63 \cdot 10^{-3})$	$1.62 \cdot 10^{-1} (4.29 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$1.93 \cdot 10^{-1} (5.23 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (6.45 \cdot 10^{-3})$	$1.74 \cdot 10^{-1} (5.49 \cdot 10^{-3})$	$1.80 \cdot 10^{-1} (6.04 \cdot 10^{-3})$	$2.50 \cdot 10^{-1} (7.94 \cdot 10^{-3})$	$1.68 \cdot 10^{-1} (4.97 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.91 \cdot 10^{-1} (5.09 \cdot 10^{-3})$

## Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.41 \cdot 10^0 (1.90 \cdot 10^{-1})$	$4.05 \cdot 10^0 (1.47 \cdot 10^{-1})$	$4.22 \cdot 10^0 (1.86 \cdot 10^{-1})$	$4.42 \cdot 10^0 (1.89 \cdot 10^{-1})$	$5.32 \cdot 10^0 (1.24 \cdot 10^{-1})$	$4.40 \cdot 10^0 (1.97 \cdot 10^{-1})$
$v = 3/2$	$3.98 \cdot 10^0 (2.08 \cdot 10^{-1})$	$4.03 \cdot 10^0 (2.36 \cdot 10^{-1})$	$4.01 \cdot 10^0 (2.63 \cdot 10^{-1})$	$4.11 \cdot 10^0 (2.11 \cdot 10^{-1})$	$4.80 \cdot 10^0 (1.54 \cdot 10^{-1})$	$4.17 \cdot 10^0 (3.06 \cdot 10^{-1})$
$v = 5/2$	$4.12 \cdot 10^0 (2.14 \cdot 10^{-1})$	$4.02 \cdot 10^0 (2.25 \cdot 10^{-1})$	$3.85 \cdot 10^0 (1.84 \cdot 10^{-1})$	$4.14 \cdot 10^0 (2.01 \cdot 10^{-1})$	$4.57 \cdot 10^0 (1.42 \cdot 10^{-1})$	$3.97 \cdot 10^0 (3.12 \cdot 10^{-1})$
$v = 7/2$	$4.17 \cdot 10^0 (2.04 \cdot 10^{-1})$	$4.07 \cdot 10^0 (2.10 \cdot 10^{-1})$	$3.73 \cdot 10^0 (1.73 \cdot 10^{-1})$	$4.20 \cdot 10^0 (1.98 \cdot 10^{-1})$	$4.46 \cdot 10^0 (1.42 \cdot 10^{-1})$	$3.46 \cdot 10^0 (2.21 \cdot 10^{-1})$
$v = 9/2$	$3.99 \cdot 10^0 (1.80 \cdot 10^{-1})$	$4.02 \cdot 10^0 (2.11 \cdot 10^{-1})$	$3.70 \cdot 10^0 (1.78 \cdot 10^{-1})$	$4.15 \cdot 10^0 (1.98 \cdot 10^{-1})$	$4.38 \cdot 10^0 (1.43 \cdot 10^{-1})$	$3.24 \cdot 10^0 (2.00 \cdot 10^{-1})$
$v = \infty$	$4.13 \cdot 10^0 (1.76 \cdot 10^{-1})$	$3.76 \cdot 10^0 (1.88 \cdot 10^{-1})$	$4.04 \cdot 10^0 (1.96 \cdot 10^{-1})$	$4.23 \cdot 10^0 (2.07 \cdot 10^{-1})$	$4.19 \cdot 10^0 (1.45 \cdot 10^{-1})$	$3.28 \cdot 10^0 (1.54 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$4.35 \cdot 10^0 (2.04 \cdot 10^{-1})$	$4.16 \cdot 10^0 (2.13 \cdot 10^{-1})$	$4.01 \cdot 10^0 (1.90 \cdot 10^{-1})$	$4.41 \cdot 10^0 (2.27 \cdot 10^{-1})$	$4.57 \cdot 10^0 (1.45 \cdot 10^{-1})$	$3.52 \cdot 10^0 (1.77 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$4.27 \cdot 10^0 (1.97 \cdot 10^{-1})$

## SM-2.2.5 Goldstein-Price, $n = 40, d = 2$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.43 \cdot 10^{-1} (9.79 \cdot 10^{-3})$	$1.51 \cdot 10^{-1} (1.03 \cdot 10^{-2})$	$1.45 \cdot 10^{-1} (9.76 \cdot 10^{-3})$	$1.52 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$2.55 \cdot 10^{-1} (1.42 \cdot 10^{-2})$	$1.43 \cdot 10^{-1} (9.63 \cdot 10^{-3})$
$v = 3/2$	$8.29 \cdot 10^{-2} (7.24 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (9.85 \cdot 10^{-3})$	$8.43 \cdot 10^{-2} (7.47 \cdot 10^{-3})$	$8.92 \cdot 10^{-2} (8.56 \cdot 10^{-3})$	$2.13 \cdot 10^{-1} (1.54 \cdot 10^{-2})$	$7.83 \cdot 10^{-2} (7.13 \cdot 10^{-3})$
$v = 5/2$	$6.86 \cdot 10^{-2} (6.94 \cdot 10^{-3})$	$8.23 \cdot 10^{-2} (8.99 \cdot 10^{-3})$	$5.58 \cdot 10^{-2} (4.14 \cdot 10^{-3})$	$6.93 \cdot 10^{-2} (6.81 \cdot 10^{-3})$	$2.00 \cdot 10^{-1} (1.59 \cdot 10^{-2})$	$5.77 \cdot 10^{-2} (4.43 \cdot 10^{-3})$
$v = 7/2$	$6.28 \cdot 10^{-2} (6.90 \cdot 10^{-3})$	$7.37 \cdot 10^{-2} (8.18 \cdot 10^{-3})$	$4.68 \cdot 10^{-2} (3.57 \cdot 10^{-3})$	$5.68 \cdot 10^{-2} (5.01 \cdot 10^{-3})$	$1.97 \cdot 10^{-1} (1.60 \cdot 10^{-2})$	$5.40 \cdot 10^{-2} (5.05 \cdot 10^{-3})$
$v = 9/2$	$5.97 \cdot 10^{-2} (6.89 \cdot 10^{-3})$	$5.53 \cdot 10^{-2} (5.01 \cdot 10^{-3})$	$4.12 \cdot 10^{-2} (3.50 \cdot 10^{-3})$	$4.71 \cdot 10^{-2} (4.14 \cdot 10^{-3})$	$1.88 \cdot 10^{-1} (1.55 \cdot 10^{-2})$	$4.43 \cdot 10^{-2} (4.49 \cdot 10^{-3})$
$v \in \{1/2, \dots, 9/2\}$	$6.03 \cdot 10^{-2} (6.90 \cdot 10^{-3})$	$5.65 \cdot 10^{-2} (4.85 \cdot 10^{-3})$	$4.12 \cdot 10^{-2} (3.50 \cdot 10^{-3})$	$4.79 \cdot 10^{-2} (4.56 \cdot 10^{-3})$	$2.08 \cdot 10^{-1} (1.44 \cdot 10^{-2})$	$4.43 \cdot 10^{-2} (4.49 \cdot 10^{-3})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.20 \cdot 10^{-1} (8.48 \cdot 10^{-3})$
$v = 3/2$	$4.73 \cdot 10^{-2} (3.67 \cdot 10^{-3})$
$v = 5/2$	$3.09 \cdot 10^{-2} (2.32 \cdot 10^{-3})$
$v = 7/2$	$2.38 \cdot 10^{-2} (1.69 \cdot 10^{-3})$
$v = 9/2$	$1.96 \cdot 10^{-2} (1.44 \cdot 10^{-3})$
$v \in \{1/2, \dots, 9/2\}$	$1.93 \cdot 10^{-2} (1.43 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$6.17 \cdot 10^{-2}$ ( $6.91 \cdot 10^{-3}$ )
linear	$8.05 \cdot 10^{-1}$ ( $3.34 \cdot 10^{-3}$ )
1NN	$2.71 \cdot 10^{-1}$ ( $9.40 \cdot 10^{-3}$ )
2NN	$2.70 \cdot 10^{-1}$ ( $9.71 \cdot 10^{-3}$ )

## Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.49 \cdot 10^{-1}$ ( $1.47 \cdot 10^{-3}$ )	$1.46 \cdot 10^{-1}$ ( $1.84 \cdot 10^{-3}$ )	$1.43 \cdot 10^{-1}$ ( $1.75 \cdot 10^{-3}$ )	$1.21 \cdot 10^{-1}$ ( $2.63 \cdot 10^{-3}$ )	$1.98 \cdot 10^{-1}$ ( $3.89 \cdot 10^{-3}$ )	$1.43 \cdot 10^{-1}$ ( $1.79 \cdot 10^{-3}$ )
$v = 3/2$	$9.24 \cdot 10^{-2}$ ( $2.09 \cdot 10^{-3}$ )	$1.02 \cdot 10^{-1}$ ( $3.35 \cdot 10^{-3}$ )	$9.44 \cdot 10^{-2}$ ( $3.11 \cdot 10^{-3}$ )	$9.17 \cdot 10^{-2}$ ( $3.13 \cdot 10^{-3}$ )	$1.63 \cdot 10^{-1}$ ( $5.88 \cdot 10^{-3}$ )	$9.10 \cdot 10^{-2}$ ( $2.79 \cdot 10^{-3}$ )
$v = 5/2$	$7.89 \cdot 10^{-2}$ ( $2.36 \cdot 10^{-3}$ )	$8.72 \cdot 10^{-2}$ ( $3.20 \cdot 10^{-3}$ )	$7.44 \cdot 10^{-2}$ ( $2.00 \cdot 10^{-3}$ )	$8.05 \cdot 10^{-2}$ ( $2.66 \cdot 10^{-3}$ )	$1.54 \cdot 10^{-1}$ ( $6.39 \cdot 10^{-3}$ )	$7.57 \cdot 10^{-2}$ ( $2.20 \cdot 10^{-3}$ )
$v = 7/2$	$7.34 \cdot 10^{-2}$ ( $2.47 \cdot 10^{-3}$ )	$8.06 \cdot 10^{-2}$ ( $3.26 \cdot 10^{-3}$ )	$6.65 \cdot 10^{-2}$ ( $2.01 \cdot 10^{-3}$ )	$7.20 \cdot 10^{-2}$ ( $2.40 \cdot 10^{-3}$ )	$1.50 \cdot 10^{-1}$ ( $6.63 \cdot 10^{-3}$ )	$6.94 \cdot 10^{-2}$ ( $2.55 \cdot 10^{-3}$ )
$v = 9/2$	$7.04 \cdot 10^{-2}$ ( $2.54 \cdot 10^{-3}$ )	$7.01 \cdot 10^{-2}$ ( $2.51 \cdot 10^{-3}$ )	$6.07 \cdot 10^{-2}$ ( $2.01 \cdot 10^{-3}$ )	$6.53 \cdot 10^{-2}$ ( $2.27 \cdot 10^{-3}$ )	$1.46 \cdot 10^{-1}$ ( $6.62 \cdot 10^{-3}$ )	$6.13 \cdot 10^{-2}$ ( $2.31 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$7.09 \cdot 10^{-2}$ ( $2.56 \cdot 10^{-3}$ )	$7.13 \cdot 10^{-2}$ ( $2.60 \cdot 10^{-3}$ )	$6.07 \cdot 10^{-2}$ ( $2.01 \cdot 10^{-3}$ )	$6.56 \cdot 10^{-2}$ ( $2.37 \cdot 10^{-3}$ )	$1.66 \cdot 10^{-1}$ ( $5.84 \cdot 10^{-3}$ )	$6.13 \cdot 10^{-2}$ ( $2.31 \cdot 10^{-3}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$7.25 \cdot 10^{-2}$ ( $2.62 \cdot 10^{-3}$ )

## Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.50 \cdot 10^0$ ( $7.64 \cdot 10^{-2}$ )	$2.56 \cdot 10^0$ ( $8.77 \cdot 10^{-2}$ )	$2.51 \cdot 10^0$ ( $8.43 \cdot 10^{-2}$ )	$3.04 \cdot 10^0$ ( $9.98 \cdot 10^{-2}$ )	$3.61 \cdot 10^0$ ( $9.07 \cdot 10^{-2}$ )	$2.50 \cdot 10^0$ ( $8.30 \cdot 10^{-2}$ )
$v = 3/2$	$1.67 \cdot 10^0$ ( $8.00 \cdot 10^{-2}$ )	$1.94 \cdot 10^0$ ( $1.09 \cdot 10^{-1}$ )	$1.77 \cdot 10^0$ ( $1.06 \cdot 10^{-1}$ )	$2.07 \cdot 10^0$ ( $1.14 \cdot 10^{-1}$ )	$2.98 \cdot 10^0$ ( $1.17 \cdot 10^{-1}$ )	$1.67 \cdot 10^0$ ( $1.03 \cdot 10^{-1}$ )
$v = 5/2$	$1.49 \cdot 10^0$ ( $8.46 \cdot 10^{-2}$ )	$1.62 \cdot 10^0$ ( $1.03 \cdot 10^{-1}$ )	$1.32 \cdot 10^0$ ( $7.04 \cdot 10^{-2}$ )	$1.57 \cdot 10^0$ ( $9.34 \cdot 10^{-2}$ )	$2.78 \cdot 10^0$ ( $1.27 \cdot 10^{-1}$ )	$1.33 \cdot 10^0$ ( $7.70 \cdot 10^{-2}$ )
$v = 7/2$	$1.42 \cdot 10^0$ ( $8.65 \cdot 10^{-2}$ )	$1.46 \cdot 10^0$ ( $9.73 \cdot 10^{-2}$ )	$1.15 \cdot 10^0$ ( $6.93 \cdot 10^{-2}$ )	$1.33 \cdot 10^0$ ( $8.05 \cdot 10^{-2}$ )	$2.70 \cdot 10^0$ ( $1.27 \cdot 10^{-1}$ )	$1.16 \cdot 10^0$ ( $8.01 \cdot 10^{-2}$ )
$v = 9/2$	$1.39 \cdot 10^0$ ( $8.69 \cdot 10^{-2}$ )	$1.21 \cdot 10^0$ ( $7.64 \cdot 10^{-2}$ )	$1.00 \cdot 10^0$ ( $6.81 \cdot 10^{-2}$ )	$1.13 \cdot 10^0$ ( $7.65 \cdot 10^{-2}$ )	$2.59 \cdot 10^0$ ( $1.23 \cdot 10^{-1}$ )	$9.54 \cdot 10^{-1}$ ( $7.18 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.41 \cdot 10^0$ ( $8.80 \cdot 10^{-2}$ )	$1.25 \cdot 10^0$ ( $7.82 \cdot 10^{-2}$ )	$1.00 \cdot 10^0$ ( $6.81 \cdot 10^{-2}$ )	$1.16 \cdot 10^0$ ( $8.11 \cdot 10^{-2}$ )	$3.01 \cdot 10^0$ ( $1.17 \cdot 10^{-1}$ )	$9.54 \cdot 10^{-1}$ ( $7.18 \cdot 10^{-2}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.43 \cdot 10^0$ ( $8.84 \cdot 10^{-2}$ )

## SM-2.2.6 Goldstein-Price, $n = 100$ , $d = 2$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.69 \cdot 10^{-2}$ ( $3.92 \cdot 10^{-3}$ )	$5.12 \cdot 10^{-2}$ ( $3.99 \cdot 10^{-3}$ )	$4.73 \cdot 10^{-2}$ ( $3.97 \cdot 10^{-3}$ )	$4.72 \cdot 10^{-2}$ ( $3.84 \cdot 10^{-3}$ )	$9.30 \cdot 10^{-2}$ ( $7.07 \cdot 10^{-3}$ )	$4.67 \cdot 10^{-2}$ ( $3.95 \cdot 10^{-3}$ )
$v = 3/2$	$1.26 \cdot 10^{-2}$ ( $1.25 \cdot 10^{-3}$ )	$1.61 \cdot 10^{-2}$ ( $1.61 \cdot 10^{-3}$ )	$1.27 \cdot 10^{-2}$ ( $1.20 \cdot 10^{-3}$ )	$1.36 \cdot 10^{-2}$ ( $1.25 \cdot 10^{-3}$ )	$5.42 \cdot 10^{-2}$ ( $5.14 \cdot 10^{-3}$ )	$1.20 \cdot 10^{-2}$ ( $1.07 \cdot 10^{-3}$ )
$v = 5/2$	$5.27 \cdot 10^{-3}$ ( $5.55 \cdot 10^{-4}$ )	$7.33 \cdot 10^{-3}$ ( $8.80 \cdot 10^{-4}$ )	$5.03 \cdot 10^{-3}$ ( $4.59 \cdot 10^{-4}$ )	$5.95 \cdot 10^{-3}$ ( $5.92 \cdot 10^{-4}$ )	$4.29 \cdot 10^{-2}$ ( $4.48 \cdot 10^{-3}$ )	$5.04 \cdot 10^{-3}$ ( $4.43 \cdot 10^{-4}$ )
$v = 7/2$	$2.55 \cdot 10^{-3}$ ( $2.74 \cdot 10^{-4}$ )	$4.30 \cdot 10^{-3}$ ( $6.41 \cdot 10^{-4}$ )	$2.54 \cdot 10^{-3}$ ( $2.64 \cdot 10^{-4}$ )	$3.09 \cdot 10^{-3}$ ( $3.70 \cdot 10^{-4}$ )	$3.72 \cdot 10^{-2}$ ( $4.50 \cdot 10^{-3}$ )	$2.72 \cdot 10^{-3}$ ( $2.94 \cdot 10^{-4}$ )
$v = 9/2$	$1.75 \cdot 10^{-3}$ ( $2.01 \cdot 10^{-4}$ )	$3.13 \cdot 10^{-3}$ ( $5.02 \cdot 10^{-4}$ )	$1.76 \cdot 10^{-3}$ ( $2.09 \cdot 10^{-4}$ )	$2.18 \cdot 10^{-3}$ ( $2.98 \cdot 10^{-4}$ )	$3.34 \cdot 10^{-2}$ ( $4.54 \cdot 10^{-3}$ )	$1.89 \cdot 10^{-3}$ ( $2.22 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.75 \cdot 10^{-3}$ ( $2.01 \cdot 10^{-4}$ )	$3.13 \cdot 10^{-3}$ ( $4.99 \cdot 10^{-4}$ )	$1.76 \cdot 10^{-3}$ ( $2.08 \cdot 10^{-4}$ )	$2.26 \cdot 10^{-3}$ ( $3.27 \cdot 10^{-4}$ )	$4.09 \cdot 10^{-2}$ ( $4.42 \cdot 10^{-3}$ )	$1.89 \cdot 10^{-3}$ ( $2.22 \cdot 10^{-4}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$4.20 \cdot 10^{-2}$ ( $3.48 \cdot 10^{-3}$ )
$v = 3/2$	$8.87 \cdot 10^{-3}$ ( $7.15 \cdot 10^{-4}$ )
$v = 5/2$	$3.07 \cdot 10^{-3}$ ( $2.17 \cdot 10^{-4}$ )
$v = 7/2$	$1.56 \cdot 10^{-3}$ ( $1.34 \cdot 10^{-4}$ )
$v = 9/2$	$1.23 \cdot 10^{-3}$ ( $1.37 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.16 \cdot 10^{-3}$ ( $1.16 \cdot 10^{-4}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.77 \cdot 10^{-3}$ ( $2.02 \cdot 10^{-4}$ )
linear	$7.89 \cdot 10^{-1}$ ( $1.70 \cdot 10^{-3}$ )
1NN	$1.37 \cdot 10^{-1}$ ( $5.16 \cdot 10^{-3}$ )
2NN	$1.27 \cdot 10^{-1}$ ( $5.14 \cdot 10^{-3}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$8.57 \cdot 10^{-2}$ ( $8.54 \cdot 10^{-4}$ )	$7.73 \cdot 10^{-2}$ ( $1.21 \cdot 10^{-3}$ )	$7.46 \cdot 10^{-2}$ ( $1.10 \cdot 10^{-3}$ )	$5.82 \cdot 10^{-2}$ ( $1.49 \cdot 10^{-3}$ )	$1.06 \cdot 10^{-1}$ ( $1.34 \cdot 10^{-3}$ )	$7.44 \cdot 10^{-2}$ ( $1.12 \cdot 10^{-3}$ )
$v = 3/2$	$3.23 \cdot 10^{-2}$ ( $7.05 \cdot 10^{-4}$ )	$3.42 \cdot 10^{-2}$ ( $1.01 \cdot 10^{-3}$ )	$3.15 \cdot 10^{-2}$ ( $7.50 \cdot 10^{-4}$ )	$3.05 \cdot 10^{-2}$ ( $8.95 \cdot 10^{-4}$ )	$6.71 \cdot 10^{-2}$ ( $1.35 \cdot 10^{-3}$ )	$3.12 \cdot 10^{-2}$ ( $7.06 \cdot 10^{-4}$ )
$v = 5/2$	$1.82 \cdot 10^{-2}$ ( $5.06 \cdot 10^{-4}$ )	$2.09 \cdot 10^{-2}$ ( $7.92 \cdot 10^{-4}$ )	$1.81 \cdot 10^{-2}$ ( $4.97 \cdot 10^{-4}$ )	$1.94 \cdot 10^{-2}$ ( $6.32 \cdot 10^{-4}$ )	$5.57 \cdot 10^{-2}$ ( $1.50 \cdot 10^{-3}$ )	$1.84 \cdot 10^{-2}$ ( $4.88 \cdot 10^{-4}$ )
$v = 7/2$	$1.17 \cdot 10^{-2}$ ( $3.52 \cdot 10^{-4}$ )	$1.43 \cdot 10^{-2}$ ( $6.89 \cdot 10^{-4}$ )	$1.18 \cdot 10^{-2}$ ( $3.65 \cdot 10^{-4}$ )	$1.27 \cdot 10^{-2}$ ( $4.63 \cdot 10^{-4}$ )	$4.97 \cdot 10^{-2}$ ( $1.72 \cdot 10^{-3}$ )	$1.22 \cdot 10^{-2}$ ( $3.89 \cdot 10^{-4}$ )
$v = 9/2$	$9.07 \cdot 10^{-3}$ ( $2.93 \cdot 10^{-4}$ )	$1.14 \cdot 10^{-2}$ ( $6.12 \cdot 10^{-4}$ )	$9.12 \cdot 10^{-3}$ ( $2.94 \cdot 10^{-4}$ )	$9.81 \cdot 10^{-3}$ ( $3.89 \cdot 10^{-4}$ )	$4.59 \cdot 10^{-2}$ ( $1.85 \cdot 10^{-3}$ )	$9.48 \cdot 10^{-3}$ ( $3.16 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, 9/2\}$	$9.07 \cdot 10^{-3}$ ( $2.93 \cdot 10^{-4}$ )	$1.15 \cdot 10^{-2}$ ( $6.09 \cdot 10^{-4}$ )	$9.14 \cdot 10^{-3}$ ( $2.92 \cdot 10^{-4}$ )	$9.89 \cdot 10^{-3}$ ( $4.15 \cdot 10^{-4}$ )	$5.75 \cdot 10^{-2}$ ( $2.69 \cdot 10^{-3}$ )	$9.48 \cdot 10^{-3}$ ( $3.16 \cdot 10^{-4}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$9.13 \cdot 10^{-3}$ ( $3.00 \cdot 10^{-4}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.41 \cdot 10^0$ ( $3.68 \cdot 10^{-2}$ )	$1.40 \cdot 10^0$ ( $4.42 \cdot 10^{-2}$ )	$1.35 \cdot 10^0$ ( $4.48 \cdot 10^{-2}$ )	$1.62 \cdot 10^0$ ( $5.60 \cdot 10^{-2}$ )	$1.93 \cdot 10^0$ ( $5.58 \cdot 10^{-2}$ )	$1.34 \cdot 10^0$ ( $4.50 \cdot 10^{-2}$ )
$v = 3/2$	$5.31 \cdot 10^{-1}$ ( $2.34 \cdot 10^{-2}$ )	$6.25 \cdot 10^{-1}$ ( $3.18 \cdot 10^{-2}$ )	$5.62 \cdot 10^{-1}$ ( $2.66 \cdot 10^{-2}$ )	$7.08 \cdot 10^{-1}$ ( $3.12 \cdot 10^{-2}$ )	$1.24 \cdot 10^0$ ( $5.15 \cdot 10^{-2}$ )	$5.44 \cdot 10^{-1}$ ( $2.50 \cdot 10^{-2}$ )
$v = 5/2$	$2.87 \cdot 10^{-1}$ ( $1.52 \cdot 10^{-2}$ )	$3.52 \cdot 10^{-1}$ ( $2.36 \cdot 10^{-2}$ )	$2.91 \cdot 10^{-1}$ ( $1.49 \cdot 10^{-2}$ )	$3.46 \cdot 10^{-1}$ ( $1.97 \cdot 10^{-2}$ )	$1.03 \cdot 10^0$ ( $4.95 \cdot 10^{-2}$ )	$2.87 \cdot 10^{-1}$ ( $1.46 \cdot 10^{-2}$ )
$v = 7/2$	$1.73 \cdot 10^{-1}$ ( $9.82 \cdot 10^{-3}$ )	$2.30 \cdot 10^{-1}$ ( $1.90 \cdot 10^{-2}$ )	$1.71 \cdot 10^{-1}$ ( $9.78 \cdot 10^{-3}$ )	$1.91 \cdot 10^{-1}$ ( $1.29 \cdot 10^{-2}$ )	$9.13 \cdot 10^{-1}$ ( $5.06 \cdot 10^{-2}$ )	$1.73 \cdot 10^{-1}$ ( $1.04 \cdot 10^{-2}$ )
$v = 9/2$	$1.32 \cdot 10^{-1}$ ( $7.96 \cdot 10^{-3}$ )	$1.79 \cdot 10^{-1}$ ( $1.57 \cdot 10^{-2}$ )	$1.27 \cdot 10^{-1}$ ( $7.86 \cdot 10^{-3}$ )	$1.35 \cdot 10^{-1}$ ( $9.66 \cdot 10^{-3}$ )	$8.34 \cdot 10^{-1}$ ( $5.10 \cdot 10^{-2}$ )	$1.30 \cdot 10^{-1}$ ( $8.54 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.32 \cdot 10^{-1}$ ( $7.96 \cdot 10^{-3}$ )	$1.81 \cdot 10^{-1}$ ( $1.58 \cdot 10^{-2}$ )	$1.27 \cdot 10^{-1}$ ( $7.85 \cdot 10^{-3}$ )	$1.37 \cdot 10^{-1}$ ( $1.07 \cdot 10^{-2}$ )	$1.03 \cdot 10^0$ ( $5.74 \cdot 10^{-2}$ )	$1.30 \cdot 10^{-1}$ ( $8.54 \cdot 10^{-3}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.33 \cdot 10^{-1}$ ( $8.05 \cdot 10^{-3}$ )

### SM-2.2.7 Mystery, $n = 20, d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.13 \cdot 10^{-1}$ ( $1.29 \cdot 10^{-2}$ )	$3.70 \cdot 10^{-1}$ ( $1.23 \cdot 10^{-2}$ )	$3.89 \cdot 10^{-1}$ ( $1.33 \cdot 10^{-2}$ )	$3.76 \cdot 10^{-1}$ ( $1.27 \cdot 10^{-2}$ )	$4.83 \cdot 10^{-1}$ ( $1.33 \cdot 10^{-2}$ )	$4.33 \cdot 10^{-1}$ ( $1.87 \cdot 10^{-2}$ )
$v = 3/2$	$4.72 \cdot 10^{-1}$ ( $1.83 \cdot 10^{-2}$ )	$5.65 \cdot 10^{-1}$ ( $2.79 \cdot 10^{-2}$ )	$5.38 \cdot 10^{-1}$ ( $2.50 \cdot 10^{-2}$ )	$5.35 \cdot 10^{-1}$ ( $2.43 \cdot 10^{-2}$ )	$5.10 \cdot 10^{-1}$ ( $1.96 \cdot 10^{-2}$ )	$6.61 \cdot 10^{-1}$ ( $4.15 \cdot 10^{-2}$ )
$v = 5/2$	$5.05 \cdot 10^{-1}$ ( $2.13 \cdot 10^{-2}$ )	$6.03 \cdot 10^{-1}$ ( $4.01 \cdot 10^{-2}$ )	$5.76 \cdot 10^{-1}$ ( $2.71 \cdot 10^{-2}$ )	$5.93 \cdot 10^{-1}$ ( $3.15 \cdot 10^{-2}$ )	$5.44 \cdot 10^{-1}$ ( $2.13 \cdot 10^{-2}$ )	$7.79 \cdot 10^{-1}$ ( $4.75 \cdot 10^{-2}$ )
$v = 7/2$	$5.15 \cdot 10^{-1}$ ( $2.10 \cdot 10^{-2}$ )	$5.99 \cdot 10^{-1}$ ( $2.60 \cdot 10^{-2}$ )	$6.04 \cdot 10^{-1}$ ( $2.77 \cdot 10^{-2}$ )	$5.92 \cdot 10^{-1}$ ( $2.66 \cdot 10^{-2}$ )	$5.61 \cdot 10^{-1}$ ( $2.07 \cdot 10^{-2}$ )	$1.04 \cdot 10^0$ ( $1.07 \cdot 10^{-1}$ )
$v = 9/2$	$5.27 \cdot 10^{-1}$ ( $2.16 \cdot 10^{-2}$ )	$5.97 \cdot 10^{-1}$ ( $2.55 \cdot 10^{-2}$ )	$6.45 \cdot 10^{-1}$ ( $3.41 \cdot 10^{-2}$ )	$5.97 \cdot 10^{-1}$ ( $2.66 \cdot 10^{-2}$ )	$5.74 \cdot 10^{-1}$ ( $2.07 \cdot 10^{-2}$ )	$1.47 \cdot 10^0$ ( $2.40 \cdot 10^{-1}$ )
$v = \infty$	$5.48 \cdot 10^{-1}$ ( $2.09 \cdot 10^{-2}$ )	$1.17 \cdot 10^0$ ( $3.76 \cdot 10^{-1}$ )	$1.09 \cdot 10^0$ ( $2.37 \cdot 10^{-1}$ )	$1.09 \cdot 10^0$ ( $3.08 \cdot 10^{-1}$ )	$6.82 \cdot 10^{-1}$ ( $2.93 \cdot 10^{-2}$ )	$2.44 \cdot 10^0$ ( $4.33 \cdot 10^{-1}$ )
$v \in \{1/2, \dots, \infty\}$	$4.68 \cdot 10^{-1}$ ( $1.63 \cdot 10^{-2}$ )	$1.01 \cdot 10^0$ ( $3.12 \cdot 10^{-1}$ )	$1.16 \cdot 10^0$ ( $3.12 \cdot 10^{-1}$ )	$9.74 \cdot 10^{-1}$ ( $3.08 \cdot 10^{-1}$ )	$5.96 \cdot 10^{-1}$ ( $2.66 \cdot 10^{-2}$ )	$2.44 \cdot 10^0$ ( $4.33 \cdot 10^{-1}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.79 \cdot 10^{-1} (5.53 \cdot 10^{-3})$
$v = 3/2$	$3.05 \cdot 10^{-1} (7.79 \cdot 10^{-3})$
$v = 5/2$	$3.16 \cdot 10^{-1} (8.26 \cdot 10^{-3})$
$v = 7/2$	$3.19 \cdot 10^{-1} (8.36 \cdot 10^{-3})$
$v = 9/2$	$3.21 \cdot 10^{-1} (8.42 \cdot 10^{-3})$
$v = \infty$	$3.27 \cdot 10^{-1} (8.55 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$2.72 \cdot 10^{-1} (5.91 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.67 \cdot 10^{-1} (1.55 \cdot 10^{-2})$
linear	$9.94 \cdot 10^{-1} (2.24 \cdot 10^{-3})$
1NN	$4.89 \cdot 10^{-1} (8.30 \cdot 10^{-3})$
2NN	$4.61 \cdot 10^{-1} (7.56 \cdot 10^{-3})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.50 \cdot 10^{-1} (6.15 \cdot 10^{-3})$	$3.30 \cdot 10^{-1} (5.91 \cdot 10^{-3})$	$3.38 \cdot 10^{-1} (6.46 \cdot 10^{-3})$	$3.32 \cdot 10^{-1} (6.16 \cdot 10^{-3})$	$3.84 \cdot 10^{-1} (6.05 \cdot 10^{-3})$	$3.58 \cdot 10^{-1} (8.35 \cdot 10^{-3})$
$v = 3/2$	$3.68 \cdot 10^{-1} (9.07 \cdot 10^{-3})$	$4.05 \cdot 10^{-1} (1.11 \cdot 10^{-2})$	$3.99 \cdot 10^{-1} (1.14 \cdot 10^{-2})$	$3.97 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$3.88 \cdot 10^{-1} (9.38 \cdot 10^{-3})$	$4.50 \cdot 10^{-1} (1.52 \cdot 10^{-2})$
$v = 5/2$	$3.82 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$4.13 \cdot 10^{-1} (1.13 \cdot 10^{-2})$	$4.14 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$4.16 \cdot 10^{-1} (1.26 \cdot 10^{-2})$	$3.99 \cdot 10^{-1} (9.98 \cdot 10^{-3})$	$4.78 \cdot 10^{-1} (1.54 \cdot 10^{-2})$
$v = 7/2$	$3.86 \cdot 10^{-1} (9.90 \cdot 10^{-3})$	$4.17 \cdot 10^{-1} (1.08 \cdot 10^{-2})$	$4.24 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$4.16 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$4.03 \cdot 10^{-1} (9.24 \cdot 10^{-3})$	$5.21 \cdot 10^{-1} (1.57 \cdot 10^{-2})$
$v = 9/2$	$3.89 \cdot 10^{-1} (9.90 \cdot 10^{-3})$	$4.15 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$4.34 \cdot 10^{-1} (1.23 \cdot 10^{-2})$	$4.16 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$4.05 \cdot 10^{-1} (8.82 \cdot 10^{-3})$	$5.51 \cdot 10^{-1} (2.01 \cdot 10^{-2})$
$v = \infty$	$3.97 \cdot 10^{-1} (9.32 \cdot 10^{-3})$	$4.65 \cdot 10^{-1} (2.21 \cdot 10^{-2})$	$4.83 \cdot 10^{-1} (2.30 \cdot 10^{-2})$	$4.67 \cdot 10^{-1} (2.31 \cdot 10^{-2})$	$4.35 \cdot 10^{-1} (9.30 \cdot 10^{-3})$	$6.49 \cdot 10^{-1} (3.40 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$3.72 \cdot 10^{-1} (7.81 \cdot 10^{-3})$	$4.38 \cdot 10^{-1} (2.34 \cdot 10^{-2})$	$4.75 \cdot 10^{-1} (2.47 \cdot 10^{-2})$	$4.35 \cdot 10^{-1} (2.35 \cdot 10^{-2})$	$4.14 \cdot 10^{-1} (9.40 \cdot 10^{-3})$	$6.62 \cdot 10^{-1} (3.49 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$3.70 \cdot 10^{-1} (7.46 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.57 \cdot 10^0 (1.33 \cdot 10^{-1})$	$3.34 \cdot 10^0 (1.13 \cdot 10^{-1})$	$3.49 \cdot 10^0 (1.24 \cdot 10^{-1})$	$3.40 \cdot 10^0 (1.26 \cdot 10^{-1})$	$3.90 \cdot 10^0 (1.39 \cdot 10^{-1})$	$3.88 \cdot 10^0 (1.68 \cdot 10^{-1})$
$v = 3/2$	$4.80 \cdot 10^0 (2.77 \cdot 10^{-1})$	$5.99 \cdot 10^0 (3.18 \cdot 10^{-1})$	$5.95 \cdot 10^0 (3.61 \cdot 10^{-1})$	$6.09 \cdot 10^0 (3.56 \cdot 10^{-1})$	$4.71 \cdot 10^0 (2.75 \cdot 10^{-1})$	$7.40 \cdot 10^0 (4.75 \cdot 10^{-1})$
$v = 5/2$	$5.44 \cdot 10^0 (3.33 \cdot 10^{-1})$	$6.40 \cdot 10^0 (3.63 \cdot 10^{-1})$	$6.60 \cdot 10^0 (4.12 \cdot 10^{-1})$	$6.79 \cdot 10^0 (4.24 \cdot 10^{-1})$	$5.11 \cdot 10^0 (2.89 \cdot 10^{-1})$	$8.43 \cdot 10^0 (5.09 \cdot 10^{-1})$
$v = 7/2$	$5.58 \cdot 10^0 (3.11 \cdot 10^{-1})$	$6.57 \cdot 10^0 (3.39 \cdot 10^{-1})$	$6.98 \cdot 10^0 (3.91 \cdot 10^{-1})$	$6.83 \cdot 10^0 (3.80 \cdot 10^{-1})$	$5.10 \cdot 10^0 (2.35 \cdot 10^{-1})$	$9.40 \cdot 10^0 (4.70 \cdot 10^{-1})$
$v = 9/2$	$5.66 \cdot 10^0 (2.97 \cdot 10^{-1})$	$6.53 \cdot 10^0 (3.16 \cdot 10^{-1})$	$7.29 \cdot 10^0 (4.00 \cdot 10^{-1})$	$6.85 \cdot 10^0 (3.57 \cdot 10^{-1})$	$5.12 \cdot 10^0 (2.08 \cdot 10^{-1})$	$1.02 \cdot 10^1 (5.64 \cdot 10^{-1})$
$v = \infty$	$5.95 \cdot 10^0 (2.67 \cdot 10^{-1})$	$8.40 \cdot 10^0 (7.68 \cdot 10^{-1})$	$9.40 \cdot 10^0 (8.55 \cdot 10^{-1})$	$8.88 \cdot 10^0 (8.85 \cdot 10^{-1})$	$5.82 \cdot 10^0 (2.01 \cdot 10^{-1})$	$1.39 \cdot 10^1 (1.19 \cdot 10^0)$
$v \in \{1/2, \dots, \infty\}$	$5.10 \cdot 10^0 (2.76 \cdot 10^{-1})$	$7.79 \cdot 10^0 (8.16 \cdot 10^{-1})$	$9.24 \cdot 10^0 (9.00 \cdot 10^{-1})$	$8.11 \cdot 10^0 (8.93 \cdot 10^{-1})$	$5.43 \cdot 10^0 (2.52 \cdot 10^{-1})$	$1.48 \cdot 10^1 (1.20 \cdot 10^0)$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$5.05 \cdot 10^0 (2.70 \cdot 10^{-1})$

### SM-2.2.8 Mystery, $n = 40, d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.93 \cdot 10^{-1}$ ( $4.75 \cdot 10^{-3}$ )	$1.83 \cdot 10^{-1}$ ( $5.40 \cdot 10^{-3}$ )	$1.75 \cdot 10^{-1}$ ( $4.73 \cdot 10^{-3}$ )	$1.78 \cdot 10^{-1}$ ( $5.01 \cdot 10^{-3}$ )	$3.13 \cdot 10^{-1}$ ( $7.26 \cdot 10^{-3}$ )	$1.75 \cdot 10^{-1}$ ( $4.84 \cdot 10^{-3}$ )
$v = 3/2$	$1.60 \cdot 10^{-1}$ ( $6.47 \cdot 10^{-3}$ )	$1.90 \cdot 10^{-1}$ ( $8.81 \cdot 10^{-3}$ )	$1.77 \cdot 10^{-1}$ ( $8.37 \cdot 10^{-3}$ )	$1.69 \cdot 10^{-1}$ ( $7.51 \cdot 10^{-3}$ )	$3.20 \cdot 10^{-1}$ ( $1.24 \cdot 10^{-2}$ )	$2.04 \cdot 10^{-1}$ ( $1.42 \cdot 10^{-2}$ )
$v = 5/2$	$1.51 \cdot 10^{-1}$ ( $7.07 \cdot 10^{-3}$ )	$1.93 \cdot 10^{-1}$ ( $1.07 \cdot 10^{-2}$ )	$2.07 \cdot 10^{-1}$ ( $1.30 \cdot 10^{-2}$ )	$1.79 \cdot 10^{-1}$ ( $1.00 \cdot 10^{-2}$ )	$3.54 \cdot 10^{-1}$ ( $1.41 \cdot 10^{-2}$ )	$3.03 \cdot 10^{-1}$ ( $3.06 \cdot 10^{-2}$ )
$v = 7/2$	$1.47 \cdot 10^{-1}$ ( $7.38 \cdot 10^{-3}$ )	$2.12 \cdot 10^{-1}$ ( $2.58 \cdot 10^{-2}$ )	$2.76 \cdot 10^{-1}$ ( $3.29 \cdot 10^{-2}$ )	$2.13 \cdot 10^{-1}$ ( $3.23 \cdot 10^{-2}$ )	$3.96 \cdot 10^{-1}$ ( $1.78 \cdot 10^{-2}$ )	$5.68 \cdot 10^{-1}$ ( $6.35 \cdot 10^{-2}$ )
$v = 9/2$	$1.46 \cdot 10^{-1}$ ( $7.56 \cdot 10^{-3}$ )	$2.31 \cdot 10^{-1}$ ( $3.37 \cdot 10^{-2}$ )	$3.74 \cdot 10^{-1}$ ( $7.22 \cdot 10^{-2}$ )	$2.48 \cdot 10^{-1}$ ( $4.03 \cdot 10^{-2}$ )	$4.32 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )	$8.79 \cdot 10^{-1}$ ( $1.11 \cdot 10^{-1}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.62 \cdot 10^{-1}$ ( $7.51 \cdot 10^{-3}$ )	$2.80 \cdot 10^{-1}$ ( $4.86 \cdot 10^{-2}$ )	$3.00 \cdot 10^{-1}$ ( $3.61 \cdot 10^{-2}$ )	$2.55 \cdot 10^{-1}$ ( $4.02 \cdot 10^{-2}$ )	$4.14 \cdot 10^{-1}$ ( $2.11 \cdot 10^{-2}$ )	$8.88 \cdot 10^{-1}$ ( $1.10 \cdot 10^{-1}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.64 \cdot 10^{-1}$ ( $4.24 \cdot 10^{-3}$ )
$v = 3/2$	$1.23 \cdot 10^{-1}$ ( $4.98 \cdot 10^{-3}$ )
$v = 5/2$	$1.08 \cdot 10^{-1}$ ( $4.30 \cdot 10^{-3}$ )
$v = 7/2$	$1.02 \cdot 10^{-1}$ ( $4.29 \cdot 10^{-3}$ )
$v = 9/2$	$9.90 \cdot 10^{-2}$ ( $4.30 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$9.77 \cdot 10^{-2}$ ( $4.05 \cdot 10^{-3}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.64 \cdot 10^{-1}$ ( $7.28 \cdot 10^{-3}$ )
linear	$9.83 \cdot 10^{-1}$ ( $1.62 \cdot 10^{-3}$ )
1NN	$3.02 \cdot 10^{-1}$ ( $5.15 \cdot 10^{-3}$ )
2NN	$2.79 \cdot 10^{-1}$ ( $5.40 \cdot 10^{-3}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.28 \cdot 10^{-1}$ ( $1.97 \cdot 10^{-3}$ )	$2.21 \cdot 10^{-1}$ ( $2.57 \cdot 10^{-3}$ )	$2.17 \cdot 10^{-1}$ ( $2.09 \cdot 10^{-3}$ )	$2.18 \cdot 10^{-1}$ ( $2.37 \cdot 10^{-3}$ )	$3.00 \cdot 10^{-1}$ ( $3.62 \cdot 10^{-3}$ )	$2.16 \cdot 10^{-1}$ ( $2.16 \cdot 10^{-3}$ )
$v = 3/2$	$1.87 \cdot 10^{-1}$ ( $2.69 \cdot 10^{-3}$ )	$1.99 \cdot 10^{-1}$ ( $4.30 \cdot 10^{-3}$ )	$1.89 \cdot 10^{-1}$ ( $3.73 \cdot 10^{-3}$ )	$1.86 \cdot 10^{-1}$ ( $3.50 \cdot 10^{-3}$ )	$2.88 \cdot 10^{-1}$ ( $6.21 \cdot 10^{-3}$ )	$2.02 \cdot 10^{-1}$ ( $7.01 \cdot 10^{-3}$ )
$v = 5/2$	$1.76 \cdot 10^{-1}$ ( $3.08 \cdot 10^{-3}$ )	$1.90 \cdot 10^{-1}$ ( $4.82 \cdot 10^{-3}$ )	$1.88 \cdot 10^{-1}$ ( $4.96 \cdot 10^{-3}$ )	$1.82 \cdot 10^{-1}$ ( $4.76 \cdot 10^{-3}$ )	$2.95 \cdot 10^{-1}$ ( $6.27 \cdot 10^{-3}$ )	$2.11 \cdot 10^{-1}$ ( $6.19 \cdot 10^{-3}$ )
$v = 7/2$	$1.70 \cdot 10^{-1}$ ( $3.29 \cdot 10^{-3}$ )	$1.86 \cdot 10^{-1}$ ( $5.17 \cdot 10^{-3}$ )	$1.94 \cdot 10^{-1}$ ( $6.06 \cdot 10^{-3}$ )	$1.82 \cdot 10^{-1}$ ( $5.73 \cdot 10^{-3}$ )	$3.06 \cdot 10^{-1}$ ( $6.96 \cdot 10^{-3}$ )	$2.47 \cdot 10^{-1}$ ( $8.74 \cdot 10^{-3}$ )
$v = 9/2$	$1.68 \cdot 10^{-1}$ ( $3.43 \cdot 10^{-3}$ )	$1.87 \cdot 10^{-1}$ ( $5.74 \cdot 10^{-3}$ )	$2.02 \cdot 10^{-1}$ ( $7.41 \cdot 10^{-3}$ )	$1.86 \cdot 10^{-1}$ ( $6.56 \cdot 10^{-3}$ )	$3.16 \cdot 10^{-1}$ ( $7.36 \cdot 10^{-3}$ )	$2.82 \cdot 10^{-1}$ ( $1.01 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.84 \cdot 10^{-1}$ ( $4.24 \cdot 10^{-3}$ )	$2.02 \cdot 10^{-1}$ ( $6.90 \cdot 10^{-3}$ )	$2.02 \cdot 10^{-1}$ ( $6.87 \cdot 10^{-3}$ )	$1.92 \cdot 10^{-1}$ ( $6.63 \cdot 10^{-3}$ )	$3.12 \cdot 10^{-1}$ ( $7.38 \cdot 10^{-3}$ )	$2.85 \cdot 10^{-1}$ ( $1.02 \cdot 10^{-2}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.86 \cdot 10^{-1}$ ( $4.14 \cdot 10^{-3}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.28 \cdot 10^0$ ( $3.44 \cdot 10^{-2}$ )	$2.24 \cdot 10^0$ ( $4.30 \cdot 10^{-2}$ )	$2.18 \cdot 10^0$ ( $4.06 \cdot 10^{-2}$ )	$2.23 \cdot 10^0$ ( $4.68 \cdot 10^{-2}$ )	$2.81 \cdot 10^0$ ( $5.36 \cdot 10^{-2}$ )	$2.18 \cdot 10^0$ ( $4.07 \cdot 10^{-2}$ )
$v = 3/2$	$2.12 \cdot 10^0$ ( $5.96 \cdot 10^{-2}$ )	$2.49 \cdot 10^0$ ( $1.07 \cdot 10^{-1}$ )	$2.35 \cdot 10^0$ ( $9.58 \cdot 10^{-2}$ )	$2.45 \cdot 10^0$ ( $1.03 \cdot 10^{-1}$ )	$3.21 \cdot 10^0$ ( $1.49 \cdot 10^{-1}$ )	$2.58 \cdot 10^0$ ( $1.52 \cdot 10^{-1}$ )
$v = 5/2$	$2.08 \cdot 10^0$ ( $7.00 \cdot 10^{-2}$ )	$2.51 \cdot 10^0$ ( $1.23 \cdot 10^{-1}$ )	$2.60 \cdot 10^0$ ( $1.42 \cdot 10^{-1}$ )	$2.72 \cdot 10^0$ ( $1.51 \cdot 10^{-1}$ )	$3.46 \cdot 10^0$ ( $1.01 \cdot 10^{-1}$ )	$3.02 \cdot 10^0$ ( $1.61 \cdot 10^{-1}$ )
$v = 7/2$	$2.07 \cdot 10^0$ ( $7.76 \cdot 10^{-2}$ )	$2.56 \cdot 10^0$ ( $1.41 \cdot 10^{-1}$ )	$2.92 \cdot 10^0$ ( $1.78 \cdot 10^{-1}$ )	$2.89 \cdot 10^0$ ( $1.86 \cdot 10^{-1}$ )	$3.75 \cdot 10^0$ ( $1.12 \cdot 10^{-1}$ )	$3.79 \cdot 10^0$ ( $2.13 \cdot 10^{-1}$ )
$v = 9/2$	$2.07 \cdot 10^0$ ( $8.28 \cdot 10^{-2}$ )	$2.66 \cdot 10^0$ ( $1.58 \cdot 10^{-1}$ )	$3.26 \cdot 10^0$ ( $2.27 \cdot 10^{-1}$ )	$3.13 \cdot 10^0$ ( $2.23 \cdot 10^{-1}$ )	$3.97 \cdot 10^0$ ( $1.15 \cdot 10^{-1}$ )	$4.56 \cdot 10^0$ ( $2.55 \cdot 10^{-1}$ )
$v \in \{1/2, \dots, 9/2\}$	$2.19 \cdot 10^0$ ( $7.89 \cdot 10^{-2}$ )	$2.88 \cdot 10^0$ ( $1.93 \cdot 10^{-1}$ )	$3.22 \cdot 10^0$ ( $2.11 \cdot 10^{-1}$ )	$3.26 \cdot 10^0$ ( $2.26 \cdot 10^{-1}$ )	$3.82 \cdot 10^0$ ( $1.25 \cdot 10^{-1}$ )	$4.63 \cdot 10^0$ ( $2.58 \cdot 10^{-1}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$2.19 \cdot 10^0$ ( $7.61 \cdot 10^{-2}$ )

### SM-2.2.9 Mystery, $n = 100, d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$



$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.00 \cdot 10^{-2}$ ( $1.27 \cdot 10^{-3}$ )	$4.90 \cdot 10^{-2}$ ( $1.23 \cdot 10^{-3}$ )	$4.84 \cdot 10^{-2}$ ( $1.17 \cdot 10^{-3}$ )	$4.88 \cdot 10^{-2}$ ( $1.19 \cdot 10^{-3}$ )	$1.60 \cdot 10^{-1}$ ( $4.00 \cdot 10^{-3}$ )	$4.80 \cdot 10^{-2}$ ( $1.14 \cdot 10^{-3}$ )
$v = 3/2$	$2.23 \cdot 10^{-2}$ ( $1.16 \cdot 10^{-3}$ )	$2.74 \cdot 10^{-2}$ ( $2.13 \cdot 10^{-3}$ )	$2.60 \cdot 10^{-2}$ ( $2.03 \cdot 10^{-3}$ )	$2.64 \cdot 10^{-2}$ ( $2.01 \cdot 10^{-3}$ )	$1.32 \cdot 10^{-1}$ ( $5.75 \cdot 10^{-3}$ )	$2.57 \cdot 10^{-2}$ ( $1.87 \cdot 10^{-3}$ )
$v = 5/2$	$1.59 \cdot 10^{-2}$ ( $1.44 \cdot 10^{-3}$ )	$2.50 \cdot 10^{-2}$ ( $4.85 \cdot 10^{-3}$ )	$2.00 \cdot 10^{-2}$ ( $2.38 \cdot 10^{-3}$ )	$1.98 \cdot 10^{-2}$ ( $2.39 \cdot 10^{-3}$ )	$1.49 \cdot 10^{-1}$ ( $8.49 \cdot 10^{-3}$ )	$2.06 \cdot 10^{-2}$ ( $2.74 \cdot 10^{-3}$ )
$v = 7/2$	$1.23 \cdot 10^{-2}$ ( $1.33 \cdot 10^{-3}$ )	$1.83 \cdot 10^{-2}$ ( $2.42 \cdot 10^{-3}$ )	$1.78 \cdot 10^{-2}$ ( $1.95 \cdot 10^{-3}$ )	$1.74 \cdot 10^{-2}$ ( $1.87 \cdot 10^{-3}$ )	$1.83 \cdot 10^{-1}$ ( $1.27 \cdot 10^{-2}$ )	$1.79 \cdot 10^{-2}$ ( $2.02 \cdot 10^{-3}$ )
$v = 9/2$	$1.00 \cdot 10^{-2}$ ( $1.13 \cdot 10^{-3}$ )	$1.70 \cdot 10^{-2}$ ( $1.55 \cdot 10^{-3}$ )	$1.69 \cdot 10^{-2}$ ( $1.24 \cdot 10^{-3}$ )	$1.64 \cdot 10^{-2}$ ( $1.34 \cdot 10^{-3}$ )	$2.27 \cdot 10^{-1}$ ( $1.76 \cdot 10^{-2}$ )	$1.89 \cdot 10^{-2}$ ( $1.87 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.00 \cdot 10^{-2}$ ( $1.13 \cdot 10^{-3}$ )	$1.90 \cdot 10^{-2}$ ( $2.04 \cdot 10^{-3}$ )	$1.69 \cdot 10^{-2}$ ( $1.24 \cdot 10^{-3}$ )	$1.65 \cdot 10^{-2}$ ( $1.34 \cdot 10^{-3}$ )	$2.20 \cdot 10^{-1}$ ( $1.64 \cdot 10^{-2}$ )	$1.89 \cdot 10^{-2}$ ( $1.87 \cdot 10^{-3}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$4.58 \cdot 10^{-2}$ ( $1.02 \cdot 10^{-3}$ )
$v = 3/2$	$1.67 \cdot 10^{-2}$ ( $8.75 \cdot 10^{-4}$ )
$v = 5/2$	$1.03 \cdot 10^{-2}$ ( $7.78 \cdot 10^{-4}$ )
$v = 7/2$	$6.93 \cdot 10^{-3}$ ( $5.76 \cdot 10^{-4}$ )
$v = 9/2$	$5.77 \cdot 10^{-3}$ ( $5.27 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, 9/2\}$	$5.75 \cdot 10^{-3}$ ( $5.19 \cdot 10^{-4}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.12 \cdot 10^{-2}$ ( $1.24 \cdot 10^{-3}$ )
linear	$9.74 \cdot 10^{-1}$ ( $6.43 \cdot 10^{-4}$ )
1NN	$1.39 \cdot 10^{-1}$ ( $1.58 \cdot 10^{-3}$ )
2NN	$1.16 \cdot 10^{-1}$ ( $1.67 \cdot 10^{-3}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.17 \cdot 10^{-1}$ ( $8.23 \cdot 10^{-4}$ )	$1.08 \cdot 10^{-1}$ ( $9.72 \cdot 10^{-4}$ )	$1.08 \cdot 10^{-1}$ ( $9.31 \cdot 10^{-4}$ )	$1.07 \cdot 10^{-1}$ ( $1.00 \cdot 10^{-3}$ )	$2.05 \cdot 10^{-1}$ ( $2.57 \cdot 10^{-3}$ )	$1.07 \cdot 10^{-1}$ ( $9.07 \cdot 10^{-4}$ )
$v = 3/2$	$5.90 \cdot 10^{-2}$ ( $7.69 \cdot 10^{-4}$ )	$5.82 \cdot 10^{-2}$ ( $1.14 \cdot 10^{-3}$ )	$5.52 \cdot 10^{-2}$ ( $1.04 \cdot 10^{-3}$ )	$5.50 \cdot 10^{-2}$ ( $1.14 \cdot 10^{-3}$ )	$1.64 \cdot 10^{-1}$ ( $3.71 \cdot 10^{-3}$ )	$5.49 \cdot 10^{-2}$ ( $9.88 \cdot 10^{-4}$ )
$v = 5/2$	$4.24 \cdot 10^{-2}$ ( $8.32 \cdot 10^{-4}$ )	$4.46 \cdot 10^{-2}$ ( $1.63 \cdot 10^{-3}$ )	$4.02 \cdot 10^{-2}$ ( $1.15 \cdot 10^{-3}$ )	$4.05 \cdot 10^{-2}$ ( $1.27 \cdot 10^{-3}$ )	$1.61 \cdot 10^{-1}$ ( $4.62 \cdot 10^{-3}$ )	$4.03 \cdot 10^{-2}$ ( $1.22 \cdot 10^{-3}$ )
$v = 7/2$	$3.43 \cdot 10^{-2}$ ( $8.18 \cdot 10^{-4}$ )	$3.70 \cdot 10^{-2}$ ( $1.26 \cdot 10^{-3}$ )	$3.35 \cdot 10^{-2}$ ( $1.02 \cdot 10^{-3}$ )	$3.49 \cdot 10^{-2}$ ( $1.16 \cdot 10^{-3}$ )	$1.67 \cdot 10^{-1}$ ( $5.53 \cdot 10^{-3}$ )	$3.38 \cdot 10^{-2}$ ( $1.10 \cdot 10^{-3}$ )
$v = 9/2$	$2.98 \cdot 10^{-2}$ ( $7.70 \cdot 10^{-4}$ )	$3.33 \cdot 10^{-2}$ ( $1.12 \cdot 10^{-3}$ )	$3.00 \cdot 10^{-2}$ ( $7.60 \cdot 10^{-4}$ )	$3.15 \cdot 10^{-2}$ ( $9.48 \cdot 10^{-4}$ )	$1.76 \cdot 10^{-1}$ ( $6.24 \cdot 10^{-3}$ )	$3.10 \cdot 10^{-2}$ ( $9.11 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, 9/2\}$	$2.98 \cdot 10^{-2}$ ( $7.70 \cdot 10^{-4}$ )	$3.47 \cdot 10^{-2}$ ( $1.41 \cdot 10^{-3}$ )	$3.00 \cdot 10^{-2}$ ( $7.60 \cdot 10^{-4}$ )	$3.16 \cdot 10^{-2}$ ( $9.57 \cdot 10^{-4}$ )	$1.76 \cdot 10^{-1}$ ( $6.13 \cdot 10^{-3}$ )	$3.10 \cdot 10^{-2}$ ( $9.11 \cdot 10^{-4}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$3.20 \cdot 10^{-2}$ ( $1.11 \cdot 10^{-3}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.30 \cdot 10^0$ ( $1.01 \cdot 10^{-2}$ )	$1.18 \cdot 10^0$ ( $2.05 \cdot 10^{-2}$ )	$1.17 \cdot 10^0$ ( $1.95 \cdot 10^{-2}$ )	$1.20 \cdot 10^0$ ( $2.35 \cdot 10^{-2}$ )	$2.10 \cdot 10^0$ ( $3.20 \cdot 10^{-2}$ )	$1.16 \cdot 10^0$ ( $1.88 \cdot 10^{-2}$ )
$v = 3/2$	$7.31 \cdot 10^{-1}$ ( $1.55 \cdot 10^{-2}$ )	$8.45 \cdot 10^{-1}$ ( $3.46 \cdot 10^{-2}$ )	$8.05 \cdot 10^{-1}$ ( $3.25 \cdot 10^{-2}$ )	$9.50 \cdot 10^{-1}$ ( $3.97 \cdot 10^{-2}$ )	$2.03 \cdot 10^0$ ( $6.02 \cdot 10^{-2}$ )	$7.96 \cdot 10^{-1}$ ( $3.06 \cdot 10^{-2}$ )
$v = 5/2$	$5.49 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )	$7.26 \cdot 10^{-1}$ ( $5.13 \cdot 10^{-2}$ )	$6.48 \cdot 10^{-1}$ ( $3.82 \cdot 10^{-2}$ )	$8.08 \cdot 10^{-1}$ ( $4.57 \cdot 10^{-2}$ )	$2.17 \cdot 10^0$ ( $7.82 \cdot 10^{-2}$ )	$6.27 \cdot 10^{-1}$ ( $3.73 \cdot 10^{-2}$ )
$v = 7/2$	$4.50 \cdot 10^{-1}$ ( $2.12 \cdot 10^{-2}$ )	$6.15 \cdot 10^{-1}$ ( $3.77 \cdot 10^{-2}$ )	$5.48 \cdot 10^{-1}$ ( $3.12 \cdot 10^{-2}$ )	$7.13 \cdot 10^{-1}$ ( $4.03 \cdot 10^{-2}$ )	$2.34 \cdot 10^0$ ( $9.22 \cdot 10^{-2}$ )	$5.25 \cdot 10^{-1}$ ( $2.80 \cdot 10^{-2}$ )
$v = 9/2$	$3.93 \cdot 10^{-1}$ ( $1.98 \cdot 10^{-2}$ )	$5.67 \cdot 10^{-1}$ ( $3.15 \cdot 10^{-2}$ )	$4.98 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )	$6.43 \cdot 10^{-1}$ ( $3.30 \cdot 10^{-2}$ )	$2.49 \cdot 10^0$ ( $1.01 \cdot 10^{-1}$ )	$4.94 \cdot 10^{-1}$ ( $1.94 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, 9/2\}$	$3.93 \cdot 10^{-1}$ ( $1.98 \cdot 10^{-2}$ )	$6.03 \cdot 10^{-1}$ ( $3.84 \cdot 10^{-2}$ )	$4.98 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )	$6.47 \cdot 10^{-1}$ ( $3.34 \cdot 10^{-2}$ )	$2.48 \cdot 10^0$ ( $1.01 \cdot 10^{-1}$ )	$4.94 \cdot 10^{-1}$ ( $1.94 \cdot 10^{-2}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$4.22 \cdot 10^{-1}$ ( $2.26 \cdot 10^{-2}$ )

### SM-2.2.10 GKLS $C^0$ , $n = 20$ , $d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.07 \cdot 10^{-1} (2.64 \cdot 10^{-2})$	$3.97 \cdot 10^{-1} (2.63 \cdot 10^{-2})$	$3.99 \cdot 10^{-1} (2.69 \cdot 10^{-2})$	$4.05 \cdot 10^{-1} (2.63 \cdot 10^{-2})$	$4.98 \cdot 10^{-1} (2.25 \cdot 10^{-2})$	$4.47 \cdot 10^{-1} (3.01 \cdot 10^{-2})$
$v = 3/2$	$4.19 \cdot 10^{-1} (2.69 \cdot 10^{-2})$	$4.93 \cdot 10^{-1} (3.06 \cdot 10^{-2})$	$4.91 \cdot 10^{-1} (3.13 \cdot 10^{-2})$	$4.65 \cdot 10^{-1} (2.92 \cdot 10^{-2})$	$4.84 \cdot 10^{-1} (2.44 \cdot 10^{-2})$	$6.24 \cdot 10^{-1} (3.85 \cdot 10^{-2})$
$v = 5/2$	$4.16 \cdot 10^{-1} (2.43 \cdot 10^{-2})$	$5.01 \cdot 10^{-1} (3.15 \cdot 10^{-2})$	$5.53 \cdot 10^{-1} (3.49 \cdot 10^{-2})$	$4.67 \cdot 10^{-1} (2.59 \cdot 10^{-2})$	$4.96 \cdot 10^{-1} (2.59 \cdot 10^{-2})$	$8.27 \cdot 10^{-1} (6.88 \cdot 10^{-2})$
$v = 7/2$	$4.39 \cdot 10^{-1} (2.65 \cdot 10^{-2})$	$5.24 \cdot 10^{-1} (3.46 \cdot 10^{-2})$	$6.60 \cdot 10^{-1} (6.73 \cdot 10^{-2})$	$4.99 \cdot 10^{-1} (3.12 \cdot 10^{-2})$	$5.12 \cdot 10^{-1} (2.93 \cdot 10^{-2})$	$1.12 \cdot 10^0 (1.48 \cdot 10^{-1})$
$v = 9/2$	$4.44 \cdot 10^{-1} (2.66 \cdot 10^{-2})$	$5.67 \cdot 10^{-1} (4.47 \cdot 10^{-2})$	$8.13 \cdot 10^{-1} (1.49 \cdot 10^{-1})$	$5.17 \cdot 10^{-1} (3.49 \cdot 10^{-2})$	$5.39 \cdot 10^{-1} (3.51 \cdot 10^{-2})$	$1.34 \cdot 10^0 (1.61 \cdot 10^{-1})$
$v = \infty$	$4.62 \cdot 10^{-1} (2.51 \cdot 10^{-2})$	$9.35 \cdot 10^{-1} (2.31 \cdot 10^{-1})$	$8.20 \cdot 10^{-1} (1.53 \cdot 10^{-1})$	$7.92 \cdot 10^{-1} (1.85 \cdot 10^{-1})$	$7.90 \cdot 10^{-1} (7.37 \cdot 10^{-2})$	$7.94 \cdot 10^0 (4.73 \cdot 10^0)$
$v \in \{1/2, \dots, \infty\}$	$4.19 \cdot 10^{-1} (2.31 \cdot 10^{-2})$	$7.08 \cdot 10^{-1} (1.48 \cdot 10^{-1})$	$9.64 \cdot 10^{-1} (1.79 \cdot 10^{-1})$	$5.36 \cdot 10^{-1} (4.56 \cdot 10^{-2})$	$6.73 \cdot 10^{-1} (5.70 \cdot 10^{-2})$	$7.53 \cdot 10^0 (4.73 \cdot 10^0)$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.89 \cdot 10^{-1} (1.51 \cdot 10^{-2})$
$v = 3/2$	$2.73 \cdot 10^{-1} (1.36 \cdot 10^{-2})$
$v = 5/2$	$2.85 \cdot 10^{-1} (1.33 \cdot 10^{-2})$
$v = 7/2$	$2.90 \cdot 10^{-1} (1.32 \cdot 10^{-2})$
$v = 9/2$	$2.92 \cdot 10^{-1} (1.30 \cdot 10^{-2})$
$v = \infty$	$3.09 \cdot 10^{-1} (1.24 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$2.57 \cdot 10^{-1} (1.29 \cdot 10^{-2})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.17 \cdot 10^{-1} (2.35 \cdot 10^{-2})$
linear	$7.54 \cdot 10^{-1} (2.10 \cdot 10^{-2})$
1NN	$4.80 \cdot 10^{-1} (2.07 \cdot 10^{-2})$
2NN	$4.35 \cdot 10^{-1} (1.73 \cdot 10^{-2})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.09 \cdot 10^{-1} (9.93 \cdot 10^{-3})$	$3.03 \cdot 10^{-1} (9.82 \cdot 10^{-3})$	$3.05 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$3.04 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$3.55 \cdot 10^{-1} (8.18 \cdot 10^{-3})$	$3.23 \cdot 10^{-1} (1.18 \cdot 10^{-2})$
$v = 3/2$	$3.13 \cdot 10^{-1} (1.08 \cdot 10^{-2})$	$3.43 \cdot 10^{-1} (1.16 \cdot 10^{-2})$	$3.44 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$3.34 \cdot 10^{-1} (1.16 \cdot 10^{-2})$	$3.43 \cdot 10^{-1} (9.33 \cdot 10^{-3})$	$3.97 \cdot 10^{-1} (1.43 \cdot 10^{-2})$
$v = 5/2$	$3.14 \cdot 10^{-1} (9.63 \cdot 10^{-3})$	$3.49 \cdot 10^{-1} (1.15 \cdot 10^{-2})$	$3.69 \cdot 10^{-1} (1.35 \cdot 10^{-2})$	$3.40 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$3.45 \cdot 10^{-1} (9.49 \cdot 10^{-3})$	$4.37 \cdot 10^{-1} (1.53 \cdot 10^{-2})$
$v = 7/2$	$3.26 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$3.54 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$3.87 \cdot 10^{-1} (1.52 \cdot 10^{-2})$	$3.49 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$3.49 \cdot 10^{-1} (9.35 \cdot 10^{-3})$	$4.78 \cdot 10^{-1} (1.89 \cdot 10^{-2})$
$v = 9/2$	$3.28 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$3.65 \cdot 10^{-1} (1.21 \cdot 10^{-2})$	$4.06 \cdot 10^{-1} (1.90 \cdot 10^{-2})$	$3.55 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$3.56 \cdot 10^{-1} (9.81 \cdot 10^{-3})$	$5.14 \cdot 10^{-1} (2.14 \cdot 10^{-2})$
$v = \infty$	$3.40 \cdot 10^{-1} (9.41 \cdot 10^{-3})$	$4.00 \cdot 10^{-1} (2.19 \cdot 10^{-2})$	$4.17 \cdot 10^{-1} (1.92 \cdot 10^{-2})$	$3.90 \cdot 10^{-1} (1.90 \cdot 10^{-2})$	$4.25 \cdot 10^{-1} (1.53 \cdot 10^{-2})$	$7.26 \cdot 10^{-1} (7.53 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$3.17 \cdot 10^{-1} (9.18 \cdot 10^{-3})$	$3.81 \cdot 10^{-1} (1.79 \cdot 10^{-2})$	$4.31 \cdot 10^{-1} (2.19 \cdot 10^{-2})$	$3.59 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$4.05 \cdot 10^{-1} (1.32 \cdot 10^{-2})$	$6.94 \cdot 10^{-1} (7.41 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$3.17 \cdot 10^{-1} (8.97 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.40 \cdot 10^0 (2.47 \cdot 10^{-1})$	$4.48 \cdot 10^0 (2.51 \cdot 10^{-1})$	$4.55 \cdot 10^0 (2.67 \cdot 10^{-1})$	$4.90 \cdot 10^0 (2.65 \cdot 10^{-1})$	$4.97 \cdot 10^0 (2.10 \cdot 10^{-1})$	$4.87 \cdot 10^0 (2.81 \cdot 10^{-1})$
$v = 3/2$	$5.17 \cdot 10^0 (3.60 \cdot 10^{-1})$	$5.82 \cdot 10^0 (3.80 \cdot 10^{-1})$	$6.00 \cdot 10^0 (3.88 \cdot 10^{-1})$	$6.09 \cdot 10^0 (3.92 \cdot 10^{-1})$	$5.12 \cdot 10^0 (3.02 \cdot 10^{-1})$	$7.04 \cdot 10^0 (4.21 \cdot 10^{-1})$
$v = 5/2$	$5.21 \cdot 10^0 (3.15 \cdot 10^{-1})$	$5.88 \cdot 10^0 (3.81 \cdot 10^{-1})$	$6.74 \cdot 10^0 (4.53 \cdot 10^{-1})$	$6.08 \cdot 10^0 (3.29 \cdot 10^{-1})$	$5.04 \cdot 10^0 (2.85 \cdot 10^{-1})$	$7.72 \cdot 10^0 (4.17 \cdot 10^{-1})$
$v = 7/2$	$5.52 \cdot 10^0 (3.30 \cdot 10^{-1})$	$5.90 \cdot 10^0 (3.44 \cdot 10^{-1})$	$7.02 \cdot 10^0 (4.36 \cdot 10^{-1})$	$6.31 \cdot 10^0 (3.33 \cdot 10^{-1})$	$4.95 \cdot 10^0 (2.44 \cdot 10^{-1})$	$8.27 \cdot 10^0 (4.05 \cdot 10^{-1})$
$v = 9/2$	$5.57 \cdot 10^0 (3.22 \cdot 10^{-1})$	$6.08 \cdot 10^0 (3.31 \cdot 10^{-1})$	$7.55 \cdot 10^0 (4.98 \cdot 10^{-1})$	$6.43 \cdot 10^0 (3.24 \cdot 10^{-1})$	$5.02 \cdot 10^0 (2.30 \cdot 10^{-1})$	$8.73 \cdot 10^0 (4.42 \cdot 10^{-1})$
$v = \infty$	$5.77 \cdot 10^0 (2.65 \cdot 10^{-1})$	$6.81 \cdot 10^0 (4.94 \cdot 10^{-1})$	$8.06 \cdot 10^0 (5.64 \cdot 10^{-1})$	$7.46 \cdot 10^0 (5.89 \cdot 10^{-1})$	$5.87 \cdot 10^0 (2.30 \cdot 10^{-1})$	$1.37 \cdot 10^1 (1.99 \cdot 10^0)$
$v \in \{1/2, \dots, \infty\}$	$5.37 \cdot 10^0 (2.80 \cdot 10^{-1})$	$6.64 \cdot 10^0 (4.00 \cdot 10^{-1})$	$8.55 \cdot 10^0 (6.04 \cdot 10^{-1})$	$6.95 \cdot 10^0 (3.42 \cdot 10^{-1})$	$5.76 \cdot 10^0 (2.42 \cdot 10^{-1})$	$1.36 \cdot 10^1 (1.99 \cdot 10^0)$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$5.22 \cdot 10^0 (2.51 \cdot 10^{-1})$

### SM-2.2.11 GKLS $C^0$ , $n = 40$ , $d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.63 \cdot 10^{-1} (9.15 \cdot 10^{-3})$	$1.68 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$1.62 \cdot 10^{-1} (1.03 \cdot 10^{-2})$	$1.66 \cdot 10^{-1} (1.13 \cdot 10^{-2})$	$2.65 \cdot 10^{-1} (1.67 \cdot 10^{-2})$	$1.66 \cdot 10^{-1} (1.20 \cdot 10^{-2})$
$v = 3/2$	$1.54 \cdot 10^{-1} (1.04 \cdot 10^{-2})$	$1.81 \cdot 10^{-1} (1.55 \cdot 10^{-2})$	$1.71 \cdot 10^{-1} (1.30 \cdot 10^{-2})$	$1.68 \cdot 10^{-1} (1.27 \cdot 10^{-2})$	$2.86 \cdot 10^{-1} (4.73 \cdot 10^{-2})$	$1.95 \cdot 10^{-1} (1.94 \cdot 10^{-2})$
$v = 5/2$	$1.55 \cdot 10^{-1} (9.44 \cdot 10^{-3})$	$2.03 \cdot 10^{-1} (1.85 \cdot 10^{-2})$	$2.11 \cdot 10^{-1} (2.23 \cdot 10^{-2})$	$1.93 \cdot 10^{-1} (1.84 \cdot 10^{-2})$	$2.52 \cdot 10^{-1} (1.92 \cdot 10^{-2})$	$3.27 \cdot 10^{-1} (4.55 \cdot 10^{-2})$
$v = 7/2$	$1.61 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$2.04 \cdot 10^{-1} (1.73 \cdot 10^{-2})$	$2.17 \cdot 10^{-1} (2.24 \cdot 10^{-2})$	$2.05 \cdot 10^{-1} (2.14 \cdot 10^{-2})$	$2.79 \cdot 10^{-1} (2.10 \cdot 10^{-2})$	$6.14 \cdot 10^{-1} (8.46 \cdot 10^{-2})$
$v = 9/2$	$1.62 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$2.26 \cdot 10^{-1} (2.27 \cdot 10^{-2})$	$2.20 \cdot 10^{-1} (2.22 \cdot 10^{-2})$	$2.19 \cdot 10^{-1} (2.46 \cdot 10^{-2})$	$3.24 \cdot 10^{-1} (2.89 \cdot 10^{-2})$	$9.10 \cdot 10^{-1} (1.37 \cdot 10^{-1})$
$v \in \{1/2, \dots, 9/2\}$	$1.55 \cdot 10^{-1} (9.57 \cdot 10^{-3})$	$2.31 \cdot 10^{-1} (2.45 \cdot 10^{-2})$	$2.20 \cdot 10^{-1} (2.21 \cdot 10^{-2})$	$1.95 \cdot 10^{-1} (1.82 \cdot 10^{-2})$	$2.82 \cdot 10^{-1} (2.25 \cdot 10^{-2})$	$9.93 \cdot 10^{-1} (1.55 \cdot 10^{-1})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.37 \cdot 10^{-1} (8.01 \cdot 10^{-3})$
$v = 3/2$	$1.06 \cdot 10^{-1} (5.92 \cdot 10^{-3})$
$v = 5/2$	$1.06 \cdot 10^{-1} (5.63 \cdot 10^{-3})$
$v = 7/2$	$1.08 \cdot 10^{-1} (5.66 \cdot 10^{-3})$
$v = 9/2$	$1.11 \cdot 10^{-1} (5.78 \cdot 10^{-3})$
$v \in \{1/2, \dots, 9/2\}$	$9.96 \cdot 10^{-2} (5.48 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.58 \cdot 10^{-1} (1.04 \cdot 10^{-2})$
linear	$7.29 \cdot 10^{-1} (2.00 \cdot 10^{-2})$
INN	$2.67 \cdot 10^{-1} (1.30 \cdot 10^{-2})$
2NN	$2.33 \cdot 10^{-1} (1.26 \cdot 10^{-2})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.98 \cdot 10^{-1} (4.33 \cdot 10^{-3})$	$1.95 \cdot 10^{-1} (4.96 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (4.48 \cdot 10^{-3})$	$1.88 \cdot 10^{-1} (5.05 \cdot 10^{-3})$	$2.53 \cdot 10^{-1} (6.97 \cdot 10^{-3})$	$1.95 \cdot 10^{-1} (5.51 \cdot 10^{-3})$
$v = 3/2$	$1.79 \cdot 10^{-1} (4.53 \cdot 10^{-3})$	$1.89 \cdot 10^{-1} (6.13 \cdot 10^{-3})$	$1.85 \cdot 10^{-1} (5.30 \cdot 10^{-3})$	$1.83 \cdot 10^{-1} (5.28 \cdot 10^{-3})$	$2.34 \cdot 10^{-1} (1.05 \cdot 10^{-2})$	$1.96 \cdot 10^{-1} (8.40 \cdot 10^{-3})$
$v = 5/2$	$1.80 \cdot 10^{-1} (4.35 \cdot 10^{-3})$	$1.97 \cdot 10^{-1} (6.50 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (7.18 \cdot 10^{-3})$	$1.92 \cdot 10^{-1} (6.10 \cdot 10^{-3})$	$2.32 \cdot 10^{-1} (8.63 \cdot 10^{-3})$	$2.25 \cdot 10^{-1} (9.45 \cdot 10^{-3})$
$v = 7/2$	$1.83 \cdot 10^{-1} (4.80 \cdot 10^{-3})$	$2.00 \cdot 10^{-1} (6.75 \cdot 10^{-3})$	$2.01 \cdot 10^{-1} (7.37 \cdot 10^{-3})$	$2.00 \cdot 10^{-1} (7.49 \cdot 10^{-3})$	$2.43 \cdot 10^{-1} (9.37 \cdot 10^{-3})$	$2.65 \cdot 10^{-1} (1.02 \cdot 10^{-2})$
$v = 9/2$	$1.84 \cdot 10^{-1} (4.84 \cdot 10^{-3})$	$2.05 \cdot 10^{-1} (7.00 \cdot 10^{-3})$	$2.03 \cdot 10^{-1} (7.41 \cdot 10^{-3})$	$2.03 \cdot 10^{-1} (6.75 \cdot 10^{-3})$	$2.58 \cdot 10^{-1} (1.14 \cdot 10^{-2})$	$3.03 \cdot 10^{-1} (1.26 \cdot 10^{-2})$
$v \in \{1/2, \dots, 9/2\}$	$1.80 \cdot 10^{-1} (4.35 \cdot 10^{-3})$	$2.04 \cdot 10^{-1} (7.01 \cdot 10^{-3})$	$2.01 \cdot 10^{-1} (7.19 \cdot 10^{-3})$	$1.94 \cdot 10^{-1} (6.41 \cdot 10^{-3})$	$2.43 \cdot 10^{-1} (9.17 \cdot 10^{-3})$	$3.11 \cdot 10^{-1} (1.36 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.82 \cdot 10^{-1} (4.52 \cdot 10^{-3})$

#### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.43 \cdot 10^0 (8.51 \cdot 10^{-2})$	$2.50 \cdot 10^0 (1.09 \cdot 10^{-1})$	$2.45 \cdot 10^0 (1.03 \cdot 10^{-1})$	$2.71 \cdot 10^0 (1.26 \cdot 10^{-1})$	$3.20 \cdot 10^0 (1.29 \cdot 10^{-1})$	$2.49 \cdot 10^0 (1.17 \cdot 10^{-1})$
$v = 3/2$	$2.31 \cdot 10^0 (1.19 \cdot 10^{-1})$	$2.73 \cdot 10^0 (1.71 \cdot 10^{-1})$	$2.62 \cdot 10^0 (1.55 \cdot 10^{-1})$	$2.92 \cdot 10^0 (1.64 \cdot 10^{-1})$	$3.08 \cdot 10^0 (1.75 \cdot 10^{-1})$	$2.74 \cdot 10^0 (1.94 \cdot 10^{-1})$
$v = 5/2$	$2.35 \cdot 10^0 (1.18 \cdot 10^{-1})$	$2.88 \cdot 10^0 (1.81 \cdot 10^{-1})$	$2.91 \cdot 10^0 (2.06 \cdot 10^{-1})$	$3.12 \cdot 10^0 (1.95 \cdot 10^{-1})$	$3.11 \cdot 10^0 (1.64 \cdot 10^{-1})$	$3.24 \cdot 10^0 (2.24 \cdot 10^{-1})$
$v = 7/2$	$2.44 \cdot 10^0 (1.36 \cdot 10^{-1})$	$2.92 \cdot 10^0 (1.83 \cdot 10^{-1})$	$3.08 \cdot 10^0 (2.23 \cdot 10^{-1})$	$3.35 \cdot 10^0 (2.30 \cdot 10^{-1})$	$3.30 \cdot 10^0 (1.71 \cdot 10^{-1})$	$3.76 \cdot 10^0 (2.25 \cdot 10^{-1})$
$v = 9/2$	$2.47 \cdot 10^0 (1.39 \cdot 10^{-1})$	$3.03 \cdot 10^0 (1.85 \cdot 10^{-1})$	$3.14 \cdot 10^0 (2.22 \cdot 10^{-1})$	$3.42 \cdot 10^0 (2.14 \cdot 10^{-1})$	$3.49 \cdot 10^0 (1.87 \cdot 10^{-1})$	$4.31 \cdot 10^0 (2.40 \cdot 10^{-1})$
$v \in \{1/2, \dots, 9/2\}$	$2.40 \cdot 10^0 (1.18 \cdot 10^{-1})$	$3.08 \cdot 10^0 (1.91 \cdot 10^{-1})$	$3.13 \cdot 10^0 (2.16 \cdot 10^{-1})$	$3.35 \cdot 10^0 (2.06 \cdot 10^{-1})$	$3.23 \cdot 10^0 (1.54 \cdot 10^{-1})$	$4.46 \cdot 10^0 (2.59 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$2.43 \cdot 10^0 (1.28 \cdot 10^{-1})$

## SM-2.2.12 GKLS $C^0$ , $n = 100$ , $d = 2$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.64 \cdot 10^{-2} (2.07 \cdot 10^{-3})$	$4.71 \cdot 10^{-2} (2.13 \cdot 10^{-3})$	$4.59 \cdot 10^{-2} (2.04 \cdot 10^{-3})$	$4.63 \cdot 10^{-2} (2.06 \cdot 10^{-3})$	$1.39 \cdot 10^{-1} (1.05 \cdot 10^{-2})$	$4.54 \cdot 10^{-2} (2.02 \cdot 10^{-3})$
$v = 3/2$	$2.90 \cdot 10^{-2} (1.28 \cdot 10^{-3})$	$3.01 \cdot 10^{-2} (1.35 \cdot 10^{-3})$	$2.96 \cdot 10^{-2} (1.28 \cdot 10^{-3})$	$2.96 \cdot 10^{-2} (1.29 \cdot 10^{-3})$	$1.47 \cdot 10^{-1} (1.82 \cdot 10^{-2})$	$3.12 \cdot 10^{-2} (1.47 \cdot 10^{-3})$
$v = 5/2$	$3.04 \cdot 10^{-2} (1.27 \cdot 10^{-3})$	$3.32 \cdot 10^{-2} (1.27 \cdot 10^{-3})$	$3.27 \cdot 10^{-2} (1.44 \cdot 10^{-3})$	$3.26 \cdot 10^{-2} (1.27 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (2.21 \cdot 10^{-2})$	$4.48 \cdot 10^{-2} (2.95 \cdot 10^{-3})$
$v = 7/2$	$3.34 \cdot 10^{-2} (1.36 \cdot 10^{-3})$	$3.69 \cdot 10^{-2} (1.38 \cdot 10^{-3})$	$3.62 \cdot 10^{-2} (1.67 \cdot 10^{-3})$	$3.60 \cdot 10^{-2} (1.39 \cdot 10^{-3})$	$2.22 \cdot 10^{-1} (3.65 \cdot 10^{-2})$	$7.70 \cdot 10^{-2} (7.12 \cdot 10^{-3})$
$v = 9/2$	$3.62 \cdot 10^{-2} (1.47 \cdot 10^{-3})$	$4.00 \cdot 10^{-2} (1.52 \cdot 10^{-3})$	$3.94 \cdot 10^{-2} (1.91 \cdot 10^{-3})$	$3.92 \cdot 10^{-2} (1.58 \cdot 10^{-3})$	$2.13 \cdot 10^{-1} (2.68 \cdot 10^{-2})$	$1.12 \cdot 10^{-1} (1.30 \cdot 10^{-2})$
$v \in \{1/2, \dots, 9/2\}$	$3.00 \cdot 10^{-2} (1.27 \cdot 10^{-3})$	$3.20 \cdot 10^{-2} (1.46 \cdot 10^{-3})$	$3.26 \cdot 10^{-2} (1.54 \cdot 10^{-3})$	$3.13 \cdot 10^{-2} (1.42 \cdot 10^{-3})$	$2.52 \cdot 10^{-1} (3.75 \cdot 10^{-2})$	$1.02 \cdot 10^{-1} (1.06 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$4.39 \cdot 10^{-2} (1.94 \cdot 10^{-3})$
$v = 3/2$	$2.63 \cdot 10^{-2} (1.11 \cdot 10^{-3})$
$v = 5/2$	$2.70 \cdot 10^{-2} (1.04 \cdot 10^{-3})$
$v = 7/2$	$2.92 \cdot 10^{-2} (1.10 \cdot 10^{-3})$
$v = 9/2$	$3.11 \cdot 10^{-2} (1.15 \cdot 10^{-3})$
$v \in \{1/2, \dots, 9/2\}$	$2.53 \cdot 10^{-2} (1.03 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$2.98 \cdot 10^{-2} (1.29 \cdot 10^{-3})$
linear	$7.28 \cdot 10^{-1} (2.00 \cdot 10^{-2})$
INN	$1.25 \cdot 10^{-1} (4.32 \cdot 10^{-3})$
2NN	$1.03 \cdot 10^{-1} (3.95 \cdot 10^{-3})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.12 \cdot 10^{-1} (2.15 \cdot 10^{-3})$	$1.05 \cdot 10^{-1} (2.09 \cdot 10^{-3})$	$1.04 \cdot 10^{-1} (2.05 \cdot 10^{-3})$	$9.94 \cdot 10^{-2} (2.04 \cdot 10^{-3})$	$1.73 \cdot 10^{-1} (5.99 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (2.04 \cdot 10^{-3})$
$v = 3/2$	$8.01 \cdot 10^{-2} (1.63 \cdot 10^{-3})$	$8.03 \cdot 10^{-2} (1.63 \cdot 10^{-3})$	$7.94 \cdot 10^{-2} (1.58 \cdot 10^{-3})$	$7.82 \cdot 10^{-2} (1.62 \cdot 10^{-3})$	$1.58 \cdot 10^{-1} (9.33 \cdot 10^{-3})$	$8.15 \cdot 10^{-2} (1.73 \cdot 10^{-3})$
$v = 5/2$	$8.16 \cdot 10^{-2} (1.61 \cdot 10^{-3})$	$8.57 \cdot 10^{-2} (1.58 \cdot 10^{-3})$	$8.39 \cdot 10^{-2} (1.68 \cdot 10^{-3})$	$8.32 \cdot 10^{-2} (1.58 \cdot 10^{-3})$	$1.69 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$9.45 \cdot 10^{-2} (2.35 \cdot 10^{-3})$
$v = 7/2$	$8.58 \cdot 10^{-2} (1.64 \cdot 10^{-3})$	$9.14 \cdot 10^{-2} (1.65 \cdot 10^{-3})$	$8.85 \cdot 10^{-2} (1.80 \cdot 10^{-3})$	$8.79 \cdot 10^{-2} (1.64 \cdot 10^{-3})$	$1.80 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$1.16 \cdot 10^{-1} (3.59 \cdot 10^{-3})$
$v = 9/2$	$8.95 \cdot 10^{-2} (1.68 \cdot 10^{-3})$	$9.56 \cdot 10^{-2} (1.74 \cdot 10^{-3})$	$9.22 \cdot 10^{-2} (1.89 \cdot 10^{-3})$	$9.17 \cdot 10^{-2} (1.74 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (8.70 \cdot 10^{-3})$	$1.34 \cdot 10^{-1} (5.39 \cdot 10^{-3})$
$v \in \{1/2, \dots, 9/2\}$	$8.10 \cdot 10^{-2} (1.63 \cdot 10^{-3})$	$8.26 \cdot 10^{-2} (1.71 \cdot 10^{-3})$	$8.30 \cdot 10^{-2} (1.81 \cdot 10^{-3})$	$8.03 \cdot 10^{-2} (1.74 \cdot 10^{-3})$	$1.95 \cdot 10^{-1} (1.13 \cdot 10^{-2})$	$1.28 \cdot 10^{-1} (4.53 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$8.08 \cdot 10^{-2} (1.63 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.33 \cdot 10^0 (2.93 \cdot 10^{-2})$	$1.30 \cdot 10^0 (3.48 \cdot 10^{-2})$	$1.28 \cdot 10^0 (3.38 \cdot 10^{-2})$	$1.50 \cdot 10^0 (4.92 \cdot 10^{-2})$	$2.24 \cdot 10^0 (1.04 \cdot 10^{-1})$	$1.27 \cdot 10^0 (3.35 \cdot 10^{-2})$
$v = 3/2$	$9.45 \cdot 10^{-1} (2.19 \cdot 10^{-2})$	$9.95 \cdot 10^{-1} (2.47 \cdot 10^{-2})$	$9.85 \cdot 10^{-1} (2.39 \cdot 10^{-2})$	$1.18 \cdot 10^0 (3.55 \cdot 10^{-2})$	$2.13 \cdot 10^0 (1.49 \cdot 10^{-1})$	$1.00 \cdot 10^0 (2.61 \cdot 10^{-2})$
$v = 5/2$	$9.80 \cdot 10^{-1} (2.16 \cdot 10^{-2})$	$1.04 \cdot 10^0 (2.18 \cdot 10^{-2})$	$1.02 \cdot 10^0 (2.37 \cdot 10^{-2})$	$1.24 \cdot 10^0 (3.36 \cdot 10^{-2})$	$2.22 \cdot 10^0 (1.50 \cdot 10^{-1})$	$1.17 \cdot 10^0 (3.78 \cdot 10^{-2})$
$v = 7/2$	$1.04 \cdot 10^0 (2.26 \cdot 10^{-2})$	$1.11 \cdot 10^0 (2.30 \cdot 10^{-2})$	$1.07 \cdot 10^0 (2.60 \cdot 10^{-2})$	$1.27 \cdot 10^0 (3.34 \cdot 10^{-2})$	$2.29 \cdot 10^0 (1.53 \cdot 10^{-1})$	$1.46 \cdot 10^0 (6.08 \cdot 10^{-2})$
$v = 9/2$	$1.09 \cdot 10^0 (2.39 \cdot 10^{-2})$	$1.16 \cdot 10^0 (2.46 \cdot 10^{-2})$	$1.11 \cdot 10^0 (2.84 \cdot 10^{-2})$	$1.30 \cdot 10^0 (3.60 \cdot 10^{-2})$	$2.22 \cdot 10^0 (1.25 \cdot 10^{-1})$	$1.69 \cdot 10^0 (1.03 \cdot 10^{-1})$
$v \in \{1/2, \dots, 9/2\}$	$9.83 \cdot 10^{-1} (2.23 \cdot 10^{-2})$	$1.03 \cdot 10^0 (2.54 \cdot 10^{-2})$	$1.04 \cdot 10^0 (2.63 \cdot 10^{-2})$	$1.25 \cdot 10^0 (3.75 \cdot 10^{-2})$	$2.57 \cdot 10^0 (1.71 \cdot 10^{-1})$	$1.59 \cdot 10^0 (6.79 \cdot 10^{-2})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$9.76 \cdot 10^{-1} (2.20 \cdot 10^{-2})$

### SM-2.2.13 GKLS $C^0$ , $n = 50$ , $d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.75 \cdot 10^{-1} (8.72 \cdot 10^{-3})$	$1.56 \cdot 10^{-1} (8.53 \cdot 10^{-3})$	$1.57 \cdot 10^{-1} (8.51 \cdot 10^{-3})$	$1.49 \cdot 10^{-1} (8.03 \cdot 10^{-3})$	$3.97 \cdot 10^{-1} (1.15 \cdot 10^{-2})$	$1.64 \cdot 10^{-1} (1.00 \cdot 10^{-2})$
$v = 3/2$	$1.50 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$1.72 \cdot 10^{-1} (1.35 \cdot 10^{-2})$	$1.69 \cdot 10^{-1} (1.45 \cdot 10^{-2})$	$1.50 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$3.54 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$2.07 \cdot 10^{-1} (2.27 \cdot 10^{-2})$
$v = 5/2$	$1.68 \cdot 10^{-1} (1.30 \cdot 10^{-2})$	$2.21 \cdot 10^{-1} (1.86 \cdot 10^{-2})$	$2.34 \cdot 10^{-1} (2.40 \cdot 10^{-2})$	$2.08 \cdot 10^{-1} (1.78 \cdot 10^{-2})$	$3.52 \cdot 10^{-1} (1.26 \cdot 10^{-2})$	$2.50 \cdot 10^{-1} (2.43 \cdot 10^{-2})$
$v = 7/2$	$1.85 \cdot 10^{-1} (1.42 \cdot 10^{-2})$	$2.60 \cdot 10^{-1} (2.12 \cdot 10^{-2})$	$2.64 \cdot 10^{-1} (2.25 \cdot 10^{-2})$	$2.42 \cdot 10^{-1} (2.03 \cdot 10^{-2})$	$3.52 \cdot 10^{-1} (1.29 \cdot 10^{-2})$	$3.01 \cdot 10^{-1} (2.95 \cdot 10^{-2})$
$v = 9/2$	$1.92 \cdot 10^{-1} (1.45 \cdot 10^{-2})$	$2.78 \cdot 10^{-1} (2.23 \cdot 10^{-2})$	$2.87 \cdot 10^{-1} (2.25 \cdot 10^{-2})$	$2.57 \cdot 10^{-1} (1.97 \cdot 10^{-2})$	$3.49 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$3.32 \cdot 10^{-1} (3.30 \cdot 10^{-2})$
$v = 11/2$	$1.96 \cdot 10^{-1} (1.47 \cdot 10^{-2})$	$3.06 \cdot 10^{-1} (2.81 \cdot 10^{-2})$	$3.06 \cdot 10^{-1} (2.99 \cdot 10^{-2})$	$2.78 \cdot 10^{-1} (2.21 \cdot 10^{-2})$	$3.50 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$3.51 \cdot 10^{-1} (3.15 \cdot 10^{-2})$
$v = 21/2$	$2.10 \cdot 10^{-1} (1.53 \cdot 10^{-2})$	$3.20 \cdot 10^{-1} (2.90 \cdot 10^{-2})$	$3.31 \cdot 10^{-1} (2.78 \cdot 10^{-2})$	$3.11 \cdot 10^{-1} (2.75 \cdot 10^{-2})$	$3.54 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$4.11 \cdot 10^{-1} (4.38 \cdot 10^{-2})$
$v = \infty$	$2.24 \cdot 10^{-1} (1.59 \cdot 10^{-2})$	$3.31 \cdot 10^{-1} (2.48 \cdot 10^{-2})$	$3.45 \cdot 10^{-1} (3.00 \cdot 10^{-2})$	$3.41 \cdot 10^{-1} (3.00 \cdot 10^{-2})$	$3.63 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$4.16 \cdot 10^{-1} (3.75 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$1.61 \cdot 10^{-1} (1.45 \cdot 10^{-2})$	$2.83 \cdot 10^{-1} (2.46 \cdot 10^{-2})$	$3.31 \cdot 10^{-1} (3.26 \cdot 10^{-2})$	$2.82 \cdot 10^{-1} (2.98 \cdot 10^{-2})$	$3.64 \cdot 10^{-1} (1.32 \cdot 10^{-2})$	$4.20 \cdot 10^{-1} (4.09 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.33 \cdot 10^{-1} (6.84 \cdot 10^{-3})$
$v = 3/2$	$1.02 \cdot 10^{-1} (6.87 \cdot 10^{-3})$
$v = 5/2$	$1.13 \cdot 10^{-1} (7.20 \cdot 10^{-3})$
$v = 7/2$	$1.21 \cdot 10^{-1} (7.35 \cdot 10^{-3})$
$v = 9/2$	$1.26 \cdot 10^{-1} (7.50 \cdot 10^{-3})$
$v = 11/2$	$1.29 \cdot 10^{-1} (7.61 \cdot 10^{-3})$
$v = 21/2$	$1.36 \cdot 10^{-1} (7.92 \cdot 10^{-3})$
$v = \infty$	$1.43 \cdot 10^{-1} (8.30 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$9.82 \cdot 10^{-2} (6.29 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.75 \cdot 10^{-1} (1.48 \cdot 10^{-2})$
linear	$4.43 \cdot 10^{-1} (2.09 \cdot 10^{-2})$
1NN	$5.15 \cdot 10^{-1} (9.18 \cdot 10^{-3})$
2NN	$4.09 \cdot 10^{-1} (8.06 \cdot 10^{-3})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.12 \cdot 10^{-1} (4.85 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (5.21 \cdot 10^{-3})$	$1.94 \cdot 10^{-1} (5.19 \cdot 10^{-3})$	$1.87 \cdot 10^{-1} (4.85 \cdot 10^{-3})$	$3.43 \cdot 10^{-1} (5.02 \cdot 10^{-3})$	$1.97 \cdot 10^{-1} (5.79 \cdot 10^{-3})$
$v = 3/2$	$1.71 \cdot 10^{-1} (8.09 \cdot 10^{-3})$	$1.88 \cdot 10^{-1} (9.41 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (9.25 \cdot 10^{-3})$	$1.71 \cdot 10^{-1} (8.45 \cdot 10^{-3})$	$3.18 \cdot 10^{-1} (5.45 \cdot 10^{-3})$	$2.01 \cdot 10^{-1} (1.17 \cdot 10^{-2})$
$v = 5/2$	$1.82 \cdot 10^{-1} (8.93 \cdot 10^{-3})$	$2.14 \cdot 10^{-1} (1.11 \cdot 10^{-2})$	$2.16 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$2.08 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$3.16 \cdot 10^{-1} (5.77 \cdot 10^{-3})$	$2.23 \cdot 10^{-1} (1.25 \cdot 10^{-2})$
$v = 7/2$	$1.95 \cdot 10^{-1} (9.39 \cdot 10^{-3})$	$2.41 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$2.39 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$2.31 \cdot 10^{-1} (1.18 \cdot 10^{-2})$	$3.15 \cdot 10^{-1} (5.91 \cdot 10^{-3})$	$2.51 \cdot 10^{-1} (1.32 \cdot 10^{-2})$
$v = 9/2$	$2.01 \cdot 10^{-1} (9.47 \cdot 10^{-3})$	$2.51 \cdot 10^{-1} (1.23 \cdot 10^{-2})$	$2.57 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$2.45 \cdot 10^{-1} (1.15 \cdot 10^{-2})$	$3.13 \cdot 10^{-1} (5.60 \cdot 10^{-3})$	$2.65 \cdot 10^{-1} (1.40 \cdot 10^{-2})$
$v = 11/2$	$2.05 \cdot 10^{-1} (9.52 \cdot 10^{-3})$	$2.64 \cdot 10^{-1} (1.35 \cdot 10^{-2})$	$2.63 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$2.55 \cdot 10^{-1} (1.21 \cdot 10^{-2})$	$3.13 \cdot 10^{-1} (5.60 \cdot 10^{-3})$	$2.79 \cdot 10^{-1} (1.43 \cdot 10^{-2})$
$v = 21/2$	$2.15 \cdot 10^{-1} (9.84 \cdot 10^{-3})$	$2.71 \cdot 10^{-1} (1.37 \cdot 10^{-2})$	$2.79 \cdot 10^{-1} (1.36 \cdot 10^{-2})$	$2.71 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$3.14 \cdot 10^{-1} (5.72 \cdot 10^{-3})$	$2.98 \cdot 10^{-1} (1.68 \cdot 10^{-2})$
$v = \infty$	$2.25 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$2.82 \cdot 10^{-1} (1.33 \cdot 10^{-2})$	$2.88 \cdot 10^{-1} (1.45 \cdot 10^{-2})$	$2.87 \cdot 10^{-1} (1.44 \cdot 10^{-2})$	$3.18 \cdot 10^{-1} (6.03 \cdot 10^{-3})$	$3.08 \cdot 10^{-1} (1.63 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$1.75 \cdot 10^{-1} (9.47 \cdot 10^{-3})$	$2.50 \cdot 10^{-1} (1.35 \cdot 10^{-2})$	$2.69 \cdot 10^{-1} (1.51 \cdot 10^{-2})$	$2.46 \cdot 10^{-1} (1.43 \cdot 10^{-2})$	$3.18 \cdot 10^{-1} (6.06 \cdot 10^{-3})$	$3.06 \cdot 10^{-1} (1.69 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.86 \cdot 10^{-1} (9.88 \cdot 10^{-3})$

#### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.35 \cdot 10^0 (6.40 \cdot 10^{-2})$	$2.36 \cdot 10^0 (7.19 \cdot 10^{-2})$	$2.36 \cdot 10^0 (7.22 \cdot 10^{-2})$	$2.50 \cdot 10^0 (8.11 \cdot 10^{-2})$	$3.34 \cdot 10^0 (6.59 \cdot 10^{-2})$	$2.40 \cdot 10^0 (7.61 \cdot 10^{-2})$
$v = 3/2$	$2.35 \cdot 10^0 (1.02 \cdot 10^{-1})$	$2.72 \cdot 10^0 (1.33 \cdot 10^{-1})$	$2.69 \cdot 10^0 (1.29 \cdot 10^{-1})$	$2.88 \cdot 10^0 (1.41 \cdot 10^{-1})$	$3.23 \cdot 10^0 (7.29 \cdot 10^{-2})$	$2.93 \cdot 10^0 (1.57 \cdot 10^{-1})$
$v = 5/2$	$2.57 \cdot 10^0 (1.15 \cdot 10^{-1})$	$3.34 \cdot 10^0 (1.71 \cdot 10^{-1})$	$3.55 \cdot 10^0 (1.99 \cdot 10^{-1})$	$3.72 \cdot 10^0 (1.89 \cdot 10^{-1})$	$3.26 \cdot 10^0 (7.82 \cdot 10^{-2})$	$3.63 \cdot 10^0 (1.90 \cdot 10^{-1})$
$v = 7/2$	$2.76 \cdot 10^0 (1.25 \cdot 10^{-1})$	$3.97 \cdot 10^0 (2.05 \cdot 10^{-1})$	$4.11 \cdot 10^0 (2.13 \cdot 10^{-1})$	$4.28 \cdot 10^0 (2.30 \cdot 10^{-1})$	$3.26 \cdot 10^0 (8.00 \cdot 10^{-2})$	$4.32 \cdot 10^0 (2.35 \cdot 10^{-1})$
$v = 9/2$	$2.85 \cdot 10^0 (1.28 \cdot 10^{-1})$	$4.27 \cdot 10^0 (2.21 \cdot 10^{-1})$	$4.58 \cdot 10^0 (2.32 \cdot 10^{-1})$	$4.62 \cdot 10^0 (2.27 \cdot 10^{-1})$	$3.25 \cdot 10^0 (7.31 \cdot 10^{-2})$	$4.63 \cdot 10^0 (2.57 \cdot 10^{-1})$
$v = 11/2$	$2.91 \cdot 10^0 (1.30 \cdot 10^{-1})$	$4.60 \cdot 10^0 (2.54 \cdot 10^{-1})$	$4.76 \cdot 10^0 (2.60 \cdot 10^{-1})$	$4.92 \cdot 10^0 (2.46 \cdot 10^{-1})$	$3.25 \cdot 10^0 (7.15 \cdot 10^{-2})$	$4.99 \cdot 10^0 (2.61 \cdot 10^{-1})$
$v = 21/2$	$3.09 \cdot 10^0 (1.38 \cdot 10^{-1})$	$4.78 \cdot 10^0 (2.68 \cdot 10^{-1})$	$5.25 \cdot 10^0 (2.62 \cdot 10^{-1})$	$5.43 \cdot 10^0 (2.77 \cdot 10^{-1})$	$3.28 \cdot 10^0 (7.06 \cdot 10^{-2})$	$5.53 \cdot 10^0 (3.27 \cdot 10^{-1})$
$v = \infty$	$3.26 \cdot 10^0 (1.44 \cdot 10^{-1})$	$5.00 \cdot 10^0 (2.48 \cdot 10^{-1})$	$5.57 \cdot 10^0 (3.02 \cdot 10^{-1})$	$5.92 \cdot 10^0 (3.29 \cdot 10^{-1})$	$3.34 \cdot 10^0 (7.23 \cdot 10^{-2})$	$5.92 \cdot 10^0 (3.35 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$2.53 \cdot 10^0 (1.30 \cdot 10^{-1})$	$4.59 \cdot 10^0 (2.69 \cdot 10^{-1})$	$5.24 \cdot 10^0 (3.08 \cdot 10^{-1})$	$5.10 \cdot 10^0 (3.19 \cdot 10^{-1})$	$3.34 \cdot 10^0 (7.22 \cdot 10^{-2})$	$5.91 \cdot 10^0 (3.40 \cdot 10^{-1})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$2.71 \cdot 10^0 (1.34 \cdot 10^{-1})$

### SM-2.2.14 GKLS $C^0$ , $n = 100$ , $d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$8.88 \cdot 10^{-2} (4.65 \cdot 10^{-3})$	$8.85 \cdot 10^{-2} (4.63 \cdot 10^{-3})$	$8.82 \cdot 10^{-2} (4.60 \cdot 10^{-3})$	$8.53 \cdot 10^{-2} (4.48 \cdot 10^{-3})$	$2.84 \cdot 10^{-1} (9.55 \cdot 10^{-3})$	$8.94 \cdot 10^{-2} (4.57 \cdot 10^{-3})$
$v = 3/2$	$8.48 \cdot 10^{-2} (4.48 \cdot 10^{-3})$	$9.37 \cdot 10^{-2} (5.06 \cdot 10^{-3})$	$9.61 \cdot 10^{-2} (5.28 \cdot 10^{-3})$	$8.86 \cdot 10^{-2} (4.69 \cdot 10^{-3})$	$2.45 \cdot 10^{-1} (1.03 \cdot 10^{-2})$	$1.10 \cdot 10^{-1} (7.90 \cdot 10^{-3})$
$v = 5/2$	$1.02 \cdot 10^{-1} (5.08 \cdot 10^{-3})$	$1.26 \cdot 10^{-1} (7.15 \cdot 10^{-3})$	$1.32 \cdot 10^{-1} (8.00 \cdot 10^{-3})$	$1.24 \cdot 10^{-1} (7.67 \cdot 10^{-3})$	$2.47 \cdot 10^{-1} (1.11 \cdot 10^{-2})$	$1.65 \cdot 10^{-1} (1.11 \cdot 10^{-2})$
$v = 7/2$	$1.12 \cdot 10^{-1} (5.41 \cdot 10^{-3})$	$1.49 \cdot 10^{-1} (8.45 \cdot 10^{-3})$	$1.52 \cdot 10^{-1} (8.92 \cdot 10^{-3})$	$1.38 \cdot 10^{-1} (7.40 \cdot 10^{-3})$	$2.46 \cdot 10^{-1} (1.14 \cdot 10^{-2})$	$2.00 \cdot 10^{-1} (1.27 \cdot 10^{-2})$
$v = 9/2$	$1.19 \cdot 10^{-1} (5.63 \cdot 10^{-3})$	$1.60 \cdot 10^{-1} (9.01 \cdot 10^{-3})$	$1.69 \cdot 10^{-1} (9.79 \cdot 10^{-3})$	$1.48 \cdot 10^{-1} (7.83 \cdot 10^{-3})$	$2.45 \cdot 10^{-1} (1.16 \cdot 10^{-2})$	$2.35 \cdot 10^{-1} (1.95 \cdot 10^{-2})$
$v = 11/2$	$1.25 \cdot 10^{-1} (5.82 \cdot 10^{-3})$	$1.72 \cdot 10^{-1} (9.72 \cdot 10^{-3})$	$1.80 \cdot 10^{-1} (1.13 \cdot 10^{-2})$	$1.55 \cdot 10^{-1} (8.06 \cdot 10^{-3})$	$2.47 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$2.37 \cdot 10^{-1} (1.76 \cdot 10^{-2})$
$v = 21/2$	$1.36 \cdot 10^{-1} (6.11 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$2.04 \cdot 10^{-1} (1.37 \cdot 10^{-2})$	$1.77 \cdot 10^{-1} (9.45 \cdot 10^{-3})$	$2.56 \cdot 10^{-1} (1.33 \cdot 10^{-2})$	$2.64 \cdot 10^{-1} (1.92 \cdot 10^{-2})$
$v = \infty$	$1.50 \cdot 10^{-1} (6.50 \cdot 10^{-3})$	$2.17 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$2.24 \cdot 10^{-1} (1.47 \cdot 10^{-2})$	$1.92 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$2.68 \cdot 10^{-1} (1.46 \cdot 10^{-2})$	$2.95 \cdot 10^{-1} (2.21 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$8.59 \cdot 10^{-2} (4.83 \cdot 10^{-3})$	$1.25 \cdot 10^{-1} (1.04 \cdot 10^{-2})$	$1.66 \cdot 10^{-1} (1.35 \cdot 10^{-2})$	$9.67 \cdot 10^{-2} (5.86 \cdot 10^{-3})$	$2.68 \cdot 10^{-1} (1.47 \cdot 10^{-2})$	$2.53 \cdot 10^{-1} (2.04 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$8.18 \cdot 10^{-2} (4.23 \cdot 10^{-3})$
$v = 3/2$	$6.81 \cdot 10^{-2} (3.57 \cdot 10^{-3})$
$v = 5/2$	$7.99 \cdot 10^{-2} (3.96 \cdot 10^{-3})$
$v = 7/2$	$8.75 \cdot 10^{-2} (4.20 \cdot 10^{-3})$
$v = 9/2$	$9.24 \cdot 10^{-2} (4.36 \cdot 10^{-3})$
$v = 11/2$	$9.56 \cdot 10^{-2} (4.46 \cdot 10^{-3})$
$v = 21/2$	$1.03 \cdot 10^{-1} (4.72 \cdot 10^{-3})$
$v = \infty$	$1.12 \cdot 10^{-1} (5.04 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$6.71 \cdot 10^{-2} (3.50 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$8.62 \cdot 10^{-2} (4.72 \cdot 10^{-3})$
linear	$4.22 \cdot 10^{-1} (1.92 \cdot 10^{-2})$
INN	$4.07 \cdot 10^{-1} (7.43 \cdot 10^{-3})$
2NN	$3.05 \cdot 10^{-1} (5.96 \cdot 10^{-3})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{CRPS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.47 \cdot 10^{-1} (3.42 \cdot 10^{-3})$	$1.40 \cdot 10^{-1} (3.60 \cdot 10^{-3})$	$1.40 \cdot 10^{-1} (3.57 \cdot 10^{-3})$	$1.31 \cdot 10^{-1} (3.42 \cdot 10^{-3})$	$2.86 \cdot 10^{-1} (4.58 \cdot 10^{-3})$	$1.41 \cdot 10^{-1} (3.54 \cdot 10^{-3})$
$v = 3/2$	$1.31 \cdot 10^{-1} (4.05 \cdot 10^{-3})$	$1.38 \cdot 10^{-1} (4.27 \cdot 10^{-3})$	$1.39 \cdot 10^{-1} (4.39 \cdot 10^{-3})$	$1.30 \cdot 10^{-1} (4.11 \cdot 10^{-3})$	$2.61 \cdot 10^{-1} (5.17 \cdot 10^{-3})$	$1.48 \cdot 10^{-1} (5.47 \cdot 10^{-3})$
$v = 5/2$	$1.46 \cdot 10^{-1} (4.34 \cdot 10^{-3})$	$1.63 \cdot 10^{-1} (5.16 \cdot 10^{-3})$	$1.64 \cdot 10^{-1} (5.52 \cdot 10^{-3})$	$1.61 \cdot 10^{-1} (5.43 \cdot 10^{-3})$	$2.61 \cdot 10^{-1} (5.81 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (6.75 \cdot 10^{-3})$
$v = 7/2$	$1.55 \cdot 10^{-1} (4.43 \cdot 10^{-3})$	$1.80 \cdot 10^{-1} (5.69 \cdot 10^{-3})$	$1.79 \cdot 10^{-1} (5.85 \cdot 10^{-3})$	$1.73 \cdot 10^{-1} (5.40 \cdot 10^{-3})$	$2.59 \cdot 10^{-1} (5.80 \cdot 10^{-3})$	$2.04 \cdot 10^{-1} (7.44 \cdot 10^{-3})$
$v = 9/2$	$1.61 \cdot 10^{-1} (4.48 \cdot 10^{-3})$	$1.89 \cdot 10^{-1} (5.88 \cdot 10^{-3})$	$1.91 \cdot 10^{-1} (6.30 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (5.48 \cdot 10^{-3})$	$2.57 \cdot 10^{-1} (5.82 \cdot 10^{-3})$	$2.18 \cdot 10^{-1} (8.49 \cdot 10^{-3})$
$v = 11/2$	$1.66 \cdot 10^{-1} (4.54 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (6.12 \cdot 10^{-3})$	$1.98 \cdot 10^{-1} (6.82 \cdot 10^{-3})$	$1.87 \cdot 10^{-1} (5.56 \cdot 10^{-3})$	$2.57 \cdot 10^{-1} (5.91 \cdot 10^{-3})$	$2.22 \cdot 10^{-1} (8.35 \cdot 10^{-3})$
$v = 21/2$	$1.76 \cdot 10^{-1} (4.58 \cdot 10^{-3})$	$2.10 \cdot 10^{-1} (6.62 \cdot 10^{-3})$	$2.11 \cdot 10^{-1} (7.43 \cdot 10^{-3})$	$2.03 \cdot 10^{-1} (6.26 \cdot 10^{-3})$	$2.60 \cdot 10^{-1} (6.32 \cdot 10^{-3})$	$2.36 \cdot 10^{-1} (8.69 \cdot 10^{-3})$
$v = \infty$	$1.87 \cdot 10^{-1} (4.70 \cdot 10^{-3})$	$2.24 \cdot 10^{-1} (7.11 \cdot 10^{-3})$	$2.24 \cdot 10^{-1} (7.59 \cdot 10^{-3})$	$2.14 \cdot 10^{-1} (6.57 \cdot 10^{-3})$	$2.63 \cdot 10^{-1} (6.67 \cdot 10^{-3})$	$2.50 \cdot 10^{-1} (9.41 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$1.32 \cdot 10^{-1} (4.21 \cdot 10^{-3})$	$1.59 \cdot 10^{-1} (6.64 \cdot 10^{-3})$	$1.81 \cdot 10^{-1} (8.02 \cdot 10^{-3})$	$1.36 \cdot 10^{-1} (4.82 \cdot 10^{-3})$	$2.63 \cdot 10^{-1} (6.68 \cdot 10^{-3})$	$2.26 \cdot 10^{-1} (9.20 \cdot 10^{-3})$

$R(\theta; S^{CRPS})$	
NLL/SPE	$1.34 \cdot 10^{-1} (4.26 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{IS})$

$R(\theta; S_{0.95}^{IS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.88 \cdot 10^0 (4.48 \cdot 10^{-2})$	$1.86 \cdot 10^0 (5.03 \cdot 10^{-2})$	$1.86 \cdot 10^0 (5.03 \cdot 10^{-2})$	$2.05 \cdot 10^0 (6.09 \cdot 10^{-2})$	$2.81 \cdot 10^0 (5.25 \cdot 10^{-2})$	$1.86 \cdot 10^0 (5.02 \cdot 10^{-2})$
$v = 3/2$	$1.88 \cdot 10^0 (5.12 \cdot 10^{-2})$	$2.02 \cdot 10^0 (5.91 \cdot 10^{-2})$	$2.06 \cdot 10^0 (5.82 \cdot 10^{-2})$	$2.33 \cdot 10^0 (6.92 \cdot 10^{-2})$	$2.70 \cdot 10^0 (6.09 \cdot 10^{-2})$	$2.16 \cdot 10^0 (7.03 \cdot 10^{-2})$
$v = 5/2$	$2.04 \cdot 10^0 (5.36 \cdot 10^{-2})$	$2.35 \cdot 10^0 (6.75 \cdot 10^{-2})$	$2.47 \cdot 10^0 (7.50 \cdot 10^{-2})$	$2.81 \cdot 10^0 (9.72 \cdot 10^{-2})$	$2.75 \cdot 10^0 (6.63 \cdot 10^{-2})$	$2.73 \cdot 10^0 (9.18 \cdot 10^{-2})$
$v = 7/2$	$2.14 \cdot 10^0 (5.45 \cdot 10^{-2})$	$2.53 \cdot 10^0 (7.35 \cdot 10^{-2})$	$2.69 \cdot 10^0 (8.31 \cdot 10^{-2})$	$2.90 \cdot 10^0 (9.26 \cdot 10^{-2})$	$2.73 \cdot 10^0 (6.50 \cdot 10^{-2})$	$2.98 \cdot 10^0 (1.02 \cdot 10^{-1})$
$v = 9/2$	$2.19 \cdot 10^0 (5.60 \cdot 10^{-2})$	$2.64 \cdot 10^0 (7.95 \cdot 10^{-2})$	$2.84 \cdot 10^0 (9.33 \cdot 10^{-2})$	$2.99 \cdot 10^0 (9.15 \cdot 10^{-2})$	$2.72 \cdot 10^0 (6.51 \cdot 10^{-2})$	$3.19 \cdot 10^0 (1.28 \cdot 10^{-1})$
$v = 11/2$	$2.25 \cdot 10^0 (5.74 \cdot 10^{-2})$	$2.75 \cdot 10^0 (8.61 \cdot 10^{-2})$	$2.94 \cdot 10^0 (9.94 \cdot 10^{-2})$	$3.05 \cdot 10^0 (9.42 \cdot 10^{-2})$	$2.73 \cdot 10^0 (6.59 \cdot 10^{-2})$	$3.20 \cdot 10^0 (1.16 \cdot 10^{-1})$
$v = 21/2$	$2.37 \cdot 10^0 (5.79 \cdot 10^{-2})$	$2.94 \cdot 10^0 (9.15 \cdot 10^{-2})$	$3.14 \cdot 10^0 (1.13 \cdot 10^{-1})$	$3.27 \cdot 10^0 (1.03 \cdot 10^{-1})$	$2.77 \cdot 10^0 (7.00 \cdot 10^{-2})$	$3.39 \cdot 10^0 (1.20 \cdot 10^{-1})$
$v = \infty$	$2.52 \cdot 10^0 (5.92 \cdot 10^{-2})$	$3.08 \cdot 10^0 (8.79 \cdot 10^{-2})$	$3.29 \cdot 10^0 (1.17 \cdot 10^{-1})$	$3.42 \cdot 10^0 (1.10 \cdot 10^{-1})$	$2.83 \cdot 10^0 (7.38 \cdot 10^{-2})$	$3.52 \cdot 10^0 (1.29 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$1.90 \cdot 10^0 (5.27 \cdot 10^{-2})$	$2.36 \cdot 10^0 (9.02 \cdot 10^{-2})$	$2.87 \cdot 10^0 (1.25 \cdot 10^{-1})$	$2.49 \cdot 10^0 (8.68 \cdot 10^{-2})$	$2.84 \cdot 10^0 (7.42 \cdot 10^{-2})$	$3.39 \cdot 10^0 (1.31 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{IS})$	
NLL/SPE	$1.93 \cdot 10^0 (5.49 \cdot 10^{-2})$

### SM-2.2.15 GKLS $C^0$ , $n = 250$ , $d = 5$

### Averages of $R(\theta; S^{SPE})$

$R(\theta; S^{SPE})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.94 \cdot 10^{-2} (2.85 \cdot 10^{-3})$	$5.02 \cdot 10^{-2} (2.99 \cdot 10^{-3})$	$5.01 \cdot 10^{-2} (2.92 \cdot 10^{-3})$	$4.91 \cdot 10^{-2} (2.85 \cdot 10^{-3})$	$1.85 \cdot 10^{-1} (7.09 \cdot 10^{-3})$	$5.03 \cdot 10^{-2} (2.85 \cdot 10^{-3})$
$v = 3/2$	$4.43 \cdot 10^{-2} (2.32 \cdot 10^{-3})$	$4.74 \cdot 10^{-2} (2.45 \cdot 10^{-3})$	$4.88 \cdot 10^{-2} (2.51 \cdot 10^{-3})$	$4.69 \cdot 10^{-2} (2.37 \cdot 10^{-3})$	$1.56 \cdot 10^{-1} (8.14 \cdot 10^{-3})$	$5.21 \cdot 10^{-2} (2.74 \cdot 10^{-3})$
$v = 5/2$	$5.23 \cdot 10^{-2} (2.63 \cdot 10^{-3})$	$6.06 \cdot 10^{-2} (3.05 \cdot 10^{-3})$	$6.09 \cdot 10^{-2} (3.06 \cdot 10^{-3})$	$5.88 \cdot 10^{-2} (2.83 \cdot 10^{-3})$	$1.50 \cdot 10^{-1} (8.25 \cdot 10^{-3})$	$7.48 \cdot 10^{-2} (5.54 \cdot 10^{-3})$
$v = 7/2$	$5.85 \cdot 10^{-2} (2.85 \cdot 10^{-3})$	$6.97 \cdot 10^{-2} (3.34 \cdot 10^{-3})$	$6.99 \cdot 10^{-2} (3.52 \cdot 10^{-3})$	$6.72 \cdot 10^{-2} (3.21 \cdot 10^{-3})$	$1.45 \cdot 10^{-1} (7.64 \cdot 10^{-3})$	$8.98 \cdot 10^{-2} (5.93 \cdot 10^{-3})$
$v = 9/2$	$6.29 \cdot 10^{-2} (3.01 \cdot 10^{-3})$	$7.51 \cdot 10^{-2} (3.58 \cdot 10^{-3})$	$7.44 \cdot 10^{-2} (3.56 \cdot 10^{-3})$	$7.26 \cdot 10^{-2} (3.43 \cdot 10^{-3})$	$1.43 \cdot 10^{-1} (7.59 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (6.23 \cdot 10^{-3})$
$v = 11/2$	$6.63 \cdot 10^{-2} (3.10 \cdot 10^{-3})$	$7.98 \cdot 10^{-2} (3.72 \cdot 10^{-3})$	$7.81 \cdot 10^{-2} (3.65 \cdot 10^{-3})$	$7.58 \cdot 10^{-2} (3.38 \cdot 10^{-3})$	$1.40 \cdot 10^{-1} (7.42 \cdot 10^{-3})$	$1.07 \cdot 10^{-1} (6.60 \cdot 10^{-3})$
$v = 21/2$	$7.48 \cdot 10^{-2} (3.35 \cdot 10^{-3})$	$8.90 \cdot 10^{-2} (3.97 \cdot 10^{-3})$	$8.68 \cdot 10^{-2} (3.89 \cdot 10^{-3})$	$8.57 \cdot 10^{-2} (3.69 \cdot 10^{-3})$	$1.35 \cdot 10^{-1} (6.85 \cdot 10^{-3})$	$1.24 \cdot 10^{-1} (7.74 \cdot 10^{-3})$
$v = \infty$	$8.66 \cdot 10^{-2} (3.66 \cdot 10^{-3})$	$1.05 \cdot 10^{-1} (4.55 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (4.44 \cdot 10^{-3})$	$1.00 \cdot 10^{-1} (4.27 \cdot 10^{-3})$	$1.52 \cdot 10^{-1} (8.09 \cdot 10^{-3})$	$1.42 \cdot 10^{-1} (8.62 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$4.38 \cdot 10^{-2} (2.27 \cdot 10^{-3})$	$4.98 \cdot 10^{-2} (2.67 \cdot 10^{-3})$	$5.20 \cdot 10^{-2} (2.64 \cdot 10^{-3})$	$4.66 \cdot 10^{-2} (2.32 \cdot 10^{-3})$	$1.52 \cdot 10^{-1} (8.09 \cdot 10^{-3})$	$1.04 \cdot 10^{-1} (8.76 \cdot 10^{-3})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$4.75 \cdot 10^{-2} (2.68 \cdot 10^{-3})$
$v = 3/2$	$3.87 \cdot 10^{-2} (1.96 \cdot 10^{-3})$
$v = 5/2$	$4.46 \cdot 10^{-2} (2.10 \cdot 10^{-3})$
$v = 7/2$	$4.96 \cdot 10^{-2} (2.27 \cdot 10^{-3})$
$v = 9/2$	$5.34 \cdot 10^{-2} (2.40 \cdot 10^{-3})$
$v = 11/2$	$5.62 \cdot 10^{-2} (2.49 \cdot 10^{-3})$
$v = 21/2$	$6.34 \cdot 10^{-2} (2.73 \cdot 10^{-3})$
$v = \infty$	$7.32 \cdot 10^{-2} (3.06 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$3.84 \cdot 10^{-2} (1.92 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.48 \cdot 10^{-2} (2.28 \cdot 10^{-3})$
linear	$4.18 \cdot 10^{-1} (1.91 \cdot 10^{-2})$
1NN	$2.93 \cdot 10^{-1} (5.81 \cdot 10^{-3})$
2NN	$2.19 \cdot 10^{-1} (4.69 \cdot 10^{-3})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.05 \cdot 10^{-1} (2.66 \cdot 10^{-3})$	$9.97 \cdot 10^{-2} (2.74 \cdot 10^{-3})$	$9.96 \cdot 10^{-2} (2.71 \cdot 10^{-3})$	$9.21 \cdot 10^{-2} (2.67 \cdot 10^{-3})$	$2.26 \cdot 10^{-1} (4.06 \cdot 10^{-3})$	$9.99 \cdot 10^{-2} (2.68 \cdot 10^{-3})$
$v = 3/2$	$9.26 \cdot 10^{-2} (2.72 \cdot 10^{-3})$	$9.61 \cdot 10^{-2} (2.77 \cdot 10^{-3})$	$9.66 \cdot 10^{-2} (2.81 \cdot 10^{-3})$	$9.12 \cdot 10^{-2} (2.75 \cdot 10^{-3})$	$2.02 \cdot 10^{-1} (5.33 \cdot 10^{-3})$	$9.98 \cdot 10^{-2} (2.91 \cdot 10^{-3})$
$v = 5/2$	$1.03 \cdot 10^{-1} (2.84 \cdot 10^{-3})$	$1.11 \cdot 10^{-1} (3.10 \cdot 10^{-3})$	$1.10 \cdot 10^{-1} (3.07 \cdot 10^{-3})$	$1.07 \cdot 10^{-1} (2.93 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (5.68 \cdot 10^{-3})$	$1.20 \cdot 10^{-1} (3.84 \cdot 10^{-3})$
$v = 7/2$	$1.10 \cdot 10^{-1} (2.90 \cdot 10^{-3})$	$1.21 \cdot 10^{-1} (3.14 \cdot 10^{-3})$	$1.20 \cdot 10^{-1} (3.23 \cdot 10^{-3})$	$1.18 \cdot 10^{-1} (3.06 \cdot 10^{-3})$	$1.92 \cdot 10^{-1} (5.46 \cdot 10^{-3})$	$1.34 \cdot 10^{-1} (4.25 \cdot 10^{-3})$
$v = 9/2$	$1.15 \cdot 10^{-1} (2.96 \cdot 10^{-3})$	$1.27 \cdot 10^{-1} (3.25 \cdot 10^{-3})$	$1.25 \cdot 10^{-1} (3.24 \cdot 10^{-3})$	$1.24 \cdot 10^{-1} (3.16 \cdot 10^{-3})$	$1.89 \cdot 10^{-1} (5.33 \cdot 10^{-3})$	$1.44 \cdot 10^{-1} (4.31 \cdot 10^{-3})$
$v = 11/2$	$1.19 \cdot 10^{-1} (2.98 \cdot 10^{-3})$	$1.32 \cdot 10^{-1} (3.26 \cdot 10^{-3})$	$1.29 \cdot 10^{-1} (3.24 \cdot 10^{-3})$	$1.28 \cdot 10^{-1} (3.17 \cdot 10^{-3})$	$1.86 \cdot 10^{-1} (5.13 \cdot 10^{-3})$	$1.48 \cdot 10^{-1} (4.34 \cdot 10^{-3})$
$v = 21/2$	$1.28 \cdot 10^{-1} (3.06 \cdot 10^{-3})$	$1.41 \cdot 10^{-1} (3.44 \cdot 10^{-3})$	$1.38 \cdot 10^{-1} (3.35 \cdot 10^{-3})$	$1.38 \cdot 10^{-1} (3.27 \cdot 10^{-3})$	$1.81 \cdot 10^{-1} (4.71 \cdot 10^{-3})$	$1.60 \cdot 10^{-1} (4.62 \cdot 10^{-3})$
$v = \infty$	$1.40 \cdot 10^{-1} (3.16 \cdot 10^{-3})$	$1.54 \cdot 10^{-1} (3.64 \cdot 10^{-3})$	$1.51 \cdot 10^{-1} (3.53 \cdot 10^{-3})$	$1.51 \cdot 10^{-1} (3.52 \cdot 10^{-3})$	$1.91 \cdot 10^{-1} (5.35 \cdot 10^{-3})$	$1.74 \cdot 10^{-1} (4.98 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$9.24 \cdot 10^{-2} (2.69 \cdot 10^{-3})$	$9.85 \cdot 10^{-2} (2.86 \cdot 10^{-3})$	$1.00 \cdot 10^{-1} (2.86 \cdot 10^{-3})$	$9.03 \cdot 10^{-2} (2.69 \cdot 10^{-3})$	$1.91 \cdot 10^{-1} (5.35 \cdot 10^{-3})$	$1.40 \cdot 10^{-1} (5.45 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$9.39 \cdot 10^{-2} (2.67 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.48 \cdot 10^0 (3.51 \cdot 10^{-2})$	$1.46 \cdot 10^0 (3.95 \cdot 10^{-2})$	$1.46 \cdot 10^0 (3.91 \cdot 10^{-2})$	$1.69 \cdot 10^0 (5.15 \cdot 10^{-2})$	$2.34 \cdot 10^0 (4.40 \cdot 10^{-2})$	$1.46 \cdot 10^0 (3.86 \cdot 10^{-2})$
$v = 3/2$	$1.37 \cdot 10^0 (3.35 \cdot 10^{-2})$	$1.44 \cdot 10^0 (3.59 \cdot 10^{-2})$	$1.46 \cdot 10^0 (3.69 \cdot 10^{-2})$	$1.73 \cdot 10^0 (4.71 \cdot 10^{-2})$	$2.19 \cdot 10^0 (5.71 \cdot 10^{-2})$	$1.51 \cdot 10^0 (3.96 \cdot 10^{-2})$
$v = 5/2$	$1.45 \cdot 10^0 (3.37 \cdot 10^{-2})$	$1.58 \cdot 10^0 (4.11 \cdot 10^{-2})$	$1.59 \cdot 10^0 (4.16 \cdot 10^{-2})$	$1.86 \cdot 10^0 (5.02 \cdot 10^{-2})$	$2.17 \cdot 10^0 (6.23 \cdot 10^{-2})$	$1.74 \cdot 10^0 (5.39 \cdot 10^{-2})$
$v = 7/2$	$1.51 \cdot 10^0 (3.42 \cdot 10^{-2})$	$1.67 \cdot 10^0 (4.14 \cdot 10^{-2})$	$1.67 \cdot 10^0 (4.43 \cdot 10^{-2})$	$1.91 \cdot 10^0 (5.12 \cdot 10^{-2})$	$2.14 \cdot 10^0 (6.06 \cdot 10^{-2})$	$1.86 \cdot 10^0 (5.47 \cdot 10^{-2})$
$v = 9/2$	$1.55 \cdot 10^0 (3.45 \cdot 10^{-2})$	$1.70 \cdot 10^0 (4.16 \cdot 10^{-2})$	$1.71 \cdot 10^0 (4.30 \cdot 10^{-2})$	$1.95 \cdot 10^0 (5.25 \cdot 10^{-2})$	$2.12 \cdot 10^0 (6.01 \cdot 10^{-2})$	$1.95 \cdot 10^0 (5.29 \cdot 10^{-2})$
$v = 11/2$	$1.58 \cdot 10^0 (3.48 \cdot 10^{-2})$	$1.74 \cdot 10^0 (4.27 \cdot 10^{-2})$	$1.74 \cdot 10^0 (4.34 \cdot 10^{-2})$	$1.96 \cdot 10^0 (5.01 \cdot 10^{-2})$	$2.11 \cdot 10^0 (5.91 \cdot 10^{-2})$	$1.99 \cdot 10^0 (5.40 \cdot 10^{-2})$
$v = 21/2$	$1.66 \cdot 10^0 (3.55 \cdot 10^{-2})$	$1.80 \cdot 10^0 (4.12 \cdot 10^{-2})$	$1.79 \cdot 10^0 (4.29 \cdot 10^{-2})$	$2.03 \cdot 10^0 (5.04 \cdot 10^{-2})$	$2.10 \cdot 10^0 (5.64 \cdot 10^{-2})$	$2.09 \cdot 10^0 (5.46 \cdot 10^{-2})$
$v = \infty$	$1.78 \cdot 10^0 (3.57 \cdot 10^{-2})$	$1.91 \cdot 10^0 (4.47 \cdot 10^{-2})$	$1.92 \cdot 10^0 (4.62 \cdot 10^{-2})$	$2.13 \cdot 10^0 (5.17 \cdot 10^{-2})$	$2.21 \cdot 10^0 (5.96 \cdot 10^{-2})$	$2.21 \cdot 10^0 (5.63 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$1.37 \cdot 10^0 (3.31 \cdot 10^{-2})$	$1.47 \cdot 10^0 (4.03 \cdot 10^{-2})$	$1.51 \cdot 10^0 (3.74 \cdot 10^{-2})$	$1.72 \cdot 10^0 (4.56 \cdot 10^{-2})$	$2.21 \cdot 10^0 (5.96 \cdot 10^{-2})$	$1.95 \cdot 10^0 (5.95 \cdot 10^{-2})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.39 \cdot 10^0 (3.36 \cdot 10^{-2})$

### SM-2.2.16 GKLS $C^1, n = 20, d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$



$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.32 \cdot 10^{-1} (2.07 \cdot 10^{-2})$	$3.36 \cdot 10^{-1} (2.13 \cdot 10^{-2})$	$3.55 \cdot 10^{-1} (2.66 \cdot 10^{-2})$	$3.34 \cdot 10^{-1} (2.08 \cdot 10^{-2})$	$4.44 \cdot 10^{-1} (2.17 \cdot 10^{-2})$	$3.69 \cdot 10^{-1} (2.53 \cdot 10^{-2})$
$v = 3/2$	$3.77 \cdot 10^{-1} (2.47 \cdot 10^{-2})$	$4.33 \cdot 10^{-1} (3.01 \cdot 10^{-2})$	$4.36 \cdot 10^{-1} (2.80 \cdot 10^{-2})$	$3.98 \cdot 10^{-1} (2.34 \cdot 10^{-2})$	$4.59 \cdot 10^{-1} (2.52 \cdot 10^{-2})$	$5.63 \cdot 10^{-1} (3.94 \cdot 10^{-2})$
$v = 5/2$	$4.06 \cdot 10^{-1} (2.60 \cdot 10^{-2})$	$4.58 \cdot 10^{-1} (2.77 \cdot 10^{-2})$	$5.13 \cdot 10^{-1} (3.31 \cdot 10^{-2})$	$4.46 \cdot 10^{-1} (2.63 \cdot 10^{-2})$	$4.65 \cdot 10^{-1} (2.43 \cdot 10^{-2})$	$6.66 \cdot 10^{-1} (4.54 \cdot 10^{-2})$
$v = 7/2$	$4.15 \cdot 10^{-1} (2.59 \cdot 10^{-2})$	$5.14 \cdot 10^{-1} (3.53 \cdot 10^{-2})$	$6.00 \cdot 10^{-1} (7.16 \cdot 10^{-2})$	$4.80 \cdot 10^{-1} (3.17 \cdot 10^{-2})$	$4.82 \cdot 10^{-1} (2.71 \cdot 10^{-2})$	$1.13 \cdot 10^0 (2.26 \cdot 10^{-1})$
$v = 9/2$	$4.21 \cdot 10^{-1} (2.58 \cdot 10^{-2})$	$5.24 \cdot 10^{-1} (3.92 \cdot 10^{-2})$	$6.55 \cdot 10^{-1} (7.90 \cdot 10^{-2})$	$4.87 \cdot 10^{-1} (3.18 \cdot 10^{-2})$	$5.21 \cdot 10^{-1} (3.39 \cdot 10^{-2})$	$1.33 \cdot 10^0 (3.16 \cdot 10^{-1})$
$v = \infty$	$4.52 \cdot 10^{-1} (2.57 \cdot 10^{-2})$	$5.63 \cdot 10^{-1} (4.37 \cdot 10^{-2})$	$1.01 \cdot 10^0 (2.10 \cdot 10^{-1})$	$6.45 \cdot 10^{-1} (1.05 \cdot 10^{-1})$	$9.76 \cdot 10^{-1} (1.66 \cdot 10^{-1})$	$2.35 \cdot 10^0 (4.57 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$4.02 \cdot 10^{-1} (2.62 \cdot 10^{-2})$	$5.21 \cdot 10^{-1} (4.43 \cdot 10^{-2})$	$1.15 \cdot 10^0 (2.64 \cdot 10^{-1})$	$5.37 \cdot 10^{-1} (7.69 \cdot 10^{-2})$	$7.85 \cdot 10^{-1} (1.58 \cdot 10^{-1})$	$2.16 \cdot 10^0 (4.41 \cdot 10^{-1})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.61 \cdot 10^{-1} (1.44 \cdot 10^{-2})$
$v = 3/2$	$2.52 \cdot 10^{-1} (1.32 \cdot 10^{-2})$
$v = 5/2$	$2.63 \cdot 10^{-1} (1.27 \cdot 10^{-2})$
$v = 7/2$	$2.70 \cdot 10^{-1} (1.27 \cdot 10^{-2})$
$v = 9/2$	$2.76 \cdot 10^{-1} (1.29 \cdot 10^{-2})$
$v = \infty$	$2.91 \cdot 10^{-1} (1.22 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$2.31 \cdot 10^{-1} (1.21 \cdot 10^{-2})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$3.80 \cdot 10^{-1} (2.30 \cdot 10^{-2})$
linear	$7.06 \cdot 10^{-1} (2.29 \cdot 10^{-2})$
1NN	$4.31 \cdot 10^{-1} (1.93 \cdot 10^{-2})$
2NN	$3.99 \cdot 10^{-1} (1.69 \cdot 10^{-2})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.65 \cdot 10^{-1} (8.13 \cdot 10^{-3})$	$2.64 \cdot 10^{-1} (8.55 \cdot 10^{-3})$	$2.70 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$2.59 \cdot 10^{-1} (8.49 \cdot 10^{-3})$	$3.29 \cdot 10^{-1} (8.53 \cdot 10^{-3})$	$2.76 \cdot 10^{-1} (9.91 \cdot 10^{-3})$
$v = 3/2$	$2.79 \cdot 10^{-1} (9.61 \cdot 10^{-3})$	$3.09 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$3.09 \cdot 10^{-1} (1.16 \cdot 10^{-2})$	$2.96 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$3.31 \cdot 10^{-1} (1.08 \cdot 10^{-2})$	$3.59 \cdot 10^{-1} (1.49 \cdot 10^{-2})$
$v = 5/2$	$2.95 \cdot 10^{-1} (9.83 \cdot 10^{-3})$	$3.24 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$3.37 \cdot 10^{-1} (1.28 \cdot 10^{-2})$	$3.21 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$3.31 \cdot 10^{-1} (1.03 \cdot 10^{-2})$	$3.89 \cdot 10^{-1} (1.53 \cdot 10^{-2})$
$v = 7/2$	$3.00 \cdot 10^{-1} (9.68 \cdot 10^{-3})$	$3.41 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$3.54 \cdot 10^{-1} (1.40 \cdot 10^{-2})$	$3.35 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$3.33 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$4.38 \cdot 10^{-1} (2.03 \cdot 10^{-2})$
$v = 9/2$	$3.04 \cdot 10^{-1} (9.57 \cdot 10^{-3})$	$3.41 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$3.68 \cdot 10^{-1} (1.58 \cdot 10^{-2})$	$3.34 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$3.43 \cdot 10^{-1} (1.08 \cdot 10^{-2})$	$4.62 \cdot 10^{-1} (2.24 \cdot 10^{-2})$
$v = \infty$	$3.23 \cdot 10^{-1} (9.24 \cdot 10^{-3})$	$3.59 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$4.17 \cdot 10^{-1} (2.40 \cdot 10^{-2})$	$3.61 \cdot 10^{-1} (1.60 \cdot 10^{-2})$	$4.26 \cdot 10^{-1} (2.00 \cdot 10^{-2})$	$5.74 \cdot 10^{-1} (3.83 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$2.91 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$3.38 \cdot 10^{-1} (1.33 \cdot 10^{-2})$	$4.15 \cdot 10^{-1} (2.73 \cdot 10^{-2})$	$3.36 \cdot 10^{-1} (1.50 \cdot 10^{-2})$	$3.90 \cdot 10^{-1} (1.70 \cdot 10^{-2})$	$5.62 \cdot 10^{-1} (3.81 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$2.89 \cdot 10^{-1} (9.51 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.78 \cdot 10^0 (1.44 \cdot 10^{-1})$	$3.99 \cdot 10^0 (1.58 \cdot 10^{-1})$	$4.14 \cdot 10^0 (2.00 \cdot 10^{-1})$	$4.23 \cdot 10^0 (1.77 \cdot 10^{-1})$	$4.64 \cdot 10^0 (1.85 \cdot 10^{-1})$	$4.18 \cdot 10^0 (1.79 \cdot 10^{-1})$
$v = 3/2$	$4.69 \cdot 10^0 (2.20 \cdot 10^{-1})$	$5.38 \cdot 10^0 (2.88 \cdot 10^{-1})$	$5.49 \cdot 10^0 (2.82 \cdot 10^{-1})$	$5.49 \cdot 10^0 (2.76 \cdot 10^{-1})$	$5.27 \cdot 10^0 (3.12 \cdot 10^{-1})$	$6.56 \cdot 10^0 (3.80 \cdot 10^{-1})$
$v = 5/2$	$5.05 \cdot 10^0 (2.26 \cdot 10^{-1})$	$5.84 \cdot 10^0 (3.37 \cdot 10^{-1})$	$6.22 \cdot 10^0 (3.86 \cdot 10^{-1})$	$6.28 \cdot 10^0 (3.59 \cdot 10^{-1})$	$5.06 \cdot 10^0 (2.63 \cdot 10^{-1})$	$7.10 \cdot 10^0 (3.94 \cdot 10^{-1})$
$v = 7/2$	$5.18 \cdot 10^0 (2.23 \cdot 10^{-1})$	$5.99 \cdot 10^0 (2.94 \cdot 10^{-1})$	$6.54 \cdot 10^0 (3.72 \cdot 10^{-1})$	$6.57 \cdot 10^0 (4.18 \cdot 10^{-1})$	$4.89 \cdot 10^0 (2.04 \cdot 10^{-1})$	$7.55 \cdot 10^0 (4.07 \cdot 10^{-1})$
$v = 9/2$	$5.26 \cdot 10^0 (2.18 \cdot 10^{-1})$	$5.88 \cdot 10^0 (2.64 \cdot 10^{-1})$	$6.82 \cdot 10^0 (3.83 \cdot 10^{-1})$	$6.30 \cdot 10^0 (3.00 \cdot 10^{-1})$	$5.03 \cdot 10^0 (2.03 \cdot 10^{-1})$	$8.06 \cdot 10^0 (4.72 \cdot 10^{-1})$
$v = \infty$	$5.68 \cdot 10^0 (2.15 \cdot 10^{-1})$	$6.27 \cdot 10^0 (2.77 \cdot 10^{-1})$	$8.25 \cdot 10^0 (6.72 \cdot 10^{-1})$	$6.94 \cdot 10^0 (4.34 \cdot 10^{-1})$	$6.00 \cdot 10^0 (2.57 \cdot 10^{-1})$	$1.03 \cdot 10^1 (7.70 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$5.13 \cdot 10^0 (2.26 \cdot 10^{-1})$	$6.29 \cdot 10^0 (3.15 \cdot 10^{-1})$	$8.42 \cdot 10^0 (6.84 \cdot 10^{-1})$	$6.93 \cdot 10^0 (4.08 \cdot 10^{-1})$	$5.60 \cdot 10^0 (2.46 \cdot 10^{-1})$	$1.03 \cdot 10^1 (8.02 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$4.94 \cdot 10^0 (2.16 \cdot 10^{-1})$

## SM-2.2.17 GKLS $C^1$ , $n = 40$ , $d = 2$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.57 \cdot 10^{-1} (9.29 \cdot 10^{-3})$	$1.52 \cdot 10^{-1} (9.28 \cdot 10^{-3})$	$1.53 \cdot 10^{-1} (9.28 \cdot 10^{-3})$	$1.48 \cdot 10^{-1} (9.23 \cdot 10^{-3})$	$2.52 \cdot 10^{-1} (1.49 \cdot 10^{-2})$	$1.59 \cdot 10^{-1} (1.05 \cdot 10^{-2})$
$v = 3/2$	$1.61 \cdot 10^{-1} (8.92 \cdot 10^{-3})$	$1.73 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$2.11 \cdot 10^{-1} (3.60 \cdot 10^{-2})$	$1.65 \cdot 10^{-1} (9.60 \cdot 10^{-3})$	$2.65 \cdot 10^{-1} (2.84 \cdot 10^{-2})$	$2.50 \cdot 10^{-1} (3.84 \cdot 10^{-2})$
$v = 5/2$	$1.69 \cdot 10^{-1} (9.00 \cdot 10^{-3})$	$2.04 \cdot 10^{-1} (1.68 \cdot 10^{-2})$	$2.09 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$2.34 \cdot 10^{-1} (5.22 \cdot 10^{-2})$	$2.92 \cdot 10^{-1} (3.05 \cdot 10^{-2})$	$3.41 \cdot 10^{-1} (5.59 \cdot 10^{-2})$
$v = 7/2$	$1.75 \cdot 10^{-1} (9.07 \cdot 10^{-3})$	$2.08 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$2.70 \cdot 10^{-1} (4.86 \cdot 10^{-2})$	$1.97 \cdot 10^{-1} (1.25 \cdot 10^{-2})$	$3.69 \cdot 10^{-1} (5.50 \cdot 10^{-2})$	$7.22 \cdot 10^{-1} (1.60 \cdot 10^{-1})$
$v = 9/2$	$1.78 \cdot 10^{-1} (9.12 \cdot 10^{-3})$	$2.16 \cdot 10^{-1} (1.42 \cdot 10^{-2})$	$2.76 \cdot 10^{-1} (4.34 \cdot 10^{-2})$	$2.05 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$4.99 \cdot 10^{-1} (8.99 \cdot 10^{-2})$	$1.46 \cdot 10^0 (4.59 \cdot 10^{-1})$
$v \in \{1/2, \dots, 9/2\}$	$1.67 \cdot 10^{-1} (9.11 \cdot 10^{-3})$	$2.08 \cdot 10^{-1} (1.87 \cdot 10^{-2})$	$2.76 \cdot 10^{-1} (5.35 \cdot 10^{-2})$	$2.30 \cdot 10^{-1} (5.28 \cdot 10^{-2})$	$4.28 \cdot 10^{-1} (7.31 \cdot 10^{-2})$	$1.50 \cdot 10^0 (4.61 \cdot 10^{-1})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.32 \cdot 10^{-1} (8.41 \cdot 10^{-3})$
$v = 3/2$	$1.14 \cdot 10^{-1} (6.83 \cdot 10^{-3})$
$v = 5/2$	$1.18 \cdot 10^{-1} (6.70 \cdot 10^{-3})$
$v = 7/2$	$1.22 \cdot 10^{-1} (6.78 \cdot 10^{-3})$
$v = 9/2$	$1.25 \cdot 10^{-1} (6.83 \cdot 10^{-3})$
$v \in \{1/2, \dots, 9/2\}$	$1.07 \cdot 10^{-1} (6.81 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.64 \cdot 10^{-1} (9.09 \cdot 10^{-3})$
linear	$6.84 \cdot 10^{-1} (2.18 \cdot 10^{-2})$
INN	$2.49 \cdot 10^{-1} (1.16 \cdot 10^{-2})$
2NN	$2.20 \cdot 10^{-1} (1.13 \cdot 10^{-2})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.87 \cdot 10^{-1} (4.50 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (4.28 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (4.33 \cdot 10^{-3})$	$1.68 \cdot 10^{-1} (4.43 \cdot 10^{-3})$	$2.44 \cdot 10^{-1} (6.79 \cdot 10^{-3})$	$1.85 \cdot 10^{-1} (5.07 \cdot 10^{-3})$
$v = 3/2$	$1.78 \cdot 10^{-1} (4.39 \cdot 10^{-3})$	$1.84 \cdot 10^{-1} (4.78 \cdot 10^{-3})$	$1.90 \cdot 10^{-1} (7.91 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (4.74 \cdot 10^{-3})$	$2.30 \cdot 10^{-1} (9.80 \cdot 10^{-3})$	$2.06 \cdot 10^{-1} (1.05 \cdot 10^{-2})$
$v = 5/2$	$1.82 \cdot 10^{-1} (4.40 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (5.35 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (5.52 \cdot 10^{-3})$	$1.95 \cdot 10^{-1} (9.65 \cdot 10^{-3})$	$2.40 \cdot 10^{-1} (1.05 \cdot 10^{-2})$	$2.23 \cdot 10^{-1} (1.02 \cdot 10^{-2})$
$v = 7/2$	$1.86 \cdot 10^{-1} (4.45 \cdot 10^{-3})$	$1.99 \cdot 10^{-1} (5.46 \cdot 10^{-3})$	$2.09 \cdot 10^{-1} (9.52 \cdot 10^{-3})$	$1.94 \cdot 10^{-1} (5.38 \cdot 10^{-3})$	$2.60 \cdot 10^{-1} (1.41 \cdot 10^{-2})$	$2.67 \cdot 10^{-1} (1.28 \cdot 10^{-2})$
$v = 9/2$	$1.87 \cdot 10^{-1} (4.46 \cdot 10^{-3})$	$2.02 \cdot 10^{-1} (5.43 \cdot 10^{-3})$	$2.12 \cdot 10^{-1} (9.04 \cdot 10^{-3})$	$1.98 \cdot 10^{-1} (5.39 \cdot 10^{-3})$	$2.88 \cdot 10^{-1} (1.94 \cdot 10^{-2})$	$3.10 \cdot 10^{-1} (1.80 \cdot 10^{-2})$
$v \in \{1/2, \dots, 9/2\}$	$1.81 \cdot 10^{-1} (4.35 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (5.59 \cdot 10^{-3})$	$2.09 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$1.87 \cdot 10^{-1} (9.94 \cdot 10^{-3})$	$2.77 \cdot 10^{-1} (1.59 \cdot 10^{-2})$	$3.17 \cdot 10^{-1} (1.95 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.82 \cdot 10^{-1} (4.24 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.50 \cdot 10^0 (8.08 \cdot 10^{-2})$	$2.50 \cdot 10^0 (8.62 \cdot 10^{-2})$	$2.50 \cdot 10^0 (8.73 \cdot 10^{-2})$	$2.65 \cdot 10^0 (1.02 \cdot 10^{-1})$	$3.19 \cdot 10^0 (1.07 \cdot 10^{-1})$	$2.56 \cdot 10^0 (9.84 \cdot 10^{-2})$
$v = 3/2$	$2.58 \cdot 10^0 (9.84 \cdot 10^{-2})$	$2.75 \cdot 10^0 (1.13 \cdot 10^{-1})$	$2.98 \cdot 10^0 (2.12 \cdot 10^{-1})$	$3.10 \cdot 10^0 (1.33 \cdot 10^{-1})$	$3.19 \cdot 10^0 (1.58 \cdot 10^{-1})$	$3.28 \cdot 10^0 (2.46 \cdot 10^{-1})$
$v = 5/2$	$2.73 \cdot 10^0 (1.07 \cdot 10^{-1})$	$2.97 \cdot 10^0 (1.33 \cdot 10^{-1})$	$3.11 \cdot 10^0 (1.37 \cdot 10^{-1})$	$3.53 \cdot 10^0 (3.12 \cdot 10^{-1})$	$3.39 \cdot 10^0 (1.69 \cdot 10^{-1})$	$3.58 \cdot 10^0 (2.96 \cdot 10^{-1})$
$v = 7/2$	$2.83 \cdot 10^0 (1.12 \cdot 10^{-1})$	$3.10 \cdot 10^0 (1.42 \cdot 10^{-1})$	$3.47 \cdot 10^0 (2.72 \cdot 10^{-1})$	$3.40 \cdot 10^0 (1.61 \cdot 10^{-1})$	$3.72 \cdot 10^0 (2.11 \cdot 10^{-1})$	$4.12 \cdot 10^0 (2.68 \cdot 10^{-1})$
$v = 9/2$	$2.88 \cdot 10^0 (1.13 \cdot 10^{-1})$	$3.19 \cdot 10^0 (1.44 \cdot 10^{-1})$	$3.54 \cdot 10^0 (2.47 \cdot 10^{-1})$	$3.50 \cdot 10^0 (1.64 \cdot 10^{-1})$	$4.05 \cdot 10^0 (2.70 \cdot 10^{-1})$	$4.60 \cdot 10^0 (2.48 \cdot 10^{-1})$
$v \in \{1/2, \dots, 9/2\}$	$2.70 \cdot 10^0 (1.02 \cdot 10^{-1})$	$3.05 \cdot 10^0 (1.43 \cdot 10^{-1})$	$3.58 \cdot 10^0 (3.03 \cdot 10^{-1})$	$3.52 \cdot 10^0 (3.15 \cdot 10^{-1})$	$3.81 \cdot 10^0 (2.21 \cdot 10^{-1})$	$4.88 \cdot 10^0 (3.59 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$2.67 \cdot 10^0 (9.77 \cdot 10^{-2})$

## SM-2.2.18 GKLS $C^1$ , $n = 100$ , $d = 2$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.03 \cdot 10^{-2}$ ( $2.93 \cdot 10^{-3}$ )	$5.07 \cdot 10^{-2}$ ( $3.04 \cdot 10^{-3}$ )	$4.98 \cdot 10^{-2}$ ( $2.96 \cdot 10^{-3}$ )	$4.96 \cdot 10^{-2}$ ( $2.93 \cdot 10^{-3}$ )	$1.44 \cdot 10^{-1}$ ( $1.10 \cdot 10^{-2}$ )	$4.94 \cdot 10^{-2}$ ( $2.90 \cdot 10^{-3}$ )
$v = 3/2$	$3.11 \cdot 10^{-2}$ ( $1.98 \cdot 10^{-3}$ )	$3.23 \cdot 10^{-2}$ ( $2.38 \cdot 10^{-3}$ )	$3.07 \cdot 10^{-2}$ ( $1.95 \cdot 10^{-3}$ )	$3.08 \cdot 10^{-2}$ ( $2.10 \cdot 10^{-3}$ )	$1.72 \cdot 10^{-1}$ ( $2.11 \cdot 10^{-2}$ )	$3.17 \cdot 10^{-2}$ ( $1.93 \cdot 10^{-3}$ )
$v = 5/2$	$2.89 \cdot 10^{-2}$ ( $1.71 \cdot 10^{-3}$ )	$3.03 \cdot 10^{-2}$ ( $1.86 \cdot 10^{-3}$ )	$3.06 \cdot 10^{-2}$ ( $1.86 \cdot 10^{-3}$ )	$2.90 \cdot 10^{-2}$ ( $1.68 \cdot 10^{-3}$ )	$1.92 \cdot 10^{-1}$ ( $2.30 \cdot 10^{-2}$ )	$4.35 \cdot 10^{-2}$ ( $3.63 \cdot 10^{-3}$ )
$v = 7/2$	$2.90 \cdot 10^{-2}$ ( $1.63 \cdot 10^{-3}$ )	$2.98 \cdot 10^{-2}$ ( $1.69 \cdot 10^{-3}$ )	$3.24 \cdot 10^{-2}$ ( $2.18 \cdot 10^{-3}$ )	$2.90 \cdot 10^{-2}$ ( $1.57 \cdot 10^{-3}$ )	$2.35 \cdot 10^{-1}$ ( $2.90 \cdot 10^{-2}$ )	$6.28 \cdot 10^{-2}$ ( $6.00 \cdot 10^{-3}$ )
$v = 9/2$	$2.96 \cdot 10^{-2}$ ( $1.61 \cdot 10^{-3}$ )	$3.07 \cdot 10^{-2}$ ( $1.67 \cdot 10^{-3}$ )	$3.38 \cdot 10^{-2}$ ( $2.40 \cdot 10^{-3}$ )	$2.98 \cdot 10^{-2}$ ( $1.54 \cdot 10^{-3}$ )	$2.30 \cdot 10^{-1}$ ( $2.66 \cdot 10^{-2}$ )	$8.16 \cdot 10^{-2}$ ( $8.38 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$2.90 \cdot 10^{-2}$ ( $1.66 \cdot 10^{-3}$ )	$3.21 \cdot 10^{-2}$ ( $2.32 \cdot 10^{-3}$ )	$3.26 \cdot 10^{-2}$ ( $2.34 \cdot 10^{-3}$ )	$3.19 \cdot 10^{-2}$ ( $2.34 \cdot 10^{-3}$ )	$2.52 \cdot 10^{-1}$ ( $2.74 \cdot 10^{-2}$ )	$7.96 \cdot 10^{-2}$ ( $8.46 \cdot 10^{-3}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$4.69 \cdot 10^{-2}$ ( $2.79 \cdot 10^{-3}$ )
$v = 3/2$	$2.65 \cdot 10^{-2}$ ( $1.66 \cdot 10^{-3}$ )
$v = 5/2$	$2.31 \cdot 10^{-2}$ ( $1.37 \cdot 10^{-3}$ )
$v = 7/2$	$2.28 \cdot 10^{-2}$ ( $1.30 \cdot 10^{-3}$ )
$v = 9/2$	$2.34 \cdot 10^{-2}$ ( $1.31 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$2.16 \cdot 10^{-2}$ ( $1.29 \cdot 10^{-3}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$3.06 \cdot 10^{-2}$ ( $2.04 \cdot 10^{-3}$ )
linear	$6.81 \cdot 10^{-1}$ ( $2.18 \cdot 10^{-2}$ )
INN	$1.19 \cdot 10^{-1}$ ( $4.89 \cdot 10^{-3}$ )
2NN	$1.01 \cdot 10^{-1}$ ( $4.64 \cdot 10^{-3}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.08 \cdot 10^{-1}$ ( $2.42 \cdot 10^{-3}$ )	$1.01 \cdot 10^{-1}$ ( $2.40 \cdot 10^{-3}$ )	$1.00 \cdot 10^{-1}$ ( $2.34 \cdot 10^{-3}$ )	$9.11 \cdot 10^{-2}$ ( $2.36 \cdot 10^{-3}$ )	$1.72 \cdot 10^{-1}$ ( $6.30 \cdot 10^{-3}$ )	$9.99 \cdot 10^{-2}$ ( $2.31 \cdot 10^{-3}$ )
$v = 3/2$	$7.73 \cdot 10^{-2}$ ( $1.99 \cdot 10^{-3}$ )	$7.61 \cdot 10^{-2}$ ( $2.22 \cdot 10^{-3}$ )	$7.46 \cdot 10^{-2}$ ( $2.02 \cdot 10^{-3}$ )	$7.26 \cdot 10^{-2}$ ( $2.14 \cdot 10^{-3}$ )	$1.68 \cdot 10^{-1}$ ( $1.07 \cdot 10^{-2}$ )	$7.56 \cdot 10^{-2}$ ( $2.03 \cdot 10^{-3}$ )
$v = 5/2$	$7.34 \cdot 10^{-2}$ ( $1.85 \cdot 10^{-3}$ )	$7.44 \cdot 10^{-2}$ ( $1.97 \cdot 10^{-3}$ )	$7.37 \cdot 10^{-2}$ ( $1.94 \cdot 10^{-3}$ )	$7.18 \cdot 10^{-2}$ ( $1.92 \cdot 10^{-3}$ )	$1.71 \cdot 10^{-1}$ ( $1.03 \cdot 10^{-2}$ )	$8.35 \cdot 10^{-2}$ ( $2.65 \cdot 10^{-3}$ )
$v = 7/2$	$7.34 \cdot 10^{-2}$ ( $1.80 \cdot 10^{-3}$ )	$7.53 \cdot 10^{-2}$ ( $1.88 \cdot 10^{-3}$ )	$7.57 \cdot 10^{-2}$ ( $2.06 \cdot 10^{-3}$ )	$7.28 \cdot 10^{-2}$ ( $1.84 \cdot 10^{-3}$ )	$1.83 \cdot 10^{-1}$ ( $1.08 \cdot 10^{-2}$ )	$9.66 \cdot 10^{-2}$ ( $3.36 \cdot 10^{-3}$ )
$v = 9/2$	$7.43 \cdot 10^{-2}$ ( $1.80 \cdot 10^{-3}$ )	$7.74 \cdot 10^{-2}$ ( $1.88 \cdot 10^{-3}$ )	$7.76 \cdot 10^{-2}$ ( $2.12 \cdot 10^{-3}$ )	$7.45 \cdot 10^{-2}$ ( $1.83 \cdot 10^{-3}$ )	$1.77 \cdot 10^{-1}$ ( $9.29 \cdot 10^{-3}$ )	$1.08 \cdot 10^{-1}$ ( $4.01 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$7.29 \cdot 10^{-2}$ ( $1.79 \cdot 10^{-3}$ )	$7.62 \cdot 10^{-2}$ ( $2.12 \cdot 10^{-3}$ )	$7.58 \cdot 10^{-2}$ ( $2.06 \cdot 10^{-3}$ )	$7.34 \cdot 10^{-2}$ ( $2.12 \cdot 10^{-3}$ )	$1.93 \cdot 10^{-1}$ ( $1.06 \cdot 10^{-2}$ )	$1.05 \cdot 10^{-1}$ ( $4.13 \cdot 10^{-3}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$7.47 \cdot 10^{-2}$ ( $1.91 \cdot 10^{-3}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.47 \cdot 10^0$ ( $3.93 \cdot 10^{-2}$ )	$1.43 \cdot 10^0$ ( $4.39 \cdot 10^{-2}$ )	$1.41 \cdot 10^0$ ( $4.32 \cdot 10^{-2}$ )	$1.69 \cdot 10^0$ ( $6.04 \cdot 10^{-2}$ )	$2.34 \cdot 10^0$ ( $1.01 \cdot 10^{-1}$ )	$1.41 \cdot 10^0$ ( $4.25 \cdot 10^{-2}$ )
$v = 3/2$	$1.02 \cdot 10^0$ ( $3.53 \cdot 10^{-2}$ )	$1.05 \cdot 10^0$ ( $4.30 \cdot 10^{-2}$ )	$1.02 \cdot 10^0$ ( $3.60 \cdot 10^{-2}$ )	$1.28 \cdot 10^0$ ( $5.52 \cdot 10^{-2}$ )	$2.40 \cdot 10^0$ ( $1.72 \cdot 10^{-1}$ )	$1.03 \cdot 10^0$ ( $3.50 \cdot 10^{-2}$ )
$v = 5/2$	$9.46 \cdot 10^{-1}$ ( $3.23 \cdot 10^{-2}$ )	$9.66 \cdot 10^{-1}$ ( $3.33 \cdot 10^{-2}$ )	$9.73 \cdot 10^{-1}$ ( $3.33 \cdot 10^{-2}$ )	$1.16 \cdot 10^0$ ( $4.15 \cdot 10^{-2}$ )	$2.44 \cdot 10^0$ ( $1.72 \cdot 10^{-1}$ )	$1.10 \cdot 10^0$ ( $4.51 \cdot 10^{-2}$ )
$v = 7/2$	$9.32 \cdot 10^{-1}$ ( $3.10 \cdot 10^{-2}$ )	$9.42 \cdot 10^{-1}$ ( $2.90 \cdot 10^{-2}$ )	$9.76 \cdot 10^{-1}$ ( $3.48 \cdot 10^{-2}$ )	$1.11 \cdot 10^0$ ( $3.74 \cdot 10^{-2}$ )	$2.49 \cdot 10^0$ ( $1.61 \cdot 10^{-1}$ )	$1.24 \cdot 10^0$ ( $5.42 \cdot 10^{-2}$ )
$v = 9/2$	$9.35 \cdot 10^{-1}$ ( $3.08 \cdot 10^{-2}$ )	$9.54 \cdot 10^{-1}$ ( $2.88 \cdot 10^{-2}$ )	$9.89 \cdot 10^{-1}$ ( $3.69 \cdot 10^{-2}$ )	$1.10 \cdot 10^0$ ( $3.68 \cdot 10^{-2}$ )	$2.34 \cdot 10^0$ ( $1.36 \cdot 10^{-1}$ )	$1.37 \cdot 10^0$ ( $6.17 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, 9/2\}$	$9.52 \cdot 10^{-1}$ ( $3.21 \cdot 10^{-2}$ )	$1.00 \cdot 10^0$ ( $4.04 \cdot 10^{-2}$ )	$9.90 \cdot 10^{-1}$ ( $3.58 \cdot 10^{-2}$ )	$1.27 \cdot 10^0$ ( $5.91 \cdot 10^{-2}$ )	$2.69 \cdot 10^0$ ( $1.72 \cdot 10^{-1}$ )	$1.35 \cdot 10^0$ ( $6.23 \cdot 10^{-2}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$9.77 \cdot 10^{-1}$ ( $3.62 \cdot 10^{-2}$ )

### SM-2.2.19 GKLS $C^1$ , $n = 50$ , $d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.18 \cdot 10^{-1} (5.29 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (4.88 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (4.91 \cdot 10^{-3})$	$9.52 \cdot 10^{-2} (4.16 \cdot 10^{-3})$	$3.56 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$1.04 \cdot 10^{-1} (6.37 \cdot 10^{-3})$
$v = 3/2$	$7.36 \cdot 10^{-2} (9.29 \cdot 10^{-3})$	$7.89 \cdot 10^{-2} (8.74 \cdot 10^{-3})$	$7.93 \cdot 10^{-2} (9.73 \cdot 10^{-3})$	$6.81 \cdot 10^{-2} (7.61 \cdot 10^{-3})$	$3.06 \cdot 10^{-1} (9.71 \cdot 10^{-3})$	$8.65 \cdot 10^{-2} (1.27 \cdot 10^{-2})$
$v = 5/2$	$8.33 \cdot 10^{-2} (1.06 \cdot 10^{-2})$	$1.00 \cdot 10^{-1} (1.23 \cdot 10^{-2})$	$1.14 \cdot 10^{-1} (1.76 \cdot 10^{-2})$	$9.57 \cdot 10^{-2} (1.39 \cdot 10^{-2})$	$2.97 \cdot 10^{-1} (9.86 \cdot 10^{-3})$	$1.12 \cdot 10^{-1} (1.71 \cdot 10^{-2})$
$v = 7/2$	$9.09 \cdot 10^{-2} (1.11 \cdot 10^{-2})$	$1.29 \cdot 10^{-1} (1.74 \cdot 10^{-2})$	$1.34 \cdot 10^{-1} (1.90 \cdot 10^{-2})$	$1.17 \cdot 10^{-1} (1.68 \cdot 10^{-2})$	$2.92 \cdot 10^{-1} (9.66 \cdot 10^{-3})$	$1.37 \cdot 10^{-1} (2.02 \cdot 10^{-2})$
$v = 9/2$	$9.56 \cdot 10^{-2} (1.14 \cdot 10^{-2})$	$1.39 \cdot 10^{-1} (1.91 \cdot 10^{-2})$	$1.50 \cdot 10^{-1} (2.08 \cdot 10^{-2})$	$1.34 \cdot 10^{-1} (1.85 \cdot 10^{-2})$	$2.90 \cdot 10^{-1} (9.61 \cdot 10^{-3})$	$1.50 \cdot 10^{-1} (2.16 \cdot 10^{-2})$
$v = 11/2$	$9.87 \cdot 10^{-2} (1.16 \cdot 10^{-2})$	$1.49 \cdot 10^{-1} (2.43 \cdot 10^{-2})$	$1.57 \cdot 10^{-1} (2.32 \cdot 10^{-2})$	$1.41 \cdot 10^{-1} (1.97 \cdot 10^{-2})$	$2.89 \cdot 10^{-1} (9.69 \cdot 10^{-3})$	$1.63 \cdot 10^{-1} (2.40 \cdot 10^{-2})$
$v = 21/2$	$1.06 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$1.64 \cdot 10^{-1} (2.43 \cdot 10^{-2})$	$1.76 \cdot 10^{-1} (2.63 \cdot 10^{-2})$	$1.71 \cdot 10^{-1} (2.70 \cdot 10^{-2})$	$2.89 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$1.81 \cdot 10^{-1} (2.68 \cdot 10^{-2})$
$v = \infty$	$1.17 \cdot 10^{-1} (1.38 \cdot 10^{-2})$	$1.70 \cdot 10^{-1} (2.22 \cdot 10^{-2})$	$1.80 \cdot 10^{-1} (2.63 \cdot 10^{-2})$	$1.76 \cdot 10^{-1} (2.71 \cdot 10^{-2})$	$2.90 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$1.90 \cdot 10^{-1} (2.79 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$7.50 \cdot 10^{-2} (1.01 \cdot 10^{-2})$	$1.47 \cdot 10^{-1} (2.20 \cdot 10^{-2})$	$1.66 \cdot 10^{-1} (2.54 \cdot 10^{-2})$	$1.37 \cdot 10^{-1} (2.33 \cdot 10^{-2})$	$2.92 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$1.95 \cdot 10^{-1} (2.96 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$8.76 \cdot 10^{-2} (3.81 \cdot 10^{-3})$
$v = 3/2$	$4.72 \cdot 10^{-2} (4.10 \cdot 10^{-3})$
$v = 5/2$	$5.18 \cdot 10^{-2} (4.71 \cdot 10^{-3})$
$v = 7/2$	$5.68 \cdot 10^{-2} (5.04 \cdot 10^{-3})$
$v = 9/2$	$5.97 \cdot 10^{-2} (5.29 \cdot 10^{-3})$
$v = 11/2$	$6.18 \cdot 10^{-2} (5.47 \cdot 10^{-3})$
$v = 21/2$	$6.59 \cdot 10^{-2} (5.85 \cdot 10^{-3})$
$v = \infty$	$6.99 \cdot 10^{-2} (6.33 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$4.33 \cdot 10^{-2} (3.64 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$8.16 \cdot 10^{-2} (1.09 \cdot 10^{-2})$
linear	$3.97 \cdot 10^{-1} (2.07 \cdot 10^{-2})$
1NN	$4.72 \cdot 10^{-1} (7.85 \cdot 10^{-3})$
2NN	$3.72 \cdot 10^{-1} (6.80 \cdot 10^{-3})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.79 \cdot 10^{-1} (3.53 \cdot 10^{-3})$	$1.58 \cdot 10^{-1} (3.64 \cdot 10^{-3})$	$1.58 \cdot 10^{-1} (3.64 \cdot 10^{-3})$	$1.53 \cdot 10^{-1} (3.03 \cdot 10^{-3})$	$3.26 \cdot 10^{-1} (4.73 \cdot 10^{-3})$	$1.59 \cdot 10^{-1} (4.19 \cdot 10^{-3})$
$v = 3/2$	$1.09 \cdot 10^{-1} (6.94 \cdot 10^{-3})$	$1.14 \cdot 10^{-1} (7.66 \cdot 10^{-3})$	$1.11 \cdot 10^{-1} (7.62 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (6.64 \cdot 10^{-3})$	$2.97 \cdot 10^{-1} (4.84 \cdot 10^{-3})$	$1.15 \cdot 10^{-1} (8.73 \cdot 10^{-3})$
$v = 5/2$	$1.09 \cdot 10^{-1} (8.33 \cdot 10^{-3})$	$1.24 \cdot 10^{-1} (9.66 \cdot 10^{-3})$	$1.26 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$1.18 \cdot 10^{-1} (9.24 \cdot 10^{-3})$	$2.91 \cdot 10^{-1} (5.04 \cdot 10^{-3})$	$1.23 \cdot 10^{-1} (1.03 \cdot 10^{-2})$
$v = 7/2$	$1.18 \cdot 10^{-1} (8.61 \cdot 10^{-3})$	$1.44 \cdot 10^{-1} (1.15 \cdot 10^{-2})$	$1.44 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$1.38 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$2.88 \cdot 10^{-1} (4.93 \cdot 10^{-3})$	$1.42 \cdot 10^{-1} (1.15 \cdot 10^{-2})$
$v = 9/2$	$1.23 \cdot 10^{-1} (8.76 \cdot 10^{-3})$	$1.53 \cdot 10^{-1} (1.21 \cdot 10^{-2})$	$1.56 \cdot 10^{-1} (1.27 \cdot 10^{-2})$	$1.52 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$2.86 \cdot 10^{-1} (4.90 \cdot 10^{-3})$	$1.53 \cdot 10^{-1} (1.21 \cdot 10^{-2})$
$v = 11/2$	$1.26 \cdot 10^{-1} (8.88 \cdot 10^{-3})$	$1.58 \cdot 10^{-1} (1.27 \cdot 10^{-2})$	$1.60 \cdot 10^{-1} (1.32 \cdot 10^{-2})$	$1.57 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$2.85 \cdot 10^{-1} (4.93 \cdot 10^{-3})$	$1.61 \cdot 10^{-1} (1.31 \cdot 10^{-2})$
$v = 21/2$	$1.33 \cdot 10^{-1} (9.25 \cdot 10^{-3})$	$1.70 \cdot 10^{-1} (1.32 \cdot 10^{-2})$	$1.72 \cdot 10^{-1} (1.41 \cdot 10^{-2})$	$1.75 \cdot 10^{-1} (1.43 \cdot 10^{-2})$	$2.84 \cdot 10^{-1} (5.13 \cdot 10^{-3})$	$1.69 \cdot 10^{-1} (1.38 \cdot 10^{-2})$
$v = \infty$	$1.41 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$1.77 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$1.79 \cdot 10^{-1} (1.42 \cdot 10^{-2})$	$1.80 \cdot 10^{-1} (1.43 \cdot 10^{-2})$	$2.83 \cdot 10^{-1} (5.32 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (1.42 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$1.03 \cdot 10^{-1} (7.92 \cdot 10^{-3})$	$1.53 \cdot 10^{-1} (1.32 \cdot 10^{-2})$	$1.66 \cdot 10^{-1} (1.42 \cdot 10^{-2})$	$1.47 \cdot 10^{-1} (1.33 \cdot 10^{-2})$	$2.85 \cdot 10^{-1} (5.39 \cdot 10^{-3})$	$1.74 \cdot 10^{-1} (1.47 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.09 \cdot 10^{-1} (8.53 \cdot 10^{-3})$

#### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.81 \cdot 10^0 (3.91 \cdot 10^{-2})$	$1.77 \cdot 10^0 (4.51 \cdot 10^{-2})$	$1.77 \cdot 10^0 (4.51 \cdot 10^{-2})$	$1.81 \cdot 10^0 (4.45 \cdot 10^{-2})$	$3.04 \cdot 10^0 (5.29 \cdot 10^{-2})$	$1.78 \cdot 10^0 (4.91 \cdot 10^{-2})$
$v = 3/2$	$1.44 \cdot 10^0 (8.32 \cdot 10^{-2})$	$1.57 \cdot 10^0 (9.41 \cdot 10^{-2})$	$1.58 \cdot 10^0 (9.70 \cdot 10^{-2})$	$1.65 \cdot 10^0 (1.02 \cdot 10^{-1})$	$2.87 \cdot 10^0 (5.51 \cdot 10^{-2})$	$1.64 \cdot 10^0 (1.10 \cdot 10^{-1})$
$v = 5/2$	$1.58 \cdot 10^0 (1.03 \cdot 10^{-1})$	$1.94 \cdot 10^0 (1.36 \cdot 10^{-1})$	$2.12 \cdot 10^0 (1.72 \cdot 10^{-1})$	$2.16 \cdot 10^0 (1.72 \cdot 10^{-1})$	$2.85 \cdot 10^0 (5.79 \cdot 10^{-2})$	$2.04 \cdot 10^0 (1.63 \cdot 10^{-1})$
$v = 7/2$	$1.70 \cdot 10^0 (1.09 \cdot 10^{-1})$	$2.43 \cdot 10^0 (1.93 \cdot 10^{-1})$	$2.55 \cdot 10^0 (2.11 \cdot 10^{-1})$	$2.63 \cdot 10^0 (2.15 \cdot 10^{-1})$	$2.82 \cdot 10^0 (5.57 \cdot 10^{-2})$	$2.46 \cdot 10^0 (2.00 \cdot 10^{-1})$
$v = 9/2$	$1.78 \cdot 10^0 (1.13 \cdot 10^{-1})$	$2.60 \cdot 10^0 (2.06 \cdot 10^{-1})$	$2.87 \cdot 10^0 (2.33 \cdot 10^{-1})$	$3.01 \cdot 10^0 (2.33 \cdot 10^{-1})$	$2.81 \cdot 10^0 (5.54 \cdot 10^{-2})$	$2.69 \cdot 10^0 (2.11 \cdot 10^{-1})$
$v = 11/2$	$1.83 \cdot 10^0 (1.15 \cdot 10^{-1})$	$2.72 \cdot 10^0 (2.17 \cdot 10^{-1})$	$2.95 \cdot 10^0 (2.50 \cdot 10^{-1})$	$3.15 \cdot 10^0 (2.53 \cdot 10^{-1})$	$2.81 \cdot 10^0 (5.58 \cdot 10^{-2})$	$2.89 \cdot 10^0 (2.34 \cdot 10^{-1})$
$v = 21/2$	$1.94 \cdot 10^0 (1.21 \cdot 10^{-1})$	$3.03 \cdot 10^0 (2.40 \cdot 10^{-1})$	$3.32 \cdot 10^0 (2.85 \cdot 10^{-1})$	$3.67 \cdot 10^0 (3.12 \cdot 10^{-1})$	$2.81 \cdot 10^0 (5.82 \cdot 10^{-2})$	$3.13 \cdot 10^0 (2.57 \cdot 10^{-1})$
$v = \infty$	$2.08 \cdot 10^0 (1.34 \cdot 10^{-1})$	$3.23 \cdot 10^0 (2.54 \cdot 10^{-1})$	$3.51 \cdot 10^0 (2.85 \cdot 10^{-1})$	$3.78 \cdot 10^0 (3.14 \cdot 10^{-1})$	$2.83 \cdot 10^0 (6.18 \cdot 10^{-2})$	$3.30 \cdot 10^0 (2.66 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$1.50 \cdot 10^0 (9.93 \cdot 10^{-2})$	$2.90 \cdot 10^0 (2.59 \cdot 10^{-1})$	$3.31 \cdot 10^0 (2.91 \cdot 10^{-1})$	$3.13 \cdot 10^0 (2.98 \cdot 10^{-1})$	$2.84 \cdot 10^0 (6.20 \cdot 10^{-2})$	$3.29 \cdot 10^0 (2.76 \cdot 10^{-1})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$1.62 \cdot 10^0 (1.16 \cdot 10^{-1})$

### SM-2.2.20 GKLS $C^1$ , $n = 100$ , $d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.09 \cdot 10^{-2} (2.86 \cdot 10^{-3})$	$5.01 \cdot 10^{-2} (2.85 \cdot 10^{-3})$	$4.97 \cdot 10^{-2} (2.80 \cdot 10^{-3})$	$4.88 \cdot 10^{-2} (2.78 \cdot 10^{-3})$	$2.40 \cdot 10^{-1} (7.22 \cdot 10^{-3})$	$4.98 \cdot 10^{-2} (2.81 \cdot 10^{-3})$
$v = 3/2$	$3.88 \cdot 10^{-2} (2.84 \cdot 10^{-3})$	$4.20 \cdot 10^{-2} (2.87 \cdot 10^{-3})$	$4.45 \cdot 10^{-2} (3.07 \cdot 10^{-3})$	$3.99 \cdot 10^{-2} (2.77 \cdot 10^{-3})$	$1.95 \cdot 10^{-1} (7.54 \cdot 10^{-3})$	$5.09 \cdot 10^{-2} (6.25 \cdot 10^{-3})$
$v = 5/2$	$4.95 \cdot 10^{-2} (3.52 \cdot 10^{-3})$	$6.21 \cdot 10^{-2} (4.55 \cdot 10^{-3})$	$6.26 \cdot 10^{-2} (4.93 \cdot 10^{-3})$	$5.73 \cdot 10^{-2} (3.95 \cdot 10^{-3})$	$1.94 \cdot 10^{-1} (9.20 \cdot 10^{-3})$	$7.31 \cdot 10^{-2} (6.32 \cdot 10^{-3})$
$v = 7/2$	$5.66 \cdot 10^{-2} (3.95 \cdot 10^{-3})$	$7.59 \cdot 10^{-2} (6.34 \cdot 10^{-3})$	$7.81 \cdot 10^{-2} (6.20 \cdot 10^{-3})$	$7.02 \cdot 10^{-2} (5.38 \cdot 10^{-3})$	$1.86 \cdot 10^{-1} (8.38 \cdot 10^{-3})$	$9.85 \cdot 10^{-2} (9.02 \cdot 10^{-3})$
$v = 9/2$	$6.11 \cdot 10^{-2} (4.23 \cdot 10^{-3})$	$8.66 \cdot 10^{-2} (7.31 \cdot 10^{-3})$	$8.60 \cdot 10^{-2} (6.71 \cdot 10^{-3})$	$7.61 \cdot 10^{-2} (5.94 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (8.02 \cdot 10^{-3})$	$1.06 \cdot 10^{-1} (9.57 \cdot 10^{-3})$
$v = 11/2$	$6.42 \cdot 10^{-2} (4.40 \cdot 10^{-3})$	$9.18 \cdot 10^{-2} (9.21 \cdot 10^{-3})$	$9.16 \cdot 10^{-2} (7.47 \cdot 10^{-3})$	$8.18 \cdot 10^{-2} (6.49 \cdot 10^{-3})$	$1.81 \cdot 10^{-1} (8.02 \cdot 10^{-3})$	$1.08 \cdot 10^{-1} (9.51 \cdot 10^{-3})$
$v = 21/2$	$7.14 \cdot 10^{-2} (4.92 \cdot 10^{-3})$	$1.06 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$1.02 \cdot 10^{-1} (8.48 \cdot 10^{-3})$	$1.02 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$1.84 \cdot 10^{-1} (8.66 \cdot 10^{-3})$	$1.24 \cdot 10^{-1} (1.12 \cdot 10^{-2})$
$v = \infty$	$7.97 \cdot 10^{-2} (5.46 \cdot 10^{-3})$	$1.10 \cdot 10^{-1} (9.69 \cdot 10^{-3})$	$1.12 \cdot 10^{-1} (9.39 \cdot 10^{-3})$	$1.08 \cdot 10^{-1} (9.42 \cdot 10^{-3})$	$1.88 \cdot 10^{-1} (9.64 \cdot 10^{-3})$	$1.31 \cdot 10^{-1} (1.18 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$3.98 \cdot 10^{-2} (3.12 \cdot 10^{-3})$	$5.75 \cdot 10^{-2} (5.38 \cdot 10^{-3})$	$8.20 \cdot 10^{-2} (8.92 \cdot 10^{-3})$	$4.72 \cdot 10^{-2} (3.49 \cdot 10^{-3})$	$1.88 \cdot 10^{-1} (9.64 \cdot 10^{-3})$	$1.14 \cdot 10^{-1} (1.06 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$4.73 \cdot 10^{-2} (2.65 \cdot 10^{-3})$
$v = 3/2$	$3.16 \cdot 10^{-2} (2.06 \cdot 10^{-3})$
$v = 5/2$	$3.77 \cdot 10^{-2} (2.42 \cdot 10^{-3})$
$v = 7/2$	$4.29 \cdot 10^{-2} (2.71 \cdot 10^{-3})$
$v = 9/2$	$4.64 \cdot 10^{-2} (2.98 \cdot 10^{-3})$
$v = 11/2$	$4.85 \cdot 10^{-2} (3.12 \cdot 10^{-3})$
$v = 21/2$	$5.32 \cdot 10^{-2} (3.37 \cdot 10^{-3})$
$v = \infty$	$5.85 \cdot 10^{-2} (3.75 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$3.10 \cdot 10^{-2} (2.02 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.13 \cdot 10^{-2} (3.37 \cdot 10^{-3})$
linear	$3.80 \cdot 10^{-1} (1.92 \cdot 10^{-2})$
INN	$3.67 \cdot 10^{-1} (6.50 \cdot 10^{-3})$
2NN	$2.73 \cdot 10^{-1} (5.28 \cdot 10^{-3})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.14 \cdot 10^{-1} (2.40 \cdot 10^{-3})$	$1.04 \cdot 10^{-1} (2.54 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (2.51 \cdot 10^{-3})$	$9.83 \cdot 10^{-2} (2.37 \cdot 10^{-3})$	$2.66 \cdot 10^{-1} (3.93 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (2.51 \cdot 10^{-3})$
$v = 3/2$	$7.97 \cdot 10^{-2} (3.15 \cdot 10^{-3})$	$8.37 \cdot 10^{-2} (3.22 \cdot 10^{-3})$	$8.50 \cdot 10^{-2} (3.32 \cdot 10^{-3})$	$7.70 \cdot 10^{-2} (3.04 \cdot 10^{-3})$	$2.35 \cdot 10^{-1} (4.53 \cdot 10^{-3})$	$8.96 \cdot 10^{-2} (4.08 \cdot 10^{-3})$
$v = 5/2$	$9.10 \cdot 10^{-2} (3.75 \cdot 10^{-3})$	$1.02 \cdot 10^{-1} (4.36 \cdot 10^{-3})$	$1.00 \cdot 10^{-1} (4.33 \cdot 10^{-3})$	$9.70 \cdot 10^{-2} (4.03 \cdot 10^{-3})$	$2.33 \cdot 10^{-1} (5.43 \cdot 10^{-3})$	$1.07 \cdot 10^{-1} (4.92 \cdot 10^{-3})$
$v = 7/2$	$9.97 \cdot 10^{-2} (3.96 \cdot 10^{-3})$	$1.14 \cdot 10^{-1} (5.08 \cdot 10^{-3})$	$1.14 \cdot 10^{-1} (4.92 \cdot 10^{-3})$	$1.10 \cdot 10^{-1} (4.50 \cdot 10^{-3})$	$2.27 \cdot 10^{-1} (5.01 \cdot 10^{-3})$	$1.25 \cdot 10^{-1} (5.93 \cdot 10^{-3})$
$v = 9/2$	$1.05 \cdot 10^{-1} (4.08 \cdot 10^{-3})$	$1.23 \cdot 10^{-1} (5.56 \cdot 10^{-3})$	$1.22 \cdot 10^{-1} (5.31 \cdot 10^{-3})$	$1.16 \cdot 10^{-1} (4.80 \cdot 10^{-3})$	$2.24 \cdot 10^{-1} (4.81 \cdot 10^{-3})$	$1.31 \cdot 10^{-1} (6.17 \cdot 10^{-3})$
$v = 11/2$	$1.09 \cdot 10^{-1} (4.15 \cdot 10^{-3})$	$1.27 \cdot 10^{-1} (5.77 \cdot 10^{-3})$	$1.27 \cdot 10^{-1} (5.61 \cdot 10^{-3})$	$1.22 \cdot 10^{-1} (5.05 \cdot 10^{-3})$	$2.23 \cdot 10^{-1} (4.81 \cdot 10^{-3})$	$1.34 \cdot 10^{-1} (6.23 \cdot 10^{-3})$
$v = 21/2$	$1.17 \cdot 10^{-1} (4.45 \cdot 10^{-3})$	$1.36 \cdot 10^{-1} (6.42 \cdot 10^{-3})$	$1.35 \cdot 10^{-1} (5.95 \cdot 10^{-3})$	$1.36 \cdot 10^{-1} (6.46 \cdot 10^{-3})$	$2.23 \cdot 10^{-1} (5.11 \cdot 10^{-3})$	$1.45 \cdot 10^{-1} (6.87 \cdot 10^{-3})$
$v = \infty$	$1.26 \cdot 10^{-1} (4.74 \cdot 10^{-3})$	$1.44 \cdot 10^{-1} (6.38 \cdot 10^{-3})$	$1.43 \cdot 10^{-1} (6.33 \cdot 10^{-3})$	$1.44 \cdot 10^{-1} (6.32 \cdot 10^{-3})$	$2.22 \cdot 10^{-1} (5.49 \cdot 10^{-3})$	$1.51 \cdot 10^{-1} (7.15 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$7.96 \cdot 10^{-2} (3.31 \cdot 10^{-3})$	$9.66 \cdot 10^{-2} (4.39 \cdot 10^{-3})$	$1.13 \cdot 10^{-1} (5.94 \cdot 10^{-3})$	$8.40 \cdot 10^{-2} (3.70 \cdot 10^{-3})$	$2.22 \cdot 10^{-1} (5.49 \cdot 10^{-3})$	$1.37 \cdot 10^{-1} (6.79 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$8.26 \cdot 10^{-2} (3.55 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.44 \cdot 10^0 (3.16 \cdot 10^{-2})$	$1.33 \cdot 10^0 (3.62 \cdot 10^{-2})$	$1.33 \cdot 10^0 (3.61 \cdot 10^{-2})$	$1.43 \cdot 10^0 (4.15 \cdot 10^{-2})$	$2.47 \cdot 10^0 (3.77 \cdot 10^{-2})$	$1.34 \cdot 10^0 (3.61 \cdot 10^{-2})$
$v = 3/2$	$1.22 \cdot 10^0 (4.25 \cdot 10^{-2})$	$1.29 \cdot 10^0 (4.49 \cdot 10^{-2})$	$1.34 \cdot 10^0 (4.83 \cdot 10^{-2})$	$1.45 \cdot 10^0 (5.56 \cdot 10^{-2})$	$2.29 \cdot 10^0 (4.73 \cdot 10^{-2})$	$1.38 \cdot 10^0 (6.12 \cdot 10^{-2})$
$v = 5/2$	$1.38 \cdot 10^0 (4.64 \cdot 10^{-2})$	$1.58 \cdot 10^0 (6.06 \cdot 10^{-2})$	$1.62 \cdot 10^0 (6.14 \cdot 10^{-2})$	$1.83 \cdot 10^0 (7.17 \cdot 10^{-2})$	$2.30 \cdot 10^0 (5.90 \cdot 10^{-2})$	$1.73 \cdot 10^0 (7.53 \cdot 10^{-2})$
$v = 7/2$	$1.48 \cdot 10^0 (4.78 \cdot 10^{-2})$	$1.73 \cdot 10^0 (7.54 \cdot 10^{-2})$	$1.80 \cdot 10^0 (7.24 \cdot 10^{-2})$	$1.96 \cdot 10^0 (8.19 \cdot 10^{-2})$	$2.25 \cdot 10^0 (5.29 \cdot 10^{-2})$	$1.94 \cdot 10^0 (8.90 \cdot 10^{-2})$
$v = 9/2$	$1.54 \cdot 10^0 (4.90 \cdot 10^{-2})$	$1.86 \cdot 10^0 (8.27 \cdot 10^{-2})$	$1.89 \cdot 10^0 (7.04 \cdot 10^{-2})$	$2.02 \cdot 10^0 (8.19 \cdot 10^{-2})$	$2.22 \cdot 10^0 (5.06 \cdot 10^{-2})$	$2.00 \cdot 10^0 (9.10 \cdot 10^{-2})$
$v = 11/2$	$1.58 \cdot 10^0 (4.95 \cdot 10^{-2})$	$1.86 \cdot 10^0 (7.94 \cdot 10^{-2})$	$1.93 \cdot 10^0 (7.77 \cdot 10^{-2})$	$2.07 \cdot 10^0 (8.62 \cdot 10^{-2})$	$2.21 \cdot 10^0 (5.06 \cdot 10^{-2})$	$2.03 \cdot 10^0 (9.08 \cdot 10^{-2})$
$v = 21/2$	$1.68 \cdot 10^0 (5.12 \cdot 10^{-2})$	$1.99 \cdot 10^0 (1.01 \cdot 10^{-1})$	$2.06 \cdot 10^0 (8.17 \cdot 10^{-2})$	$2.32 \cdot 10^0 (1.29 \cdot 10^{-1})$	$2.22 \cdot 10^0 (5.40 \cdot 10^{-2})$	$2.18 \cdot 10^0 (9.93 \cdot 10^{-2})$
$v = \infty$	$1.80 \cdot 10^0 (5.39 \cdot 10^{-2})$	$2.06 \cdot 10^0 (8.82 \cdot 10^{-2})$	$2.16 \cdot 10^0 (8.92 \cdot 10^{-2})$	$2.40 \cdot 10^0 (1.10 \cdot 10^{-1})$	$2.23 \cdot 10^0 (5.81 \cdot 10^{-2})$	$2.24 \cdot 10^0 (1.03 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$1.25 \cdot 10^0 (4.37 \cdot 10^{-2})$	$1.56 \cdot 10^0 (6.88 \cdot 10^{-2})$	$1.86 \cdot 10^0 (9.26 \cdot 10^{-2})$	$1.64 \cdot 10^0 (7.03 \cdot 10^{-2})$	$2.23 \cdot 10^0 (5.81 \cdot 10^{-2})$	$2.13 \cdot 10^0 (9.87 \cdot 10^{-2})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.28 \cdot 10^0 (4.53 \cdot 10^{-2})$

### SM-2.2.21 GKLS $C^1$ , $n = 250$ , $d = 5$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.59 \cdot 10^{-2} (1.68 \cdot 10^{-3})$	$2.62 \cdot 10^{-2} (1.75 \cdot 10^{-3})$	$2.62 \cdot 10^{-2} (1.71 \cdot 10^{-3})$	$2.56 \cdot 10^{-2} (1.66 \cdot 10^{-3})$	$1.50 \cdot 10^{-1} (5.68 \cdot 10^{-3})$	$2.63 \cdot 10^{-2} (1.68 \cdot 10^{-3})$
$v = 3/2$	$2.12 \cdot 10^{-2} (1.60 \cdot 10^{-3})$	$2.28 \cdot 10^{-2} (1.65 \cdot 10^{-3})$	$2.40 \cdot 10^{-2} (1.83 \cdot 10^{-3})$	$2.25 \cdot 10^{-2} (1.80 \cdot 10^{-3})$	$1.16 \cdot 10^{-1} (5.59 \cdot 10^{-3})$	$2.56 \cdot 10^{-2} (1.93 \cdot 10^{-3})$
$v = 5/2$	$2.63 \cdot 10^{-2} (2.00 \cdot 10^{-3})$	$3.10 \cdot 10^{-2} (2.39 \cdot 10^{-3})$	$3.28 \cdot 10^{-2} (2.77 \cdot 10^{-3})$	$3.11 \cdot 10^{-2} (2.63 \cdot 10^{-3})$	$1.06 \cdot 10^{-1} (5.44 \cdot 10^{-3})$	$3.63 \cdot 10^{-2} (2.79 \cdot 10^{-3})$
$v = 7/2$	$3.01 \cdot 10^{-2} (2.22 \cdot 10^{-3})$	$3.58 \cdot 10^{-2} (2.66 \cdot 10^{-3})$	$3.80 \cdot 10^{-2} (2.88 \cdot 10^{-3})$	$3.61 \cdot 10^{-2} (2.86 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (5.58 \cdot 10^{-3})$	$4.36 \cdot 10^{-2} (3.24 \cdot 10^{-3})$
$v = 9/2$	$3.29 \cdot 10^{-2} (2.35 \cdot 10^{-3})$	$3.87 \cdot 10^{-2} (2.81 \cdot 10^{-3})$	$4.22 \cdot 10^{-2} (3.16 \cdot 10^{-3})$	$3.86 \cdot 10^{-2} (2.81 \cdot 10^{-3})$	$9.70 \cdot 10^{-2} (5.59 \cdot 10^{-3})$	$4.89 \cdot 10^{-2} (3.72 \cdot 10^{-3})$
$v = 11/2$	$3.50 \cdot 10^{-2} (2.44 \cdot 10^{-3})$	$4.19 \cdot 10^{-2} (3.00 \cdot 10^{-3})$	$4.41 \cdot 10^{-2} (3.26 \cdot 10^{-3})$	$4.11 \cdot 10^{-2} (2.94 \cdot 10^{-3})$	$9.18 \cdot 10^{-2} (5.27 \cdot 10^{-3})$	$5.31 \cdot 10^{-2} (4.05 \cdot 10^{-3})$
$v = 21/2$	$4.07 \cdot 10^{-2} (2.70 \cdot 10^{-3})$	$4.90 \cdot 10^{-2} (3.39 \cdot 10^{-3})$	$5.04 \cdot 10^{-2} (3.74 \cdot 10^{-3})$	$4.70 \cdot 10^{-2} (3.26 \cdot 10^{-3})$	$7.95 \cdot 10^{-2} (4.63 \cdot 10^{-3})$	$6.18 \cdot 10^{-2} (4.52 \cdot 10^{-3})$
$v = \infty$	$4.81 \cdot 10^{-2} (3.08 \cdot 10^{-3})$	$5.80 \cdot 10^{-2} (3.71 \cdot 10^{-3})$	$5.89 \cdot 10^{-2} (4.11 \cdot 10^{-3})$	$5.54 \cdot 10^{-2} (3.58 \cdot 10^{-3})$	$8.03 \cdot 10^{-2} (5.57 \cdot 10^{-3})$	$7.32 \cdot 10^{-2} (5.37 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$2.18 \cdot 10^{-2} (1.89 \cdot 10^{-3})$	$2.50 \cdot 10^{-2} (2.25 \cdot 10^{-3})$	$3.20 \cdot 10^{-2} (2.95 \cdot 10^{-3})$	$2.29 \cdot 10^{-2} (2.13 \cdot 10^{-3})$	$8.03 \cdot 10^{-2} (5.57 \cdot 10^{-3})$	$5.26 \cdot 10^{-2} (4.88 \cdot 10^{-3})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.50 \cdot 10^{-2} (1.59 \cdot 10^{-3})$
$v = 3/2$	$1.81 \cdot 10^{-2} (1.25 \cdot 10^{-3})$
$v = 5/2$	$2.12 \cdot 10^{-2} (1.33 \cdot 10^{-3})$
$v = 7/2$	$2.42 \cdot 10^{-2} (1.47 \cdot 10^{-3})$
$v = 9/2$	$2.63 \cdot 10^{-2} (1.57 \cdot 10^{-3})$
$v = 11/2$	$2.80 \cdot 10^{-2} (1.66 \cdot 10^{-3})$
$v = 21/2$	$3.22 \cdot 10^{-2} (1.89 \cdot 10^{-3})$
$v = \infty$	$3.79 \cdot 10^{-2} (2.22 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$1.78 \cdot 10^{-2} (1.20 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$2.18 \cdot 10^{-2} (1.91 \cdot 10^{-3})$
linear	$3.78 \cdot 10^{-1} (1.91 \cdot 10^{-2})$
1NN	$2.64 \cdot 10^{-1} (5.03 \cdot 10^{-3})$
2NN	$1.97 \cdot 10^{-1} (4.07 \cdot 10^{-3})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$7.64 \cdot 10^{-2} (1.81 \cdot 10^{-3})$	$6.87 \cdot 10^{-2} (1.90 \cdot 10^{-3})$	$6.86 \cdot 10^{-2} (1.87 \cdot 10^{-3})$	$6.31 \cdot 10^{-2} (1.71 \cdot 10^{-3})$	$2.06 \cdot 10^{-1} (3.69 \cdot 10^{-3})$	$6.88 \cdot 10^{-2} (1.86 \cdot 10^{-3})$
$v = 3/2$	$5.61 \cdot 10^{-2} (2.14 \cdot 10^{-3})$	$5.87 \cdot 10^{-2} (2.17 \cdot 10^{-3})$	$5.93 \cdot 10^{-2} (2.29 \cdot 10^{-3})$	$5.35 \cdot 10^{-2} (2.16 \cdot 10^{-3})$	$1.76 \cdot 10^{-1} (4.34 \cdot 10^{-3})$	$6.14 \cdot 10^{-2} (2.41 \cdot 10^{-3})$
$v = 5/2$	$6.36 \cdot 10^{-2} (2.40 \cdot 10^{-3})$	$6.98 \cdot 10^{-2} (2.72 \cdot 10^{-3})$	$7.02 \cdot 10^{-2} (2.81 \cdot 10^{-3})$	$6.69 \cdot 10^{-2} (2.70 \cdot 10^{-3})$	$1.65 \cdot 10^{-1} (4.55 \cdot 10^{-3})$	$7.37 \cdot 10^{-2} (3.04 \cdot 10^{-3})$
$v = 7/2$	$6.97 \cdot 10^{-2} (2.52 \cdot 10^{-3})$	$7.73 \cdot 10^{-2} (2.83 \cdot 10^{-3})$	$7.77 \cdot 10^{-2} (2.91 \cdot 10^{-3})$	$7.54 \cdot 10^{-2} (2.86 \cdot 10^{-3})$	$1.59 \cdot 10^{-1} (4.73 \cdot 10^{-3})$	$8.34 \cdot 10^{-2} (3.34 \cdot 10^{-3})$
$v = 9/2$	$7.42 \cdot 10^{-2} (2.61 \cdot 10^{-3})$	$8.16 \cdot 10^{-2} (2.91 \cdot 10^{-3})$	$8.31 \cdot 10^{-2} (3.09 \cdot 10^{-3})$	$8.00 \cdot 10^{-2} (2.84 \cdot 10^{-3})$	$1.54 \cdot 10^{-1} (4.67 \cdot 10^{-3})$	$8.92 \cdot 10^{-2} (3.54 \cdot 10^{-3})$
$v = 11/2$	$7.73 \cdot 10^{-2} (2.66 \cdot 10^{-3})$	$8.56 \cdot 10^{-2} (3.05 \cdot 10^{-3})$	$8.59 \cdot 10^{-2} (3.12 \cdot 10^{-3})$	$8.36 \cdot 10^{-2} (2.95 \cdot 10^{-3})$	$1.48 \cdot 10^{-1} (4.36 \cdot 10^{-3})$	$9.39 \cdot 10^{-2} (3.67 \cdot 10^{-3})$
$v = 21/2$	$8.51 \cdot 10^{-2} (2.85 \cdot 10^{-3})$	$9.38 \cdot 10^{-2} (3.30 \cdot 10^{-3})$	$9.37 \cdot 10^{-2} (3.44 \cdot 10^{-3})$	$9.14 \cdot 10^{-2} (3.14 \cdot 10^{-3})$	$1.32 \cdot 10^{-1} (3.75 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (3.95 \cdot 10^{-3})$
$v = \infty$	$9.44 \cdot 10^{-2} (3.11 \cdot 10^{-3})$	$1.04 \cdot 10^{-1} (3.47 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (3.67 \cdot 10^{-3})$	$1.02 \cdot 10^{-1} (3.43 \cdot 10^{-3})$	$1.29 \cdot 10^{-1} (4.34 \cdot 10^{-3})$	$1.13 \cdot 10^{-1} (4.33 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$5.65 \cdot 10^{-2} (2.20 \cdot 10^{-3})$	$6.08 \cdot 10^{-2} (2.35 \cdot 10^{-3})$	$6.72 \cdot 10^{-2} (2.83 \cdot 10^{-3})$	$5.39 \cdot 10^{-2} (2.19 \cdot 10^{-3})$	$1.29 \cdot 10^{-1} (4.34 \cdot 10^{-3})$	$8.81 \cdot 10^{-2} (4.37 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$5.70 \cdot 10^{-2} (2.23 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.08 \cdot 10^0 (2.60 \cdot 10^{-2})$	$1.00 \cdot 10^0 (2.87 \cdot 10^{-2})$	$1.00 \cdot 10^0 (2.86 \cdot 10^{-2})$	$1.11 \cdot 10^0 (3.53 \cdot 10^{-2})$	$2.00 \cdot 10^0 (3.53 \cdot 10^{-2})$	$1.01 \cdot 10^0 (2.86 \cdot 10^{-2})$
$v = 3/2$	$9.16 \cdot 10^{-1} (3.08 \cdot 10^{-2})$	$9.58 \cdot 10^{-1} (3.18 \cdot 10^{-2})$	$9.82 \cdot 10^{-1} (3.53 \cdot 10^{-2})$	$1.13 \cdot 10^0 (4.47 \cdot 10^{-2})$	$1.79 \cdot 10^0 (4.63 \cdot 10^{-2})$	$1.01 \cdot 10^0 (3.72 \cdot 10^{-2})$
$v = 5/2$	$1.00 \cdot 10^0 (3.44 \cdot 10^{-2})$	$1.10 \cdot 10^0 (4.06 \cdot 10^{-2})$	$1.13 \cdot 10^0 (4.33 \cdot 10^{-2})$	$1.30 \cdot 10^0 (5.42 \cdot 10^{-2})$	$1.72 \cdot 10^0 (5.03 \cdot 10^{-2})$	$1.18 \cdot 10^0 (4.44 \cdot 10^{-2})$
$v = 7/2$	$1.06 \cdot 10^0 (3.58 \cdot 10^{-2})$	$1.16 \cdot 10^0 (4.05 \cdot 10^{-2})$	$1.19 \cdot 10^0 (4.36 \cdot 10^{-2})$	$1.35 \cdot 10^0 (5.35 \cdot 10^{-2})$	$1.68 \cdot 10^0 (5.19 \cdot 10^{-2})$	$1.26 \cdot 10^0 (4.56 \cdot 10^{-2})$
$v = 9/2$	$1.10 \cdot 10^0 (3.66 \cdot 10^{-2})$	$1.18 \cdot 10^0 (4.02 \cdot 10^{-2})$	$1.24 \cdot 10^0 (4.56 \cdot 10^{-2})$	$1.36 \cdot 10^0 (4.93 \cdot 10^{-2})$	$1.64 \cdot 10^0 (5.19 \cdot 10^{-2})$	$1.31 \cdot 10^0 (4.69 \cdot 10^{-2})$
$v = 11/2$	$1.12 \cdot 10^0 (3.70 \cdot 10^{-2})$	$1.22 \cdot 10^0 (4.21 \cdot 10^{-2})$	$1.26 \cdot 10^0 (4.59 \cdot 10^{-2})$	$1.38 \cdot 10^0 (5.00 \cdot 10^{-2})$	$1.59 \cdot 10^0 (5.00 \cdot 10^{-2})$	$1.34 \cdot 10^0 (4.89 \cdot 10^{-2})$
$v = 21/2$	$1.20 \cdot 10^0 (3.83 \cdot 10^{-2})$	$1.28 \cdot 10^0 (4.43 \cdot 10^{-2})$	$1.30 \cdot 10^0 (4.89 \cdot 10^{-2})$	$1.43 \cdot 10^0 (5.12 \cdot 10^{-2})$	$1.51 \cdot 10^0 (4.91 \cdot 10^{-2})$	$1.42 \cdot 10^0 (5.10 \cdot 10^{-2})$
$v = \infty$	$1.30 \cdot 10^0 (4.03 \cdot 10^{-2})$	$1.37 \cdot 10^0 (4.43 \cdot 10^{-2})$	$1.37 \cdot 10^0 (4.86 \cdot 10^{-2})$	$1.52 \cdot 10^0 (5.27 \cdot 10^{-2})$	$1.52 \cdot 10^0 (5.62 \cdot 10^{-2})$	$1.50 \cdot 10^0 (5.56 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$9.28 \cdot 10^{-1} (3.38 \cdot 10^{-2})$	$9.95 \cdot 10^{-1} (3.66 \cdot 10^{-2})$	$1.11 \cdot 10^0 (4.43 \cdot 10^{-2})$	$1.13 \cdot 10^0 (4.58 \cdot 10^{-2})$	$1.52 \cdot 10^0 (5.62 \cdot 10^{-2})$	$1.31 \cdot 10^0 (5.27 \cdot 10^{-2})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$9.32 \cdot 10^{-1} (3.41 \cdot 10^{-2})$

### SM-2.2.22 GKLS $C^2$ , $n = 20$ , $d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.56 \cdot 10^{-1} (2.06 \cdot 10^{-2})$	$3.59 \cdot 10^{-1} (2.09 \cdot 10^{-2})$	$3.79 \cdot 10^{-1} (2.67 \cdot 10^{-2})$	$3.51 \cdot 10^{-1} (1.86 \cdot 10^{-2})$	$4.59 \cdot 10^{-1} (2.13 \cdot 10^{-2})$	$3.96 \cdot 10^{-1} (2.56 \cdot 10^{-2})$
$v = 3/2$	$4.03 \cdot 10^{-1} (2.50 \cdot 10^{-2})$	$4.54 \cdot 10^{-1} (2.89 \cdot 10^{-2})$	$4.76 \cdot 10^{-1} (3.14 \cdot 10^{-2})$	$4.15 \cdot 10^{-1} (2.15 \cdot 10^{-2})$	$4.84 \cdot 10^{-1} (2.55 \cdot 10^{-2})$	$5.95 \cdot 10^{-1} (4.06 \cdot 10^{-2})$
$v = 5/2$	$4.38 \cdot 10^{-1} (2.70 \cdot 10^{-2})$	$5.11 \cdot 10^{-1} (3.63 \cdot 10^{-2})$	$5.62 \cdot 10^{-1} (4.00 \cdot 10^{-2})$	$4.52 \cdot 10^{-1} (2.46 \cdot 10^{-2})$	$4.95 \cdot 10^{-1} (2.54 \cdot 10^{-2})$	$7.35 \cdot 10^{-1} (5.61 \cdot 10^{-2})$
$v = 7/2$	$4.51 \cdot 10^{-1} (2.72 \cdot 10^{-2})$	$5.52 \cdot 10^{-1} (4.46 \cdot 10^{-2})$	$6.71 \cdot 10^{-1} (8.36 \cdot 10^{-2})$	$5.36 \cdot 10^{-1} (4.64 \cdot 10^{-2})$	$5.22 \cdot 10^{-1} (2.87 \cdot 10^{-2})$	$9.98 \cdot 10^{-1} (1.07 \cdot 10^{-1})$
$v = 9/2$	$4.56 \cdot 10^{-1} (2.72 \cdot 10^{-2})$	$5.57 \cdot 10^{-1} (4.45 \cdot 10^{-2})$	$6.90 \cdot 10^{-1} (7.91 \cdot 10^{-2})$	$5.76 \cdot 10^{-1} (6.68 \cdot 10^{-2})$	$5.68 \cdot 10^{-1} (3.73 \cdot 10^{-2})$	$1.49 \cdot 10^0 (3.71 \cdot 10^{-1})$
$v = \infty$	$4.77 \cdot 10^{-1} (2.68 \cdot 10^{-2})$	$7.30 \cdot 10^{-1} (1.12 \cdot 10^{-1})$	$8.84 \cdot 10^{-1} (1.34 \cdot 10^{-1})$	$6.49 \cdot 10^{-1} (8.39 \cdot 10^{-2})$	$1.17 \cdot 10^0 (2.62 \cdot 10^{-1})$	$1.90 \cdot 10^0 (4.10 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$4.31 \cdot 10^{-1} (2.77 \cdot 10^{-2})$	$5.92 \cdot 10^{-1} (6.18 \cdot 10^{-2})$	$1.03 \cdot 10^0 (2.16 \cdot 10^{-1})$	$6.22 \cdot 10^{-1} (8.42 \cdot 10^{-2})$	$9.49 \cdot 10^{-1} (2.56 \cdot 10^{-1})$	$1.80 \cdot 10^0 (3.83 \cdot 10^{-1})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.81 \cdot 10^{-1} (1.48 \cdot 10^{-2})$
$v = 3/2$	$2.75 \cdot 10^{-1} (1.36 \cdot 10^{-2})$
$v = 5/2$	$2.84 \cdot 10^{-1} (1.31 \cdot 10^{-2})$
$v = 7/2$	$2.93 \cdot 10^{-1} (1.31 \cdot 10^{-2})$
$v = 9/2$	$2.97 \cdot 10^{-1} (1.34 \cdot 10^{-2})$
$v = \infty$	$3.14 \cdot 10^{-1} (1.25 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$2.52 \cdot 10^{-1} (1.26 \cdot 10^{-2})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.08 \cdot 10^{-1} (2.29 \cdot 10^{-2})$
linear	$7.12 \cdot 10^{-1} (2.27 \cdot 10^{-2})$
1NN	$4.53 \cdot 10^{-1} (2.02 \cdot 10^{-2})$
2NN	$4.20 \cdot 10^{-1} (1.73 \cdot 10^{-2})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.70 \cdot 10^{-1} (8.16 \cdot 10^{-3})$	$2.70 \cdot 10^{-1} (8.41 \cdot 10^{-3})$	$2.75 \cdot 10^{-1} (9.99 \cdot 10^{-3})$	$2.63 \cdot 10^{-1} (7.87 \cdot 10^{-3})$	$3.32 \cdot 10^{-1} (8.31 \cdot 10^{-3})$	$2.83 \cdot 10^{-1} (1.00 \cdot 10^{-2})$
$v = 3/2$	$2.86 \cdot 10^{-1} (9.70 \cdot 10^{-3})$	$3.14 \cdot 10^{-1} (1.13 \cdot 10^{-2})$	$3.20 \cdot 10^{-1} (1.30 \cdot 10^{-2})$	$2.98 \cdot 10^{-1} (9.76 \cdot 10^{-3})$	$3.38 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$3.63 \cdot 10^{-1} (1.46 \cdot 10^{-2})$
$v = 5/2$	$3.02 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$3.33 \cdot 10^{-1} (1.14 \cdot 10^{-2})$	$3.45 \cdot 10^{-1} (1.32 \cdot 10^{-2})$	$3.20 \cdot 10^{-1} (1.08 \cdot 10^{-2})$	$3.40 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$3.96 \cdot 10^{-1} (1.61 \cdot 10^{-2})$
$v = 7/2$	$3.10 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$3.45 \cdot 10^{-1} (1.21 \cdot 10^{-2})$	$3.61 \cdot 10^{-1} (1.48 \cdot 10^{-2})$	$3.40 \cdot 10^{-1} (1.26 \cdot 10^{-2})$	$3.46 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$4.33 \cdot 10^{-1} (1.90 \cdot 10^{-2})$
$v = 9/2$	$3.12 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$3.47 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$3.70 \cdot 10^{-1} (1.61 \cdot 10^{-2})$	$3.47 \cdot 10^{-1} (1.46 \cdot 10^{-2})$	$3.58 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$4.73 \cdot 10^{-1} (2.48 \cdot 10^{-2})$
$v = \infty$	$3.26 \cdot 10^{-1} (9.35 \cdot 10^{-3})$	$3.78 \cdot 10^{-1} (1.75 \cdot 10^{-2})$	$4.08 \cdot 10^{-1} (2.09 \cdot 10^{-2})$	$3.68 \cdot 10^{-1} (1.71 \cdot 10^{-2})$	$4.45 \cdot 10^{-1} (2.37 \cdot 10^{-2})$	$5.35 \cdot 10^{-1} (3.16 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$2.96 \cdot 10^{-1} (1.04 \cdot 10^{-2})$	$3.51 \cdot 10^{-1} (1.45 \cdot 10^{-2})$	$4.07 \cdot 10^{-1} (2.51 \cdot 10^{-2})$	$3.53 \cdot 10^{-1} (1.89 \cdot 10^{-2})$	$4.03 \cdot 10^{-1} (2.07 \cdot 10^{-2})$	$5.31 \cdot 10^{-1} (2.89 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$2.96 \cdot 10^{-1} (9.69 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.97 \cdot 10^0 (1.41 \cdot 10^{-1})$	$4.18 \cdot 10^0 (1.48 \cdot 10^{-1})$	$4.34 \cdot 10^0 (1.99 \cdot 10^{-1})$	$4.45 \cdot 10^0 (1.62 \cdot 10^{-1})$	$4.72 \cdot 10^0 (1.73 \cdot 10^{-1})$	$4.41 \cdot 10^0 (1.75 \cdot 10^{-1})$
$v = 3/2$	$4.98 \cdot 10^0 (2.19 \cdot 10^{-1})$	$5.68 \cdot 10^0 (2.82 \cdot 10^{-1})$	$6.05 \cdot 10^0 (3.55 \cdot 10^{-1})$	$5.69 \cdot 10^0 (2.66 \cdot 10^{-1})$	$5.53 \cdot 10^0 (3.15 \cdot 10^{-1})$	$6.69 \cdot 10^0 (3.38 \cdot 10^{-1})$
$v = 5/2$	$5.40 \cdot 10^0 (2.30 \cdot 10^{-1})$	$6.06 \cdot 10^0 (2.93 \cdot 10^{-1})$	$6.51 \cdot 10^0 (3.89 \cdot 10^{-1})$	$6.32 \cdot 10^0 (3.50 \cdot 10^{-1})$	$5.32 \cdot 10^0 (2.60 \cdot 10^{-1})$	$7.17 \cdot 10^0 (3.69 \cdot 10^{-1})$
$v = 7/2$	$5.59 \cdot 10^0 (2.33 \cdot 10^{-1})$	$6.34 \cdot 10^0 (3.05 \cdot 10^{-1})$	$6.78 \cdot 10^0 (3.70 \cdot 10^{-1})$	$6.72 \cdot 10^0 (3.67 \cdot 10^{-1})$	$5.29 \cdot 10^0 (2.28 \cdot 10^{-1})$	$7.62 \cdot 10^0 (3.79 \cdot 10^{-1})$
$v = 9/2$	$5.63 \cdot 10^0 (2.25 \cdot 10^{-1})$	$6.28 \cdot 10^0 (2.88 \cdot 10^{-1})$	$7.00 \cdot 10^0 (4.13 \cdot 10^{-1})$	$6.88 \cdot 10^0 (4.24 \cdot 10^{-1})$	$5.44 \cdot 10^0 (2.24 \cdot 10^{-1})$	$8.25 \cdot 10^0 (5.00 \cdot 10^{-1})$
$v = \infty$	$5.91 \cdot 10^0 (2.10 \cdot 10^{-1})$	$6.75 \cdot 10^0 (3.84 \cdot 10^{-1})$	$7.95 \cdot 10^0 (5.38 \cdot 10^{-1})$	$7.41 \cdot 10^0 (5.25 \cdot 10^{-1})$	$6.41 \cdot 10^0 (2.93 \cdot 10^{-1})$	$9.66 \cdot 10^0 (6.66 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$5.36 \cdot 10^0 (2.24 \cdot 10^{-1})$	$6.72 \cdot 10^0 (3.51 \cdot 10^{-1})$	$8.27 \cdot 10^0 (5.69 \cdot 10^{-1})$	$7.72 \cdot 10^0 (5.63 \cdot 10^{-1})$	$5.90 \cdot 10^0 (2.78 \cdot 10^{-1})$	$9.79 \cdot 10^0 (6.49 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$5.32 \cdot 10^0 (2.27 \cdot 10^{-1})$



### SM-2.2.23 GKLS $C^2$ , $n = 40$ , $d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.77 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$1.74 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$1.74 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$1.67 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$2.63 \cdot 10^{-1} (1.46 \cdot 10^{-2})$	$1.81 \cdot 10^{-1} (1.13 \cdot 10^{-2})$
$v = 3/2$	$1.86 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$2.03 \cdot 10^{-1} (1.14 \cdot 10^{-2})$	$2.10 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$1.91 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$2.86 \cdot 10^{-1} (2.91 \cdot 10^{-2})$	$2.98 \cdot 10^{-1} (4.18 \cdot 10^{-2})$
$v = 5/2$	$1.97 \cdot 10^{-1} (1.03 \cdot 10^{-2})$	$2.34 \cdot 10^{-1} (1.51 \cdot 10^{-2})$	$2.92 \cdot 10^{-1} (5.45 \cdot 10^{-2})$	$2.12 \cdot 10^{-1} (1.26 \cdot 10^{-2})$	$3.21 \cdot 10^{-1} (3.19 \cdot 10^{-2})$	$3.44 \cdot 10^{-1} (2.79 \cdot 10^{-2})$
$v = 7/2$	$2.03 \cdot 10^{-1} (1.03 \cdot 10^{-2})$	$2.50 \cdot 10^{-1} (1.72 \cdot 10^{-2})$	$3.15 \cdot 10^{-1} (5.06 \cdot 10^{-2})$	$2.31 \cdot 10^{-1} (1.52 \cdot 10^{-2})$	$4.09 \cdot 10^{-1} (5.80 \cdot 10^{-2})$	$8.19 \cdot 10^{-1} (2.20 \cdot 10^{-1})$
$v = 9/2$	$2.09 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$2.52 \cdot 10^{-1} (1.74 \cdot 10^{-2})$	$3.30 \cdot 10^{-1} (4.57 \cdot 10^{-2})$	$2.40 \cdot 10^{-1} (1.67 \cdot 10^{-2})$	$5.71 \cdot 10^{-1} (1.06 \cdot 10^{-1})$	$1.55 \cdot 10^0 (5.61 \cdot 10^{-1})$
$v \in \{1/2, \dots, 9/2\}$	$1.93 \cdot 10^{-1} (1.05 \cdot 10^{-2})$	$2.26 \cdot 10^{-1} (1.81 \cdot 10^{-2})$	$3.22 \cdot 10^{-1} (5.67 \cdot 10^{-2})$	$2.03 \cdot 10^{-1} (1.49 \cdot 10^{-2})$	$4.51 \cdot 10^{-1} (7.99 \cdot 10^{-2})$	$1.56 \cdot 10^0 (5.61 \cdot 10^{-1})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.48 \cdot 10^{-1} (9.07 \cdot 10^{-3})$
$v = 3/2$	$1.34 \cdot 10^{-1} (7.67 \cdot 10^{-3})$
$v = 5/2$	$1.39 \cdot 10^{-1} (7.58 \cdot 10^{-3})$
$v = 7/2$	$1.43 \cdot 10^{-1} (7.65 \cdot 10^{-3})$
$v = 9/2$	$1.46 \cdot 10^{-1} (7.70 \cdot 10^{-3})$
$v \in \{1/2, \dots, 9/2\}$	$1.26 \cdot 10^{-1} (7.71 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.88 \cdot 10^{-1} (1.03 \cdot 10^{-2})$
linear	$6.90 \cdot 10^{-1} (2.15 \cdot 10^{-2})$
INN	$2.70 \cdot 10^{-1} (1.24 \cdot 10^{-2})$
2NN	$2.40 \cdot 10^{-1} (1.18 \cdot 10^{-2})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.97 \cdot 10^{-1} (4.68 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (4.48 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (4.54 \cdot 10^{-3})$	$1.76 \cdot 10^{-1} (4.54 \cdot 10^{-3})$	$2.48 \cdot 10^{-1} (6.46 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (5.24 \cdot 10^{-3})$
$v = 3/2$	$1.90 \cdot 10^{-1} (4.64 \cdot 10^{-3})$	$1.99 \cdot 10^{-1} (5.08 \cdot 10^{-3})$	$1.99 \cdot 10^{-1} (5.22 \cdot 10^{-3})$	$1.88 \cdot 10^{-1} (4.87 \cdot 10^{-3})$	$2.39 \cdot 10^{-1} (9.67 \cdot 10^{-3})$	$2.26 \cdot 10^{-1} (1.15 \cdot 10^{-2})$
$v = 5/2$	$1.95 \cdot 10^{-1} (4.69 \cdot 10^{-3})$	$2.08 \cdot 10^{-1} (5.88 \cdot 10^{-3})$	$2.17 \cdot 10^{-1} (9.56 \cdot 10^{-3})$	$1.99 \cdot 10^{-1} (5.35 \cdot 10^{-3})$	$2.52 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$2.36 \cdot 10^{-1} (7.27 \cdot 10^{-3})$
$v = 7/2$	$1.98 \cdot 10^{-1} (4.65 \cdot 10^{-3})$	$2.14 \cdot 10^{-1} (5.96 \cdot 10^{-3})$	$2.25 \cdot 10^{-1} (9.43 \cdot 10^{-3})$	$2.06 \cdot 10^{-1} (5.49 \cdot 10^{-3})$	$2.76 \cdot 10^{-1} (1.48 \cdot 10^{-2})$	$2.79 \cdot 10^{-1} (1.23 \cdot 10^{-2})$
$v = 9/2$	$2.02 \cdot 10^{-1} (4.79 \cdot 10^{-3})$	$2.15 \cdot 10^{-1} (5.71 \cdot 10^{-3})$	$2.29 \cdot 10^{-1} (9.07 \cdot 10^{-3})$	$2.09 \cdot 10^{-1} (5.57 \cdot 10^{-3})$	$3.07 \cdot 10^{-1} (2.08 \cdot 10^{-2})$	$3.23 \cdot 10^{-1} (1.89 \cdot 10^{-2})$
$v \in \{1/2, \dots, 9/2\}$	$1.94 \cdot 10^{-1} (4.70 \cdot 10^{-3})$	$2.04 \cdot 10^{-1} (5.99 \cdot 10^{-3})$	$2.24 \cdot 10^{-1} (1.04 \cdot 10^{-2})$	$1.88 \cdot 10^{-1} (5.82 \cdot 10^{-3})$	$2.85 \cdot 10^{-1} (1.60 \cdot 10^{-2})$	$3.25 \cdot 10^{-1} (1.91 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.93 \cdot 10^{-1} (4.52 \cdot 10^{-3})$

#### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.68 \cdot 10^0 (8.49 \cdot 10^{-2})$	$2.71 \cdot 10^0 (8.94 \cdot 10^{-2})$	$2.71 \cdot 10^0 (9.10 \cdot 10^{-2})$	$2.93 \cdot 10^0 (1.08 \cdot 10^{-1})$	$3.32 \cdot 10^0 (1.08 \cdot 10^{-1})$	$2.77 \cdot 10^0 (1.02 \cdot 10^{-1})$
$v = 3/2$	$2.85 \cdot 10^0 (1.04 \cdot 10^{-1})$	$3.06 \cdot 10^0 (1.19 \cdot 10^{-1})$	$3.11 \cdot 10^0 (1.21 \cdot 10^{-1})$	$3.41 \cdot 10^0 (1.41 \cdot 10^{-1})$	$3.37 \cdot 10^0 (1.55 \cdot 10^{-1})$	$3.72 \cdot 10^0 (2.63 \cdot 10^{-1})$
$v = 5/2$	$3.03 \cdot 10^0 (1.12 \cdot 10^{-1})$	$3.27 \cdot 10^0 (1.38 \cdot 10^{-1})$	$3.65 \cdot 10^0 (2.81 \cdot 10^{-1})$	$3.56 \cdot 10^0 (1.57 \cdot 10^{-1})$	$3.62 \cdot 10^0 (1.69 \cdot 10^{-1})$	$3.78 \cdot 10^0 (1.51 \cdot 10^{-1})$
$v = 7/2$	$3.13 \cdot 10^0 (1.17 \cdot 10^{-1})$	$3.41 \cdot 10^0 (1.48 \cdot 10^{-1})$	$3.81 \cdot 10^0 (2.58 \cdot 10^{-1})$	$3.71 \cdot 10^0 (1.62 \cdot 10^{-1})$	$3.99 \cdot 10^0 (2.17 \cdot 10^{-1})$	$4.30 \cdot 10^0 (1.76 \cdot 10^{-1})$
$v = 9/2$	$3.21 \cdot 10^0 (1.20 \cdot 10^{-1})$	$3.44 \cdot 10^0 (1.46 \cdot 10^{-1})$	$3.91 \cdot 10^0 (2.37 \cdot 10^{-1})$	$3.80 \cdot 10^0 (1.67 \cdot 10^{-1})$	$4.37 \cdot 10^0 (2.85 \cdot 10^{-1})$	$4.80 \cdot 10^0 (2.32 \cdot 10^{-1})$
$v \in \{1/2, \dots, 9/2\}$	$2.97 \cdot 10^0 (1.08 \cdot 10^{-1})$	$3.26 \cdot 10^0 (1.39 \cdot 10^{-1})$	$3.87 \cdot 10^0 (2.93 \cdot 10^{-1})$	$3.52 \cdot 10^0 (1.61 \cdot 10^{-1})$	$4.00 \cdot 10^0 (2.22 \cdot 10^{-1})$	$4.90 \cdot 10^0 (2.44 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$2.94 \cdot 10^0 (1.06 \cdot 10^{-1})$

## SM-2.2.24 GKLS $C^2$ , $n = 100$ , $d = 2$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$6.16 \cdot 10^{-2}$ ( $3.48 \cdot 10^{-3}$ )	$6.25 \cdot 10^{-2}$ ( $3.73 \cdot 10^{-3}$ )	$6.10 \cdot 10^{-2}$ ( $3.53 \cdot 10^{-3}$ )	$6.04 \cdot 10^{-2}$ ( $3.46 \cdot 10^{-3}$ )	$1.57 \cdot 10^{-1}$ ( $1.15 \cdot 10^{-2}$ )	$6.06 \cdot 10^{-2}$ ( $3.45 \cdot 10^{-3}$ )
$v = 3/2$	$4.21 \cdot 10^{-2}$ ( $2.62 \cdot 10^{-3}$ )	$4.56 \cdot 10^{-2}$ ( $3.69 \cdot 10^{-3}$ )	$4.19 \cdot 10^{-2}$ ( $2.63 \cdot 10^{-3}$ )	$4.20 \cdot 10^{-2}$ ( $2.89 \cdot 10^{-3}$ )	$1.89 \cdot 10^{-1}$ ( $2.15 \cdot 10^{-2}$ )	$4.44 \cdot 10^{-2}$ ( $2.77 \cdot 10^{-3}$ )
$v = 5/2$	$4.04 \cdot 10^{-2}$ ( $2.38 \cdot 10^{-3}$ )	$4.48 \cdot 10^{-2}$ ( $3.37 \cdot 10^{-3}$ )	$4.39 \cdot 10^{-2}$ ( $3.15 \cdot 10^{-3}$ )	$4.00 \cdot 10^{-2}$ ( $2.35 \cdot 10^{-3}$ )	$2.22 \cdot 10^{-1}$ ( $2.70 \cdot 10^{-2}$ )	$6.19 \cdot 10^{-2}$ ( $5.83 \cdot 10^{-3}$ )
$v = 7/2$	$4.10 \cdot 10^{-2}$ ( $2.32 \cdot 10^{-3}$ )	$4.43 \cdot 10^{-2}$ ( $3.15 \cdot 10^{-3}$ )	$4.68 \cdot 10^{-2}$ ( $3.63 \cdot 10^{-3}$ )	$4.19 \cdot 10^{-2}$ ( $2.61 \cdot 10^{-3}$ )	$2.77 \cdot 10^{-1}$ ( $3.33 \cdot 10^{-2}$ )	$9.26 \cdot 10^{-2}$ ( $9.15 \cdot 10^{-3}$ )
$v = 9/2$	$4.20 \cdot 10^{-2}$ ( $2.31 \cdot 10^{-3}$ )	$4.52 \cdot 10^{-2}$ ( $3.07 \cdot 10^{-3}$ )	$5.05 \cdot 10^{-2}$ ( $4.32 \cdot 10^{-3}$ )	$4.35 \cdot 10^{-2}$ ( $2.62 \cdot 10^{-3}$ )	$2.87 \cdot 10^{-1}$ ( $3.29 \cdot 10^{-2}$ )	$1.20 \cdot 10^{-1}$ ( $1.10 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, 9/2\}$	$4.11 \cdot 10^{-2}$ ( $2.34 \cdot 10^{-3}$ )	$4.49 \cdot 10^{-2}$ ( $3.38 \cdot 10^{-3}$ )	$4.78 \cdot 10^{-2}$ ( $4.10 \cdot 10^{-3}$ )	$4.19 \cdot 10^{-2}$ ( $2.71 \cdot 10^{-3}$ )	$2.84 \cdot 10^{-1}$ ( $3.21 \cdot 10^{-2}$ )	$1.19 \cdot 10^{-1}$ ( $1.10 \cdot 10^{-2}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$5.66 \cdot 10^{-2}$ ( $3.28 \cdot 10^{-3}$ )
$v = 3/2$	$3.49 \cdot 10^{-2}$ ( $2.14 \cdot 10^{-3}$ )
$v = 5/2$	$3.17 \cdot 10^{-2}$ ( $1.92 \cdot 10^{-3}$ )
$v = 7/2$	$3.17 \cdot 10^{-2}$ ( $1.87 \cdot 10^{-3}$ )
$v = 9/2$	$3.22 \cdot 10^{-2}$ ( $1.86 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$3.01 \cdot 10^{-2}$ ( $1.82 \cdot 10^{-3}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.29 \cdot 10^{-2}$ ( $2.87 \cdot 10^{-3}$ )
linear	$6.87 \cdot 10^{-1}$ ( $2.15 \cdot 10^{-2}$ )
INN	$1.32 \cdot 10^{-1}$ ( $5.50 \cdot 10^{-3}$ )
2NN	$1.13 \cdot 10^{-1}$ ( $5.22 \cdot 10^{-3}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.17 \cdot 10^{-1}$ ( $2.59 \cdot 10^{-3}$ )	$1.11 \cdot 10^{-1}$ ( $2.59 \cdot 10^{-3}$ )	$1.10 \cdot 10^{-1}$ ( $2.50 \cdot 10^{-3}$ )	$9.91 \cdot 10^{-2}$ ( $2.51 \cdot 10^{-3}$ )	$1.79 \cdot 10^{-1}$ ( $6.27 \cdot 10^{-3}$ )	$1.10 \cdot 10^{-1}$ ( $2.47 \cdot 10^{-3}$ )
$v = 3/2$	$8.90 \cdot 10^{-2}$ ( $2.22 \cdot 10^{-3}$ )	$8.92 \cdot 10^{-2}$ ( $2.69 \cdot 10^{-3}$ )	$8.67 \cdot 10^{-2}$ ( $2.26 \cdot 10^{-3}$ )	$8.39 \cdot 10^{-2}$ ( $2.42 \cdot 10^{-3}$ )	$1.78 \cdot 10^{-1}$ ( $1.05 \cdot 10^{-2}$ )	$8.87 \cdot 10^{-2}$ ( $2.36 \cdot 10^{-3}$ )
$v = 5/2$	$8.64 \cdot 10^{-2}$ ( $2.12 \cdot 10^{-3}$ )	$8.92 \cdot 10^{-2}$ ( $2.49 \cdot 10^{-3}$ )	$8.73 \cdot 10^{-2}$ ( $2.29 \cdot 10^{-3}$ )	$8.41 \cdot 10^{-2}$ ( $2.15 \cdot 10^{-3}$ )	$1.85 \cdot 10^{-1}$ ( $1.07 \cdot 10^{-2}$ )	$9.95 \cdot 10^{-2}$ ( $2.91 \cdot 10^{-3}$ )
$v = 7/2$	$8.71 \cdot 10^{-2}$ ( $2.08 \cdot 10^{-3}$ )	$9.05 \cdot 10^{-2}$ ( $2.40 \cdot 10^{-3}$ )	$9.00 \cdot 10^{-2}$ ( $2.33 \cdot 10^{-3}$ )	$8.68 \cdot 10^{-2}$ ( $2.19 \cdot 10^{-3}$ )	$2.01 \cdot 10^{-1}$ ( $1.14 \cdot 10^{-2}$ )	$1.18 \cdot 10^{-1}$ ( $3.79 \cdot 10^{-3}$ )
$v = 9/2$	$8.85 \cdot 10^{-2}$ ( $2.07 \cdot 10^{-3}$ )	$9.27 \cdot 10^{-2}$ ( $2.37 \cdot 10^{-3}$ )	$9.33 \cdot 10^{-2}$ ( $2.50 \cdot 10^{-3}$ )	$8.90 \cdot 10^{-2}$ ( $2.17 \cdot 10^{-3}$ )	$2.00 \cdot 10^{-1}$ ( $1.01 \cdot 10^{-2}$ )	$1.34 \cdot 10^{-1}$ ( $4.53 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, 9/2\}$	$8.65 \cdot 10^{-2}$ ( $2.09 \cdot 10^{-3}$ )	$8.94 \cdot 10^{-2}$ ( $2.43 \cdot 10^{-3}$ )	$9.06 \cdot 10^{-2}$ ( $2.43 \cdot 10^{-3}$ )	$8.45 \cdot 10^{-2}$ ( $2.29 \cdot 10^{-3}$ )	$2.05 \cdot 10^{-1}$ ( $1.05 \cdot 10^{-2}$ )	$1.32 \cdot 10^{-1}$ ( $4.71 \cdot 10^{-3}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$8.80 \cdot 10^{-2}$ ( $2.24 \cdot 10^{-3}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.64 \cdot 10^0$ ( $4.40 \cdot 10^{-2}$ )	$1.62 \cdot 10^0$ ( $4.96 \cdot 10^{-2}$ )	$1.60 \cdot 10^0$ ( $4.80 \cdot 10^{-2}$ )	$1.94 \cdot 10^0$ ( $6.73 \cdot 10^{-2}$ )	$2.49 \cdot 10^0$ ( $1.03 \cdot 10^{-1}$ )	$1.59 \cdot 10^0$ ( $4.73 \cdot 10^{-2}$ )
$v = 3/2$	$1.24 \cdot 10^0$ ( $4.35 \cdot 10^{-2}$ )	$1.29 \cdot 10^0$ ( $5.71 \cdot 10^{-2}$ )	$1.24 \cdot 10^0$ ( $4.46 \cdot 10^{-2}$ )	$1.57 \cdot 10^0$ ( $6.68 \cdot 10^{-2}$ )	$2.60 \cdot 10^0$ ( $1.72 \cdot 10^{-1}$ )	$1.27 \cdot 10^0$ ( $4.42 \cdot 10^{-2}$ )
$v = 5/2$	$1.19 \cdot 10^0$ ( $4.22 \cdot 10^{-2}$ )	$1.25 \cdot 10^0$ ( $5.21 \cdot 10^{-2}$ )	$1.23 \cdot 10^0$ ( $4.65 \cdot 10^{-2}$ )	$1.44 \cdot 10^0$ ( $5.11 \cdot 10^{-2}$ )	$2.70 \cdot 10^0$ ( $1.82 \cdot 10^{-1}$ )	$1.40 \cdot 10^0$ ( $5.78 \cdot 10^{-2}$ )
$v = 7/2$	$1.19 \cdot 10^0$ ( $4.11 \cdot 10^{-2}$ )	$1.23 \cdot 10^0$ ( $4.97 \cdot 10^{-2}$ )	$1.24 \cdot 10^0$ ( $4.74 \cdot 10^{-2}$ )	$1.43 \cdot 10^0$ ( $5.56 \cdot 10^{-2}$ )	$2.81 \cdot 10^0$ ( $1.72 \cdot 10^{-1}$ )	$1.63 \cdot 10^0$ ( $7.03 \cdot 10^{-2}$ )
$v = 9/2$	$1.20 \cdot 10^0$ ( $4.05 \cdot 10^{-2}$ )	$1.24 \cdot 10^0$ ( $4.83 \cdot 10^{-2}$ )	$1.30 \cdot 10^0$ ( $5.72 \cdot 10^{-2}$ )	$1.43 \cdot 10^0$ ( $5.61 \cdot 10^{-2}$ )	$2.72 \cdot 10^0$ ( $1.48 \cdot 10^{-1}$ )	$1.79 \cdot 10^0$ ( $7.54 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, 9/2\}$	$1.21 \cdot 10^0$ ( $4.13 \cdot 10^{-2}$ )	$1.25 \cdot 10^0$ ( $5.27 \cdot 10^{-2}$ )	$1.27 \cdot 10^0$ ( $5.02 \cdot 10^{-2}$ )	$1.54 \cdot 10^0$ ( $6.21 \cdot 10^{-2}$ )	$2.93 \cdot 10^0$ ( $1.75 \cdot 10^{-1}$ )	$1.80 \cdot 10^0$ ( $7.77 \cdot 10^{-2}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.22 \cdot 10^0$ ( $4.53 \cdot 10^{-2}$ )

### SM-2.2.25 GKLS $C^2$ , $n = 50$ , $d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.14 \cdot 10^{-1}$ ( $5.12 \cdot 10^{-3}$ )	$9.80 \cdot 10^{-2}$ ( $4.84 \cdot 10^{-3}$ )	$9.85 \cdot 10^{-2}$ ( $4.90 \cdot 10^{-3}$ )	$9.34 \cdot 10^{-2}$ ( $4.25 \cdot 10^{-3}$ )	$3.52 \cdot 10^{-1}$ ( $9.45 \cdot 10^{-3}$ )	$1.04 \cdot 10^{-1}$ ( $6.99 \cdot 10^{-3}$ )
$v = 3/2$	$7.06 \cdot 10^{-2}$ ( $9.70 \cdot 10^{-3}$ )	$7.69 \cdot 10^{-2}$ ( $9.22 \cdot 10^{-3}$ )	$8.00 \cdot 10^{-2}$ ( $1.08 \cdot 10^{-2}$ )	$6.61 \cdot 10^{-2}$ ( $8.03 \cdot 10^{-3}$ )	$3.03 \cdot 10^{-1}$ ( $9.39 \cdot 10^{-3}$ )	$9.36 \cdot 10^{-2}$ ( $1.79 \cdot 10^{-2}$ )
$v = 5/2$	$7.97 \cdot 10^{-2}$ ( $1.11 \cdot 10^{-2}$ )	$9.61 \cdot 10^{-2}$ ( $1.28 \cdot 10^{-2}$ )	$1.10 \cdot 10^{-1}$ ( $1.85 \cdot 10^{-2}$ )	$8.35 \cdot 10^{-2}$ ( $1.10 \cdot 10^{-2}$ )	$2.95 \cdot 10^{-1}$ ( $9.76 \cdot 10^{-3}$ )	$1.02 \cdot 10^{-1}$ ( $1.70 \cdot 10^{-2}$ )
$v = 7/2$	$8.65 \cdot 10^{-2}$ ( $1.17 \cdot 10^{-2}$ )	$1.25 \cdot 10^{-1}$ ( $2.05 \cdot 10^{-2}$ )	$1.27 \cdot 10^{-1}$ ( $2.16 \cdot 10^{-2}$ )	$1.12 \cdot 10^{-1}$ ( $1.76 \cdot 10^{-2}$ )	$2.90 \cdot 10^{-1}$ ( $9.62 \cdot 10^{-3}$ )	$1.26 \cdot 10^{-1}$ ( $2.02 \cdot 10^{-2}$ )
$v = 9/2$	$9.08 \cdot 10^{-2}$ ( $1.20 \cdot 10^{-2}$ )	$1.37 \cdot 10^{-1}$ ( $2.35 \cdot 10^{-2}$ )	$1.39 \cdot 10^{-1}$ ( $2.51 \cdot 10^{-2}$ )	$1.26 \cdot 10^{-1}$ ( $2.30 \cdot 10^{-2}$ )	$2.87 \cdot 10^{-1}$ ( $9.56 \cdot 10^{-3}$ )	$1.49 \cdot 10^{-1}$ ( $2.42 \cdot 10^{-2}$ )
$v = 11/2$	$9.35 \cdot 10^{-2}$ ( $1.22 \cdot 10^{-2}$ )	$1.50 \cdot 10^{-1}$ ( $2.61 \cdot 10^{-2}$ )	$1.54 \cdot 10^{-1}$ ( $2.70 \cdot 10^{-2}$ )	$1.35 \cdot 10^{-1}$ ( $2.43 \cdot 10^{-2}$ )	$2.86 \cdot 10^{-1}$ ( $9.66 \cdot 10^{-3}$ )	$1.47 \cdot 10^{-1}$ ( $2.52 \cdot 10^{-2}$ )
$v = 21/2$	$1.03 \cdot 10^{-1}$ ( $1.38 \cdot 10^{-2}$ )	$1.61 \cdot 10^{-1}$ ( $2.83 \cdot 10^{-2}$ )	$1.68 \cdot 10^{-1}$ ( $2.96 \cdot 10^{-2}$ )	$1.55 \cdot 10^{-1}$ ( $2.73 \cdot 10^{-2}$ )	$2.84 \cdot 10^{-1}$ ( $9.93 \cdot 10^{-3}$ )	$1.81 \cdot 10^{-1}$ ( $3.05 \cdot 10^{-2}$ )
$v = \infty$	$1.09 \cdot 10^{-1}$ ( $1.44 \cdot 10^{-2}$ )	$1.79 \cdot 10^{-1}$ ( $3.00 \cdot 10^{-2}$ )	$1.76 \cdot 10^{-1}$ ( $3.02 \cdot 10^{-2}$ )	$1.65 \cdot 10^{-1}$ ( $2.28 \cdot 10^{-2}$ )	$2.88 \cdot 10^{-1}$ ( $1.08 \cdot 10^{-2}$ )	$2.08 \cdot 10^{-1}$ ( $3.88 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, \infty\}$	$7.16 \cdot 10^{-2}$ ( $1.07 \cdot 10^{-2}$ )	$1.49 \cdot 10^{-1}$ ( $2.86 \cdot 10^{-2}$ )	$1.45 \cdot 10^{-1}$ ( $2.10 \cdot 10^{-2}$ )	$1.17 \cdot 10^{-1}$ ( $2.01 \cdot 10^{-2}$ )	$2.90 \cdot 10^{-1}$ ( $1.09 \cdot 10^{-2}$ )	$1.87 \cdot 10^{-1}$ ( $3.55 \cdot 10^{-2}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$8.54 \cdot 10^{-2}$ ( $3.64 \cdot 10^{-3}$ )
$v = 3/2$	$4.52 \cdot 10^{-2}$ ( $4.24 \cdot 10^{-3}$ )
$v = 5/2$	$4.93 \cdot 10^{-2}$ ( $4.98 \cdot 10^{-3}$ )
$v = 7/2$	$5.40 \cdot 10^{-2}$ ( $5.36 \cdot 10^{-3}$ )
$v = 9/2$	$5.69 \cdot 10^{-2}$ ( $5.68 \cdot 10^{-3}$ )
$v = 11/2$	$5.82 \cdot 10^{-2}$ ( $5.78 \cdot 10^{-3}$ )
$v = 21/2$	$6.22 \cdot 10^{-2}$ ( $6.17 \cdot 10^{-3}$ )
$v = \infty$	$6.54 \cdot 10^{-2}$ ( $6.69 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, \infty\}$	$4.06 \cdot 10^{-2}$ ( $3.70 \cdot 10^{-3}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$7.09 \cdot 10^{-2}$ ( $8.30 \cdot 10^{-3}$ )
linear	$3.93 \cdot 10^{-1}$ ( $2.07 \cdot 10^{-2}$ )
1NN	$4.70 \cdot 10^{-1}$ ( $8.11 \cdot 10^{-3}$ )
2NN	$3.71 \cdot 10^{-1}$ ( $6.86 \cdot 10^{-3}$ )

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.75 \cdot 10^{-1}$ ( $3.45 \cdot 10^{-3}$ )	$1.55 \cdot 10^{-1}$ ( $3.59 \cdot 10^{-3}$ )	$1.55 \cdot 10^{-1}$ ( $3.63 \cdot 10^{-3}$ )	$1.50 \cdot 10^{-1}$ ( $3.04 \cdot 10^{-3}$ )	$3.24 \cdot 10^{-1}$ ( $4.50 \cdot 10^{-3}$ )	$1.57 \cdot 10^{-1}$ ( $4.51 \cdot 10^{-3}$ )
$v = 3/2$	$1.02 \cdot 10^{-1}$ ( $7.10 \cdot 10^{-3}$ )	$1.08 \cdot 10^{-1}$ ( $7.95 \cdot 10^{-3}$ )	$1.07 \cdot 10^{-1}$ ( $8.32 \cdot 10^{-3}$ )	$9.70 \cdot 10^{-2}$ ( $7.00 \cdot 10^{-3}$ )	$2.95 \cdot 10^{-1}$ ( $4.75 \cdot 10^{-3}$ )	$1.11 \cdot 10^{-1}$ ( $9.96 \cdot 10^{-3}$ )
$v = 5/2$	$9.96 \cdot 10^{-2}$ ( $8.69 \cdot 10^{-3}$ )	$1.13 \cdot 10^{-1}$ ( $1.01 \cdot 10^{-2}$ )	$1.15 \cdot 10^{-1}$ ( $1.12 \cdot 10^{-2}$ )	$1.04 \cdot 10^{-1}$ ( $9.17 \cdot 10^{-3}$ )	$2.90 \cdot 10^{-1}$ ( $5.00 \cdot 10^{-3}$ )	$1.10 \cdot 10^{-1}$ ( $1.03 \cdot 10^{-2}$ )
$v = 7/2$	$1.07 \cdot 10^{-1}$ ( $9.03 \cdot 10^{-3}$ )	$1.31 \cdot 10^{-1}$ ( $1.19 \cdot 10^{-2}$ )	$1.28 \cdot 10^{-1}$ ( $1.25 \cdot 10^{-2}$ )	$1.24 \cdot 10^{-1}$ ( $1.12 \cdot 10^{-2}$ )	$2.86 \cdot 10^{-1}$ ( $4.93 \cdot 10^{-3}$ )	$1.24 \cdot 10^{-1}$ ( $1.15 \cdot 10^{-2}$ )
$v = 9/2$	$1.11 \cdot 10^{-1}$ ( $9.20 \cdot 10^{-3}$ )	$1.38 \cdot 10^{-1}$ ( $1.25 \cdot 10^{-2}$ )	$1.34 \cdot 10^{-1}$ ( $1.27 \cdot 10^{-2}$ )	$1.33 \cdot 10^{-1}$ ( $1.24 \cdot 10^{-2}$ )	$2.84 \cdot 10^{-1}$ ( $4.89 \cdot 10^{-3}$ )	$1.37 \cdot 10^{-1}$ ( $1.28 \cdot 10^{-2}$ )
$v = 11/2$	$1.14 \cdot 10^{-1}$ ( $9.33 \cdot 10^{-3}$ )	$1.46 \cdot 10^{-1}$ ( $1.35 \cdot 10^{-2}$ )	$1.44 \cdot 10^{-1}$ ( $1.36 \cdot 10^{-2}$ )	$1.41 \cdot 10^{-1}$ ( $1.29 \cdot 10^{-2}$ )	$2.83 \cdot 10^{-1}$ ( $4.93 \cdot 10^{-3}$ )	$1.39 \cdot 10^{-1}$ ( $1.31 \cdot 10^{-2}$ )
$v = 21/2$	$1.21 \cdot 10^{-1}$ ( $1.01 \cdot 10^{-2}$ )	$1.54 \cdot 10^{-1}$ ( $1.42 \cdot 10^{-2}$ )	$1.55 \cdot 10^{-1}$ ( $1.46 \cdot 10^{-2}$ )	$1.54 \cdot 10^{-1}$ ( $1.41 \cdot 10^{-2}$ )	$2.81 \cdot 10^{-1}$ ( $5.01 \cdot 10^{-3}$ )	$1.55 \cdot 10^{-1}$ ( $1.51 \cdot 10^{-2}$ )
$v = \infty$	$1.26 \cdot 10^{-1}$ ( $1.04 \cdot 10^{-2}$ )	$1.65 \cdot 10^{-1}$ ( $1.51 \cdot 10^{-2}$ )	$1.61 \cdot 10^{-1}$ ( $1.49 \cdot 10^{-2}$ )	$1.63 \cdot 10^{-1}$ ( $1.44 \cdot 10^{-2}$ )	$2.82 \cdot 10^{-1}$ ( $5.36 \cdot 10^{-3}$ )	$1.65 \cdot 10^{-1}$ ( $1.65 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, \infty\}$	$9.41 \cdot 10^{-2}$ ( $8.24 \cdot 10^{-3}$ )	$1.40 \cdot 10^{-1}$ ( $1.43 \cdot 10^{-2}$ )	$1.43 \cdot 10^{-1}$ ( $1.31 \cdot 10^{-2}$ )	$1.25 \cdot 10^{-1}$ ( $1.23 \cdot 10^{-2}$ )	$2.83 \cdot 10^{-1}$ ( $5.44 \cdot 10^{-3}$ )	$1.56 \cdot 10^{-1}$ ( $1.59 \cdot 10^{-2}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$9.61 \cdot 10^{-2}$ ( $8.10 \cdot 10^{-3}$ )

#### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.79 \cdot 10^0 (3.79 \cdot 10^{-2})$	$1.73 \cdot 10^0 (4.27 \cdot 10^{-2})$	$1.74 \cdot 10^0 (4.28 \cdot 10^{-2})$	$1.77 \cdot 10^0 (4.30 \cdot 10^{-2})$	$3.01 \cdot 10^0 (4.70 \cdot 10^{-2})$	$1.76 \cdot 10^0 (4.93 \cdot 10^{-2})$
$v = 3/2$	$1.37 \cdot 10^0 (8.61 \cdot 10^{-2})$	$1.48 \cdot 10^0 (9.53 \cdot 10^{-2})$	$1.53 \cdot 10^0 (1.10 \cdot 10^{-1})$	$1.56 \cdot 10^0 (1.10 \cdot 10^{-1})$	$2.85 \cdot 10^0 (5.12 \cdot 10^{-2})$	$1.60 \cdot 10^0 (1.33 \cdot 10^{-1})$
$v = 5/2$	$1.48 \cdot 10^0 (1.08 \cdot 10^{-1})$	$1.79 \cdot 10^0 (1.38 \cdot 10^{-1})$	$1.96 \cdot 10^0 (1.81 \cdot 10^{-1})$	$1.90 \cdot 10^0 (1.50 \cdot 10^{-1})$	$2.83 \cdot 10^0 (5.59 \cdot 10^{-2})$	$1.85 \cdot 10^0 (1.62 \cdot 10^{-1})$
$v = 7/2$	$1.58 \cdot 10^0 (1.16 \cdot 10^{-1})$	$2.15 \cdot 10^0 (1.79 \cdot 10^{-1})$	$2.25 \cdot 10^0 (2.08 \cdot 10^{-1})$	$2.41 \cdot 10^0 (2.24 \cdot 10^{-1})$	$2.81 \cdot 10^0 (5.48 \cdot 10^{-2})$	$2.15 \cdot 10^0 (1.94 \cdot 10^{-1})$
$v = 9/2$	$1.65 \cdot 10^0 (1.19 \cdot 10^{-1})$	$2.35 \cdot 10^0 (2.07 \cdot 10^{-1})$	$2.42 \cdot 10^0 (2.27 \cdot 10^{-1})$	$2.63 \cdot 10^0 (2.49 \cdot 10^{-1})$	$2.79 \cdot 10^0 (5.43 \cdot 10^{-2})$	$2.46 \cdot 10^0 (2.24 \cdot 10^{-1})$
$v = 11/2$	$1.69 \cdot 10^0 (1.22 \cdot 10^{-1})$	$2.54 \cdot 10^0 (2.29 \cdot 10^{-1})$	$2.66 \cdot 10^0 (2.49 \cdot 10^{-1})$	$2.84 \cdot 10^0 (2.68 \cdot 10^{-1})$	$2.79 \cdot 10^0 (5.48 \cdot 10^{-2})$	$2.48 \cdot 10^0 (2.27 \cdot 10^{-1})$
$v = 21/2$	$1.81 \cdot 10^0 (1.33 \cdot 10^{-1})$	$2.78 \cdot 10^0 (2.62 \cdot 10^{-1})$	$2.90 \cdot 10^0 (2.74 \cdot 10^{-1})$	$3.25 \cdot 10^0 (3.12 \cdot 10^{-1})$	$2.78 \cdot 10^0 (5.63 \cdot 10^{-2})$	$2.91 \cdot 10^0 (2.83 \cdot 10^{-1})$
$v = \infty$	$1.91 \cdot 10^0 (1.40 \cdot 10^{-1})$	$3.11 \cdot 10^0 (2.88 \cdot 10^{-1})$	$3.18 \cdot 10^0 (3.03 \cdot 10^{-1})$	$3.50 \cdot 10^0 (3.08 \cdot 10^{-1})$	$2.82 \cdot 10^0 (6.06 \cdot 10^{-2})$	$3.13 \cdot 10^0 (3.10 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$1.41 \cdot 10^0 (1.08 \cdot 10^{-1})$	$2.67 \cdot 10^0 (2.84 \cdot 10^{-1})$	$2.83 \cdot 10^0 (2.56 \cdot 10^{-1})$	$2.65 \cdot 10^0 (2.79 \cdot 10^{-1})$	$2.83 \cdot 10^0 (6.09 \cdot 10^{-2})$	$2.99 \cdot 10^0 (3.02 \cdot 10^{-1})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$1.44 \cdot 10^0 (1.04 \cdot 10^{-1})$

### SM-2.2.26 GKLS $C^2$ , $n = 100$ , $d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.99 \cdot 10^{-2} (2.90 \cdot 10^{-3})$	$4.91 \cdot 10^{-2} (2.89 \cdot 10^{-3})$	$4.88 \cdot 10^{-2} (2.85 \cdot 10^{-3})$	$4.78 \cdot 10^{-2} (2.82 \cdot 10^{-3})$	$2.38 \cdot 10^{-1} (7.15 \cdot 10^{-3})$	$4.91 \cdot 10^{-2} (2.91 \cdot 10^{-3})$
$v = 3/2$	$3.94 \cdot 10^{-2} (3.30 \cdot 10^{-3})$	$4.35 \cdot 10^{-2} (3.46 \cdot 10^{-3})$	$4.43 \cdot 10^{-2} (3.43 \cdot 10^{-3})$	$4.06 \cdot 10^{-2} (3.15 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (7.56 \cdot 10^{-3})$	$4.63 \cdot 10^{-2} (3.81 \cdot 10^{-3})$
$v = 5/2$	$5.07 \cdot 10^{-2} (4.23 \cdot 10^{-3})$	$6.33 \cdot 10^{-2} (5.69 \cdot 10^{-3})$	$6.33 \cdot 10^{-2} (5.38 \cdot 10^{-3})$	$5.88 \cdot 10^{-2} (4.84 \cdot 10^{-3})$	$1.91 \cdot 10^{-1} (9.01 \cdot 10^{-3})$	$7.18 \cdot 10^{-2} (6.80 \cdot 10^{-3})$
$v = 7/2$	$5.77 \cdot 10^{-2} (4.74 \cdot 10^{-3})$	$6.91 \cdot 10^{-2} (5.35 \cdot 10^{-3})$	$7.62 \cdot 10^{-2} (6.73 \cdot 10^{-3})$	$7.09 \cdot 10^{-2} (6.26 \cdot 10^{-3})$	$1.84 \cdot 10^{-1} (8.22 \cdot 10^{-3})$	$9.41 \cdot 10^{-2} (1.25 \cdot 10^{-2})$
$v = 9/2$	$6.18 \cdot 10^{-2} (5.04 \cdot 10^{-3})$	$7.95 \cdot 10^{-2} (6.87 \cdot 10^{-3})$	$8.58 \cdot 10^{-2} (8.04 \cdot 10^{-3})$	$7.62 \cdot 10^{-2} (6.89 \cdot 10^{-3})$	$1.80 \cdot 10^{-1} (7.87 \cdot 10^{-3})$	$1.00 \cdot 10^{-1} (1.08 \cdot 10^{-2})$
$v = 11/2$	$6.52 \cdot 10^{-2} (5.33 \cdot 10^{-3})$	$9.16 \cdot 10^{-2} (1.09 \cdot 10^{-2})$	$8.54 \cdot 10^{-2} (7.91 \cdot 10^{-3})$	$8.32 \cdot 10^{-2} (7.72 \cdot 10^{-3})$	$1.79 \cdot 10^{-1} (7.86 \cdot 10^{-3})$	$1.08 \cdot 10^{-1} (1.16 \cdot 10^{-2})$
$v = 21/2$	$6.96 \cdot 10^{-2} (5.29 \cdot 10^{-3})$	$1.00 \cdot 10^{-1} (1.18 \cdot 10^{-2})$	$1.03 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$9.30 \cdot 10^{-2} (1.01 \cdot 10^{-2})$	$1.81 \cdot 10^{-1} (8.44 \cdot 10^{-3})$	$1.19 \cdot 10^{-1} (1.26 \cdot 10^{-2})$
$v = \infty$	$7.66 \cdot 10^{-2} (5.77 \cdot 10^{-3})$	$1.10 \cdot 10^{-1} (1.28 \cdot 10^{-2})$	$1.18 \cdot 10^{-1} (1.33 \cdot 10^{-2})$	$9.85 \cdot 10^{-2} (8.72 \cdot 10^{-3})$	$1.85 \cdot 10^{-1} (9.56 \cdot 10^{-3})$	$1.30 \cdot 10^{-1} (1.40 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$4.05 \cdot 10^{-2} (3.67 \cdot 10^{-3})$	$5.90 \cdot 10^{-2} (5.93 \cdot 10^{-3})$	$7.30 \cdot 10^{-2} (6.72 \cdot 10^{-3})$	$4.66 \cdot 10^{-2} (3.71 \cdot 10^{-3})$	$1.85 \cdot 10^{-1} (9.56 \cdot 10^{-3})$	$1.11 \cdot 10^{-1} (1.23 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$4.64 \cdot 10^{-2} (2.70 \cdot 10^{-3})$
$v = 3/2$	$3.16 \cdot 10^{-2} (2.28 \cdot 10^{-3})$
$v = 5/2$	$3.75 \cdot 10^{-2} (2.67 \cdot 10^{-3})$
$v = 7/2$	$4.25 \cdot 10^{-2} (3.05 \cdot 10^{-3})$
$v = 9/2$	$4.55 \cdot 10^{-2} (3.20 \cdot 10^{-3})$
$v = 11/2$	$4.77 \cdot 10^{-2} (3.41 \cdot 10^{-3})$
$v = 21/2$	$5.20 \cdot 10^{-2} (3.71 \cdot 10^{-3})$
$v = \infty$	$5.68 \cdot 10^{-2} (4.12 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$3.07 \cdot 10^{-2} (2.19 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.14 \cdot 10^{-2} (3.77 \cdot 10^{-3})$
linear	$3.77 \cdot 10^{-1} (1.92 \cdot 10^{-2})$
INN	$3.66 \cdot 10^{-1} (6.66 \cdot 10^{-3})$
2NN	$2.72 \cdot 10^{-1} (5.41 \cdot 10^{-3})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.11 \cdot 10^{-1} (2.37 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (2.53 \cdot 10^{-3})$	$1.00 \cdot 10^{-1} (2.51 \cdot 10^{-3})$	$9.53 \cdot 10^{-2} (2.31 \cdot 10^{-3})$	$2.64 \cdot 10^{-1} (3.92 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (2.53 \cdot 10^{-3})$
$v = 3/2$	$7.57 \cdot 10^{-2} (3.39 \cdot 10^{-3})$	$8.04 \cdot 10^{-2} (3.60 \cdot 10^{-3})$	$8.03 \cdot 10^{-2} (3.55 \cdot 10^{-3})$	$7.29 \cdot 10^{-2} (3.19 \cdot 10^{-3})$	$2.34 \cdot 10^{-1} (4.56 \cdot 10^{-3})$	$8.30 \cdot 10^{-2} (3.71 \cdot 10^{-3})$
$v = 5/2$	$8.62 \cdot 10^{-2} (4.16 \cdot 10^{-3})$	$9.62 \cdot 10^{-2} (4.80 \cdot 10^{-3})$	$9.46 \cdot 10^{-2} (4.62 \cdot 10^{-3})$	$9.15 \cdot 10^{-2} (4.46 \cdot 10^{-3})$	$2.31 \cdot 10^{-1} (5.39 \cdot 10^{-3})$	$9.91 \cdot 10^{-2} (5.14 \cdot 10^{-3})$
$v = 7/2$	$9.46 \cdot 10^{-2} (4.43 \cdot 10^{-3})$	$1.05 \cdot 10^{-1} (4.97 \cdot 10^{-3})$	$1.06 \cdot 10^{-1} (5.32 \cdot 10^{-3})$	$1.04 \cdot 10^{-1} (4.99 \cdot 10^{-3})$	$2.25 \cdot 10^{-1} (4.97 \cdot 10^{-3})$	$1.14 \cdot 10^{-1} (6.28 \cdot 10^{-3})$
$v = 9/2$	$9.97 \cdot 10^{-2} (4.58 \cdot 10^{-3})$	$1.13 \cdot 10^{-1} (5.43 \cdot 10^{-3})$	$1.13 \cdot 10^{-1} (5.85 \cdot 10^{-3})$	$1.09 \cdot 10^{-1} (5.30 \cdot 10^{-3})$	$2.22 \cdot 10^{-1} (4.77 \cdot 10^{-3})$	$1.20 \cdot 10^{-1} (6.43 \cdot 10^{-3})$
$v = 11/2$	$1.04 \cdot 10^{-1} (4.79 \cdot 10^{-3})$	$1.18 \cdot 10^{-1} (6.08 \cdot 10^{-3})$	$1.15 \cdot 10^{-1} (5.68 \cdot 10^{-3})$	$1.16 \cdot 10^{-1} (5.84 \cdot 10^{-3})$	$2.21 \cdot 10^{-1} (4.77 \cdot 10^{-3})$	$1.25 \cdot 10^{-1} (6.75 \cdot 10^{-3})$
$v = 21/2$	$1.10 \cdot 10^{-1} (4.92 \cdot 10^{-3})$	$1.27 \cdot 10^{-1} (6.52 \cdot 10^{-3})$	$1.26 \cdot 10^{-1} (6.67 \cdot 10^{-3})$	$1.25 \cdot 10^{-1} (6.25 \cdot 10^{-3})$	$2.20 \cdot 10^{-1} (5.05 \cdot 10^{-3})$	$1.34 \cdot 10^{-1} (7.24 \cdot 10^{-3})$
$v = \infty$	$1.18 \cdot 10^{-1} (5.20 \cdot 10^{-3})$	$1.35 \cdot 10^{-1} (6.94 \cdot 10^{-3})$	$1.36 \cdot 10^{-1} (7.37 \cdot 10^{-3})$	$1.32 \cdot 10^{-1} (6.53 \cdot 10^{-3})$	$2.20 \cdot 10^{-1} (5.49 \cdot 10^{-3})$	$1.40 \cdot 10^{-1} (7.71 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$7.52 \cdot 10^{-2} (3.64 \cdot 10^{-3})$	$9.26 \cdot 10^{-2} (4.95 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (5.61 \cdot 10^{-3})$	$7.81 \cdot 10^{-2} (3.76 \cdot 10^{-3})$	$2.20 \cdot 10^{-1} (5.49 \cdot 10^{-3})$	$1.25 \cdot 10^{-1} (7.21 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$7.72 \cdot 10^{-2} (3.75 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.43 \cdot 10^0 (3.21 \cdot 10^{-2})$	$1.31 \cdot 10^0 (3.64 \cdot 10^{-2})$	$1.31 \cdot 10^0 (3.64 \cdot 10^{-2})$	$1.40 \cdot 10^0 (4.12 \cdot 10^{-2})$	$2.46 \cdot 10^0 (3.70 \cdot 10^{-2})$	$1.31 \cdot 10^0 (3.66 \cdot 10^{-2})$
$v = 3/2$	$1.20 \cdot 10^0 (4.78 \cdot 10^{-2})$	$1.27 \cdot 10^0 (5.28 \cdot 10^{-2})$	$1.30 \cdot 10^0 (5.33 \cdot 10^{-2})$	$1.42 \cdot 10^0 (6.21 \cdot 10^{-2})$	$2.28 \cdot 10^0 (4.76 \cdot 10^{-2})$	$1.31 \cdot 10^0 (5.45 \cdot 10^{-2})$
$v = 5/2$	$1.37 \cdot 10^0 (5.48 \cdot 10^{-2})$	$1.58 \cdot 10^0 (7.46 \cdot 10^{-2})$	$1.59 \cdot 10^0 (6.94 \cdot 10^{-2})$	$1.81 \cdot 10^0 (8.45 \cdot 10^{-2})$	$2.29 \cdot 10^0 (5.85 \cdot 10^{-2})$	$1.66 \cdot 10^0 (8.02 \cdot 10^{-2})$
$v = 7/2$	$1.47 \cdot 10^0 (5.70 \cdot 10^{-2})$	$1.63 \cdot 10^0 (7.08 \cdot 10^{-2})$	$1.75 \cdot 10^0 (8.08 \cdot 10^{-2})$	$1.94 \cdot 10^0 (9.34 \cdot 10^{-2})$	$2.24 \cdot 10^0 (5.30 \cdot 10^{-2})$	$1.85 \cdot 10^0 (1.08 \cdot 10^{-1})$
$v = 9/2$	$1.52 \cdot 10^0 (5.83 \cdot 10^{-2})$	$1.72 \cdot 10^0 (7.68 \cdot 10^{-2})$	$1.84 \cdot 10^0 (9.12 \cdot 10^{-2})$	$1.98 \cdot 10^0 (9.46 \cdot 10^{-2})$	$2.20 \cdot 10^0 (5.11 \cdot 10^{-2})$	$1.90 \cdot 10^0 (9.76 \cdot 10^{-2})$
$v = 11/2$	$1.57 \cdot 10^0 (5.98 \cdot 10^{-2})$	$1.80 \cdot 10^0 (9.00 \cdot 10^{-2})$	$1.83 \cdot 10^0 (8.52 \cdot 10^{-2})$	$2.05 \cdot 10^0 (1.01 \cdot 10^{-1})$	$2.20 \cdot 10^0 (5.14 \cdot 10^{-2})$	$1.97 \cdot 10^0 (1.03 \cdot 10^{-1})$
$v = 21/2$	$1.64 \cdot 10^0 (5.83 \cdot 10^{-2})$	$1.90 \cdot 10^0 (9.89 \cdot 10^{-2})$	$1.98 \cdot 10^0 (1.05 \cdot 10^{-1})$	$2.16 \cdot 10^0 (1.15 \cdot 10^{-1})$	$2.21 \cdot 10^0 (5.52 \cdot 10^{-2})$	$2.06 \cdot 10^0 (1.08 \cdot 10^{-1})$
$v = \infty$	$1.75 \cdot 10^0 (6.06 \cdot 10^{-2})$	$1.99 \cdot 10^0 (1.07 \cdot 10^{-1})$	$2.14 \cdot 10^0 (1.14 \cdot 10^{-1})$	$2.26 \cdot 10^0 (1.12 \cdot 10^{-1})$	$2.22 \cdot 10^0 (5.97 \cdot 10^{-2})$	$2.15 \cdot 10^0 (1.13 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$1.23 \cdot 10^0 (4.99 \cdot 10^{-2})$	$1.51 \cdot 10^0 (7.21 \cdot 10^{-2})$	$1.75 \cdot 10^0 (8.75 \cdot 10^{-2})$	$1.58 \cdot 10^0 (7.18 \cdot 10^{-2})$	$2.22 \cdot 10^0 (5.97 \cdot 10^{-2})$	$2.00 \cdot 10^0 (1.09 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.25 \cdot 10^0 (5.18 \cdot 10^{-2})$

### SM-2.2.27 GKLS $C^2$ , $n = 250$ , $d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.60 \cdot 10^{-2} (1.68 \cdot 10^{-3})$	$2.64 \cdot 10^{-2} (1.77 \cdot 10^{-3})$	$2.64 \cdot 10^{-2} (1.73 \cdot 10^{-3})$	$2.58 \cdot 10^{-2} (1.68 \cdot 10^{-3})$	$1.48 \cdot 10^{-1} (5.53 \cdot 10^{-3})$	$2.66 \cdot 10^{-2} (1.70 \cdot 10^{-3})$
$v = 3/2$	$2.27 \cdot 10^{-2} (1.77 \cdot 10^{-3})$	$2.48 \cdot 10^{-2} (2.01 \cdot 10^{-3})$	$2.62 \cdot 10^{-2} (2.21 \cdot 10^{-3})$	$2.43 \cdot 10^{-2} (2.13 \cdot 10^{-3})$	$1.14 \cdot 10^{-1} (5.18 \cdot 10^{-3})$	$2.81 \cdot 10^{-2} (2.24 \cdot 10^{-3})$
$v = 5/2$	$2.82 \cdot 10^{-2} (2.27 \cdot 10^{-3})$	$3.51 \cdot 10^{-2} (3.12 \cdot 10^{-3})$	$3.54 \cdot 10^{-2} (3.10 \cdot 10^{-3})$	$3.38 \cdot 10^{-2} (3.08 \cdot 10^{-3})$	$1.05 \cdot 10^{-1} (5.45 \cdot 10^{-3})$	$3.81 \cdot 10^{-2} (3.04 \cdot 10^{-3})$
$v = 7/2$	$3.22 \cdot 10^{-2} (2.53 \cdot 10^{-3})$	$3.98 \cdot 10^{-2} (3.82 \cdot 10^{-3})$	$4.03 \cdot 10^{-2} (3.52 \cdot 10^{-3})$	$3.86 \cdot 10^{-2} (3.43 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (5.70 \cdot 10^{-3})$	$4.46 \cdot 10^{-2} (3.49 \cdot 10^{-3})$
$v = 9/2$	$3.50 \cdot 10^{-2} (2.69 \cdot 10^{-3})$	$4.25 \cdot 10^{-2} (3.96 \cdot 10^{-3})$	$4.46 \cdot 10^{-2} (3.81 \cdot 10^{-3})$	$4.14 \cdot 10^{-2} (3.53 \cdot 10^{-3})$	$9.71 \cdot 10^{-2} (5.78 \cdot 10^{-3})$	$5.18 \cdot 10^{-2} (4.16 \cdot 10^{-3})$
$v = 11/2$	$3.72 \cdot 10^{-2} (2.79 \cdot 10^{-3})$	$4.51 \cdot 10^{-2} (4.40 \cdot 10^{-3})$	$4.67 \cdot 10^{-2} (3.96 \cdot 10^{-3})$	$4.34 \cdot 10^{-2} (3.66 \cdot 10^{-3})$	$9.24 \cdot 10^{-2} (5.56 \cdot 10^{-3})$	$5.33 \cdot 10^{-2} (4.54 \cdot 10^{-3})$
$v = 21/2$	$4.25 \cdot 10^{-2} (3.10 \cdot 10^{-3})$	$5.23 \cdot 10^{-2} (4.86 \cdot 10^{-3})$	$5.26 \cdot 10^{-2} (4.46 \cdot 10^{-3})$	$5.04 \cdot 10^{-2} (4.12 \cdot 10^{-3})$	$8.12 \cdot 10^{-2} (5.24 \cdot 10^{-3})$	$6.46 \cdot 10^{-2} (5.39 \cdot 10^{-3})$
$v = \infty$	$4.96 \cdot 10^{-2} (3.52 \cdot 10^{-3})$	$6.08 \cdot 10^{-2} (5.37 \cdot 10^{-3})$	$6.06 \cdot 10^{-2} (5.11 \cdot 10^{-3})$	$5.79 \cdot 10^{-2} (4.50 \cdot 10^{-3})$	$8.29 \cdot 10^{-2} (6.69 \cdot 10^{-3})$	$7.36 \cdot 10^{-2} (6.57 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$2.33 \cdot 10^{-2} (2.22 \cdot 10^{-3})$	$2.90 \cdot 10^{-2} (3.90 \cdot 10^{-3})$	$3.42 \cdot 10^{-2} (3.40 \cdot 10^{-3})$	$2.53 \cdot 10^{-2} (2.67 \cdot 10^{-3})$	$8.29 \cdot 10^{-2} (6.69 \cdot 10^{-3})$	$5.89 \cdot 10^{-2} (5.66 \cdot 10^{-3})$

$R^*(S^{SPE})$	
$v = 1/2$	$2.51 \cdot 10^{-2} (1.59 \cdot 10^{-3})$
$v = 3/2$	$1.90 \cdot 10^{-2} (1.36 \cdot 10^{-3})$
$v = 5/2$	$2.23 \cdot 10^{-2} (1.49 \cdot 10^{-3})$
$v = 7/2$	$2.51 \cdot 10^{-2} (1.63 \cdot 10^{-3})$
$v = 9/2$	$2.72 \cdot 10^{-2} (1.74 \cdot 10^{-3})$
$v = 11/2$	$2.88 \cdot 10^{-2} (1.83 \cdot 10^{-3})$
$v = 21/2$	$3.28 \cdot 10^{-2} (2.07 \cdot 10^{-3})$
$v = \infty$	$3.81 \cdot 10^{-2} (2.42 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$1.87 \cdot 10^{-2} (1.34 \cdot 10^{-3})$

$R(\theta; S^{SPE})$	
NLL/SPE	$2.40 \cdot 10^{-2} (2.41 \cdot 10^{-3})$
linear	$3.75 \cdot 10^{-1} (1.91 \cdot 10^{-2})$
1NN	$2.63 \cdot 10^{-1} (5.13 \cdot 10^{-3})$
2NN	$1.97 \cdot 10^{-1} (4.11 \cdot 10^{-3})$

### Averages of $R(\theta; S^{CRPS})$

$R(\theta; S^{CRPS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$7.49 \cdot 10^{-2} (1.85 \cdot 10^{-3})$	$6.71 \cdot 10^{-2} (1.95 \cdot 10^{-3})$	$6.70 \cdot 10^{-2} (1.93 \cdot 10^{-3})$	$6.08 \cdot 10^{-2} (1.67 \cdot 10^{-3})$	$2.04 \cdot 10^{-1} (3.60 \cdot 10^{-3})$	$6.73 \cdot 10^{-2} (1.93 \cdot 10^{-3})$
$v = 3/2$	$5.45 \cdot 10^{-2} (2.34 \cdot 10^{-3})$	$5.75 \cdot 10^{-2} (2.47 \cdot 10^{-3})$	$5.79 \cdot 10^{-2} (2.54 \cdot 10^{-3})$	$5.11 \cdot 10^{-2} (2.33 \cdot 10^{-3})$	$1.73 \cdot 10^{-1} (4.09 \cdot 10^{-3})$	$6.01 \cdot 10^{-2} (2.67 \cdot 10^{-3})$
$v = 5/2$	$6.17 \cdot 10^{-2} (2.65 \cdot 10^{-3})$	$6.89 \cdot 10^{-2} (3.11 \cdot 10^{-3})$	$6.81 \cdot 10^{-2} (3.04 \cdot 10^{-3})$	$6.46 \cdot 10^{-2} (2.92 \cdot 10^{-3})$	$1.64 \cdot 10^{-1} (4.53 \cdot 10^{-3})$	$7.14 \cdot 10^{-2} (3.22 \cdot 10^{-3})$
$v = 7/2$	$6.78 \cdot 10^{-2} (2.80 \cdot 10^{-3})$	$7.57 \cdot 10^{-2} (3.33 \cdot 10^{-3})$	$7.50 \cdot 10^{-2} (3.29 \cdot 10^{-3})$	$7.25 \cdot 10^{-2} (3.17 \cdot 10^{-3})$	$1.58 \cdot 10^{-1} (4.78 \cdot 10^{-3})$	$7.99 \cdot 10^{-2} (3.57 \cdot 10^{-3})$
$v = 9/2$	$7.21 \cdot 10^{-2} (2.89 \cdot 10^{-3})$	$7.99 \cdot 10^{-2} (3.42 \cdot 10^{-3})$	$8.01 \cdot 10^{-2} (3.48 \cdot 10^{-3})$	$7.75 \cdot 10^{-2} (3.29 \cdot 10^{-3})$	$1.52 \cdot 10^{-1} (4.76 \cdot 10^{-3})$	$8.72 \cdot 10^{-2} (3.92 \cdot 10^{-3})$
$v = 11/2$	$7.53 \cdot 10^{-2} (2.96 \cdot 10^{-3})$	$8.29 \cdot 10^{-2} (3.50 \cdot 10^{-3})$	$8.30 \cdot 10^{-2} (3.58 \cdot 10^{-3})$	$8.04 \cdot 10^{-2} (3.37 \cdot 10^{-3})$	$1.46 \cdot 10^{-1} (4.50 \cdot 10^{-3})$	$8.93 \cdot 10^{-2} (4.04 \cdot 10^{-3})$
$v = 21/2$	$8.25 \cdot 10^{-2} (3.16 \cdot 10^{-3})$	$9.08 \cdot 10^{-2} (3.71 \cdot 10^{-3})$	$9.02 \cdot 10^{-2} (3.80 \cdot 10^{-3})$	$8.88 \cdot 10^{-2} (3.63 \cdot 10^{-3})$	$1.31 \cdot 10^{-1} (4.10 \cdot 10^{-3})$	$9.95 \cdot 10^{-2} (4.44 \cdot 10^{-3})$
$v = \infty$	$9.12 \cdot 10^{-2} (3.45 \cdot 10^{-3})$	$9.96 \cdot 10^{-2} (3.98 \cdot 10^{-3})$	$9.83 \cdot 10^{-2} (4.15 \cdot 10^{-3})$	$9.81 \cdot 10^{-2} (3.96 \cdot 10^{-3})$	$1.26 \cdot 10^{-1} (4.92 \cdot 10^{-3})$	$1.07 \cdot 10^{-1} (4.70 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$5.47 \cdot 10^{-2} (2.43 \cdot 10^{-3})$	$6.04 \cdot 10^{-2} (2.80 \cdot 10^{-3})$	$6.53 \cdot 10^{-2} (3.11 \cdot 10^{-3})$	$5.18 \cdot 10^{-2} (2.43 \cdot 10^{-3})$	$1.26 \cdot 10^{-1} (4.92 \cdot 10^{-3})$	$8.88 \cdot 10^{-2} (4.72 \cdot 10^{-3})$

$R(\theta; S^{CRPS})$	
NLL/SPE	$5.59 \cdot 10^{-2} (2.55 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{IS})$

$R(\theta; S_{0.95}^{IS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.08 \cdot 10^0 (2.71 \cdot 10^{-2})$	$9.95 \cdot 10^{-1} (2.95 \cdot 10^{-2})$	$9.95 \cdot 10^{-1} (2.95 \cdot 10^{-2})$	$1.08 \cdot 10^0 (3.60 \cdot 10^{-2})$	$1.99 \cdot 10^0 (3.54 \cdot 10^{-2})$	$9.99 \cdot 10^{-1} (2.95 \cdot 10^{-2})$
$v = 3/2$	$9.29 \cdot 10^{-1} (3.46 \cdot 10^{-2})$	$9.77 \cdot 10^{-1} (3.80 \cdot 10^{-2})$	$1.00 \cdot 10^0 (4.06 \cdot 10^{-2})$	$1.14 \cdot 10^0 (5.17 \cdot 10^{-2})$	$1.78 \cdot 10^0 (4.54 \cdot 10^{-2})$	$1.03 \cdot 10^0 (4.18 \cdot 10^{-2})$
$v = 5/2$	$1.02 \cdot 10^0 (3.87 \cdot 10^{-2})$	$1.14 \cdot 10^0 (4.86 \cdot 10^{-2})$	$1.15 \cdot 10^0 (4.70 \cdot 10^{-2})$	$1.33 \cdot 10^0 (6.00 \cdot 10^{-2})$	$1.72 \cdot 10^0 (5.16 \cdot 10^{-2})$	$1.19 \cdot 10^0 (4.73 \cdot 10^{-2})$
$v = 7/2$	$1.08 \cdot 10^0 (4.05 \cdot 10^{-2})$	$1.19 \cdot 10^0 (4.99 \cdot 10^{-2})$	$1.20 \cdot 10^0 (4.94 \cdot 10^{-2})$	$1.37 \cdot 10^0 (6.07 \cdot 10^{-2})$	$1.68 \cdot 10^0 (5.41 \cdot 10^{-2})$	$1.26 \cdot 10^0 (4.89 \cdot 10^{-2})$
$v = 9/2$	$1.12 \cdot 10^0 (4.15 \cdot 10^{-2})$	$1.21 \cdot 10^0 (4.97 \cdot 10^{-2})$	$1.24 \cdot 10^0 (5.09 \cdot 10^{-2})$	$1.39 \cdot 10^0 (5.96 \cdot 10^{-2})$	$1.64 \cdot 10^0 (5.49 \cdot 10^{-2})$	$1.32 \cdot 10^0 (5.27 \cdot 10^{-2})$
$v = 11/2$	$1.15 \cdot 10^0 (4.19 \cdot 10^{-2})$	$1.23 \cdot 10^0 (5.09 \cdot 10^{-2})$	$1.25 \cdot 10^0 (5.14 \cdot 10^{-2})$	$1.40 \cdot 10^0 (5.99 \cdot 10^{-2})$	$1.60 \cdot 10^0 (5.37 \cdot 10^{-2})$	$1.33 \cdot 10^0 (5.46 \cdot 10^{-2})$
$v = 21/2$	$1.23 \cdot 10^0 (4.39 \cdot 10^{-2})$	$1.30 \cdot 10^0 (5.35 \cdot 10^{-2})$	$1.31 \cdot 10^0 (5.42 \cdot 10^{-2})$	$1.47 \cdot 10^0 (6.21 \cdot 10^{-2})$	$1.52 \cdot 10^0 (5.53 \cdot 10^{-2})$	$1.42 \cdot 10^0 (5.74 \cdot 10^{-2})$
$v = \infty$	$1.33 \cdot 10^0 (4.65 \cdot 10^{-2})$	$1.37 \cdot 10^0 (5.66 \cdot 10^{-2})$	$1.38 \cdot 10^0 (5.81 \cdot 10^{-2})$	$1.54 \cdot 10^0 (6.33 \cdot 10^{-2})$	$1.53 \cdot 10^0 (6.49 \cdot 10^{-2})$	$1.49 \cdot 10^0 (6.44 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$9.43 \cdot 10^{-1} (3.86 \cdot 10^{-2})$	$1.03 \cdot 10^0 (4.49 \cdot 10^{-2})$	$1.11 \cdot 10^0 (4.82 \cdot 10^{-2})$	$1.15 \cdot 10^0 (5.26 \cdot 10^{-2})$	$1.53 \cdot 10^0 (6.49 \cdot 10^{-2})$	$1.36 \cdot 10^0 (6.09 \cdot 10^{-2})$

$R(\theta; S_{0.95}^{IS})$	
NLL/SPE	$9.57 \cdot 10^{-1} (4.11 \cdot 10^{-2})$

### SM-2.2.28 Rosenbrock, $n = 20, d = 2$

#### Averages of $R(\theta; S^{SPE})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.06 \cdot 10^{-1}$ ( $1.51 \cdot 10^{-1}$ )	$3.10 \cdot 10^{-1}$ ( $6.38 \cdot 10^{-2}$ )	$4.11 \cdot 10^{-1}$ ( $1.09 \cdot 10^{-1}$ )	$2.91 \cdot 10^{-1}$ ( $6.08 \cdot 10^{-2}$ )	$3.46 \cdot 10^{-1}$ ( $5.28 \cdot 10^{-2}$ )	$4.43 \cdot 10^{-1}$ ( $1.18 \cdot 10^{-1}$ )
$v = 3/2$	$2.91 \cdot 10^{-1}$ ( $8.88 \cdot 10^{-2}$ )	$1.70 \cdot 10^{-1}$ ( $6.11 \cdot 10^{-2}$ )	$1.46 \cdot 10^{-1}$ ( $2.58 \cdot 10^{-2}$ )	$1.11 \cdot 10^{-1}$ ( $2.25 \cdot 10^{-2}$ )	$2.99 \cdot 10^{-1}$ ( $4.16 \cdot 10^{-2}$ )	$2.24 \cdot 10^{-1}$ ( $5.32 \cdot 10^{-2}$ )
$v = 5/2$	$1.16 \cdot 10^{-1}$ ( $2.99 \cdot 10^{-2}$ )	$2.25 \cdot 10^{-1}$ ( $9.13 \cdot 10^{-2}$ )	$6.93 \cdot 10^{-2}$ ( $1.70 \cdot 10^{-2}$ )	$8.22 \cdot 10^{-2}$ ( $1.96 \cdot 10^{-2}$ )	$2.84 \cdot 10^{-1}$ ( $3.85 \cdot 10^{-2}$ )	$7.42 \cdot 10^{-2}$ ( $1.80 \cdot 10^{-2}$ )
$v = 7/2$	$9.05 \cdot 10^{-2}$ ( $2.92 \cdot 10^{-2}$ )	$7.82 \cdot 10^{-2}$ ( $3.59 \cdot 10^{-2}$ )	$2.94 \cdot 10^{-2}$ ( $7.52 \cdot 10^{-3}$ )	$3.11 \cdot 10^{-2}$ ( $6.70 \cdot 10^{-3}$ )	$2.69 \cdot 10^{-1}$ ( $3.66 \cdot 10^{-2}$ )	$5.59 \cdot 10^{-2}$ ( $2.61 \cdot 10^{-2}$ )
$v = 9/2$	$1.70 \cdot 10^{-1}$ ( $8.49 \cdot 10^{-2}$ )	$1.28 \cdot 10^{-2}$ ( $2.86 \cdot 10^{-3}$ )	$1.12 \cdot 10^{-2}$ ( $2.29 \cdot 10^{-3}$ )	$1.25 \cdot 10^{-2}$ ( $3.01 \cdot 10^{-3}$ )	$2.59 \cdot 10^{-1}$ ( $3.55 \cdot 10^{-2}$ )	$4.57 \cdot 10^{-2}$ ( $2.85 \cdot 10^{-2}$ )
$v = \infty$	$2.29 \cdot 10^{-1}$ ( $8.78 \cdot 10^{-2}$ )	$6.57 \cdot 10^{-3}$ ( $1.53 \cdot 10^{-3}$ )	$5.28 \cdot 10^{-3}$ ( $1.35 \cdot 10^{-3}$ )	$6.23 \cdot 10^{-3}$ ( $1.26 \cdot 10^{-3}$ )	$2.09 \cdot 10^{-1}$ ( $2.78 \cdot 10^{-2}$ )	$5.90 \cdot 10^{-2}$ ( $3.74 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, \infty\}$	$1.01 \cdot 10^{-1}$ ( $4.05 \cdot 10^{-2}$ )	$6.57 \cdot 10^{-3}$ ( $1.53 \cdot 10^{-3}$ )	$5.61 \cdot 10^{-3}$ ( $1.50 \cdot 10^{-3}$ )	$6.23 \cdot 10^{-3}$ ( $1.26 \cdot 10^{-3}$ )	$2.93 \cdot 10^{-1}$ ( $4.31 \cdot 10^{-2}$ )	$5.06 \cdot 10^{-2}$ ( $3.70 \cdot 10^{-2}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.36 \cdot 10^{-1}$ ( $5.65 \cdot 10^{-2}$ )
$v = 3/2$	$8.47 \cdot 10^{-2}$ ( $2.46 \cdot 10^{-2}$ )
$v = 5/2$	$3.47 \cdot 10^{-2}$ ( $1.05 \cdot 10^{-2}$ )
$v = 7/2$	$1.75 \cdot 10^{-2}$ ( $4.89 \cdot 10^{-3}$ )
$v = 9/2$	$8.25 \cdot 10^{-3}$ ( $2.19 \cdot 10^{-3}$ )
$v = \infty$	$2.32 \cdot 10^{-3}$ ( $3.58 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$1.92 \cdot 10^{-3}$ ( $3.60 \cdot 10^{-4}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.09 \cdot 10^{-1}$ ( $4.05 \cdot 10^{-2}$ )
linear	$1.04 \cdot 10^0$ ( $2.39 \cdot 10^{-2}$ )
1NN	$4.28 \cdot 10^{-1}$ ( $5.11 \cdot 10^{-2}$ )
2NN	$4.42 \cdot 10^{-1}$ ( $6.57 \cdot 10^{-2}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.73 \cdot 10^{-1}$ ( $3.83 \cdot 10^{-2}$ )	$2.26 \cdot 10^{-1}$ ( $1.16 \cdot 10^{-2}$ )	$2.47 \cdot 10^{-1}$ ( $2.96 \cdot 10^{-2}$ )	$1.91 \cdot 10^{-1}$ ( $1.63 \cdot 10^{-2}$ )	$2.66 \cdot 10^{-1}$ ( $1.09 \cdot 10^{-2}$ )	$2.53 \cdot 10^{-1}$ ( $3.25 \cdot 10^{-2}$ )
$v = 3/2$	$1.68 \cdot 10^{-1}$ ( $2.99 \cdot 10^{-2}$ )	$1.39 \cdot 10^{-1}$ ( $2.98 \cdot 10^{-2}$ )	$1.24 \cdot 10^{-1}$ ( $1.32 \cdot 10^{-2}$ )	$1.11 \cdot 10^{-1}$ ( $8.80 \cdot 10^{-3}$ )	$2.41 \cdot 10^{-1}$ ( $1.68 \cdot 10^{-2}$ )	$1.61 \cdot 10^{-1}$ ( $2.76 \cdot 10^{-2}$ )
$v = 5/2$	$9.94 \cdot 10^{-2}$ ( $1.20 \cdot 10^{-2}$ )	$1.59 \cdot 10^{-1}$ ( $4.55 \cdot 10^{-2}$ )	$8.17 \cdot 10^{-2}$ ( $1.37 \cdot 10^{-2}$ )	$9.12 \cdot 10^{-2}$ ( $1.55 \cdot 10^{-2}$ )	$2.35 \cdot 10^{-1}$ ( $1.97 \cdot 10^{-2}$ )	$8.56 \cdot 10^{-2}$ ( $1.43 \cdot 10^{-2}$ )
$v = 7/2$	$8.46 \cdot 10^{-2}$ ( $1.37 \cdot 10^{-2}$ )	$8.10 \cdot 10^{-2}$ ( $2.48 \cdot 10^{-2}$ )	$5.18 \cdot 10^{-2}$ ( $8.08 \cdot 10^{-3}$ )	$5.29 \cdot 10^{-2}$ ( $7.24 \cdot 10^{-3}$ )	$2.27 \cdot 10^{-1}$ ( $2.08 \cdot 10^{-2}$ )	$7.07 \cdot 10^{-2}$ ( $1.96 \cdot 10^{-2}$ )
$v = 9/2$	$1.09 \cdot 10^{-1}$ ( $3.09 \cdot 10^{-2}$ )	$3.14 \cdot 10^{-2}$ ( $2.39 \cdot 10^{-3}$ )	$3.14 \cdot 10^{-2}$ ( $2.30 \cdot 10^{-3}$ )	$2.98 \cdot 10^{-2}$ ( $2.65 \cdot 10^{-3}$ )	$2.21 \cdot 10^{-1}$ ( $2.16 \cdot 10^{-2}$ )	$6.09 \cdot 10^{-2}$ ( $2.12 \cdot 10^{-2}$ )
$v = \infty$	$1.48 \cdot 10^{-1}$ ( $3.47 \cdot 10^{-2}$ )	$2.38 \cdot 10^{-2}$ ( $2.49 \cdot 10^{-3}$ )	$2.18 \cdot 10^{-2}$ ( $2.21 \cdot 10^{-3}$ )	$2.49 \cdot 10^{-2}$ ( $3.02 \cdot 10^{-3}$ )	$1.98 \cdot 10^{-1}$ ( $2.54 \cdot 10^{-2}$ )	$6.68 \cdot 10^{-2}$ ( $2.66 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, \infty\}$	$8.76 \cdot 10^{-2}$ ( $2.21 \cdot 10^{-2}$ )	$2.38 \cdot 10^{-2}$ ( $2.49 \cdot 10^{-3}$ )	$2.28 \cdot 10^{-2}$ ( $2.63 \cdot 10^{-3}$ )	$2.49 \cdot 10^{-2}$ ( $3.02 \cdot 10^{-3}$ )	$2.44 \cdot 10^{-1}$ ( $1.56 \cdot 10^{-2}$ )	$5.88 \cdot 10^{-2}$ ( $2.53 \cdot 10^{-2}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$9.50 \cdot 10^{-2}$ ( $2.27 \cdot 10^{-2}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.56 \cdot 10^0$ ( $8.92 \cdot 10^{-1}$ )	$4.68 \cdot 10^0$ ( $5.14 \cdot 10^{-1}$ )	$4.84 \cdot 10^0$ ( $7.05 \cdot 10^{-1}$ )	$4.62 \cdot 10^0$ ( $7.18 \cdot 10^{-1}$ )	$5.10 \cdot 10^0$ ( $3.78 \cdot 10^{-1}$ )	$4.85 \cdot 10^0$ ( $7.18 \cdot 10^{-1}$ )
$v = 3/2$	$3.31 \cdot 10^0$ ( $6.83 \cdot 10^{-1}$ )	$2.58 \cdot 10^0$ ( $7.55 \cdot 10^{-1}$ )	$2.03 \cdot 10^0$ ( $3.25 \cdot 10^{-1}$ )	$2.03 \cdot 10^0$ ( $3.71 \cdot 10^{-1}$ )	$4.86 \cdot 10^0$ ( $4.32 \cdot 10^{-1}$ )	$2.84 \cdot 10^0$ ( $5.80 \cdot 10^{-1}$ )
$v = 5/2$	$1.62 \cdot 10^0$ ( $3.58 \cdot 10^{-1}$ )	$3.39 \cdot 10^0$ ( $1.20 \cdot 10^0$ )	$1.38 \cdot 10^0$ ( $3.77 \cdot 10^{-1}$ )	$1.40 \cdot 10^0$ ( $3.46 \cdot 10^{-1}$ )	$4.65 \cdot 10^0$ ( $4.54 \cdot 10^{-1}$ )	$1.47 \cdot 10^0$ ( $3.97 \cdot 10^{-1}$ )
$v = 7/2$	$1.36 \cdot 10^0$ ( $3.33 \cdot 10^{-1}$ )	$1.50 \cdot 10^0$ ( $6.96 \cdot 10^{-1}$ )	$7.82 \cdot 10^{-1}$ ( $2.34 \cdot 10^{-1}$ )	$6.64 \cdot 10^{-1}$ ( $1.59 \cdot 10^{-1}$ )	$4.26 \cdot 10^0$ ( $4.02 \cdot 10^{-1}$ )	$1.36 \cdot 10^0$ ( $6.09 \cdot 10^{-1}$ )
$v = 9/2$	$2.02 \cdot 10^0$ ( $7.22 \cdot 10^{-1}$ )	$3.18 \cdot 10^{-1}$ ( $5.33 \cdot 10^{-2}$ )	$4.14 \cdot 10^{-1}$ ( $5.03 \cdot 10^{-2}$ )	$3.23 \cdot 10^{-1}$ ( $5.05 \cdot 10^{-2}$ )	$4.01 \cdot 10^0$ ( $3.92 \cdot 10^{-1}$ )	$1.28 \cdot 10^0$ ( $6.78 \cdot 10^{-1}$ )
$v = \infty$	$3.07 \cdot 10^0$ ( $8.20 \cdot 10^{-1}$ )	$2.79 \cdot 10^{-1}$ ( $4.43 \cdot 10^{-2}$ )	$3.24 \cdot 10^{-1}$ ( $5.17 \cdot 10^{-2}$ )	$4.35 \cdot 10^{-1}$ ( $9.70 \cdot 10^{-2}$ )	$2.91 \cdot 10^0$ ( $3.55 \cdot 10^{-1}$ )	$1.58 \cdot 10^0$ ( $8.77 \cdot 10^{-1}$ )
$v \in \{1/2, \dots, \infty\}$	$1.68 \cdot 10^0$ ( $5.83 \cdot 10^{-1}$ )	$2.79 \cdot 10^{-1}$ ( $4.43 \cdot 10^{-2}$ )	$3.48 \cdot 10^{-1}$ ( $5.93 \cdot 10^{-2}$ )	$4.35 \cdot 10^{-1}$ ( $9.70 \cdot 10^{-2}$ )	$4.64 \cdot 10^0$ ( $4.55 \cdot 10^{-1}$ )	$1.44 \cdot 10^0$ ( $8.46 \cdot 10^{-1}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.87 \cdot 10^0$ ( $5.94 \cdot 10^{-1}$ )

## SM-2.2.29 Rosenbrock, $n = 40, d = 2$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$9.36 \cdot 10^{-2} (2.22 \cdot 10^{-2})$	$9.26 \cdot 10^{-2} (2.14 \cdot 10^{-2})$	$9.19 \cdot 10^{-2} (2.16 \cdot 10^{-2})$	$9.25 \cdot 10^{-2} (2.17 \cdot 10^{-2})$	$2.09 \cdot 10^{-1} (3.23 \cdot 10^{-2})$	$9.22 \cdot 10^{-2} (2.17 \cdot 10^{-2})$
$v = 3/2$	$1.65 \cdot 10^{-2} (5.53 \cdot 10^{-3})$	$1.44 \cdot 10^{-2} (4.72 \cdot 10^{-3})$	$1.44 \cdot 10^{-2} (4.74 \cdot 10^{-3})$	$1.42 \cdot 10^{-2} (4.66 \cdot 10^{-3})$	$4.56 \cdot 10^{-1} (2.17 \cdot 10^{-1})$	$1.48 \cdot 10^{-2} (4.87 \cdot 10^{-3})$
$v = 5/2$	$3.01 \cdot 10^{-3} (1.12 \cdot 10^{-3})$	$2.86 \cdot 10^{-3} (9.27 \cdot 10^{-4})$	$3.00 \cdot 10^{-3} (9.95 \cdot 10^{-4})$	$2.95 \cdot 10^{-3} (9.73 \cdot 10^{-4})$	$1.18 \cdot 10^{-1} (2.88 \cdot 10^{-2})$	$3.17 \cdot 10^{-3} (1.07 \cdot 10^{-3})$
$v = 7/2$	$1.25 \cdot 10^{-3} (3.38 \cdot 10^{-4})$	$1.34 \cdot 10^{-3} (3.58 \cdot 10^{-4})$	$1.36 \cdot 10^{-3} (3.72 \cdot 10^{-4})$	$1.33 \cdot 10^{-3} (3.66 \cdot 10^{-4})$	$1.04 \cdot 10^{-1} (2.79 \cdot 10^{-2})$	$1.19 \cdot 10^{-3} (3.42 \cdot 10^{-4})$
$v = 9/2$	$1.00 \cdot 10^{-3} (2.34 \cdot 10^{-4})$	$9.82 \cdot 10^{-4} (2.38 \cdot 10^{-4})$	$8.90 \cdot 10^{-4} (2.17 \cdot 10^{-4})$	$8.94 \cdot 10^{-4} (2.17 \cdot 10^{-4})$	$9.66 \cdot 10^{-2} (2.71 \cdot 10^{-2})$	$9.36 \cdot 10^{-4} (2.22 \cdot 10^{-4})$
$v \in \{1/2, \dots, 9/2\}$	$1.01 \cdot 10^{-3} (2.42 \cdot 10^{-4})$	$9.73 \cdot 10^{-4} (2.36 \cdot 10^{-4})$	$1.06 \cdot 10^{-3} (2.63 \cdot 10^{-4})$	$1.03 \cdot 10^{-3} (2.52 \cdot 10^{-4})$	$1.10 \cdot 10^{-1} (2.67 \cdot 10^{-2})$	$9.36 \cdot 10^{-4} (2.22 \cdot 10^{-4})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$8.90 \cdot 10^{-2} (2.11 \cdot 10^{-2})$
$v = 3/2$	$1.36 \cdot 10^{-2} (4.50 \cdot 10^{-3})$
$v = 5/2$	$2.58 \cdot 10^{-3} (8.98 \cdot 10^{-4})$
$v = 7/2$	$1.09 \cdot 10^{-3} (3.19 \cdot 10^{-4})$
$v = 9/2$	$8.61 \cdot 10^{-4} (2.18 \cdot 10^{-4})$
$v \in \{1/2, \dots, 9/2\}$	$8.53 \cdot 10^{-4} (2.19 \cdot 10^{-4})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.04 \cdot 10^{-3} (2.42 \cdot 10^{-4})$
linear	$9.41 \cdot 10^{-1} (1.56 \cdot 10^{-2})$
INN	$2.51 \cdot 10^{-1} (2.31 \cdot 10^{-2})$
2NN	$2.46 \cdot 10^{-1} (2.35 \cdot 10^{-2})$

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.17 \cdot 10^{-1} (5.77 \cdot 10^{-3})$	$1.06 \cdot 10^{-1} (6.36 \cdot 10^{-3})$	$1.05 \cdot 10^{-1} (6.23 \cdot 10^{-3})$	$8.74 \cdot 10^{-2} (7.16 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (7.71 \cdot 10^{-3})$	$1.05 \cdot 10^{-1} (6.16 \cdot 10^{-3})$
$v = 3/2$	$3.22 \cdot 10^{-2} (3.15 \cdot 10^{-3})$	$2.93 \cdot 10^{-2} (3.06 \cdot 10^{-3})$	$2.90 \cdot 10^{-2} (2.96 \cdot 10^{-3})$	$2.71 \cdot 10^{-2} (2.99 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (3.92 \cdot 10^{-2})$	$2.92 \cdot 10^{-2} (3.03 \cdot 10^{-3})$
$v = 5/2$	$1.12 \cdot 10^{-2} (1.23 \cdot 10^{-3})$	$1.11 \cdot 10^{-2} (1.16 \cdot 10^{-3})$	$1.12 \cdot 10^{-2} (1.28 \cdot 10^{-3})$	$1.11 \cdot 10^{-2} (1.31 \cdot 10^{-3})$	$1.14 \cdot 10^{-1} (6.94 \cdot 10^{-3})$	$1.18 \cdot 10^{-2} (1.31 \cdot 10^{-3})$
$v = 7/2$	$7.11 \cdot 10^{-3} (6.53 \cdot 10^{-4})$	$7.51 \cdot 10^{-3} (7.64 \cdot 10^{-4})$	$7.59 \cdot 10^{-3} (8.10 \cdot 10^{-4})$	$7.47 \cdot 10^{-3} (7.90 \cdot 10^{-4})$	$1.01 \cdot 10^{-1} (7.37 \cdot 10^{-3})$	$7.24 \cdot 10^{-3} (6.88 \cdot 10^{-4})$
$v = 9/2$	$6.51 \cdot 10^{-3} (6.16 \cdot 10^{-4})$	$6.50 \cdot 10^{-3} (6.48 \cdot 10^{-4})$	$6.42 \cdot 10^{-3} (6.10 \cdot 10^{-4})$	$6.38 \cdot 10^{-3} (5.81 \cdot 10^{-4})$	$9.42 \cdot 10^{-2} (7.87 \cdot 10^{-3})$	$6.51 \cdot 10^{-3} (6.23 \cdot 10^{-4})$
$v \in \{1/2, \dots, 9/2\}$	$6.44 \cdot 10^{-3} (6.36 \cdot 10^{-4})$	$6.45 \cdot 10^{-3} (6.71 \cdot 10^{-4})$	$6.77 \cdot 10^{-3} (7.23 \cdot 10^{-4})$	$6.53 \cdot 10^{-3} (6.90 \cdot 10^{-4})$	$1.08 \cdot 10^{-1} (8.75 \cdot 10^{-3})$	$6.51 \cdot 10^{-3} (6.23 \cdot 10^{-4})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$6.53 \cdot 10^{-3} (6.55 \cdot 10^{-4})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.05 \cdot 10^0 (1.77 \cdot 10^{-1})$	$1.99 \cdot 10^0 (1.99 \cdot 10^{-1})$	$1.98 \cdot 10^0 (1.99 \cdot 10^{-1})$	$1.98 \cdot 10^0 (2.73 \cdot 10^{-1})$	$3.36 \cdot 10^0 (2.06 \cdot 10^{-1})$	$1.98 \cdot 10^0 (1.98 \cdot 10^{-1})$
$v = 3/2$	$5.67 \cdot 10^{-1} (8.46 \cdot 10^{-2})$	$5.70 \cdot 10^{-1} (8.62 \cdot 10^{-2})$	$5.65 \cdot 10^{-1} (8.33 \cdot 10^{-2})$	$6.36 \cdot 10^{-1} (1.15 \cdot 10^{-1})$	$3.30 \cdot 10^0 (4.83 \cdot 10^{-1})$	$5.63 \cdot 10^{-1} (8.26 \cdot 10^{-2})$
$v = 5/2$	$1.85 \cdot 10^{-1} (3.08 \cdot 10^{-2})$	$2.11 \cdot 10^{-1} (3.23 \cdot 10^{-2})$	$2.15 \cdot 10^{-1} (3.42 \cdot 10^{-2})$	$2.39 \cdot 10^{-1} (4.35 \cdot 10^{-2})$	$2.10 \cdot 10^0 (2.36 \cdot 10^{-1})$	$2.04 \cdot 10^{-1} (3.20 \cdot 10^{-2})$
$v = 7/2$	$1.15 \cdot 10^{-1} (1.55 \cdot 10^{-2})$	$1.35 \cdot 10^{-1} (2.04 \cdot 10^{-2})$	$1.40 \cdot 10^{-1} (2.20 \cdot 10^{-2})$	$1.34 \cdot 10^{-1} (2.16 \cdot 10^{-2})$	$1.89 \cdot 10^0 (2.47 \cdot 10^{-1})$	$1.20 \cdot 10^{-1} (1.71 \cdot 10^{-2})$
$v = 9/2$	$1.04 \cdot 10^{-1} (1.37 \cdot 10^{-2})$	$1.05 \cdot 10^{-1} (1.61 \cdot 10^{-2})$	$1.03 \cdot 10^{-1} (1.44 \cdot 10^{-2})$	$9.49 \cdot 10^{-2} (1.34 \cdot 10^{-2})$	$1.79 \cdot 10^0 (2.46 \cdot 10^{-1})$	$1.05 \cdot 10^{-1} (1.47 \cdot 10^{-2})$
$v \in \{1/2, \dots, 9/2\}$	$1.04 \cdot 10^{-1} (1.46 \cdot 10^{-2})$	$1.09 \cdot 10^{-1} (1.75 \cdot 10^{-2})$	$1.23 \cdot 10^{-1} (2.04 \cdot 10^{-2})$	$1.13 \cdot 10^{-1} (1.75 \cdot 10^{-2})$	$1.99 \cdot 10^0 (2.33 \cdot 10^{-1})$	$1.05 \cdot 10^{-1} (1.47 \cdot 10^{-2})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.05 \cdot 10^{-1} (1.47 \cdot 10^{-2})$



### SM-2.2.30 Rosenbrock, $n = 100, d = 2$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.73 \cdot 10^{-3} (7.77 \cdot 10^{-4})$	$6.92 \cdot 10^{-3} (1.03 \cdot 10^{-3})$	$5.63 \cdot 10^{-3} (7.44 \cdot 10^{-4})$	$5.67 \cdot 10^{-3} (7.45 \cdot 10^{-4})$	$5.53 \cdot 10^{-2} (1.63 \cdot 10^{-2})$	$5.76 \cdot 10^{-3} (8.12 \cdot 10^{-4})$
$v = 3/2$	$4.43 \cdot 10^{-4} (9.87 \cdot 10^{-5})$	$2.92 \cdot 10^{-4} (3.83 \cdot 10^{-5})$	$4.55 \cdot 10^{-4} (9.98 \cdot 10^{-5})$	$3.33 \cdot 10^{-4} (5.12 \cdot 10^{-5})$	$2.56 \cdot 10^{-2} (1.08 \cdot 10^{-2})$	$3.78 \cdot 10^{-4} (6.06 \cdot 10^{-5})$
$v = 5/2$	$4.69 \cdot 10^{-5} (6.82 \cdot 10^{-6})$	$4.92 \cdot 10^{-5} (5.12 \cdot 10^{-6})$	$4.73 \cdot 10^{-5} (6.72 \cdot 10^{-6})$	$4.62 \cdot 10^{-5} (6.05 \cdot 10^{-6})$	$8.59 \cdot 10^{-3} (1.85 \cdot 10^{-3})$	$6.04 \cdot 10^{-5} (7.67 \cdot 10^{-6})$
$v = 7/2$	$4.35 \cdot 10^{-5} (5.96 \cdot 10^{-6})$	$5.22 \cdot 10^{-5} (9.21 \cdot 10^{-6})$	$4.48 \cdot 10^{-5} (6.23 \cdot 10^{-6})$	$4.93 \cdot 10^{-5} (8.13 \cdot 10^{-6})$	$6.86 \cdot 10^{-3} (1.74 \cdot 10^{-3})$	$4.68 \cdot 10^{-5} (7.02 \cdot 10^{-6})$
$v = 9/2$	$6.23 \cdot 10^{-5} (1.02 \cdot 10^{-5})$	$9.40 \cdot 10^{-5} (2.56 \cdot 10^{-5})$	$6.46 \cdot 10^{-5} (1.01 \cdot 10^{-5})$	$8.93 \cdot 10^{-5} (2.30 \cdot 10^{-5})$	$6.00 \cdot 10^{-3} (1.59 \cdot 10^{-3})$	$6.80 \cdot 10^{-5} (1.13 \cdot 10^{-5})$
$v \in \{1/2, \dots, 9/2\}$	$3.55 \cdot 10^{-5} (5.04 \cdot 10^{-6})$	$4.48 \cdot 10^{-5} (8.83 \cdot 10^{-6})$	$5.97 \cdot 10^{-5} (8.50 \cdot 10^{-6})$	$3.99 \cdot 10^{-5} (7.90 \cdot 10^{-6})$	$8.73 \cdot 10^{-3} (2.23 \cdot 10^{-3})$	$6.80 \cdot 10^{-5} (1.13 \cdot 10^{-5})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$5.17 \cdot 10^{-3} (7.00 \cdot 10^{-4})$
$v = 3/2$	$2.62 \cdot 10^{-4} (3.50 \cdot 10^{-5})$
$v = 5/2$	$3.54 \cdot 10^{-5} (3.11 \cdot 10^{-6})$
$v = 7/2$	$3.72 \cdot 10^{-5} (3.78 \cdot 10^{-6})$
$v = 9/2$	$6.10 \cdot 10^{-5} (9.97 \cdot 10^{-6})$
$v \in \{1/2, \dots, 9/2\}$	$2.85 \cdot 10^{-5} (2.45 \cdot 10^{-6})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.00 \cdot 10^{-5} (6.96 \cdot 10^{-6})$
linear	$9.23 \cdot 10^{-1} (1.34 \cdot 10^{-2})$
INN	$1.09 \cdot 10^{-1} (6.00 \cdot 10^{-3})$
2NN	$9.51 \cdot 10^{-2} (5.23 \cdot 10^{-3})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.64 \cdot 10^{-2} (2.95 \cdot 10^{-3})$	$4.38 \cdot 10^{-2} (3.89 \cdot 10^{-3})$	$4.14 \cdot 10^{-2} (3.20 \cdot 10^{-3})$	$2.43 \cdot 10^{-2} (1.29 \cdot 10^{-3})$	$1.11 \cdot 10^{-1} (1.12 \cdot 10^{-2})$	$4.17 \cdot 10^{-2} (3.32 \cdot 10^{-3})$
$v = 3/2$	$8.41 \cdot 10^{-3} (7.59 \cdot 10^{-4})$	$6.61 \cdot 10^{-3} (6.10 \cdot 10^{-4})$	$6.91 \cdot 10^{-3} (7.62 \cdot 10^{-4})$	$5.40 \cdot 10^{-3} (4.87 \cdot 10^{-4})$	$5.77 \cdot 10^{-2} (7.09 \cdot 10^{-3})$	$6.92 \cdot 10^{-3} (6.44 \cdot 10^{-4})$
$v = 5/2$	$2.35 \cdot 10^{-3} (1.15 \cdot 10^{-4})$	$2.23 \cdot 10^{-3} (1.03 \cdot 10^{-4})$	$2.08 \cdot 10^{-3} (1.19 \cdot 10^{-4})$	$1.88 \cdot 10^{-3} (1.22 \cdot 10^{-4})$	$3.30 \cdot 10^{-2} (3.65 \cdot 10^{-3})$	$2.43 \cdot 10^{-3} (1.64 \cdot 10^{-4})$
$v = 7/2$	$1.99 \cdot 10^{-3} (1.15 \cdot 10^{-4})$	$2.07 \cdot 10^{-3} (1.64 \cdot 10^{-4})$	$1.96 \cdot 10^{-3} (1.27 \cdot 10^{-4})$	$1.94 \cdot 10^{-3} (1.58 \cdot 10^{-4})$	$2.63 \cdot 10^{-2} (3.54 \cdot 10^{-3})$	$2.01 \cdot 10^{-3} (1.42 \cdot 10^{-4})$
$v = 9/2$	$2.22 \cdot 10^{-3} (1.84 \cdot 10^{-4})$	$2.52 \cdot 10^{-3} (2.93 \cdot 10^{-4})$	$2.24 \cdot 10^{-3} (1.84 \cdot 10^{-4})$	$2.43 \cdot 10^{-3} (2.88 \cdot 10^{-4})$	$2.36 \cdot 10^{-2} (3.02 \cdot 10^{-3})$	$2.29 \cdot 10^{-3} (1.95 \cdot 10^{-4})$
$v \in \{1/2, \dots, 9/2\}$	$1.86 \cdot 10^{-3} (9.26 \cdot 10^{-5})$	$1.91 \cdot 10^{-3} (1.58 \cdot 10^{-4})$	$2.18 \cdot 10^{-3} (1.64 \cdot 10^{-4})$	$1.67 \cdot 10^{-3} (1.56 \cdot 10^{-4})$	$3.19 \cdot 10^{-2} (4.44 \cdot 10^{-3})$	$2.29 \cdot 10^{-3} (1.95 \cdot 10^{-4})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$2.01 \cdot 10^{-3} (1.36 \cdot 10^{-4})$

#### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$8.68 \cdot 10^{-1} (4.43 \cdot 10^{-2})$	$6.59 \cdot 10^{-1} (5.86 \cdot 10^{-2})$	$6.33 \cdot 10^{-1} (5.10 \cdot 10^{-2})$	$5.22 \cdot 10^{-1} (4.06 \cdot 10^{-2})$	$1.78 \cdot 10^0 (2.24 \cdot 10^{-1})$	$6.31 \cdot 10^{-1} (5.14 \cdot 10^{-2})$
$v = 3/2$	$1.22 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$1.03 \cdot 10^{-1} (8.93 \cdot 10^{-3})$	$1.12 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$1.16 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$8.97 \cdot 10^{-1} (1.37 \cdot 10^{-1})$	$1.10 \cdot 10^{-1} (9.12 \cdot 10^{-3})$
$v = 5/2$	$3.33 \cdot 10^{-2} (1.66 \cdot 10^{-3})$	$3.51 \cdot 10^{-2} (1.75 \cdot 10^{-3})$	$3.43 \cdot 10^{-2} (2.24 \cdot 10^{-3})$	$3.87 \cdot 10^{-2} (2.69 \cdot 10^{-3})$	$5.16 \cdot 10^{-1} (6.74 \cdot 10^{-2})$	$4.00 \cdot 10^{-2} (2.26 \cdot 10^{-3})$
$v = 7/2$	$2.60 \cdot 10^{-2} (1.46 \cdot 10^{-3})$	$3.28 \cdot 10^{-2} (3.84 \cdot 10^{-3})$	$2.99 \cdot 10^{-2} (2.71 \cdot 10^{-3})$	$3.44 \cdot 10^{-2} (4.31 \cdot 10^{-3})$	$4.30 \cdot 10^{-1} (6.54 \cdot 10^{-2})$	$2.99 \cdot 10^{-2} (2.64 \cdot 10^{-3})$
$v = 9/2$	$2.85 \cdot 10^{-2} (2.23 \cdot 10^{-3})$	$4.12 \cdot 10^{-2} (7.42 \cdot 10^{-3})$	$3.20 \cdot 10^{-2} (2.82 \cdot 10^{-3})$	$4.12 \cdot 10^{-2} (8.03 \cdot 10^{-3})$	$3.87 \cdot 10^{-1} (5.99 \cdot 10^{-2})$	$3.33 \cdot 10^{-2} (3.25 \cdot 10^{-3})$
$v \in \{1/2, \dots, 9/2\}$	$2.62 \cdot 10^{-2} (1.43 \cdot 10^{-3})$	$3.18 \cdot 10^{-2} (3.43 \cdot 10^{-3})$	$3.18 \cdot 10^{-2} (2.71 \cdot 10^{-3})$	$3.19 \cdot 10^{-2} (3.96 \cdot 10^{-3})$	$4.97 \cdot 10^{-1} (7.67 \cdot 10^{-2})$	$3.33 \cdot 10^{-2} (3.25 \cdot 10^{-3})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$2.74 \cdot 10^{-2} (1.56 \cdot 10^{-3})$

### SM-2.2.31 Rosenbrock, $n = 50, d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$6.90 \cdot 10^{-1} (2.27 \cdot 10^{-2})$	$6.56 \cdot 10^{-1} (2.52 \cdot 10^{-2})$	$6.49 \cdot 10^{-1} (2.58 \cdot 10^{-2})$	$6.19 \cdot 10^{-1} (2.10 \cdot 10^{-2})$	$8.23 \cdot 10^{-1} (1.15 \cdot 10^{-2})$	$6.31 \cdot 10^{-1} (2.67 \cdot 10^{-2})$
$v = 3/2$	$6.17 \cdot 10^{-1} (3.67 \cdot 10^{-2})$	$6.23 \cdot 10^{-1} (7.11 \cdot 10^{-2})$	$6.00 \cdot 10^{-1} (6.10 \cdot 10^{-2})$	$5.76 \cdot 10^{-1} (6.00 \cdot 10^{-2})$	$7.96 \cdot 10^{-1} (1.83 \cdot 10^{-2})$	$6.79 \cdot 10^{-1} (8.49 \cdot 10^{-2})$
$v = 5/2$	$6.59 \cdot 10^{-1} (3.27 \cdot 10^{-2})$	$6.70 \cdot 10^{-1} (5.24 \cdot 10^{-2})$	$5.90 \cdot 10^{-1} (4.55 \cdot 10^{-2})$	$6.47 \cdot 10^{-1} (5.69 \cdot 10^{-2})$	$7.89 \cdot 10^{-1} (1.86 \cdot 10^{-2})$	$6.34 \cdot 10^{-1} (4.00 \cdot 10^{-2})$
$v = 7/2$	$7.00 \cdot 10^{-1} (2.75 \cdot 10^{-2})$	$8.24 \cdot 10^{-1} (6.82 \cdot 10^{-2})$	$7.58 \cdot 10^{-1} (4.31 \cdot 10^{-2})$	$8.06 \cdot 10^{-1} (5.79 \cdot 10^{-2})$	$7.90 \cdot 10^{-1} (2.02 \cdot 10^{-2})$	$8.02 \cdot 10^{-1} (8.47 \cdot 10^{-2})$
$v = 9/2$	$7.17 \cdot 10^{-1} (2.91 \cdot 10^{-2})$	$8.73 \cdot 10^{-1} (7.65 \cdot 10^{-2})$	$8.14 \cdot 10^{-1} (5.01 \cdot 10^{-2})$	$8.43 \cdot 10^{-1} (5.18 \cdot 10^{-2})$	$7.92 \cdot 10^{-1} (2.15 \cdot 10^{-2})$	$7.92 \cdot 10^{-1} (6.02 \cdot 10^{-2})$
$v = 11/2$	$7.23 \cdot 10^{-1} (2.86 \cdot 10^{-2})$	$9.14 \cdot 10^{-1} (7.91 \cdot 10^{-2})$	$8.50 \cdot 10^{-1} (5.59 \cdot 10^{-2})$	$8.82 \cdot 10^{-1} (6.16 \cdot 10^{-2})$	$7.94 \cdot 10^{-1} (2.26 \cdot 10^{-2})$	$8.80 \cdot 10^{-1} (7.73 \cdot 10^{-2})$
$v = 21/2$	$7.28 \cdot 10^{-1} (3.14 \cdot 10^{-2})$	$9.96 \cdot 10^{-1} (9.46 \cdot 10^{-2})$	$9.42 \cdot 10^{-1} (8.01 \cdot 10^{-2})$	$1.01 \cdot 10^0 (8.37 \cdot 10^{-2})$	$8.03 \cdot 10^{-1} (2.65 \cdot 10^{-2})$	$9.80 \cdot 10^{-1} (1.01 \cdot 10^{-1})$
$v = \infty$	$7.72 \cdot 10^{-1} (2.44 \cdot 10^{-2})$	$9.95 \cdot 10^{-1} (9.25 \cdot 10^{-2})$	$9.91 \cdot 10^{-1} (9.52 \cdot 10^{-2})$	$9.96 \cdot 10^{-1} (8.71 \cdot 10^{-2})$	$8.21 \cdot 10^{-1} (3.47 \cdot 10^{-2})$	$1.03 \cdot 10^0 (1.01 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$7.02 \cdot 10^{-1} (4.66 \cdot 10^{-2})$	$1.00 \cdot 10^0 (1.05 \cdot 10^{-1})$	$1.01 \cdot 10^0 (8.86 \cdot 10^{-2})$	$1.03 \cdot 10^0 (9.99 \cdot 10^{-2})$	$8.11 \cdot 10^{-1} (3.18 \cdot 10^{-2})$	$1.02 \cdot 10^0 (9.17 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$5.15 \cdot 10^{-1} (1.08 \cdot 10^{-2})$
$v = 3/2$	$3.38 \cdot 10^{-1} (1.28 \cdot 10^{-2})$
$v = 5/2$	$3.18 \cdot 10^{-1} (1.62 \cdot 10^{-2})$
$v = 7/2$	$3.47 \cdot 10^{-1} (1.66 \cdot 10^{-2})$
$v = 9/2$	$3.59 \cdot 10^{-1} (1.64 \cdot 10^{-2})$
$v = 11/2$	$3.66 \cdot 10^{-1} (1.63 \cdot 10^{-2})$
$v = 21/2$	$3.80 \cdot 10^{-1} (1.70 \cdot 10^{-2})$
$v = \infty$	$3.96 \cdot 10^{-1} (1.77 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$3.14 \cdot 10^{-1} (1.53 \cdot 10^{-2})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$6.71 \cdot 10^{-1} (5.03 \cdot 10^{-2})$
linear	$1.01 \cdot 10^0 (1.09 \cdot 10^{-2})$
1NN	$9.18 \cdot 10^{-1} (1.17 \cdot 10^{-2})$
2NN	$8.17 \cdot 10^{-1} (1.23 \cdot 10^{-2})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.08 \cdot 10^{-1} (8.87 \cdot 10^{-3})$	$3.98 \cdot 10^{-1} (9.79 \cdot 10^{-3})$	$3.94 \cdot 10^{-1} (9.03 \cdot 10^{-3})$	$3.83 \cdot 10^{-1} (8.28 \cdot 10^{-3})$	$4.52 \cdot 10^{-1} (5.76 \cdot 10^{-3})$	$3.86 \cdot 10^{-1} (1.01 \cdot 10^{-2})$
$v = 3/2$	$3.82 \cdot 10^{-1} (1.54 \cdot 10^{-2})$	$3.89 \cdot 10^{-1} (2.69 \cdot 10^{-2})$	$3.82 \cdot 10^{-1} (2.44 \cdot 10^{-2})$	$3.74 \cdot 10^{-1} (2.44 \cdot 10^{-2})$	$4.41 \cdot 10^{-1} (8.31 \cdot 10^{-3})$	$4.07 \cdot 10^{-1} (2.94 \cdot 10^{-2})$
$v = 5/2$	$4.00 \cdot 10^{-1} (1.33 \cdot 10^{-2})$	$4.17 \cdot 10^{-1} (1.80 \cdot 10^{-2})$	$3.90 \cdot 10^{-1} (1.66 \cdot 10^{-2})$	$4.15 \cdot 10^{-1} (2.17 \cdot 10^{-2})$	$4.37 \cdot 10^{-1} (8.14 \cdot 10^{-3})$	$4.06 \cdot 10^{-1} (1.59 \cdot 10^{-2})$
$v = 7/2$	$4.14 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$4.78 \cdot 10^{-1} (2.24 \cdot 10^{-2})$	$4.62 \cdot 10^{-1} (1.62 \cdot 10^{-2})$	$4.73 \cdot 10^{-1} (1.87 \cdot 10^{-2})$	$4.37 \cdot 10^{-1} (8.70 \cdot 10^{-3})$	$4.59 \cdot 10^{-1} (2.56 \cdot 10^{-2})$
$v = 9/2$	$4.20 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$4.92 \cdot 10^{-1} (2.45 \cdot 10^{-2})$	$4.80 \cdot 10^{-1} (1.70 \cdot 10^{-2})$	$4.93 \cdot 10^{-1} (1.77 \cdot 10^{-2})$	$4.38 \cdot 10^{-1} (9.14 \cdot 10^{-3})$	$4.58 \cdot 10^{-1} (2.09 \cdot 10^{-2})$
$v = 11/2$	$4.23 \cdot 10^{-1} (1.08 \cdot 10^{-2})$	$5.12 \cdot 10^{-1} (2.54 \cdot 10^{-2})$	$4.92 \cdot 10^{-1} (1.82 \cdot 10^{-2})$	$5.09 \cdot 10^{-1} (2.15 \cdot 10^{-2})$	$4.38 \cdot 10^{-1} (9.49 \cdot 10^{-3})$	$4.88 \cdot 10^{-1} (2.11 \cdot 10^{-2})$
$v = 21/2$	$4.26 \cdot 10^{-1} (1.15 \cdot 10^{-2})$	$5.37 \cdot 10^{-1} (2.75 \cdot 10^{-2})$	$5.14 \cdot 10^{-1} (2.28 \cdot 10^{-2})$	$5.47 \cdot 10^{-1} (2.58 \cdot 10^{-2})$	$4.41 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$5.11 \cdot 10^{-1} (2.51 \cdot 10^{-2})$
$v = \infty$	$4.43 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$5.36 \cdot 10^{-1} (2.62 \cdot 10^{-2})$	$5.27 \cdot 10^{-1} (2.65 \cdot 10^{-2})$	$5.47 \cdot 10^{-1} (2.67 \cdot 10^{-2})$	$4.47 \cdot 10^{-1} (1.29 \cdot 10^{-2})$	$5.25 \cdot 10^{-1} (2.55 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$4.16 \cdot 10^{-1} (1.80 \cdot 10^{-2})$	$5.36 \cdot 10^{-1} (3.18 \cdot 10^{-2})$	$5.44 \cdot 10^{-1} (2.48 \cdot 10^{-2})$	$5.59 \cdot 10^{-1} (2.99 \cdot 10^{-2})$	$4.44 \cdot 10^{-1} (1.15 \cdot 10^{-2})$	$5.28 \cdot 10^{-1} (2.55 \cdot 10^{-2})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$4.05 \cdot 10^{-1} (1.95 \cdot 10^{-2})$

#### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.50 \cdot 10^0 (1.77 \cdot 10^{-1})$	$5.45 \cdot 10^0 (1.96 \cdot 10^{-1})$	$5.46 \cdot 10^0 (2.12 \cdot 10^{-1})$	$5.58 \cdot 10^0 (2.35 \cdot 10^{-1})$	$6.08 \cdot 10^0 (1.00 \cdot 10^{-1})$	$5.31 \cdot 10^0 (2.17 \cdot 10^{-1})$
$v = 3/2$	$5.45 \cdot 10^0 (3.40 \cdot 10^{-1})$	$5.45 \cdot 10^0 (5.41 \cdot 10^{-1})$	$5.65 \cdot 10^0 (5.75 \cdot 10^{-1})$	$5.59 \cdot 10^0 (6.03 \cdot 10^{-1})$	$5.97 \cdot 10^0 (1.19 \cdot 10^{-1})$	$6.02 \cdot 10^0 (7.11 \cdot 10^{-1})$
$v = 5/2$	$5.84 \cdot 10^0 (3.46 \cdot 10^{-1})$	$6.09 \cdot 10^0 (4.23 \cdot 10^{-1})$	$6.02 \cdot 10^0 (5.83 \cdot 10^{-1})$	$6.95 \cdot 10^0 (6.76 \cdot 10^{-1})$	$5.94 \cdot 10^0 (1.16 \cdot 10^{-1})$	$6.31 \cdot 10^0 (5.48 \cdot 10^{-1})$
$v = 7/2$	$6.10 \cdot 10^0 (2.76 \cdot 10^{-1})$	$8.18 \cdot 10^0 (5.88 \cdot 10^{-1})$	$8.19 \cdot 10^0 (5.84 \cdot 10^{-1})$	$8.53 \cdot 10^0 (5.53 \cdot 10^{-1})$	$5.92 \cdot 10^0 (1.19 \cdot 10^{-1})$	$7.83 \cdot 10^0 (7.24 \cdot 10^{-1})$
$v = 9/2$	$6.26 \cdot 10^0 (2.91 \cdot 10^{-1})$	$8.65 \cdot 10^0 (5.09 \cdot 10^{-1})$	$8.67 \cdot 10^0 (5.89 \cdot 10^{-1})$	$9.41 \cdot 10^0 (5.36 \cdot 10^{-1})$	$5.91 \cdot 10^0 (1.22 \cdot 10^{-1})$	$7.73 \cdot 10^0 (7.09 \cdot 10^{-1})$
$v = 11/2$	$6.32 \cdot 10^0 (2.92 \cdot 10^{-1})$	$9.34 \cdot 10^0 (4.83 \cdot 10^{-1})$	$9.29 \cdot 10^0 (5.84 \cdot 10^{-1})$	$1.01 \cdot 10^1 (5.69 \cdot 10^{-1})$	$5.91 \cdot 10^0 (1.25 \cdot 10^{-1})$	$8.62 \cdot 10^0 (5.77 \cdot 10^{-1})$
$v = 21/2$	$6.43 \cdot 10^0 (3.05 \cdot 10^{-1})$	$1.01 \cdot 10^1 (5.20 \cdot 10^{-1})$	$9.97 \cdot 10^0 (5.83 \cdot 10^{-1})$	$1.12 \cdot 10^1 (6.32 \cdot 10^{-1})$	$5.90 \cdot 10^0 (1.33 \cdot 10^{-1})$	$9.14 \cdot 10^0 (6.23 \cdot 10^{-1})$
$v = \infty$	$6.91 \cdot 10^0 (3.10 \cdot 10^{-1})$	$1.03 \cdot 10^1 (5.09 \cdot 10^{-1})$	$1.02 \cdot 10^1 (6.11 \cdot 10^{-1})$	$1.16 \cdot 10^1 (6.98 \cdot 10^{-1})$	$5.91 \cdot 10^0 (1.49 \cdot 10^{-1})$	$9.57 \cdot 10^0 (5.88 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$6.36 \cdot 10^0 (4.44 \cdot 10^{-1})$	$1.03 \cdot 10^1 (6.24 \cdot 10^{-1})$	$1.10 \cdot 10^1 (5.10 \cdot 10^{-1})$	$1.17 \cdot 10^1 (5.87 \cdot 10^{-1})$	$5.96 \cdot 10^0 (1.56 \cdot 10^{-1})$	$9.88 \cdot 10^0 (6.15 \cdot 10^{-1})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$6.05 \cdot 10^0 (4.87 \cdot 10^{-1})$

### SM-2.2.32 Rosenbrock, $n = 100, d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.45 \cdot 10^{-1} (1.23 \cdot 10^{-2})$	$3.50 \cdot 10^{-1} (1.29 \cdot 10^{-2})$	$3.43 \cdot 10^{-1} (1.21 \cdot 10^{-2})$	$3.35 \cdot 10^{-1} (1.36 \cdot 10^{-2})$	$8.08 \cdot 10^{-1} (2.54 \cdot 10^{-2})$	$3.37 \cdot 10^{-1} (1.24 \cdot 10^{-2})$
$v = 3/2$	$2.16 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$2.35 \cdot 10^{-1} (1.95 \cdot 10^{-2})$	$2.01 \cdot 10^{-1} (9.55 \cdot 10^{-3})$	$2.04 \cdot 10^{-1} (9.38 \cdot 10^{-3})$	$7.80 \cdot 10^{-1} (3.78 \cdot 10^{-2})$	$2.09 \cdot 10^{-1} (1.19 \cdot 10^{-2})$
$v = 5/2$	$2.07 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$1.88 \cdot 10^{-1} (1.51 \cdot 10^{-2})$	$1.72 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$1.85 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$8.56 \cdot 10^{-1} (7.72 \cdot 10^{-2})$	$1.90 \cdot 10^{-1} (1.81 \cdot 10^{-2})$
$v = 7/2$	$2.19 \cdot 10^{-1} (1.35 \cdot 10^{-2})$	$1.71 \cdot 10^{-1} (1.63 \cdot 10^{-2})$	$1.56 \cdot 10^{-1} (1.48 \cdot 10^{-2})$	$1.61 \cdot 10^{-1} (1.49 \cdot 10^{-2})$	$7.52 \cdot 10^{-1} (3.81 \cdot 10^{-2})$	$1.65 \cdot 10^{-1} (1.51 \cdot 10^{-2})$
$v = 9/2$	$2.31 \cdot 10^{-1} (1.48 \cdot 10^{-2})$	$1.57 \cdot 10^{-1} (1.65 \cdot 10^{-2})$	$1.33 \cdot 10^{-1} (1.21 \cdot 10^{-2})$	$1.46 \cdot 10^{-1} (1.27 \cdot 10^{-2})$	$8.21 \cdot 10^{-1} (7.65 \cdot 10^{-2})$	$1.52 \cdot 10^{-1} (1.44 \cdot 10^{-2})$
$v = 11/2$	$2.41 \cdot 10^{-1} (1.61 \cdot 10^{-2})$	$1.51 \cdot 10^{-1} (1.79 \cdot 10^{-2})$	$1.46 \cdot 10^{-1} (1.49 \cdot 10^{-2})$	$1.40 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$8.11 \cdot 10^{-1} (7.41 \cdot 10^{-2})$	$1.53 \cdot 10^{-1} (1.49 \cdot 10^{-2})$
$v = 21/2$	$2.63 \cdot 10^{-1} (1.75 \cdot 10^{-2})$	$1.68 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$1.69 \cdot 10^{-1} (1.61 \cdot 10^{-2})$	$1.68 \cdot 10^{-1} (1.27 \cdot 10^{-2})$	$8.00 \cdot 10^{-1} (7.43 \cdot 10^{-2})$	$1.71 \cdot 10^{-1} (2.00 \cdot 10^{-2})$
$v = \infty$	$2.99 \cdot 10^{-1} (1.90 \cdot 10^{-2})$	$2.02 \cdot 10^{-1} (1.15 \cdot 10^{-2})$	$2.01 \cdot 10^{-1} (1.43 \cdot 10^{-2})$	$2.06 \cdot 10^{-1} (1.18 \cdot 10^{-2})$	$7.85 \cdot 10^{-1} (7.35 \cdot 10^{-2})$	$2.05 \cdot 10^{-1} (2.09 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$2.09 \cdot 10^{-1} (1.21 \cdot 10^{-2})$	$1.52 \cdot 10^{-1} (1.29 \cdot 10^{-2})$	$1.49 \cdot 10^{-1} (1.25 \cdot 10^{-2})$	$1.47 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$7.80 \cdot 10^{-1} (4.85 \cdot 10^{-2})$	$1.67 \cdot 10^{-1} (1.32 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$3.15 \cdot 10^{-1} (1.36 \cdot 10^{-2})$
$v = 3/2$	$1.79 \cdot 10^{-1} (9.61 \cdot 10^{-3})$
$v = 5/2$	$1.41 \cdot 10^{-1} (9.27 \cdot 10^{-3})$
$v = 7/2$	$1.14 \cdot 10^{-1} (9.13 \cdot 10^{-3})$
$v = 9/2$	$9.81 \cdot 10^{-2} (9.23 \cdot 10^{-3})$
$v = 11/2$	$9.41 \cdot 10^{-2} (9.04 \cdot 10^{-3})$
$v = 21/2$	$9.91 \cdot 10^{-2} (8.73 \cdot 10^{-3})$
$v = \infty$	$1.15 \cdot 10^{-1} (8.44 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$9.17 \cdot 10^{-2} (8.76 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$2.11 \cdot 10^{-1} (1.20 \cdot 10^{-2})$
linear	$9.82 \cdot 10^{-1} (7.19 \cdot 10^{-3})$
1NN	$8.29 \cdot 10^{-1} (1.15 \cdot 10^{-2})$
2NN	$6.68 \cdot 10^{-1} (1.04 \cdot 10^{-2})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.84 \cdot 10^{-1} (3.15 \cdot 10^{-3})$	$2.83 \cdot 10^{-1} (4.06 \cdot 10^{-3})$	$2.79 \cdot 10^{-1} (3.36 \cdot 10^{-3})$	$2.68 \cdot 10^{-1} (3.91 \cdot 10^{-3})$	$4.56 \cdot 10^{-1} (9.95 \cdot 10^{-3})$	$2.77 \cdot 10^{-1} (2.82 \cdot 10^{-3})$
$v = 3/2$	$2.10 \cdot 10^{-1} (4.66 \cdot 10^{-3})$	$2.20 \cdot 10^{-1} (8.24 \cdot 10^{-3})$	$2.04 \cdot 10^{-1} (4.36 \cdot 10^{-3})$	$2.07 \cdot 10^{-1} (4.51 \cdot 10^{-3})$	$4.45 \cdot 10^{-1} (1.57 \cdot 10^{-2})$	$2.08 \cdot 10^{-1} (5.51 \cdot 10^{-3})$
$v = 5/2$	$2.06 \cdot 10^{-1} (5.92 \cdot 10^{-3})$	$2.00 \cdot 10^{-1} (8.19 \cdot 10^{-3})$	$1.91 \cdot 10^{-1} (7.08 \cdot 10^{-3})$	$1.99 \cdot 10^{-1} (7.53 \cdot 10^{-3})$	$4.72 \cdot 10^{-1} (2.99 \cdot 10^{-2})$	$2.00 \cdot 10^{-1} (1.05 \cdot 10^{-2})$
$v = 7/2$	$2.14 \cdot 10^{-1} (6.63 \cdot 10^{-3})$	$1.91 \cdot 10^{-1} (9.67 \cdot 10^{-3})$	$1.84 \cdot 10^{-1} (9.64 \cdot 10^{-3})$	$1.86 \cdot 10^{-1} (9.30 \cdot 10^{-3})$	$4.34 \cdot 10^{-1} (1.66 \cdot 10^{-2})$	$1.90 \cdot 10^{-1} (9.37 \cdot 10^{-3})$
$v = 9/2$	$2.21 \cdot 10^{-1} (7.21 \cdot 10^{-3})$	$1.84 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$1.70 \cdot 10^{-1} (8.65 \cdot 10^{-3})$	$1.78 \cdot 10^{-1} (8.52 \cdot 10^{-3})$	$4.57 \cdot 10^{-1} (2.94 \cdot 10^{-2})$	$1.83 \cdot 10^{-1} (1.02 \cdot 10^{-2})$
$v = 11/2$	$2.27 \cdot 10^{-1} (7.78 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (1.13 \cdot 10^{-2})$	$1.80 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$1.76 \cdot 10^{-1} (8.78 \cdot 10^{-3})$	$4.54 \cdot 10^{-1} (2.87 \cdot 10^{-2})$	$1.85 \cdot 10^{-1} (1.08 \cdot 10^{-2})$
$v = 21/2$	$2.39 \cdot 10^{-1} (7.87 \cdot 10^{-3})$	$1.94 \cdot 10^{-1} (8.16 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (1.11 \cdot 10^{-2})$	$1.96 \cdot 10^{-1} (8.64 \cdot 10^{-3})$	$4.49 \cdot 10^{-1} (2.88 \cdot 10^{-2})$	$1.96 \cdot 10^{-1} (1.13 \cdot 10^{-2})$
$v = \infty$	$2.58 \cdot 10^{-1} (8.07 \cdot 10^{-3})$	$2.13 \cdot 10^{-1} (7.10 \cdot 10^{-3})$	$2.16 \cdot 10^{-1} (9.33 \cdot 10^{-3})$	$2.21 \cdot 10^{-1} (7.38 \cdot 10^{-3})$	$4.43 \cdot 10^{-1} (2.86 \cdot 10^{-2})$	$2.17 \cdot 10^{-1} (1.05 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$2.07 \cdot 10^{-1} (5.72 \cdot 10^{-3})$	$1.85 \cdot 10^{-1} (9.61 \cdot 10^{-3})$	$1.84 \cdot 10^{-1} (9.34 \cdot 10^{-3})$	$1.83 \cdot 10^{-1} (9.86 \cdot 10^{-3})$	$4.43 \cdot 10^{-1} (1.97 \cdot 10^{-2})$	$1.95 \cdot 10^{-1} (9.47 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$2.09 \cdot 10^{-1} (5.78 \cdot 10^{-3})$

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.58 \cdot 10^0 (8.30 \cdot 10^{-2})$	$3.60 \cdot 10^0 (9.89 \cdot 10^{-2})$	$3.56 \cdot 10^0 (9.72 \cdot 10^{-2})$	$3.65 \cdot 10^0 (1.41 \cdot 10^{-1})$	$5.67 \cdot 10^0 (1.03 \cdot 10^{-1})$	$3.55 \cdot 10^0 (9.61 \cdot 10^{-2})$
$v = 3/2$	$2.48 \cdot 10^0 (9.91 \cdot 10^{-2})$	$2.60 \cdot 10^0 (1.50 \cdot 10^{-1})$	$2.36 \cdot 10^0 (8.54 \cdot 10^{-2})$	$2.47 \cdot 10^0 (1.02 \cdot 10^{-1})$	$5.68 \cdot 10^0 (1.83 \cdot 10^{-1})$	$2.43 \cdot 10^0 (1.12 \cdot 10^{-1})$
$v = 5/2$	$2.38 \cdot 10^0 (1.15 \cdot 10^{-1})$	$2.16 \cdot 10^0 (1.34 \cdot 10^{-1})$	$2.02 \cdot 10^0 (1.29 \cdot 10^{-1})$	$2.23 \cdot 10^0 (1.34 \cdot 10^{-1})$	$6.11 \cdot 10^0 (3.96 \cdot 10^{-1})$	$2.17 \cdot 10^0 (2.07 \cdot 10^{-1})$
$v = 7/2$	$2.48 \cdot 10^0 (1.24 \cdot 10^{-1})$	$2.04 \cdot 10^0 (1.83 \cdot 10^{-1})$	$1.96 \cdot 10^0 (2.00 \cdot 10^{-1})$	$2.07 \cdot 10^0 (1.86 \cdot 10^{-1})$	$5.50 \cdot 10^0 (1.58 \cdot 10^{-1})$	$1.96 \cdot 10^0 (1.98 \cdot 10^{-1})$
$v = 9/2$	$2.58 \cdot 10^0 (1.37 \cdot 10^{-1})$	$1.95 \cdot 10^0 (2.19 \cdot 10^{-1})$	$1.77 \cdot 10^0 (2.07 \cdot 10^{-1})$	$2.02 \cdot 10^0 (2.02 \cdot 10^{-1})$	$5.84 \cdot 10^0 (3.60 \cdot 10^{-1})$	$1.93 \cdot 10^0 (2.33 \cdot 10^{-1})$
$v = 11/2$	$2.67 \cdot 10^0 (1.50 \cdot 10^{-1})$	$1.94 \cdot 10^0 (2.52 \cdot 10^{-1})$	$1.95 \cdot 10^0 (2.50 \cdot 10^{-1})$	$2.00 \cdot 10^0 (2.23 \cdot 10^{-1})$	$5.77 \cdot 10^0 (3.43 \cdot 10^{-1})$	$1.97 \cdot 10^0 (2.47 \cdot 10^{-1})$
$v = 21/2$	$2.83 \cdot 10^0 (1.69 \cdot 10^{-1})$	$2.15 \cdot 10^0 (2.65 \cdot 10^{-1})$	$2.26 \cdot 10^0 (3.01 \cdot 10^{-1})$	$2.39 \cdot 10^0 (2.55 \cdot 10^{-1})$	$5.68 \cdot 10^0 (3.34 \cdot 10^{-1})$	$2.16 \cdot 10^0 (2.75 \cdot 10^{-1})$
$v = \infty$	$3.12 \cdot 10^0 (1.83 \cdot 10^{-1})$	$2.39 \cdot 10^0 (2.40 \cdot 10^{-1})$	$2.58 \cdot 10^0 (2.81 \cdot 10^{-1})$	$2.83 \cdot 10^0 (2.61 \cdot 10^{-1})$	$5.55 \cdot 10^0 (3.23 \cdot 10^{-1})$	$2.48 \cdot 10^0 (2.88 \cdot 10^{-1})$
$v \in \{1/2, \dots, \infty\}$	$2.42 \cdot 10^0 (1.19 \cdot 10^{-1})$	$2.05 \cdot 10^0 (2.84 \cdot 10^{-1})$	$2.08 \cdot 10^0 (2.78 \cdot 10^{-1})$	$2.20 \cdot 10^0 (3.00 \cdot 10^{-1})$	$5.59 \cdot 10^0 (2.35 \cdot 10^{-1})$	$2.18 \cdot 10^0 (2.75 \cdot 10^{-1})$

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$2.44 \cdot 10^0 (1.24 \cdot 10^{-1})$

### SM-2.2.33 Rosenbrock, $n = 250, d = 5$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.77 \cdot 10^{-1} (7.60 \cdot 10^{-3})$	$1.78 \cdot 10^{-1} (7.84 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (7.60 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (7.63 \cdot 10^{-3})$	$5.21 \cdot 10^{-1} (1.51 \cdot 10^{-2})$	$1.77 \cdot 10^{-1} (7.44 \cdot 10^{-3})$
$v = 3/2$	$6.50 \cdot 10^{-2} (3.95 \cdot 10^{-3})$	$6.59 \cdot 10^{-2} (4.13 \cdot 10^{-3})$	$6.36 \cdot 10^{-2} (3.81 \cdot 10^{-3})$	$6.40 \cdot 10^{-2} (3.88 \cdot 10^{-3})$	$4.27 \cdot 10^{-1} (2.64 \cdot 10^{-2})$	$6.30 \cdot 10^{-2} (3.69 \cdot 10^{-3})$
$v = 5/2$	$2.79 \cdot 10^{-2} (2.03 \cdot 10^{-3})$	$2.77 \cdot 10^{-2} (2.05 \cdot 10^{-3})$	$2.52 \cdot 10^{-2} (1.55 \cdot 10^{-3})$	$2.56 \cdot 10^{-2} (1.64 \cdot 10^{-3})$	$3.79 \cdot 10^{-1} (2.57 \cdot 10^{-2})$	$2.51 \cdot 10^{-2} (1.52 \cdot 10^{-3})$
$v = 7/2$	$1.01 \cdot 10^{-2} (8.06 \cdot 10^{-4})$	$9.29 \cdot 10^{-3} (6.58 \cdot 10^{-4})$	$8.79 \cdot 10^{-3} (5.65 \cdot 10^{-4})$	$8.80 \cdot 10^{-3} (5.52 \cdot 10^{-4})$	$3.66 \cdot 10^{-1} (2.80 \cdot 10^{-2})$	$8.94 \cdot 10^{-3} (5.99 \cdot 10^{-4})$
$v = 9/2$	$5.01 \cdot 10^{-3} (3.74 \cdot 10^{-4})$	$4.27 \cdot 10^{-3} (2.62 \cdot 10^{-4})$	$4.18 \cdot 10^{-3} (2.62 \cdot 10^{-4})$	$4.13 \cdot 10^{-3} (2.47 \cdot 10^{-4})$	$3.57 \cdot 10^{-1} (2.85 \cdot 10^{-2})$	$4.42 \cdot 10^{-3} (2.86 \cdot 10^{-4})$
$v = 11/2$	$3.84 \cdot 10^{-3} (2.66 \cdot 10^{-4})$	$3.41 \cdot 10^{-3} (1.90 \cdot 10^{-4})$	$3.38 \cdot 10^{-3} (2.02 \cdot 10^{-4})$	$3.32 \cdot 10^{-3} (1.89 \cdot 10^{-4})$	$3.50 \cdot 10^{-1} (2.85 \cdot 10^{-2})$	$3.80 \cdot 10^{-3} (2.97 \cdot 10^{-4})$
$v = 21/2$	$4.14 \cdot 10^{-3} (1.57 \cdot 10^{-4})$	$4.33 \cdot 10^{-3} (2.06 \cdot 10^{-4})$	$4.26 \cdot 10^{-3} (2.34 \cdot 10^{-4})$	$4.16 \cdot 10^{-3} (1.49 \cdot 10^{-4})$	$3.33 \cdot 10^{-1} (2.81 \cdot 10^{-2})$	$5.89 \cdot 10^{-3} (4.60 \cdot 10^{-4})$
$v = \infty$	$4.75 \cdot 10^{-3} (1.24 \cdot 10^{-4})$	$5.38 \cdot 10^{-3} (2.75 \cdot 10^{-4})$	$5.17 \cdot 10^{-3} (3.68 \cdot 10^{-4})$	$4.83 \cdot 10^{-3} (1.39 \cdot 10^{-4})$	$2.83 \cdot 10^{-1} (1.28 \cdot 10^{-2})$	$8.53 \cdot 10^{-3} (7.28 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$3.98 \cdot 10^{-3} (2.69 \cdot 10^{-4})$	$3.72 \cdot 10^{-3} (2.93 \cdot 10^{-4})$	$3.38 \cdot 10^{-3} (2.02 \cdot 10^{-4})$	$3.32 \cdot 10^{-3} (1.89 \cdot 10^{-4})$	$2.97 \cdot 10^{-1} (2.24 \cdot 10^{-2})$	$3.80 \cdot 10^{-3} (2.97 \cdot 10^{-4})$

$R^*(S^{SPE})$	
$v = 1/2$	$1.73 \cdot 10^{-1} (7.60 \cdot 10^{-3})$
$v = 3/2$	$6.05 \cdot 10^{-2} (3.58 \cdot 10^{-3})$
$v = 5/2$	$2.37 \cdot 10^{-2} (1.50 \cdot 10^{-3})$
$v = 7/2$	$8.25 \cdot 10^{-3} (5.34 \cdot 10^{-4})$
$v = 9/2$	$4.04 \cdot 10^{-3} (2.54 \cdot 10^{-4})$
$v = 11/2$	$3.24 \cdot 10^{-3} (1.94 \cdot 10^{-4})$
$v = 21/2$	$3.92 \cdot 10^{-3} (1.64 \cdot 10^{-4})$
$v = \infty$	$4.56 \cdot 10^{-3} (1.31 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$3.24 \cdot 10^{-3} (1.94 \cdot 10^{-4})$

$R(\theta; S^{SPE})$	
NLL/SPE	$3.76 \cdot 10^{-3} (2.04 \cdot 10^{-4})$
linear	$9.66 \cdot 10^{-1} (5.49 \cdot 10^{-3})$
1NN	$6.23 \cdot 10^{-1} (1.04 \cdot 10^{-2})$
2NN	$4.96 \cdot 10^{-1} (8.71 \cdot 10^{-3})$

### Averages of $R(\theta; S^{CRPS})$

$R(\theta; S^{CRPS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.01 \cdot 10^{-1} (2.05 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (2.38 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (2.35 \cdot 10^{-3})$	$1.86 \cdot 10^{-1} (2.58 \cdot 10^{-3})$	$3.53 \cdot 10^{-1} (8.42 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (2.31 \cdot 10^{-3})$
$v = 3/2$	$1.10 \cdot 10^{-1} (1.93 \cdot 10^{-3})$	$1.09 \cdot 10^{-1} (2.47 \cdot 10^{-3})$	$1.07 \cdot 10^{-1} (2.01 \cdot 10^{-3})$	$1.06 \cdot 10^{-1} (2.18 \cdot 10^{-3})$	$3.08 \cdot 10^{-1} (1.28 \cdot 10^{-2})$	$1.06 \cdot 10^{-1} (1.91 \cdot 10^{-3})$
$v = 5/2$	$6.69 \cdot 10^{-2} (1.63 \cdot 10^{-3})$	$6.72 \cdot 10^{-2} (2.34 \cdot 10^{-3})$	$6.33 \cdot 10^{-2} (1.53 \cdot 10^{-3})$	$6.39 \cdot 10^{-2} (1.64 \cdot 10^{-3})$	$2.85 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$6.32 \cdot 10^{-2} (1.47 \cdot 10^{-3})$
$v = 7/2$	$3.85 \cdot 10^{-2} (1.03 \cdot 10^{-3})$	$3.71 \cdot 10^{-2} (1.14 \cdot 10^{-3})$	$3.58 \cdot 10^{-2} (9.81 \cdot 10^{-4})$	$3.58 \cdot 10^{-2} (9.83 \cdot 10^{-4})$	$2.78 \cdot 10^{-1} (1.31 \cdot 10^{-2})$	$3.61 \cdot 10^{-2} (9.94 \cdot 10^{-4})$
$v = 9/2$	$2.65 \cdot 10^{-2} (7.62 \cdot 10^{-4})$	$2.45 \cdot 10^{-2} (6.63 \cdot 10^{-4})$	$2.41 \cdot 10^{-2} (6.04 \cdot 10^{-4})$	$2.40 \cdot 10^{-2} (5.91 \cdot 10^{-4})$	$2.73 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$2.48 \cdot 10^{-2} (6.93 \cdot 10^{-4})$
$v = 11/2$	$2.30 \cdot 10^{-2} (6.44 \cdot 10^{-4})$	$2.18 \cdot 10^{-2} (5.42 \cdot 10^{-4})$	$2.16 \cdot 10^{-2} (4.94 \cdot 10^{-4})$	$2.15 \cdot 10^{-2} (5.03 \cdot 10^{-4})$	$2.69 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$2.29 \cdot 10^{-2} (8.32 \cdot 10^{-4})$
$v = 21/2$	$2.49 \cdot 10^{-2} (4.06 \cdot 10^{-4})$	$2.53 \cdot 10^{-2} (5.59 \cdot 10^{-4})$	$2.49 \cdot 10^{-2} (5.13 \cdot 10^{-4})$	$2.48 \cdot 10^{-2} (3.85 \cdot 10^{-4})$	$2.60 \cdot 10^{-1} (1.32 \cdot 10^{-2})$	$2.94 \cdot 10^{-2} (1.01 \cdot 10^{-3})$
$v = \infty$	$2.78 \cdot 10^{-2} (3.64 \cdot 10^{-4})$	$2.85 \cdot 10^{-2} (6.87 \cdot 10^{-4})$	$2.80 \cdot 10^{-2} (7.66 \cdot 10^{-4})$	$2.72 \cdot 10^{-2} (3.61 \cdot 10^{-4})$	$2.35 \cdot 10^{-1} (6.69 \cdot 10^{-3})$	$3.59 \cdot 10^{-2} (1.50 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$2.38 \cdot 10^{-2} (7.19 \cdot 10^{-4})$	$2.29 \cdot 10^{-2} (1.01 \cdot 10^{-3})$	$2.16 \cdot 10^{-2} (4.94 \cdot 10^{-4})$	$2.15 \cdot 10^{-2} (5.03 \cdot 10^{-4})$	$2.42 \cdot 10^{-1} (1.13 \cdot 10^{-2})$	$2.29 \cdot 10^{-2} (8.32 \cdot 10^{-4})$

$R(\theta; S^{CRPS})$	
NLL/SPE	$2.31 \cdot 10^{-2} (5.87 \cdot 10^{-4})$

### Averages of $R(\theta; S_{0.95}^{IS})$

$R(\theta; S_{0.95}^{IS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.62 \cdot 10^0 (5.40 \cdot 10^{-2})$	$2.55 \cdot 10^0 (7.49 \cdot 10^{-2})$	$2.54 \cdot 10^0 (7.24 \cdot 10^{-2})$	$2.66 \cdot 10^0 (9.20 \cdot 10^{-2})$	$4.59 \cdot 10^0 (5.37 \cdot 10^{-2})$	$2.54 \cdot 10^0 (7.04 \cdot 10^{-2})$
$v = 3/2$	$1.29 \cdot 10^0 (4.63 \cdot 10^{-2})$	$1.31 \cdot 10^0 (5.96 \cdot 10^{-2})$	$1.29 \cdot 10^0 (5.62 \cdot 10^{-2})$	$1.34 \cdot 10^0 (5.91 \cdot 10^{-2})$	$4.13 \cdot 10^0 (1.28 \cdot 10^{-1})$	$1.28 \cdot 10^0 (5.28 \cdot 10^{-2})$
$v = 5/2$	$7.29 \cdot 10^{-1} (2.82 \cdot 10^{-2})$	$7.46 \cdot 10^{-1} (3.42 \cdot 10^{-2})$	$7.11 \cdot 10^{-1} (2.75 \cdot 10^{-2})$	$7.16 \cdot 10^{-1} (2.79 \cdot 10^{-2})$	$3.81 \cdot 10^0 (1.27 \cdot 10^{-1})$	$7.04 \cdot 10^{-1} (2.54 \cdot 10^{-2})$
$v = 7/2$	$3.94 \cdot 10^{-1} (1.14 \cdot 10^{-2})$	$3.91 \cdot 10^{-1} (1.44 \cdot 10^{-2})$	$3.80 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$3.75 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$3.68 \cdot 10^0 (1.36 \cdot 10^{-1})$	$3.82 \cdot 10^{-1} (1.29 \cdot 10^{-2})$
$v = 9/2$	$2.62 \cdot 10^{-1} (6.13 \cdot 10^{-3})$	$2.49 \cdot 10^{-1} (7.67 \cdot 10^{-3})$	$2.46 \cdot 10^{-1} (7.77 \cdot 10^{-3})$	$2.43 \cdot 10^{-1} (7.90 \cdot 10^{-3})$	$3.58 \cdot 10^0 (1.35 \cdot 10^{-1})$	$2.56 \cdot 10^{-1} (8.62 \cdot 10^{-3})$
$v = 11/2$	$2.21 \cdot 10^{-1} (5.32 \cdot 10^{-3})$	$2.16 \cdot 10^{-1} (6.78 \cdot 10^{-3})$	$2.15 \cdot 10^{-1} (7.10 \cdot 10^{-3})$	$2.13 \cdot 10^{-1} (7.60 \cdot 10^{-3})$	$3.51 \cdot 10^0 (1.34 \cdot 10^{-1})$	$2.34 \cdot 10^{-1} (1.20 \cdot 10^{-2})$
$v = 21/2$	$2.56 \cdot 10^{-1} (9.50 \cdot 10^{-3})$	$2.41 \cdot 10^{-1} (8.09 \cdot 10^{-3})$	$2.39 \cdot 10^{-1} (9.15 \cdot 10^{-3})$	$2.35 \cdot 10^{-1} (7.19 \cdot 10^{-3})$	$3.32 \cdot 10^0 (1.30 \cdot 10^{-1})$	$2.95 \cdot 10^{-1} (1.65 \cdot 10^{-2})$
$v = \infty$	$3.19 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$2.66 \cdot 10^{-1} (9.30 \cdot 10^{-3})$	$2.64 \cdot 10^{-1} (1.31 \cdot 10^{-2})$	$2.51 \cdot 10^{-1} (5.27 \cdot 10^{-3})$	$2.95 \cdot 10^0 (6.73 \cdot 10^{-2})$	$3.54 \cdot 10^{-1} (2.26 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$2.42 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$2.26 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$2.15 \cdot 10^{-1} (7.10 \cdot 10^{-3})$	$2.13 \cdot 10^{-1} (7.60 \cdot 10^{-3})$	$3.06 \cdot 10^0 (1.38 \cdot 10^{-1})$	$2.34 \cdot 10^{-1} (1.20 \cdot 10^{-2})$

$R(\theta; S_{0.95}^{IS})$	
NLL/SPE	$2.24 \cdot 10^{-1} (5.38 \cdot 10^{-3})$

### SM-2.2.34 Borehole, $n = 80, d = 8$

#### Averages of $R(\theta; S^{SPE})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$8.97 \cdot 10^{-3}$ ( $4.54 \cdot 10^{-4}$ )	$9.23 \cdot 10^{-3}$ ( $4.40 \cdot 10^{-4}$ )	$8.82 \cdot 10^{-3}$ ( $4.34 \cdot 10^{-4}$ )	$8.66 \cdot 10^{-3}$ ( $4.39 \cdot 10^{-4}$ )	$1.18 \cdot 10^{-1}$ ( $3.99 \cdot 10^{-3}$ )	$8.61 \cdot 10^{-3}$ ( $4.31 \cdot 10^{-4}$ )
$v = 3/2$	$6.32 \cdot 10^{-4}$ ( $4.13 \cdot 10^{-5}$ )	$7.02 \cdot 10^{-4}$ ( $4.57 \cdot 10^{-5}$ )	$6.61 \cdot 10^{-4}$ ( $4.37 \cdot 10^{-5}$ )	$6.43 \cdot 10^{-4}$ ( $4.39 \cdot 10^{-5}$ )	$1.18 \cdot 10^{-1}$ ( $6.78 \cdot 10^{-3}$ )	$6.76 \cdot 10^{-4}$ ( $4.50 \cdot 10^{-5}$ )
$v = 5/2$	$2.27 \cdot 10^{-4}$ ( $1.47 \cdot 10^{-5}$ )	$2.69 \cdot 10^{-4}$ ( $2.04 \cdot 10^{-5}$ )	$2.44 \cdot 10^{-4}$ ( $1.70 \cdot 10^{-5}$ )	$2.46 \cdot 10^{-4}$ ( $1.84 \cdot 10^{-5}$ )	$1.25 \cdot 10^{-1}$ ( $8.50 \cdot 10^{-3}$ )	$2.59 \cdot 10^{-4}$ ( $1.77 \cdot 10^{-5}$ )
$v = 7/2$	$1.90 \cdot 10^{-4}$ ( $1.12 \cdot 10^{-5}$ )	$2.14 \cdot 10^{-4}$ ( $1.18 \cdot 10^{-5}$ )	$2.08 \cdot 10^{-4}$ ( $1.11 \cdot 10^{-5}$ )	$1.97 \cdot 10^{-4}$ ( $1.06 \cdot 10^{-5}$ )	$1.05 \cdot 10^{-1}$ ( $6.89 \cdot 10^{-3}$ )	$2.28 \cdot 10^{-4}$ ( $1.27 \cdot 10^{-5}$ )
$v = 9/2$	$1.80 \cdot 10^{-4}$ ( $9.94 \cdot 10^{-6}$ )	$2.18 \cdot 10^{-4}$ ( $1.21 \cdot 10^{-5}$ )	$2.15 \cdot 10^{-4}$ ( $1.16 \cdot 10^{-5}$ )	$2.03 \cdot 10^{-4}$ ( $1.01 \cdot 10^{-5}$ )	$9.18 \cdot 10^{-2}$ ( $6.43 \cdot 10^{-3}$ )	$2.35 \cdot 10^{-4}$ ( $1.30 \cdot 10^{-5}$ )
$v = 17/2$	$1.86 \cdot 10^{-4}$ ( $9.16 \cdot 10^{-6}$ )	$2.48 \cdot 10^{-4}$ ( $1.44 \cdot 10^{-5}$ )	$2.35 \cdot 10^{-4}$ ( $1.25 \cdot 10^{-5}$ )	$2.25 \cdot 10^{-4}$ ( $1.19 \cdot 10^{-5}$ )	$7.27 \cdot 10^{-2}$ ( $5.85 \cdot 10^{-3}$ )	$2.68 \cdot 10^{-4}$ ( $1.43 \cdot 10^{-5}$ )
$v = 33/2$	$2.01 \cdot 10^{-4}$ ( $9.31 \cdot 10^{-6}$ )	$2.54 \cdot 10^{-4}$ ( $1.42 \cdot 10^{-5}$ )	$2.50 \cdot 10^{-4}$ ( $1.30 \cdot 10^{-5}$ )	$2.40 \cdot 10^{-4}$ ( $1.25 \cdot 10^{-5}$ )	$6.08 \cdot 10^{-2}$ ( $5.27 \cdot 10^{-3}$ )	$2.97 \cdot 10^{-4}$ ( $1.53 \cdot 10^{-5}$ )
$v = \infty$	$2.06 \cdot 10^{-4}$ ( $9.14 \cdot 10^{-6}$ )	$2.74 \cdot 10^{-4}$ ( $1.89 \cdot 10^{-5}$ )	$2.52 \cdot 10^{-4}$ ( $1.30 \cdot 10^{-5}$ )	$2.45 \cdot 10^{-4}$ ( $1.25 \cdot 10^{-5}$ )	$5.32 \cdot 10^{-2}$ ( $4.95 \cdot 10^{-3}$ )	$3.20 \cdot 10^{-4}$ ( $2.08 \cdot 10^{-5}$ )
$v \in \{1/2, \dots, \infty\}$	$1.92 \cdot 10^{-4}$ ( $1.02 \cdot 10^{-5}$ )	$2.52 \cdot 10^{-4}$ ( $1.85 \cdot 10^{-5}$ )	$2.31 \cdot 10^{-4}$ ( $1.21 \cdot 10^{-5}$ )	$2.19 \cdot 10^{-4}$ ( $1.22 \cdot 10^{-5}$ )	$5.32 \cdot 10^{-2}$ ( $4.95 \cdot 10^{-3}$ )	$2.56 \cdot 10^{-4}$ ( $1.46 \cdot 10^{-5}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$7.84 \cdot 10^{-3}$ ( $4.29 \cdot 10^{-4}$ )
$v = 3/2$	$5.02 \cdot 10^{-4}$ ( $3.73 \cdot 10^{-5}$ )
$v = 5/2$	$1.61 \cdot 10^{-4}$ ( $1.13 \cdot 10^{-5}$ )
$v = 7/2$	$1.23 \cdot 10^{-4}$ ( $6.49 \cdot 10^{-6}$ )
$v = 9/2$	$1.14 \cdot 10^{-4}$ ( $5.05 \cdot 10^{-6}$ )
$v = 17/2$	$1.14 \cdot 10^{-4}$ ( $4.22 \cdot 10^{-6}$ )
$v = 33/2$	$1.21 \cdot 10^{-4}$ ( $4.29 \cdot 10^{-6}$ )
$v = \infty$	$1.21 \cdot 10^{-4}$ ( $4.12 \cdot 10^{-6}$ )
$v \in \{1/2, \dots, \infty\}$	$1.05 \cdot 10^{-4}$ ( $4.29 \cdot 10^{-6}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.94 \cdot 10^{-4}$ ( $1.04 \cdot 10^{-5}$ )
linear	$1.42 \cdot 10^{-1}$ ( $6.06 \cdot 10^{-4}$ )
1NN	$4.85 \cdot 10^{-1}$ ( $1.69 \cdot 10^{-3}$ )
2NN	$3.40 \cdot 10^{-1}$ ( $2.45 \cdot 10^{-3}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.30 \cdot 10^{-2}$ ( $3.91 \cdot 10^{-4}$ )	$4.07 \cdot 10^{-2}$ ( $5.91 \cdot 10^{-4}$ )	$3.97 \cdot 10^{-2}$ ( $5.71 \cdot 10^{-4}$ )	$3.62 \cdot 10^{-2}$ ( $7.03 \cdot 10^{-4}$ )	$1.80 \cdot 10^{-1}$ ( $3.32 \cdot 10^{-3}$ )	$3.93 \cdot 10^{-2}$ ( $5.60 \cdot 10^{-4}$ )
$v = 3/2$	$1.14 \cdot 10^{-2}$ ( $1.31 \cdot 10^{-4}$ )	$1.03 \cdot 10^{-2}$ ( $2.07 \cdot 10^{-4}$ )	$9.88 \cdot 10^{-3}$ ( $1.92 \cdot 10^{-4}$ )	$9.44 \cdot 10^{-3}$ ( $2.09 \cdot 10^{-4}$ )	$1.75 \cdot 10^{-1}$ ( $5.93 \cdot 10^{-3}$ )	$1.00 \cdot 10^{-2}$ ( $1.96 \cdot 10^{-4}$ )
$v = 5/2$	$6.45 \cdot 10^{-3}$ ( $8.66 \cdot 10^{-5}$ )	$6.62 \cdot 10^{-3}$ ( $1.48 \cdot 10^{-4}$ )	$6.25 \cdot 10^{-3}$ ( $1.25 \cdot 10^{-4}$ )	$6.17 \cdot 10^{-3}$ ( $1.39 \cdot 10^{-4}$ )	$1.76 \cdot 10^{-1}$ ( $7.04 \cdot 10^{-3}$ )	$6.40 \cdot 10^{-3}$ ( $1.34 \cdot 10^{-4}$ )
$v = 7/2$	$5.97 \cdot 10^{-3}$ ( $8.54 \cdot 10^{-5}$ )	$6.42 \cdot 10^{-3}$ ( $1.42 \cdot 10^{-4}$ )	$6.26 \cdot 10^{-3}$ ( $1.25 \cdot 10^{-4}$ )	$6.08 \cdot 10^{-3}$ ( $1.16 \cdot 10^{-4}$ )	$1.59 \cdot 10^{-1}$ ( $6.29 \cdot 10^{-3}$ )	$6.52 \cdot 10^{-3}$ ( $1.44 \cdot 10^{-4}$ )
$v = 9/2$	$5.88 \cdot 10^{-3}$ ( $8.79 \cdot 10^{-5}$ )	$6.74 \cdot 10^{-3}$ ( $1.71 \cdot 10^{-4}$ )	$6.59 \cdot 10^{-3}$ ( $1.62 \cdot 10^{-4}$ )	$6.46 \cdot 10^{-3}$ ( $1.34 \cdot 10^{-4}$ )	$1.47 \cdot 10^{-1}$ ( $6.05 \cdot 10^{-3}$ )	$6.86 \cdot 10^{-3}$ ( $1.67 \cdot 10^{-4}$ )
$v = 17/2$	$6.12 \cdot 10^{-3}$ ( $9.34 \cdot 10^{-5}$ )	$7.42 \cdot 10^{-3}$ ( $2.10 \cdot 10^{-4}$ )	$7.23 \cdot 10^{-3}$ ( $1.86 \cdot 10^{-4}$ )	$7.14 \cdot 10^{-3}$ ( $1.84 \cdot 10^{-4}$ )	$1.26 \cdot 10^{-1}$ ( $5.77 \cdot 10^{-3}$ )	$7.60 \cdot 10^{-3}$ ( $1.84 \cdot 10^{-4}$ )
$v = 33/2$	$6.44 \cdot 10^{-3}$ ( $9.73 \cdot 10^{-5}$ )	$7.60 \cdot 10^{-3}$ ( $2.04 \cdot 10^{-4}$ )	$7.55 \cdot 10^{-3}$ ( $1.88 \cdot 10^{-4}$ )	$7.46 \cdot 10^{-3}$ ( $1.89 \cdot 10^{-4}$ )	$1.13 \cdot 10^{-1}$ ( $5.40 \cdot 10^{-3}$ )	$8.07 \cdot 10^{-3}$ ( $1.91 \cdot 10^{-4}$ )
$v = \infty$	$6.58 \cdot 10^{-3}$ ( $9.96 \cdot 10^{-5}$ )	$7.90 \cdot 10^{-3}$ ( $2.43 \cdot 10^{-4}$ )	$7.62 \cdot 10^{-3}$ ( $1.87 \cdot 10^{-4}$ )	$7.61 \cdot 10^{-3}$ ( $1.89 \cdot 10^{-4}$ )	$1.04 \cdot 10^{-1}$ ( $5.19 \cdot 10^{-3}$ )	$8.38 \cdot 10^{-3}$ ( $2.31 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$6.07 \cdot 10^{-3}$ ( $8.68 \cdot 10^{-5}$ )	$7.38 \cdot 10^{-3}$ ( $2.47 \cdot 10^{-4}$ )	$7.04 \cdot 10^{-3}$ ( $1.82 \cdot 10^{-4}$ )	$6.83 \cdot 10^{-3}$ ( $1.89 \cdot 10^{-4}$ )	$1.04 \cdot 10^{-1}$ ( $5.19 \cdot 10^{-3}$ )	$7.28 \cdot 10^{-3}$ ( $2.01 \cdot 10^{-4}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$6.19 \cdot 10^{-3}$ ( $9.89 \cdot 10^{-5}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$7.58 \cdot 10^{-1} (6.58 \cdot 10^{-3})$	$6.26 \cdot 10^{-1} (1.70 \cdot 10^{-2})$	$6.11 \cdot 10^{-1} (1.67 \cdot 10^{-2})$	$6.83 \cdot 10^{-1} (2.16 \cdot 10^{-2})$	$2.00 \cdot 10^0 (4.02 \cdot 10^{-2})$	$6.03 \cdot 10^{-1} (1.64 \cdot 10^{-2})$
$v = 3/2$	$1.56 \cdot 10^{-1} (2.86 \cdot 10^{-3})$	$1.53 \cdot 10^{-1} (5.99 \cdot 10^{-3})$	$1.48 \cdot 10^{-1} (5.83 \cdot 10^{-3})$	$1.62 \cdot 10^{-1} (6.34 \cdot 10^{-3})$	$2.05 \cdot 10^0 (7.57 \cdot 10^{-2})$	$1.49 \cdot 10^{-1} (5.82 \cdot 10^{-3})$
$v = 5/2$	$8.26 \cdot 10^{-2} (1.92 \cdot 10^{-3})$	$9.21 \cdot 10^{-2} (4.04 \cdot 10^{-3})$	$8.73 \cdot 10^{-2} (3.63 \cdot 10^{-3})$	$9.77 \cdot 10^{-2} (3.91 \cdot 10^{-3})$	$2.08 \cdot 10^0 (8.73 \cdot 10^{-2})$	$9.01 \cdot 10^{-2} (3.77 \cdot 10^{-3})$
$v = 7/2$	$7.17 \cdot 10^{-2} (1.79 \cdot 10^{-3})$	$8.11 \cdot 10^{-2} (3.41 \cdot 10^{-3})$	$8.01 \cdot 10^{-2} (3.08 \cdot 10^{-3})$	$8.37 \cdot 10^{-2} (3.09 \cdot 10^{-3})$	$1.85 \cdot 10^0 (7.45 \cdot 10^{-2})$	$8.42 \cdot 10^{-2} (3.48 \cdot 10^{-3})$
$v = 9/2$	$6.78 \cdot 10^{-2} (1.82 \cdot 10^{-3})$	$8.38 \cdot 10^{-2} (3.86 \cdot 10^{-3})$	$8.42 \cdot 10^{-2} (3.81 \cdot 10^{-3})$	$8.71 \cdot 10^{-2} (3.32 \cdot 10^{-3})$	$1.70 \cdot 10^0 (7.13 \cdot 10^{-2})$	$8.75 \cdot 10^{-2} (3.86 \cdot 10^{-3})$
$v = 17/2$	$6.72 \cdot 10^{-2} (1.99 \cdot 10^{-3})$	$9.47 \cdot 10^{-2} (5.01 \cdot 10^{-3})$	$9.31 \cdot 10^{-2} (4.66 \cdot 10^{-3})$	$9.76 \cdot 10^{-2} (4.84 \cdot 10^{-3})$	$1.46 \cdot 10^0 (6.77 \cdot 10^{-2})$	$9.88 \cdot 10^{-2} (4.76 \cdot 10^{-3})$
$v = 33/2$	$6.98 \cdot 10^{-2} (2.12 \cdot 10^{-3})$	$9.54 \cdot 10^{-2} (4.99 \cdot 10^{-3})$	$9.73 \cdot 10^{-2} (4.89 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (5.07 \cdot 10^{-3})$	$1.30 \cdot 10^0 (6.30 \cdot 10^{-2})$	$1.06 \cdot 10^{-1} (5.16 \cdot 10^{-3})$
$v = \infty$	$7.09 \cdot 10^{-2} (2.21 \cdot 10^{-3})$	$1.02 \cdot 10^{-1} (5.93 \cdot 10^{-3})$	$9.85 \cdot 10^{-2} (5.04 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (5.08 \cdot 10^{-3})$	$1.20 \cdot 10^0 (6.06 \cdot 10^{-2})$	$1.11 \cdot 10^{-1} (5.82 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$7.18 \cdot 10^{-2} (1.88 \cdot 10^{-3})$	$9.80 \cdot 10^{-2} (5.82 \cdot 10^{-3})$	$9.44 \cdot 10^{-2} (4.45 \cdot 10^{-3})$	$9.92 \cdot 10^{-2} (4.80 \cdot 10^{-3})$	$1.20 \cdot 10^0 (6.06 \cdot 10^{-2})$	$9.87 \cdot 10^{-2} (5.12 \cdot 10^{-3})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$7.09 \cdot 10^{-2} (2.07 \cdot 10^{-3})$

### SM-2.2.35 Borehole, $n = 160, d = 8$

#### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.92 \cdot 10^{-3} (2.05 \cdot 10^{-4})$	$4.10 \cdot 10^{-3} (2.01 \cdot 10^{-4})$	$3.94 \cdot 10^{-3} (2.03 \cdot 10^{-4})$	$3.86 \cdot 10^{-3} (2.07 \cdot 10^{-4})$	$6.59 \cdot 10^{-2} (3.34 \cdot 10^{-3})$	$3.85 \cdot 10^{-3} (2.04 \cdot 10^{-4})$
$v = 3/2$	$1.86 \cdot 10^{-4} (1.18 \cdot 10^{-5})$	$2.17 \cdot 10^{-4} (1.22 \cdot 10^{-5})$	$2.00 \cdot 10^{-4} (1.16 \cdot 10^{-5})$	$1.91 \cdot 10^{-4} (1.18 \cdot 10^{-5})$	$7.04 \cdot 10^{-2} (6.07 \cdot 10^{-3})$	$1.95 \cdot 10^{-4} (1.18 \cdot 10^{-5})$
$v = 5/2$	$5.10 \cdot 10^{-5} (2.62 \cdot 10^{-6})$	$5.83 \cdot 10^{-5} (2.98 \cdot 10^{-6})$	$5.39 \cdot 10^{-5} (2.76 \cdot 10^{-6})$	$5.37 \cdot 10^{-5} (2.82 \cdot 10^{-6})$	$5.61 \cdot 10^{-2} (4.75 \cdot 10^{-3})$	$5.37 \cdot 10^{-5} (2.84 \cdot 10^{-6})$
$v = 7/2$	$3.64 \cdot 10^{-5} (1.51 \cdot 10^{-6})$	$3.80 \cdot 10^{-5} (1.49 \cdot 10^{-6})$	$3.62 \cdot 10^{-5} (1.46 \cdot 10^{-6})$	$3.66 \cdot 10^{-5} (1.54 \cdot 10^{-6})$	$2.00 \cdot 10^{-2} (1.83 \cdot 10^{-3})$	$3.73 \cdot 10^{-5} (1.55 \cdot 10^{-6})$
$v = 9/2$	$3.63 \cdot 10^{-5} (1.30 \cdot 10^{-6})$	$3.82 \cdot 10^{-5} (1.42 \cdot 10^{-6})$	$3.64 \cdot 10^{-5} (1.32 \cdot 10^{-6})$	$3.69 \cdot 10^{-5} (1.39 \cdot 10^{-6})$	$1.13 \cdot 10^{-2} (1.00 \cdot 10^{-3})$	$3.87 \cdot 10^{-5} (1.48 \cdot 10^{-6})$
$v = 17/2$	$4.06 \cdot 10^{-5} (1.18 \cdot 10^{-6})$	$4.26 \cdot 10^{-5} (1.42 \cdot 10^{-6})$	$4.15 \cdot 10^{-5} (1.33 \cdot 10^{-6})$	$4.21 \cdot 10^{-5} (1.44 \cdot 10^{-6})$	$6.53 \cdot 10^{-3} (5.00 \cdot 10^{-4})$	$4.60 \cdot 10^{-5} (1.76 \cdot 10^{-6})$
$v = 33/2$	$4.23 \cdot 10^{-5} (1.15 \cdot 10^{-6})$	$4.54 \cdot 10^{-5} (1.50 \cdot 10^{-6})$	$4.37 \cdot 10^{-5} (1.40 \cdot 10^{-6})$	$4.35 \cdot 10^{-5} (1.38 \cdot 10^{-6})$	$5.44 \cdot 10^{-3} (3.97 \cdot 10^{-4})$	$4.93 \cdot 10^{-5} (1.94 \cdot 10^{-6})$
$v = \infty$	$4.78 \cdot 10^{-5} (1.25 \cdot 10^{-6})$	$5.16 \cdot 10^{-5} (1.66 \cdot 10^{-6})$	$5.04 \cdot 10^{-5} (1.57 \cdot 10^{-6})$	$5.09 \cdot 10^{-5} (1.66 \cdot 10^{-6})$	$4.56 \cdot 10^{-3} (3.17 \cdot 10^{-4})$	$5.55 \cdot 10^{-5} (2.07 \cdot 10^{-6})$
$v \in \{1/2, \dots, \infty\}$	$3.64 \cdot 10^{-5} (1.51 \cdot 10^{-6})$	$3.96 \cdot 10^{-5} (1.51 \cdot 10^{-6})$	$3.70 \cdot 10^{-5} (1.43 \cdot 10^{-6})$	$3.78 \cdot 10^{-5} (1.64 \cdot 10^{-6})$	$4.56 \cdot 10^{-3} (3.17 \cdot 10^{-4})$	$3.88 \cdot 10^{-5} (1.85 \cdot 10^{-6})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$3.59 \cdot 10^{-3} (1.99 \cdot 10^{-4})$
$v = 3/2$	$1.56 \cdot 10^{-4} (1.05 \cdot 10^{-5})$
$v = 5/2$	$3.99 \cdot 10^{-5} (1.98 \cdot 10^{-6})$
$v = 7/2$	$2.78 \cdot 10^{-5} (1.10 \cdot 10^{-6})$
$v = 9/2$	$2.80 \cdot 10^{-5} (9.61 \cdot 10^{-7})$
$v = 17/2$	$3.18 \cdot 10^{-5} (9.13 \cdot 10^{-7})$
$v = 33/2$	$3.31 \cdot 10^{-5} (8.94 \cdot 10^{-7})$
$v = \infty$	$3.78 \cdot 10^{-5} (9.68 \cdot 10^{-7})$
$v \in \{1/2, \dots, \infty\}$	$2.66 \cdot 10^{-5} (9.51 \cdot 10^{-7})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$3.88 \cdot 10^{-5} (1.47 \cdot 10^{-6})$
linear	$1.36 \cdot 10^{-1} (2.85 \cdot 10^{-4})$
INN	$4.00 \cdot 10^{-1} (1.24 \cdot 10^{-3})$
2NN	$2.79 \cdot 10^{-1} (1.35 \cdot 10^{-3})$

#### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; \mathcal{S}^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.51 \cdot 10^{-2}$ ( $2.04 \cdot 10^{-4}$ )	$2.52 \cdot 10^{-2}$ ( $3.24 \cdot 10^{-4}$ )	$2.47 \cdot 10^{-2}$ ( $3.25 \cdot 10^{-4}$ )	$2.22 \cdot 10^{-2}$ ( $4.16 \cdot 10^{-4}$ )	$1.29 \cdot 10^{-1}$ ( $3.72 \cdot 10^{-3}$ )	$2.45 \cdot 10^{-2}$ ( $3.28 \cdot 10^{-4}$ )
$v = 3/2$	$6.05 \cdot 10^{-3}$ ( $5.42 \cdot 10^{-5}$ )	$5.13 \cdot 10^{-3}$ ( $8.15 \cdot 10^{-5}$ )	$4.88 \cdot 10^{-3}$ ( $7.48 \cdot 10^{-5}$ )	$4.56 \cdot 10^{-3}$ ( $8.67 \cdot 10^{-5}$ )	$1.25 \cdot 10^{-1}$ ( $6.81 \cdot 10^{-3}$ )	$4.84 \cdot 10^{-3}$ ( $7.42 \cdot 10^{-5}$ )
$v = 5/2$	$2.96 \cdot 10^{-3}$ ( $2.49 \cdot 10^{-5}$ )	$2.78 \cdot 10^{-3}$ ( $4.23 \cdot 10^{-5}$ )	$2.64 \cdot 10^{-3}$ ( $3.74 \cdot 10^{-5}$ )	$2.58 \cdot 10^{-3}$ ( $3.99 \cdot 10^{-5}$ )	$1.09 \cdot 10^{-1}$ ( $6.10 \cdot 10^{-3}$ )	$2.63 \cdot 10^{-3}$ ( $3.63 \cdot 10^{-5}$ )
$v = 7/2$	$2.51 \cdot 10^{-3}$ ( $2.14 \cdot 10^{-5}$ )	$2.47 \cdot 10^{-3}$ ( $3.17 \cdot 10^{-5}$ )	$2.37 \cdot 10^{-3}$ ( $2.95 \cdot 10^{-5}$ )	$2.37 \cdot 10^{-3}$ ( $3.18 \cdot 10^{-5}$ )	$6.20 \cdot 10^{-2}$ ( $3.24 \cdot 10^{-3}$ )	$2.39 \cdot 10^{-3}$ ( $3.04 \cdot 10^{-5}$ )
$v = 9/2$	$2.55 \cdot 10^{-3}$ ( $2.18 \cdot 10^{-5}$ )	$2.60 \cdot 10^{-3}$ ( $3.46 \cdot 10^{-5}$ )	$2.51 \cdot 10^{-3}$ ( $3.10 \cdot 10^{-5}$ )	$2.52 \cdot 10^{-3}$ ( $3.31 \cdot 10^{-5}$ )	$4.56 \cdot 10^{-2}$ ( $2.07 \cdot 10^{-3}$ )	$2.57 \cdot 10^{-3}$ ( $3.42 \cdot 10^{-5}$ )
$v = 17/2$	$2.79 \cdot 10^{-3}$ ( $2.49 \cdot 10^{-5}$ )	$2.88 \cdot 10^{-3}$ ( $3.73 \cdot 10^{-5}$ )	$2.83 \cdot 10^{-3}$ ( $3.47 \cdot 10^{-5}$ )	$2.85 \cdot 10^{-3}$ ( $3.70 \cdot 10^{-5}$ )	$3.45 \cdot 10^{-2}$ ( $1.25 \cdot 10^{-3}$ )	$2.94 \cdot 10^{-3}$ ( $4.32 \cdot 10^{-5}$ )
$v = 33/2$	$2.89 \cdot 10^{-3}$ ( $2.65 \cdot 10^{-5}$ )	$3.02 \cdot 10^{-3}$ ( $4.30 \cdot 10^{-5}$ )	$2.94 \cdot 10^{-3}$ ( $3.75 \cdot 10^{-5}$ )	$2.96 \cdot 10^{-3}$ ( $3.85 \cdot 10^{-5}$ )	$3.15 \cdot 10^{-2}$ ( $1.09 \cdot 10^{-3}$ )	$3.10 \cdot 10^{-3}$ ( $5.05 \cdot 10^{-5}$ )
$v = \infty$	$3.11 \cdot 10^{-3}$ ( $2.92 \cdot 10^{-5}$ )	$3.26 \cdot 10^{-3}$ ( $4.62 \cdot 10^{-5}$ )	$3.21 \cdot 10^{-3}$ ( $4.36 \cdot 10^{-5}$ )	$3.23 \cdot 10^{-3}$ ( $4.55 \cdot 10^{-5}$ )	$2.88 \cdot 10^{-2}$ ( $9.55 \cdot 10^{-4}$ )	$3.32 \cdot 10^{-3}$ ( $5.13 \cdot 10^{-5}$ )
$v \in \{1/2, \dots, \infty\}$	$2.51 \cdot 10^{-3}$ ( $2.14 \cdot 10^{-5}$ )	$2.60 \cdot 10^{-3}$ ( $4.09 \cdot 10^{-5}$ )	$2.44 \cdot 10^{-3}$ ( $3.47 \cdot 10^{-5}$ )	$2.41 \cdot 10^{-3}$ ( $3.46 \cdot 10^{-5}$ )	$2.88 \cdot 10^{-2}$ ( $9.55 \cdot 10^{-4}$ )	$2.45 \cdot 10^{-3}$ ( $3.57 \cdot 10^{-5}$ )

$R(\theta; \mathcal{S}^{\text{CRPS}})$	
NLL/SPE	$2.62 \cdot 10^{-3}$ ( $2.96 \cdot 10^{-5}$ )

### Averages of $R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$

$R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.24 \cdot 10^{-1}$ ( $4.30 \cdot 10^{-3}$ )	$4.12 \cdot 10^{-1}$ ( $1.08 \cdot 10^{-2}$ )	$4.03 \cdot 10^{-1}$ ( $1.08 \cdot 10^{-2}$ )	$4.54 \cdot 10^{-1}$ ( $1.40 \cdot 10^{-2}$ )	$1.46 \cdot 10^0$ ( $3.81 \cdot 10^{-2}$ )	$3.98 \cdot 10^{-1}$ ( $1.09 \cdot 10^{-2}$ )
$v = 3/2$	$8.81 \cdot 10^{-2}$ ( $1.24 \cdot 10^{-3}$ )	$8.41 \cdot 10^{-2}$ ( $2.56 \cdot 10^{-3}$ )	$8.04 \cdot 10^{-2}$ ( $2.48 \cdot 10^{-3}$ )	$8.94 \cdot 10^{-2}$ ( $2.91 \cdot 10^{-3}$ )	$1.44 \cdot 10^0$ ( $7.41 \cdot 10^{-2}$ )	$7.89 \cdot 10^{-2}$ ( $2.50 \cdot 10^{-3}$ )
$v = 5/2$	$4.02 \cdot 10^{-2}$ ( $5.79 \cdot 10^{-4}$ )	$4.15 \cdot 10^{-2}$ ( $1.20 \cdot 10^{-3}$ )	$3.98 \cdot 10^{-2}$ ( $1.13 \cdot 10^{-3}$ )	$4.48 \cdot 10^{-2}$ ( $1.21 \cdot 10^{-3}$ )	$1.26 \cdot 10^0$ ( $6.66 \cdot 10^{-2}$ )	$3.96 \cdot 10^{-2}$ ( $1.12 \cdot 10^{-3}$ )
$v = 7/2$	$3.17 \cdot 10^{-2}$ ( $4.46 \cdot 10^{-4}$ )	$3.24 \cdot 10^{-2}$ ( $7.77 \cdot 10^{-4}$ )	$3.18 \cdot 10^{-2}$ ( $7.87 \cdot 10^{-4}$ )	$3.49 \cdot 10^{-2}$ ( $8.76 \cdot 10^{-4}$ )	$7.53 \cdot 10^{-1}$ ( $3.74 \cdot 10^{-2}$ )	$3.23 \cdot 10^{-2}$ ( $8.07 \cdot 10^{-4}$ )
$v = 9/2$	$3.03 \cdot 10^{-2}$ ( $4.37 \cdot 10^{-4}$ )	$3.17 \cdot 10^{-2}$ ( $7.67 \cdot 10^{-4}$ )	$3.13 \cdot 10^{-2}$ ( $7.55 \cdot 10^{-4}$ )	$3.38 \cdot 10^{-2}$ ( $8.42 \cdot 10^{-4}$ )	$5.75 \cdot 10^{-1}$ ( $2.60 \cdot 10^{-2}$ )	$3.25 \cdot 10^{-2}$ ( $7.97 \cdot 10^{-4}$ )
$v = 17/2$	$3.03 \cdot 10^{-2}$ ( $4.71 \cdot 10^{-4}$ )	$3.27 \cdot 10^{-2}$ ( $7.56 \cdot 10^{-4}$ )	$3.26 \cdot 10^{-2}$ ( $7.41 \cdot 10^{-4}$ )	$3.45 \cdot 10^{-2}$ ( $8.42 \cdot 10^{-4}$ )	$4.35 \cdot 10^{-1}$ ( $1.75 \cdot 10^{-2}$ )	$3.47 \cdot 10^{-2}$ ( $8.78 \cdot 10^{-4}$ )
$v = 33/2$	$3.06 \cdot 10^{-2}$ ( $5.02 \cdot 10^{-4}$ )	$3.36 \cdot 10^{-2}$ ( $7.87 \cdot 10^{-4}$ )	$3.32 \cdot 10^{-2}$ ( $7.58 \cdot 10^{-4}$ )	$3.45 \cdot 10^{-2}$ ( $8.15 \cdot 10^{-4}$ )	$3.91 \cdot 10^{-1}$ ( $1.53 \cdot 10^{-2}$ )	$3.58 \cdot 10^{-2}$ ( $9.26 \cdot 10^{-4}$ )
$v = \infty$	$3.26 \cdot 10^{-2}$ ( $5.55 \cdot 10^{-4}$ )	$3.57 \cdot 10^{-2}$ ( $8.32 \cdot 10^{-4}$ )	$3.54 \cdot 10^{-2}$ ( $7.87 \cdot 10^{-4}$ )	$3.71 \cdot 10^{-2}$ ( $9.02 \cdot 10^{-4}$ )	$3.52 \cdot 10^{-1}$ ( $1.34 \cdot 10^{-2}$ )	$3.75 \cdot 10^{-2}$ ( $9.28 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$3.17 \cdot 10^{-2}$ ( $4.46 \cdot 10^{-4}$ )	$3.29 \cdot 10^{-2}$ ( $8.19 \cdot 10^{-4}$ )	$3.21 \cdot 10^{-2}$ ( $7.90 \cdot 10^{-4}$ )	$3.57 \cdot 10^{-2}$ ( $9.60 \cdot 10^{-4}$ )	$3.52 \cdot 10^{-1}$ ( $1.34 \cdot 10^{-2}$ )	$3.30 \cdot 10^{-2}$ ( $9.12 \cdot 10^{-4}$ )

$R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$	
NLL/SPE	$3.16 \cdot 10^{-2}$ ( $5.12 \cdot 10^{-4}$ )

### SM-2.2.36 Borehole, $n = 400, d = 8$

### Averages of $R(\theta; \mathcal{S}^{\text{SPE}})$

$R(\theta; \mathcal{S}^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.09 \cdot 10^{-3}$ ( $4.58 \cdot 10^{-5}$ )	$1.15 \cdot 10^{-3}$ ( $4.74 \cdot 10^{-5}$ )	$1.11 \cdot 10^{-3}$ ( $4.60 \cdot 10^{-5}$ )	$1.08 \cdot 10^{-3}$ ( $4.61 \cdot 10^{-5}$ )	$1.74 \cdot 10^{-2}$ ( $1.66 \cdot 10^{-3}$ )	$1.08 \cdot 10^{-3}$ ( $4.55 \cdot 10^{-5}$ )
$v = 3/2$	$3.31 \cdot 10^{-5}$ ( $1.73 \cdot 10^{-6}$ )	$3.72 \cdot 10^{-5}$ ( $1.86 \cdot 10^{-6}$ )	$3.50 \cdot 10^{-5}$ ( $1.77 \cdot 10^{-6}$ )	$3.41 \cdot 10^{-5}$ ( $1.83 \cdot 10^{-6}$ )	$1.96 \cdot 10^{-2}$ ( $3.32 \cdot 10^{-3}$ )	$3.53 \cdot 10^{-5}$ ( $1.78 \cdot 10^{-6}$ )
$v = 5/2$	$7.40 \cdot 10^{-6}$ ( $3.55 \cdot 10^{-7}$ )	$8.25 \cdot 10^{-6}$ ( $3.78 \cdot 10^{-7}$ )	$7.91 \cdot 10^{-6}$ ( $3.61 \cdot 10^{-7}$ )	$7.73 \cdot 10^{-6}$ ( $3.63 \cdot 10^{-7}$ )	$4.57 \cdot 10^{-3}$ ( $6.34 \cdot 10^{-4}$ )	$8.03 \cdot 10^{-6}$ ( $3.70 \cdot 10^{-7}$ )
$v = 7/2$	$4.85 \cdot 10^{-6}$ ( $2.05 \cdot 10^{-7}$ )	$4.90 \cdot 10^{-6}$ ( $1.88 \cdot 10^{-7}$ )	$4.82 \cdot 10^{-6}$ ( $1.93 \cdot 10^{-7}$ )	$4.84 \cdot 10^{-6}$ ( $1.94 \cdot 10^{-7}$ )	$1.41 \cdot 10^{-3}$ ( $1.22 \cdot 10^{-4}$ )	$5.01 \cdot 10^{-6}$ ( $2.08 \cdot 10^{-7}$ )
$v = 9/2$	$4.39 \cdot 10^{-6}$ ( $1.66 \cdot 10^{-7}$ )	$4.25 \cdot 10^{-6}$ ( $1.49 \cdot 10^{-7}$ )	$4.27 \cdot 10^{-6}$ ( $1.61 \cdot 10^{-7}$ )	$4.28 \cdot 10^{-6}$ ( $1.60 \cdot 10^{-7}$ )	$1.01 \cdot 10^{-3}$ ( $8.46 \cdot 10^{-5}$ )	$4.51 \cdot 10^{-6}$ ( $1.75 \cdot 10^{-7}$ )
$v = 17/2$	$5.17 \cdot 10^{-6}$ ( $1.52 \cdot 10^{-7}$ )	$5.30 \cdot 10^{-6}$ ( $1.56 \cdot 10^{-7}$ )	$5.34 \cdot 10^{-6}$ ( $1.68 \cdot 10^{-7}$ )	$5.31 \cdot 10^{-6}$ ( $1.63 \cdot 10^{-7}$ )	$6.55 \cdot 10^{-4}$ ( $5.15 \cdot 10^{-5}$ )	$5.75 \cdot 10^{-6}$ ( $1.98 \cdot 10^{-7}$ )
$v = 33/2$	$5.98 \cdot 10^{-6}$ ( $1.54 \cdot 10^{-7}$ )	$6.25 \cdot 10^{-6}$ ( $1.65 \cdot 10^{-7}$ )	$6.20 \cdot 10^{-6}$ ( $1.73 \cdot 10^{-7}$ )	$6.18 \cdot 10^{-6}$ ( $1.64 \cdot 10^{-7}$ )	$5.46 \cdot 10^{-4}$ ( $4.03 \cdot 10^{-5}$ )	$6.64 \cdot 10^{-6}$ ( $2.19 \cdot 10^{-7}$ )
$v = \infty$	$7.51 \cdot 10^{-6}$ ( $1.70 \cdot 10^{-7}$ )	$7.78 \cdot 10^{-6}$ ( $1.88 \cdot 10^{-7}$ )	$7.72 \cdot 10^{-6}$ ( $1.78 \cdot 10^{-7}$ )	$7.71 \cdot 10^{-6}$ ( $1.79 \cdot 10^{-7}$ )	$4.71 \cdot 10^{-4}$ ( $3.14 \cdot 10^{-5}$ )	$8.28 \cdot 10^{-6}$ ( $2.52 \cdot 10^{-7}$ )
$v \in \{1/2, \dots, \infty\}$	$4.85 \cdot 10^{-6}$ ( $2.05 \cdot 10^{-7}$ )	$4.38 \cdot 10^{-6}$ ( $1.53 \cdot 10^{-7}$ )	$4.39 \cdot 10^{-6}$ ( $1.63 \cdot 10^{-7}$ )	$4.61 \cdot 10^{-6}$ ( $1.70 \cdot 10^{-7}$ )	$4.71 \cdot 10^{-4}$ ( $3.14 \cdot 10^{-5}$ )	$4.75 \cdot 10^{-6}$ ( $1.85 \cdot 10^{-7}$ )



$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.02 \cdot 10^{-3}$ ( $4.43 \cdot 10^{-5}$ )
$v = 3/2$	$2.84 \cdot 10^{-5}$ ( $1.54 \cdot 10^{-6}$ )
$v = 5/2$	$6.46 \cdot 10^{-6}$ ( $3.14 \cdot 10^{-7}$ )
$v = 7/2$	$4.29 \cdot 10^{-6}$ ( $1.73 \cdot 10^{-7}$ )
$v = 9/2$	$3.92 \cdot 10^{-6}$ ( $1.34 \cdot 10^{-7}$ )
$v = 17/2$	$4.85 \cdot 10^{-6}$ ( $1.24 \cdot 10^{-7}$ )
$v = 33/2$	$5.61 \cdot 10^{-6}$ ( $1.23 \cdot 10^{-7}$ )
$v = \infty$	$6.96 \cdot 10^{-6}$ ( $1.33 \cdot 10^{-7}$ )
$v \in \{1/2, \dots, \infty\}$	$3.86 \cdot 10^{-6}$ ( $1.34 \cdot 10^{-7}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$4.51 \cdot 10^{-6}$ ( $1.69 \cdot 10^{-7}$ )
linear	$1.33 \cdot 10^{-1}$ ( $1.06 \cdot 10^{-4}$ )
1NN	$3.09 \cdot 10^{-1}$ ( $6.87 \cdot 10^{-4}$ )
2NN	$2.08 \cdot 10^{-1}$ ( $8.08 \cdot 10^{-4}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.02 \cdot 10^{-2}$ ( $5.82 \cdot 10^{-5}$ )	$1.29 \cdot 10^{-2}$ ( $1.18 \cdot 10^{-4}$ )	$1.27 \cdot 10^{-2}$ ( $1.10 \cdot 10^{-4}$ )	$1.10 \cdot 10^{-2}$ ( $1.36 \cdot 10^{-4}$ )	$6.02 \cdot 10^{-2}$ ( $2.86 \cdot 10^{-3}$ )	$1.26 \cdot 10^{-2}$ ( $1.09 \cdot 10^{-4}$ )
$v = 3/2$	$2.77 \cdot 10^{-3}$ ( $1.36 \cdot 10^{-5}$ )	$2.04 \cdot 10^{-3}$ ( $2.51 \cdot 10^{-5}$ )	$1.97 \cdot 10^{-3}$ ( $2.27 \cdot 10^{-5}$ )	$1.81 \cdot 10^{-3}$ ( $2.49 \cdot 10^{-5}$ )	$5.44 \cdot 10^{-2}$ ( $5.16 \cdot 10^{-3}$ )	$1.99 \cdot 10^{-3}$ ( $2.33 \cdot 10^{-5}$ )
$v = 5/2$	$1.17 \cdot 10^{-3}$ ( $6.94 \cdot 10^{-6}$ )	$9.73 \cdot 10^{-4}$ ( $1.16 \cdot 10^{-5}$ )	$9.43 \cdot 10^{-4}$ ( $1.04 \cdot 10^{-5}$ )	$8.88 \cdot 10^{-4}$ ( $1.12 \cdot 10^{-5}$ )	$2.66 \cdot 10^{-2}$ ( $1.91 \cdot 10^{-3}$ )	$9.51 \cdot 10^{-4}$ ( $1.04 \cdot 10^{-5}$ )
$v = 7/2$	$9.01 \cdot 10^{-4}$ ( $6.07 \cdot 10^{-6}$ )	$8.02 \cdot 10^{-4}$ ( $7.98 \cdot 10^{-6}$ )	$7.86 \cdot 10^{-4}$ ( $7.77 \cdot 10^{-6}$ )	$7.71 \cdot 10^{-4}$ ( $8.31 \cdot 10^{-6}$ )	$1.51 \cdot 10^{-2}$ ( $6.84 \cdot 10^{-4}$ )	$7.95 \cdot 10^{-4}$ ( $8.13 \cdot 10^{-6}$ )
$v = 9/2$	$8.50 \cdot 10^{-4}$ ( $6.10 \cdot 10^{-6}$ )	$7.94 \cdot 10^{-4}$ ( $7.25 \cdot 10^{-6}$ )	$7.84 \cdot 10^{-4}$ ( $7.41 \cdot 10^{-6}$ )	$7.76 \cdot 10^{-4}$ ( $7.80 \cdot 10^{-6}$ )	$1.27 \cdot 10^{-2}$ ( $5.40 \cdot 10^{-4}$ )	$8.00 \cdot 10^{-4}$ ( $8.09 \cdot 10^{-6}$ )
$v = 17/2$	$9.50 \cdot 10^{-4}$ ( $6.89 \cdot 10^{-6}$ )	$9.64 \cdot 10^{-4}$ ( $8.59 \cdot 10^{-6}$ )	$9.55 \cdot 10^{-4}$ ( $8.11 \cdot 10^{-6}$ )	$9.49 \cdot 10^{-4}$ ( $8.50 \cdot 10^{-6}$ )	$1.02 \cdot 10^{-2}$ ( $3.60 \cdot 10^{-4}$ )	$9.81 \cdot 10^{-4}$ ( $9.64 \cdot 10^{-6}$ )
$v = 33/2$	$1.04 \cdot 10^{-3}$ ( $7.39 \cdot 10^{-6}$ )	$1.08 \cdot 10^{-3}$ ( $9.77 \cdot 10^{-6}$ )	$1.06 \cdot 10^{-3}$ ( $8.56 \cdot 10^{-6}$ )	$1.06 \cdot 10^{-3}$ ( $9.61 \cdot 10^{-6}$ )	$9.33 \cdot 10^{-3}$ ( $3.01 \cdot 10^{-4}$ )	$1.08 \cdot 10^{-3}$ ( $1.02 \cdot 10^{-5}$ )
$v = \infty$	$1.18 \cdot 10^{-3}$ ( $8.02 \cdot 10^{-6}$ )	$1.22 \cdot 10^{-3}$ ( $1.08 \cdot 10^{-5}$ )	$1.21 \cdot 10^{-3}$ ( $8.88 \cdot 10^{-6}$ )	$1.20 \cdot 10^{-3}$ ( $9.82 \cdot 10^{-6}$ )	$8.76 \cdot 10^{-3}$ ( $2.51 \cdot 10^{-4}$ )	$1.23 \cdot 10^{-3}$ ( $1.12 \cdot 10^{-5}$ )
$v \in \{1/2, \dots, \infty\}$	$9.01 \cdot 10^{-4}$ ( $6.07 \cdot 10^{-6}$ )	$8.02 \cdot 10^{-4}$ ( $9.11 \cdot 10^{-6}$ )	$7.85 \cdot 10^{-4}$ ( $7.34 \cdot 10^{-6}$ )	$7.72 \cdot 10^{-4}$ ( $7.97 \cdot 10^{-6}$ )	$8.76 \cdot 10^{-3}$ ( $2.51 \cdot 10^{-4}$ )	$8.00 \cdot 10^{-4}$ ( $8.09 \cdot 10^{-6}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$8.65 \cdot 10^{-4}$ ( $7.53 \cdot 10^{-6}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.09 \cdot 10^{-1}$ ( $1.50 \cdot 10^{-3}$ )	$2.12 \cdot 10^{-1}$ ( $3.77 \cdot 10^{-3}$ )	$2.07 \cdot 10^{-1}$ ( $3.68 \cdot 10^{-3}$ )	$2.34 \cdot 10^{-1}$ ( $4.76 \cdot 10^{-3}$ )	$7.40 \cdot 10^{-1}$ ( $2.57 \cdot 10^{-2}$ )	$2.06 \cdot 10^{-1}$ ( $3.66 \cdot 10^{-3}$ )
$v = 3/2$	$4.14 \cdot 10^{-2}$ ( $3.33 \cdot 10^{-4}$ )	$3.33 \cdot 10^{-2}$ ( $7.44 \cdot 10^{-4}$ )	$3.22 \cdot 10^{-2}$ ( $7.23 \cdot 10^{-4}$ )	$3.67 \cdot 10^{-2}$ ( $8.71 \cdot 10^{-4}$ )	$6.36 \cdot 10^{-1}$ ( $5.36 \cdot 10^{-2}$ )	$3.23 \cdot 10^{-2}$ ( $7.33 \cdot 10^{-4}$ )
$v = 5/2$	$1.67 \cdot 10^{-2}$ ( $1.58 \cdot 10^{-4}$ )	$1.52 \cdot 10^{-2}$ ( $3.18 \cdot 10^{-4}$ )	$1.49 \cdot 10^{-2}$ ( $3.01 \cdot 10^{-4}$ )	$1.75 \cdot 10^{-2}$ ( $3.71 \cdot 10^{-4}$ )	$3.39 \cdot 10^{-1}$ ( $2.20 \cdot 10^{-2}$ )	$1.50 \cdot 10^{-2}$ ( $3.02 \cdot 10^{-4}$ )
$v = 7/2$	$1.20 \cdot 10^{-2}$ ( $1.29 \cdot 10^{-4}$ )	$1.11 \cdot 10^{-2}$ ( $2.12 \cdot 10^{-4}$ )	$1.11 \cdot 10^{-2}$ ( $2.17 \cdot 10^{-4}$ )	$1.25 \cdot 10^{-2}$ ( $2.53 \cdot 10^{-4}$ )	$1.97 \cdot 10^{-1}$ ( $8.38 \cdot 10^{-3}$ )	$1.13 \cdot 10^{-2}$ ( $2.27 \cdot 10^{-4}$ )
$v = 9/2$	$1.07 \cdot 10^{-2}$ ( $1.22 \cdot 10^{-4}$ )	$9.96 \cdot 10^{-3}$ ( $1.73 \cdot 10^{-4}$ )	$1.00 \cdot 10^{-2}$ ( $1.86 \cdot 10^{-4}$ )	$1.08 \cdot 10^{-2}$ ( $2.12 \cdot 10^{-4}$ )	$1.64 \cdot 10^{-1}$ ( $6.72 \cdot 10^{-3}$ )	$1.03 \cdot 10^{-2}$ ( $1.98 \cdot 10^{-4}$ )
$v = 17/2$	$1.06 \cdot 10^{-2}$ ( $1.32 \cdot 10^{-4}$ )	$1.06 \cdot 10^{-2}$ ( $1.63 \cdot 10^{-4}$ )	$1.07 \cdot 10^{-2}$ ( $1.76 \cdot 10^{-4}$ )	$1.11 \cdot 10^{-2}$ ( $1.97 \cdot 10^{-4}$ )	$1.30 \cdot 10^{-1}$ ( $4.78 \cdot 10^{-3}$ )	$1.12 \cdot 10^{-2}$ ( $2.00 \cdot 10^{-4}$ )
$v = 33/2$	$1.11 \cdot 10^{-2}$ ( $1.41 \cdot 10^{-4}$ )	$1.14 \cdot 10^{-2}$ ( $1.68 \cdot 10^{-4}$ )	$1.14 \cdot 10^{-2}$ ( $1.74 \cdot 10^{-4}$ )	$1.18 \cdot 10^{-2}$ ( $1.96 \cdot 10^{-4}$ )	$1.17 \cdot 10^{-1}$ ( $4.04 \cdot 10^{-3}$ )	$1.18 \cdot 10^{-2}$ ( $2.06 \cdot 10^{-4}$ )
$v = \infty$	$1.24 \cdot 10^{-2}$ ( $1.56 \cdot 10^{-4}$ )	$1.27 \cdot 10^{-2}$ ( $1.76 \cdot 10^{-4}$ )	$1.27 \cdot 10^{-2}$ ( $1.70 \cdot 10^{-4}$ )	$1.30 \cdot 10^{-2}$ ( $2.03 \cdot 10^{-4}$ )	$1.07 \cdot 10^{-1}$ ( $3.36 \cdot 10^{-3}$ )	$1.31 \cdot 10^{-2}$ ( $2.14 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$1.20 \cdot 10^{-2}$ ( $1.29 \cdot 10^{-4}$ )	$1.02 \cdot 10^{-2}$ ( $1.81 \cdot 10^{-4}$ )	$1.03 \cdot 10^{-2}$ ( $1.93 \cdot 10^{-4}$ )	$1.20 \cdot 10^{-2}$ ( $2.33 \cdot 10^{-4}$ )	$1.07 \cdot 10^{-1}$ ( $3.36 \cdot 10^{-3}$ )	$1.08 \cdot 10^{-2}$ ( $2.10 \cdot 10^{-4}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.08 \cdot 10^{-2}$ ( $1.27 \cdot 10^{-4}$ )

### SM-2.2.37 $\text{Re}(S_{2\pi}), n = 60, d = 6$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.56 \cdot 10^{-2}$ (7.27 · 10 <sup>-4</sup> )	$2.57 \cdot 10^{-2}$ (7.42 · 10 <sup>-4</sup> )	$2.41 \cdot 10^{-2}$ (6.95 · 10 <sup>-4</sup> )	$2.41 \cdot 10^{-2}$ (7.22 · 10 <sup>-4</sup> )	$3.42 \cdot 10^{-1}$ (6.76 · 10 <sup>-3</sup> )	$2.34 \cdot 10^{-2}$ (6.92 · 10 <sup>-4</sup> )
$v = 3/2$	$8.23 \cdot 10^{-5}$ (2.79 · 10 <sup>-6</sup> )	$9.83 \cdot 10^{-5}$ (6.82 · 10 <sup>-6</sup> )	$7.48 \cdot 10^{-5}$ (3.20 · 10 <sup>-6</sup> )	$7.90 \cdot 10^{-5}$ (4.19 · 10 <sup>-6</sup> )	$3.04 \cdot 10^{-1}$ (8.73 · 10 <sup>-3</sup> )	$7.41 \cdot 10^{-5}$ (2.38 · 10 <sup>-6</sup> )
$v = 5/2$	$2.27 \cdot 10^{-5}$ (6.91 · 10 <sup>-7</sup> )	$2.91 \cdot 10^{-5}$ (2.84 · 10 <sup>-6</sup> )	$2.21 \cdot 10^{-5}$ (9.60 · 10 <sup>-7</sup> )	$2.23 \cdot 10^{-5}$ (9.77 · 10 <sup>-7</sup> )	$2.94 \cdot 10^{-1}$ (9.47 · 10 <sup>-3</sup> )	$2.38 \cdot 10^{-5}$ (1.09 · 10 <sup>-6</sup> )
$v = 7/2$	$2.88 \cdot 10^{-5}$ (8.15 · 10 <sup>-7</sup> )	$4.06 \cdot 10^{-5}$ (3.68 · 10 <sup>-6</sup> )	$2.93 \cdot 10^{-5}$ (1.36 · 10 <sup>-6</sup> )	$2.95 \cdot 10^{-5}$ (1.44 · 10 <sup>-6</sup> )	$2.79 \cdot 10^{-1}$ (8.90 · 10 <sup>-3</sup> )	$4.28 \cdot 10^{-5}$ (4.92 · 10 <sup>-6</sup> )
$v = 9/2$	$4.22 \cdot 10^{-5}$ (1.81 · 10 <sup>-6</sup> )	$5.92 \cdot 10^{-5}$ (5.32 · 10 <sup>-6</sup> )	$4.32 \cdot 10^{-5}$ (2.92 · 10 <sup>-6</sup> )	$4.61 \cdot 10^{-5}$ (3.18 · 10 <sup>-6</sup> )	$2.65 \cdot 10^{-1}$ (8.75 · 10 <sup>-3</sup> )	$7.42 \cdot 10^{-5}$ (1.30 · 10 <sup>-5</sup> )
$v = 13/2$	$5.53 \cdot 10^{-5}$ (3.00 · 10 <sup>-6</sup> )	$8.08 \cdot 10^{-5}$ (7.40 · 10 <sup>-6</sup> )	$6.29 \cdot 10^{-5}$ (5.15 · 10 <sup>-6</sup> )	$6.91 \cdot 10^{-5}$ (5.97 · 10 <sup>-6</sup> )	$2.53 \cdot 10^{-1}$ (8.95 · 10 <sup>-3</sup> )	$9.70 \cdot 10^{-5}$ (1.26 · 10 <sup>-5</sup> )
$v = 25/2$	$6.39 \cdot 10^{-5}$ (3.58 · 10 <sup>-6</sup> )	$9.79 \cdot 10^{-5}$ (8.38 · 10 <sup>-6</sup> )	$8.35 \cdot 10^{-5}$ (6.93 · 10 <sup>-6</sup> )	$8.91 \cdot 10^{-5}$ (7.51 · 10 <sup>-6</sup> )	$2.40 \cdot 10^{-1}$ (9.35 · 10 <sup>-3</sup> )	$1.73 \cdot 10^{-4}$ (2.85 · 10 <sup>-5</sup> )
$v = \infty$	$7.23 \cdot 10^{-5}$ (4.14 · 10 <sup>-6</sup> )	$1.24 \cdot 10^{-4}$ (1.10 · 10 <sup>-5</sup> )	$1.01 \cdot 10^{-4}$ (7.78 · 10 <sup>-6</sup> )	$1.07 \cdot 10^{-4}$ (8.65 · 10 <sup>-6</sup> )	$2.24 \cdot 10^{-1}$ (9.85 · 10 <sup>-3</sup> )	$2.09 \cdot 10^{-4}$ (3.86 · 10 <sup>-5</sup> )
$v \in \{1/2, \dots, \infty\}$	$2.28 \cdot 10^{-5}$ (7.10 · 10 <sup>-7</sup> )	$3.44 \cdot 10^{-5}$ (3.31 · 10 <sup>-6</sup> )	$2.29 \cdot 10^{-5}$ (9.76 · 10 <sup>-7</sup> )	$2.32 \cdot 10^{-5}$ (1.01 · 10 <sup>-6</sup> )	$2.31 \cdot 10^{-1}$ (1.03 · 10 <sup>-2</sup> )	$2.68 \cdot 10^{-5}$ (1.44 · 10 <sup>-6</sup> )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.21 \cdot 10^{-2}$ (6.54 · 10 <sup>-4</sup> )
$v = 3/2$	$6.47 \cdot 10^{-5}$ (2.02 · 10 <sup>-6</sup> )
$v = 5/2$	$1.74 \cdot 10^{-5}$ (7.30 · 10 <sup>-7</sup> )
$v = 7/2$	$2.11 \cdot 10^{-5}$ (7.61 · 10 <sup>-7</sup> )
$v = 9/2$	$2.90 \cdot 10^{-5}$ (1.14 · 10 <sup>-6</sup> )
$v = 13/2$	$3.67 \cdot 10^{-5}$ (1.39 · 10 <sup>-6</sup> )
$v = 25/2$	$4.55 \cdot 10^{-5}$ (1.67 · 10 <sup>-6</sup> )
$v = \infty$	$5.31 \cdot 10^{-5}$ (1.90 · 10 <sup>-6</sup> )
$v \in \{1/2, \dots, \infty\}$	$1.68 \cdot 10^{-5}$ (7.01 · 10 <sup>-7</sup> )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$2.50 \cdot 10^{-5}$ (1.02 · 10 <sup>-6</sup> )
linear	$6.23 \cdot 10^{-1}$ (3.57 · 10 <sup>-3</sup> )
1NN	$4.35 \cdot 10^{-1}$ (2.40 · 10 <sup>-3</sup> )
2NN	$3.40 \cdot 10^{-1}$ (3.35 · 10 <sup>-3</sup> )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$8.71 \cdot 10^{-2}$ (5.88 · 10 <sup>-4</sup> )	$7.29 \cdot 10^{-2}$ (1.03 · 10 <sup>-3</sup> )	$7.00 \cdot 10^{-2}$ (9.59 · 10 <sup>-4</sup> )	$6.95 \cdot 10^{-2}$ (1.08 · 10 <sup>-3</sup> )	$3.08 \cdot 10^{-1}$ (3.39 · 10 <sup>-3</sup> )	$6.89 \cdot 10^{-2}$ (9.53 · 10 <sup>-4</sup> )
$v = 3/2$	$4.31 \cdot 10^{-3}$ (3.98 · 10 <sup>-5</sup> )	$4.11 \cdot 10^{-3}$ (1.26 · 10 <sup>-4</sup> )	$3.57 \cdot 10^{-3}$ (6.59 · 10 <sup>-5</sup> )	$3.66 \cdot 10^{-3}$ (7.80 · 10 <sup>-5</sup> )	$2.82 \cdot 10^{-1}$ (4.94 · 10 <sup>-3</sup> )	$3.54 \cdot 10^{-3}$ (5.35 · 10 <sup>-5</sup> )
$v = 5/2$	$1.90 \cdot 10^{-3}$ (1.94 · 10 <sup>-5</sup> )	$1.87 \cdot 10^{-3}$ (6.37 · 10 <sup>-5</sup> )	$1.59 \cdot 10^{-3}$ (2.74 · 10 <sup>-5</sup> )	$1.63 \cdot 10^{-3}$ (3.15 · 10 <sup>-5</sup> )	$2.75 \cdot 10^{-1}$ (5.57 · 10 <sup>-3</sup> )	$1.66 \cdot 10^{-3}$ (3.05 · 10 <sup>-5</sup> )
$v = 7/2$	$2.10 \cdot 10^{-3}$ (3.32 · 10 <sup>-5</sup> )	$2.28 \cdot 10^{-3}$ (8.58 · 10 <sup>-5</sup> )	$1.94 \cdot 10^{-3}$ (4.03 · 10 <sup>-5</sup> )	$1.98 \cdot 10^{-3}$ (4.46 · 10 <sup>-5</sup> )	$2.65 \cdot 10^{-1}$ (5.00 · 10 <sup>-3</sup> )	$2.24 \cdot 10^{-3}$ (7.31 · 10 <sup>-5</sup> )
$v = 9/2$	$2.61 \cdot 10^{-3}$ (6.21 · 10 <sup>-5</sup> )	$2.88 \cdot 10^{-3}$ (1.08 · 10 <sup>-4</sup> )	$2.51 \cdot 10^{-3}$ (7.25 · 10 <sup>-5</sup> )	$2.60 \cdot 10^{-3}$ (7.39 · 10 <sup>-5</sup> )	$2.55 \cdot 10^{-1}$ (4.79 · 10 <sup>-3</sup> )	$3.00 \cdot 10^{-3}$ (1.40 · 10 <sup>-4</sup> )
$v = 13/2$	$3.04 \cdot 10^{-3}$ (8.58 · 10 <sup>-5</sup> )	$3.45 \cdot 10^{-3}$ (1.29 · 10 <sup>-4</sup> )	$3.09 \cdot 10^{-3}$ (9.90 · 10 <sup>-5</sup> )	$3.27 \cdot 10^{-3}$ (1.16 · 10 <sup>-4</sup> )	$2.45 \cdot 10^{-1}$ (4.87 · 10 <sup>-3</sup> )	$3.61 \cdot 10^{-3}$ (1.61 · 10 <sup>-4</sup> )
$v = 25/2$	$3.32 \cdot 10^{-3}$ (9.19 · 10 <sup>-5</sup> )	$3.94 \cdot 10^{-3}$ (1.41 · 10 <sup>-4</sup> )	$3.66 \cdot 10^{-3}$ (1.31 · 10 <sup>-4</sup> )	$3.78 \cdot 10^{-3}$ (1.35 · 10 <sup>-4</sup> )	$2.34 \cdot 10^{-1}$ (5.11 · 10 <sup>-3</sup> )	$4.61 \cdot 10^{-3}$ (2.33 · 10 <sup>-4</sup> )
$v = \infty$	$3.57 \cdot 10^{-3}$ (9.76 · 10 <sup>-5</sup> )	$4.40 \cdot 10^{-3}$ (1.53 · 10 <sup>-4</sup> )	$4.07 \cdot 10^{-3}$ (1.40 · 10 <sup>-4</sup> )	$4.18 \cdot 10^{-3}$ (1.46 · 10 <sup>-4</sup> )	$2.21 \cdot 10^{-1}$ (5.44 · 10 <sup>-3</sup> )	$5.14 \cdot 10^{-3}$ (2.64 · 10 <sup>-4</sup> )
$v \in \{1/2, \dots, \infty\}$	$1.90 \cdot 10^{-3}$ (2.02 · 10 <sup>-5</sup> )	$2.07 \cdot 10^{-3}$ (8.97 · 10 <sup>-5</sup> )	$1.63 \cdot 10^{-3}$ (2.74 · 10 <sup>-5</sup> )	$1.68 \cdot 10^{-3}$ (3.28 · 10 <sup>-5</sup> )	$2.26 \cdot 10^{-1}$ (5.79 · 10 <sup>-3</sup> )	$1.79 \cdot 10^{-3}$ (4.89 · 10 <sup>-5</sup> )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.98 \cdot 10^{-3}$ (3.63 · 10 <sup>-5</sup> )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.05 \cdot 10^0 (2.19 \cdot 10^{-3})$	$9.32 \cdot 10^{-1} (1.96 \cdot 10^{-2})$	$9.02 \cdot 10^{-1} (1.89 \cdot 10^{-2})$	$1.03 \cdot 10^0 (2.44 \cdot 10^{-2})$	$3.09 \cdot 10^0 (3.91 \cdot 10^{-2})$	$8.88 \cdot 10^{-1} (1.86 \cdot 10^{-2})$
$v = 3/2$	$5.05 \cdot 10^{-2} (2.14 \cdot 10^{-4})$	$4.65 \cdot 10^{-2} (2.12 \cdot 10^{-3})$	$3.97 \cdot 10^{-2} (1.27 \cdot 10^{-3})$	$4.09 \cdot 10^{-2} (1.45 \cdot 10^{-3})$	$2.88 \cdot 10^0 (5.27 \cdot 10^{-2})$	$3.91 \cdot 10^{-2} (1.05 \cdot 10^{-3})$
$v = 5/2$	$2.31 \cdot 10^{-2} (3.64 \cdot 10^{-4})$	$2.45 \cdot 10^{-2} (1.04 \cdot 10^{-3})$	$2.04 \cdot 10^{-2} (7.92 \cdot 10^{-4})$	$2.15 \cdot 10^{-2} (8.18 \cdot 10^{-4})$	$2.80 \cdot 10^0 (5.97 \cdot 10^{-2})$	$2.11 \cdot 10^{-2} (8.25 \cdot 10^{-4})$
$v = 7/2$	$2.47 \cdot 10^{-2} (4.84 \cdot 10^{-4})$	$2.79 \cdot 10^{-2} (1.28 \cdot 10^{-3})$	$2.32 \cdot 10^{-2} (8.87 \cdot 10^{-4})$	$2.42 \cdot 10^{-2} (9.35 \cdot 10^{-4})$	$2.65 \cdot 10^0 (5.04 \cdot 10^{-2})$	$2.66 \cdot 10^{-2} (1.07 \cdot 10^{-3})$
$v = 9/2$	$3.00 \cdot 10^{-2} (7.58 \cdot 10^{-4})$	$3.47 \cdot 10^{-2} (1.79 \cdot 10^{-3})$	$3.03 \cdot 10^{-2} (1.49 \cdot 10^{-3})$	$3.18 \cdot 10^{-2} (1.43 \cdot 10^{-3})$	$2.53 \cdot 10^0 (4.99 \cdot 10^{-2})$	$3.75 \cdot 10^{-2} (2.49 \cdot 10^{-3})$
$v = 13/2$	$3.57 \cdot 10^{-2} (1.23 \cdot 10^{-3})$	$4.17 \cdot 10^{-2} (2.29 \cdot 10^{-3})$	$3.75 \cdot 10^{-2} (1.93 \cdot 10^{-3})$	$4.26 \cdot 10^{-2} (2.49 \cdot 10^{-3})$	$2.41 \cdot 10^0 (5.20 \cdot 10^{-2})$	$4.54 \cdot 10^{-2} (2.87 \cdot 10^{-3})$
$v = 25/2$	$3.97 \cdot 10^{-2} (1.52 \cdot 10^{-3})$	$4.73 \cdot 10^{-2} (2.64 \cdot 10^{-3})$	$4.66 \cdot 10^{-2} (2.87 \cdot 10^{-3})$	$5.03 \cdot 10^{-2} (3.06 \cdot 10^{-3})$	$2.28 \cdot 10^0 (5.50 \cdot 10^{-2})$	$6.14 \cdot 10^{-2} (4.39 \cdot 10^{-3})$
$v = \infty$	$4.39 \cdot 10^{-2} (1.77 \cdot 10^{-3})$	$5.33 \cdot 10^{-2} (2.85 \cdot 10^{-3})$	$5.19 \cdot 10^{-2} (3.07 \cdot 10^{-3})$	$5.66 \cdot 10^{-2} (3.45 \cdot 10^{-3})$	$2.12 \cdot 10^0 (5.79 \cdot 10^{-2})$	$6.99 \cdot 10^{-2} (5.25 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$2.32 \cdot 10^{-2} (3.80 \cdot 10^{-4})$	$2.73 \cdot 10^{-2} (1.33 \cdot 10^{-3})$	$2.10 \cdot 10^{-2} (8.02 \cdot 10^{-4})$	$2.20 \cdot 10^{-2} (8.19 \cdot 10^{-4})$	$2.18 \cdot 10^0 (6.22 \cdot 10^{-2})$	$2.27 \cdot 10^{-2} (1.03 \cdot 10^{-3})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$2.41 \cdot 10^{-2} (5.53 \cdot 10^{-4})$

**SM-2.2.38**  $\text{Re}(S_{2\pi}), n = 120, d = 6$

**Averages of  $R(\theta; S^{\text{SPE}})$**

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.08 \cdot 10^{-2} (3.12 \cdot 10^{-4})$	$1.12 \cdot 10^{-2} (3.21 \cdot 10^{-4})$	$1.07 \cdot 10^{-2} (3.09 \cdot 10^{-4})$	$1.06 \cdot 10^{-2} (3.05 \cdot 10^{-4})$	$2.16 \cdot 10^{-1} (4.39 \cdot 10^{-3})$	$1.04 \cdot 10^{-2} (3.03 \cdot 10^{-4})$
$v = 3/2$	$3.27 \cdot 10^{-5} (1.07 \cdot 10^{-6})$	$3.11 \cdot 10^{-5} (1.08 \cdot 10^{-6})$	$2.78 \cdot 10^{-5} (9.29 \cdot 10^{-7})$	$2.85 \cdot 10^{-5} (9.75 \cdot 10^{-7})$	$1.71 \cdot 10^{-1} (4.84 \cdot 10^{-3})$	$2.79 \cdot 10^{-5} (9.25 \cdot 10^{-7})$
$v = 5/2$	$1.17 \cdot 10^{-5} (5.81 \cdot 10^{-7})$	$1.23 \cdot 10^{-5} (6.41 \cdot 10^{-7})$	$1.29 \cdot 10^{-5} (6.14 \cdot 10^{-7})$	$1.14 \cdot 10^{-5} (5.26 \cdot 10^{-7})$	$1.59 \cdot 10^{-1} (5.00 \cdot 10^{-3})$	$1.66 \cdot 10^{-5} (1.18 \cdot 10^{-6})$
$v = 7/2$	$1.54 \cdot 10^{-5} (5.81 \cdot 10^{-7})$	$1.80 \cdot 10^{-5} (8.56 \cdot 10^{-7})$	$1.81 \cdot 10^{-5} (7.94 \cdot 10^{-7})$	$1.62 \cdot 10^{-5} (6.91 \cdot 10^{-7})$	$1.36 \cdot 10^{-1} (4.01 \cdot 10^{-3})$	$2.32 \cdot 10^{-5} (1.22 \cdot 10^{-6})$
$v = 9/2$	$1.90 \cdot 10^{-5} (6.24 \cdot 10^{-7})$	$2.32 \cdot 10^{-5} (1.06 \cdot 10^{-6})$	$2.30 \cdot 10^{-5} (1.04 \cdot 10^{-6})$	$2.11 \cdot 10^{-5} (9.76 \cdot 10^{-7})$	$1.23 \cdot 10^{-1} (3.80 \cdot 10^{-3})$	$3.13 \cdot 10^{-5} (1.97 \cdot 10^{-6})$
$v = 13/2$	$2.36 \cdot 10^{-5} (7.45 \cdot 10^{-7})$	$3.07 \cdot 10^{-5} (1.87 \cdot 10^{-6})$	$2.90 \cdot 10^{-5} (1.46 \cdot 10^{-6})$	$2.68 \cdot 10^{-5} (1.34 \cdot 10^{-6})$	$1.12 \cdot 10^{-1} (3.71 \cdot 10^{-3})$	$4.00 \cdot 10^{-5} (2.94 \cdot 10^{-6})$
$v = 25/2$	$2.60 \cdot 10^{-5} (8.71 \cdot 10^{-7})$	$3.30 \cdot 10^{-5} (2.53 \cdot 10^{-6})$	$3.04 \cdot 10^{-5} (1.55 \cdot 10^{-6})$	$2.97 \cdot 10^{-5} (1.55 \cdot 10^{-6})$	$1.03 \cdot 10^{-1} (3.62 \cdot 10^{-3})$	$4.11 \cdot 10^{-5} (3.02 \cdot 10^{-6})$
$v = \infty$	$2.94 \cdot 10^{-5} (9.91 \cdot 10^{-7})$	$3.77 \cdot 10^{-5} (3.29 \cdot 10^{-6})$	$3.33 \cdot 10^{-5} (1.65 \cdot 10^{-6})$	$3.18 \cdot 10^{-5} (1.59 \cdot 10^{-6})$	$9.23 \cdot 10^{-2} (3.51 \cdot 10^{-3})$	$4.31 \cdot 10^{-5} (3.20 \cdot 10^{-6})$
$v \in \{1/2, \dots, \infty\}$	$1.17 \cdot 10^{-5} (5.81 \cdot 10^{-7})$	$1.27 \cdot 10^{-5} (6.91 \cdot 10^{-7})$	$1.29 \cdot 10^{-5} (6.17 \cdot 10^{-7})$	$1.15 \cdot 10^{-5} (5.30 \cdot 10^{-7})$	$9.23 \cdot 10^{-2} (3.51 \cdot 10^{-3})$	$1.74 \cdot 10^{-5} (1.16 \cdot 10^{-6})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$9.74 \cdot 10^{-3} (2.84 \cdot 10^{-4})$
$v = 3/2$	$2.45 \cdot 10^{-5} (8.20 \cdot 10^{-7})$
$v = 5/2$	$8.68 \cdot 10^{-6} (4.00 \cdot 10^{-7})$
$v = 7/2$	$1.26 \cdot 10^{-5} (4.51 \cdot 10^{-7})$
$v = 9/2$	$1.60 \cdot 10^{-5} (5.20 \cdot 10^{-7})$
$v = 13/2$	$1.99 \cdot 10^{-5} (6.24 \cdot 10^{-7})$
$v = 25/2$	$2.16 \cdot 10^{-5} (7.03 \cdot 10^{-7})$
$v = \infty$	$2.43 \cdot 10^{-5} (8.00 \cdot 10^{-7})$
$v \in \{1/2, \dots, \infty\}$	$8.68 \cdot 10^{-6} (4.00 \cdot 10^{-7})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.21 \cdot 10^{-5} (6.15 \cdot 10^{-7})$
linear	$5.94 \cdot 10^{-1} (1.70 \cdot 10^{-3})$
INN	$3.47 \cdot 10^{-1} (1.43 \cdot 10^{-3})$
2NN	$2.64 \cdot 10^{-1} (1.89 \cdot 10^{-3})$

**Averages of  $R(\theta; S^{\text{CRPS}})$**

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.67 \cdot 10^{-2}$ ( $3.19 \cdot 10^{-4}$ )	$4.45 \cdot 10^{-2}$ ( $6.07 \cdot 10^{-4}$ )	$4.33 \cdot 10^{-2}$ ( $5.72 \cdot 10^{-4}$ )	$4.24 \cdot 10^{-2}$ ( $6.35 \cdot 10^{-4}$ )	$2.34 \cdot 10^{-1}$ ( $2.81 \cdot 10^{-3}$ )	$4.27 \cdot 10^{-2}$ ( $5.57 \cdot 10^{-4}$ )
$v = 3/2$	$2.60 \cdot 10^{-3}$ ( $1.61 \cdot 10^{-5}$ )	$2.09 \cdot 10^{-3}$ ( $2.84 \cdot 10^{-5}$ )	$1.96 \cdot 10^{-3}$ ( $2.27 \cdot 10^{-5}$ )	$1.98 \cdot 10^{-3}$ ( $2.30 \cdot 10^{-5}$ )	$1.97 \cdot 10^{-1}$ ( $3.79 \cdot 10^{-3}$ )	$1.96 \cdot 10^{-3}$ ( $2.25 \cdot 10^{-5}$ )
$v = 5/2$	$1.24 \cdot 10^{-3}$ ( $1.19 \cdot 10^{-5}$ )	$1.11 \cdot 10^{-3}$ ( $1.97 \cdot 10^{-5}$ )	$1.06 \cdot 10^{-3}$ ( $1.77 \cdot 10^{-5}$ )	$1.04 \cdot 10^{-3}$ ( $1.57 \cdot 10^{-5}$ )	$1.86 \cdot 10^{-1}$ ( $4.13 \cdot 10^{-3}$ )	$1.16 \cdot 10^{-3}$ ( $2.77 \cdot 10^{-5}$ )
$v = 7/2$	$1.39 \cdot 10^{-3}$ ( $1.40 \cdot 10^{-5}$ )	$1.37 \cdot 10^{-3}$ ( $2.60 \cdot 10^{-5}$ )	$1.31 \cdot 10^{-3}$ ( $2.15 \cdot 10^{-5}$ )	$1.29 \cdot 10^{-3}$ ( $2.03 \cdot 10^{-5}$ )	$1.62 \cdot 10^{-1}$ ( $3.00 \cdot 10^{-3}$ )	$1.47 \cdot 10^{-3}$ ( $3.02 \cdot 10^{-5}$ )
$v = 9/2$	$1.56 \cdot 10^{-3}$ ( $1.86 \cdot 10^{-5}$ )	$1.60 \cdot 10^{-3}$ ( $3.11 \cdot 10^{-5}$ )	$1.53 \cdot 10^{-3}$ ( $2.67 \cdot 10^{-5}$ )	$1.51 \cdot 10^{-3}$ ( $2.85 \cdot 10^{-5}$ )	$1.47 \cdot 10^{-1}$ ( $2.68 \cdot 10^{-3}$ )	$1.75 \cdot 10^{-3}$ ( $4.25 \cdot 10^{-5}$ )
$v = 13/2$	$1.76 \cdot 10^{-3}$ ( $2.59 \cdot 10^{-5}$ )	$1.86 \cdot 10^{-3}$ ( $4.56 \cdot 10^{-5}$ )	$1.77 \cdot 10^{-3}$ ( $3.57 \cdot 10^{-5}$ )	$1.74 \cdot 10^{-3}$ ( $3.56 \cdot 10^{-5}$ )	$1.35 \cdot 10^{-1}$ ( $2.61 \cdot 10^{-3}$ )	$2.04 \cdot 10^{-3}$ ( $5.94 \cdot 10^{-5}$ )
$v = 25/2$	$1.86 \cdot 10^{-3}$ ( $3.11 \cdot 10^{-5}$ )	$1.94 \cdot 10^{-3}$ ( $5.19 \cdot 10^{-5}$ )	$1.86 \cdot 10^{-3}$ ( $4.12 \cdot 10^{-5}$ )	$1.86 \cdot 10^{-3}$ ( $4.30 \cdot 10^{-5}$ )	$1.24 \cdot 10^{-1}$ ( $2.60 \cdot 10^{-3}$ )	$2.12 \cdot 10^{-3}$ ( $6.28 \cdot 10^{-5}$ )
$v = \infty$	$2.00 \cdot 10^{-3}$ ( $3.55 \cdot 10^{-5}$ )	$2.08 \cdot 10^{-3}$ ( $5.84 \cdot 10^{-5}$ )	$1.99 \cdot 10^{-3}$ ( $4.57 \cdot 10^{-5}$ )	$1.97 \cdot 10^{-3}$ ( $4.50 \cdot 10^{-5}$ )	$1.14 \cdot 10^{-1}$ ( $2.60 \cdot 10^{-3}$ )	$2.22 \cdot 10^{-3}$ ( $6.61 \cdot 10^{-5}$ )
$v \in \{1/2, \dots, \infty\}$	$1.24 \cdot 10^{-3}$ ( $1.19 \cdot 10^{-5}$ )	$1.13 \cdot 10^{-3}$ ( $2.32 \cdot 10^{-5}$ )	$1.06 \cdot 10^{-3}$ ( $1.79 \cdot 10^{-5}$ )	$1.04 \cdot 10^{-3}$ ( $1.58 \cdot 10^{-5}$ )	$1.14 \cdot 10^{-1}$ ( $2.60 \cdot 10^{-3}$ )	$1.21 \cdot 10^{-3}$ ( $3.24 \cdot 10^{-5}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.26 \cdot 10^{-3}$ ( $1.42 \cdot 10^{-5}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$7.37 \cdot 10^{-1}$ ( $1.47 \cdot 10^{-3}$ )	$6.46 \cdot 10^{-1}$ ( $1.32 \cdot 10^{-2}$ )	$6.29 \cdot 10^{-1}$ ( $1.29 \cdot 10^{-2}$ )	$7.32 \cdot 10^{-1}$ ( $1.70 \cdot 10^{-2}$ )	$2.56 \cdot 10^0$ ( $3.34 \cdot 10^{-2}$ )	$6.18 \cdot 10^{-1}$ ( $1.26 \cdot 10^{-2}$ )
$v = 3/2$	$3.23 \cdot 10^{-2}$ ( $2.33 \cdot 10^{-4}$ )	$2.71 \cdot 10^{-2}$ ( $7.95 \cdot 10^{-4}$ )	$2.52 \cdot 10^{-2}$ ( $7.13 \cdot 10^{-4}$ )	$2.57 \cdot 10^{-2}$ ( $7.73 \cdot 10^{-4}$ )	$2.20 \cdot 10^0$ ( $3.90 \cdot 10^{-2}$ )	$2.49 \cdot 10^{-2}$ ( $6.91 \cdot 10^{-4}$ )
$v = 5/2$	$1.61 \cdot 10^{-2}$ ( $4.52 \cdot 10^{-4}$ )	$1.54 \cdot 10^{-2}$ ( $5.96 \cdot 10^{-4}$ )	$1.46 \cdot 10^{-2}$ ( $5.25 \cdot 10^{-4}$ )	$1.42 \cdot 10^{-2}$ ( $5.58 \cdot 10^{-4}$ )	$2.04 \cdot 10^0$ ( $4.11 \cdot 10^{-2}$ )	$1.58 \cdot 10^{-2}$ ( $5.98 \cdot 10^{-4}$ )
$v = 7/2$	$1.69 \cdot 10^{-2}$ ( $4.78 \cdot 10^{-4}$ )	$1.69 \cdot 10^{-2}$ ( $6.05 \cdot 10^{-4}$ )	$1.59 \cdot 10^{-2}$ ( $5.28 \cdot 10^{-4}$ )	$1.55 \cdot 10^{-2}$ ( $5.50 \cdot 10^{-4}$ )	$1.80 \cdot 10^0$ ( $4.24 \cdot 10^{-2}$ )	$1.81 \cdot 10^{-2}$ ( $6.60 \cdot 10^{-4}$ )
$v = 9/2$	$1.82 \cdot 10^{-2}$ ( $4.70 \cdot 10^{-4}$ )	$1.87 \cdot 10^{-2}$ ( $6.02 \cdot 10^{-4}$ )	$1.77 \cdot 10^{-2}$ ( $5.32 \cdot 10^{-4}$ )	$1.75 \cdot 10^{-2}$ ( $5.70 \cdot 10^{-4}$ )	$1.69 \cdot 10^0$ ( $4.45 \cdot 10^{-2}$ )	$2.07 \cdot 10^{-2}$ ( $7.29 \cdot 10^{-4}$ )
$v = 13/2$	$2.00 \cdot 10^{-2}$ ( $4.48 \cdot 10^{-4}$ )	$2.11 \cdot 10^{-2}$ ( $6.19 \cdot 10^{-4}$ )	$2.01 \cdot 10^{-2}$ ( $5.22 \cdot 10^{-4}$ )	$1.98 \cdot 10^{-2}$ ( $5.76 \cdot 10^{-4}$ )	$1.60 \cdot 10^0$ ( $4.55 \cdot 10^{-2}$ )	$2.35 \cdot 10^{-2}$ ( $7.69 \cdot 10^{-4}$ )
$v = 25/2$	$2.11 \cdot 10^{-2}$ ( $4.78 \cdot 10^{-4}$ )	$2.18 \cdot 10^{-2}$ ( $6.26 \cdot 10^{-4}$ )	$2.07 \cdot 10^{-2}$ ( $4.97 \cdot 10^{-4}$ )	$2.15 \cdot 10^{-2}$ ( $6.19 \cdot 10^{-4}$ )	$1.48 \cdot 10^0$ ( $4.55 \cdot 10^{-2}$ )	$2.40 \cdot 10^{-2}$ ( $7.61 \cdot 10^{-4}$ )
$v = \infty$	$2.24 \cdot 10^{-2}$ ( $4.84 \cdot 10^{-4}$ )	$2.29 \cdot 10^{-2}$ ( $6.35 \cdot 10^{-4}$ )	$2.18 \cdot 10^{-2}$ ( $5.10 \cdot 10^{-4}$ )	$2.23 \cdot 10^{-2}$ ( $5.67 \cdot 10^{-4}$ )	$1.35 \cdot 10^0$ ( $4.42 \cdot 10^{-2}$ )	$2.43 \cdot 10^{-2}$ ( $7.03 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$1.61 \cdot 10^{-2}$ ( $4.52 \cdot 10^{-4}$ )	$1.55 \cdot 10^{-2}$ ( $5.99 \cdot 10^{-4}$ )	$1.47 \cdot 10^{-2}$ ( $5.32 \cdot 10^{-4}$ )	$1.42 \cdot 10^{-2}$ ( $5.56 \cdot 10^{-4}$ )	$1.35 \cdot 10^0$ ( $4.42 \cdot 10^{-2}$ )	$1.63 \cdot 10^{-2}$ ( $6.58 \cdot 10^{-4}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.62 \cdot 10^{-2}$ ( $4.58 \cdot 10^{-4}$ )

### SM-2.2.39 $\text{Re}(S_{2\pi}), n = 300, d = 6$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.32 \cdot 10^{-3}$ ( $7.25 \cdot 10^{-5}$ )	$3.53 \cdot 10^{-3}$ ( $8.00 \cdot 10^{-5}$ )	$3.32 \cdot 10^{-3}$ ( $7.17 \cdot 10^{-5}$ )	$3.28 \cdot 10^{-3}$ ( $7.28 \cdot 10^{-5}$ )	$1.01 \cdot 10^{-1}$ ( $2.32 \cdot 10^{-3}$ )	$3.24 \cdot 10^{-3}$ ( $7.09 \cdot 10^{-5}$ )
$v = 3/2$	$7.56 \cdot 10^{-6}$ ( $1.75 \cdot 10^{-7}$ )	$6.74 \cdot 10^{-6}$ ( $1.70 \cdot 10^{-7}$ )	$6.65 \cdot 10^{-6}$ ( $1.72 \cdot 10^{-7}$ )	$6.54 \cdot 10^{-6}$ ( $1.66 \cdot 10^{-7}$ )	$7.33 \cdot 10^{-2}$ ( $2.45 \cdot 10^{-3}$ )	$7.07 \cdot 10^{-6}$ ( $1.85 \cdot 10^{-7}$ )
$v = 5/2$	$3.42 \cdot 10^{-6}$ ( $9.20 \cdot 10^{-8}$ )	$4.32 \cdot 10^{-6}$ ( $1.58 \cdot 10^{-7}$ )	$6.30 \cdot 10^{-6}$ ( $3.59 \cdot 10^{-7}$ )	$4.39 \cdot 10^{-6}$ ( $2.20 \cdot 10^{-7}$ )	$6.20 \cdot 10^{-2}$ ( $2.14 \cdot 10^{-3}$ )	$9.93 \cdot 10^{-6}$ ( $8.50 \cdot 10^{-7}$ )
$v = 7/2$	$7.07 \cdot 10^{-6}$ ( $2.15 \cdot 10^{-7}$ )	$9.02 \cdot 10^{-6}$ ( $3.59 \cdot 10^{-7}$ )	$1.32 \cdot 10^{-5}$ ( $6.84 \cdot 10^{-7}$ )	$9.32 \cdot 10^{-6}$ ( $4.02 \cdot 10^{-7}$ )	$5.00 \cdot 10^{-2}$ ( $1.48 \cdot 10^{-3}$ )	$2.10 \cdot 10^{-5}$ ( $1.41 \cdot 10^{-6}$ )
$v = 9/2$	$1.15 \cdot 10^{-5}$ ( $3.39 \cdot 10^{-7}$ )	$1.50 \cdot 10^{-5}$ ( $6.19 \cdot 10^{-7}$ )	$2.12 \cdot 10^{-5}$ ( $1.03 \cdot 10^{-6}$ )	$1.45 \cdot 10^{-5}$ ( $5.26 \cdot 10^{-7}$ )	$4.52 \cdot 10^{-2}$ ( $1.35 \cdot 10^{-3}$ )	$3.36 \cdot 10^{-5}$ ( $2.45 \cdot 10^{-6}$ )
$v = 13/2$	$1.97 \cdot 10^{-5}$ ( $5.79 \cdot 10^{-7}$ )	$2.44 \cdot 10^{-5}$ ( $1.17 \cdot 10^{-6}$ )	$3.10 \cdot 10^{-5}$ ( $1.63 \cdot 10^{-6}$ )	$2.29 \cdot 10^{-5}$ ( $7.41 \cdot 10^{-7}$ )	$4.06 \cdot 10^{-2}$ ( $1.26 \cdot 10^{-3}$ )	$4.78 \cdot 10^{-5}$ ( $3.56 \cdot 10^{-6}$ )
$v = 25/2$	$2.97 \cdot 10^{-5}$ ( $8.82 \cdot 10^{-7}$ )	$3.75 \cdot 10^{-5}$ ( $2.04 \cdot 10^{-6}$ )	$4.16 \cdot 10^{-5}$ ( $2.26 \cdot 10^{-6}$ )	$3.41 \cdot 10^{-5}$ ( $1.25 \cdot 10^{-6}$ )	$3.58 \cdot 10^{-2}$ ( $1.16 \cdot 10^{-3}$ )	$6.87 \cdot 10^{-5}$ ( $6.69 \cdot 10^{-6}$ )
$v = \infty$	$3.93 \cdot 10^{-5}$ ( $1.16 \cdot 10^{-6}$ )	$4.92 \cdot 10^{-5}$ ( $2.65 \cdot 10^{-6}$ )	$5.26 \cdot 10^{-5}$ ( $2.81 \cdot 10^{-6}$ )	$4.37 \cdot 10^{-5}$ ( $1.36 \cdot 10^{-6}$ )	$3.02 \cdot 10^{-2}$ ( $1.02 \cdot 10^{-3}$ )	$9.28 \cdot 10^{-5}$ ( $1.04 \cdot 10^{-5}$ )
$v \in \{1/2, \dots, \infty\}$	$3.42 \cdot 10^{-6}$ ( $9.20 \cdot 10^{-8}$ )	$4.37 \cdot 10^{-6}$ ( $1.58 \cdot 10^{-7}$ )	$6.30 \cdot 10^{-6}$ ( $3.59 \cdot 10^{-7}$ )	$4.39 \cdot 10^{-6}$ ( $2.20 \cdot 10^{-7}$ )	$3.02 \cdot 10^{-2}$ ( $1.02 \cdot 10^{-3}$ )	$1.01 \cdot 10^{-5}$ ( $8.50 \cdot 10^{-7}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$3.09 \cdot 10^{-3}$ ( $6.90 \cdot 10^{-5}$ )
$v = 3/2$	$5.89 \cdot 10^{-6}$ ( $1.34 \cdot 10^{-7}$ )
$v = 5/2$	$2.96 \cdot 10^{-6}$ ( $8.28 \cdot 10^{-8}$ )
$v = 7/2$	$6.24 \cdot 10^{-6}$ ( $1.69 \cdot 10^{-7}$ )
$v = 9/2$	$1.07 \cdot 10^{-5}$ ( $2.99 \cdot 10^{-7}$ )
$v = 13/2$	$1.89 \cdot 10^{-5}$ ( $5.51 \cdot 10^{-7}$ )
$v = 25/2$	$2.83 \cdot 10^{-5}$ ( $8.24 \cdot 10^{-7}$ )
$v = \infty$	$3.68 \cdot 10^{-5}$ ( $1.02 \cdot 10^{-6}$ )
$v \in \{1/2, \dots, \infty\}$	$2.95 \cdot 10^{-6}$ ( $8.17 \cdot 10^{-8}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$3.47 \cdot 10^{-6}$ ( $1.09 \cdot 10^{-7}$ )
linear	$5.75 \cdot 10^{-1}$ ( $5.55 \cdot 10^{-4}$ )
1NN	$2.57 \cdot 10^{-1}$ ( $6.37 \cdot 10^{-4}$ )
2NN	$1.88 \cdot 10^{-1}$ ( $8.83 \cdot 10^{-4}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$3.24 \cdot 10^{-2}$ ( $1.01 \cdot 10^{-4}$ )	$2.26 \cdot 10^{-2}$ ( $2.29 \cdot 10^{-4}$ )	$2.19 \cdot 10^{-2}$ ( $2.01 \cdot 10^{-4}$ )	$2.08 \cdot 10^{-2}$ ( $2.40 \cdot 10^{-4}$ )	$1.48 \cdot 10^{-1}$ ( $2.18 \cdot 10^{-3}$ )	$2.17 \cdot 10^{-2}$ ( $1.98 \cdot 10^{-4}$ )
$v = 3/2$	$1.21 \cdot 10^{-3}$ ( $5.67 \cdot 10^{-6}$ )	$9.09 \cdot 10^{-4}$ ( $8.75 \cdot 10^{-6}$ )	$8.88 \cdot 10^{-4}$ ( $8.37 \cdot 10^{-6}$ )	$8.89 \cdot 10^{-4}$ ( $8.25 \cdot 10^{-6}$ )	$1.14 \cdot 10^{-1}$ ( $3.13 \cdot 10^{-3}$ )	$9.08 \cdot 10^{-4}$ ( $8.63 \cdot 10^{-6}$ )
$v = 5/2$	$5.91 \cdot 10^{-4}$ ( $3.59 \cdot 10^{-6}$ )	$5.94 \cdot 10^{-4}$ ( $9.96 \cdot 10^{-6}$ )	$5.53 \cdot 10^{-4}$ ( $9.45 \cdot 10^{-6}$ )	$5.32 \cdot 10^{-4}$ ( $7.38 \cdot 10^{-6}$ )	$9.90 \cdot 10^{-2}$ ( $2.92 \cdot 10^{-3}$ )	$6.59 \cdot 10^{-4}$ ( $1.53 \cdot 10^{-5}$ )
$v = 7/2$	$7.64 \cdot 10^{-4}$ ( $6.27 \cdot 10^{-6}$ )	$8.41 \cdot 10^{-4}$ ( $1.51 \cdot 10^{-5}$ )	$8.01 \cdot 10^{-4}$ ( $1.23 \cdot 10^{-5}$ )	$7.63 \cdot 10^{-4}$ ( $9.33 \cdot 10^{-6}$ )	$7.94 \cdot 10^{-2}$ ( $1.55 \cdot 10^{-3}$ )	$1.01 \cdot 10^{-3}$ ( $2.40 \cdot 10^{-5}$ )
$v = 9/2$	$9.89 \cdot 10^{-4}$ ( $1.01 \cdot 10^{-5}$ )	$1.11 \cdot 10^{-3}$ ( $2.32 \cdot 10^{-5}$ )	$1.06 \cdot 10^{-3}$ ( $1.55 \cdot 10^{-5}$ )	$1.01 \cdot 10^{-3}$ ( $1.20 \cdot 10^{-5}$ )	$7.15 \cdot 10^{-2}$ ( $1.32 \cdot 10^{-3}$ )	$1.33 \cdot 10^{-3}$ ( $3.60 \cdot 10^{-5}$ )
$v = 13/2$	$1.36 \cdot 10^{-3}$ ( $1.74 \cdot 10^{-5}$ )	$1.47 \cdot 10^{-3}$ ( $3.09 \cdot 10^{-5}$ )	$1.43 \cdot 10^{-3}$ ( $2.27 \cdot 10^{-5}$ )	$1.36 \cdot 10^{-3}$ ( $1.85 \cdot 10^{-5}$ )	$6.45 \cdot 10^{-2}$ ( $1.24 \cdot 10^{-3}$ )	$1.72 \cdot 10^{-3}$ ( $4.79 \cdot 10^{-5}$ )
$v = 25/2$	$1.74 \cdot 10^{-3}$ ( $2.40 \cdot 10^{-5}$ )	$1.85 \cdot 10^{-3}$ ( $4.05 \cdot 10^{-5}$ )	$1.78 \cdot 10^{-3}$ ( $2.90 \cdot 10^{-5}$ )	$1.73 \cdot 10^{-3}$ ( $2.68 \cdot 10^{-5}$ )	$5.76 \cdot 10^{-2}$ ( $1.18 \cdot 10^{-3}$ )	$2.14 \cdot 10^{-3}$ ( $7.20 \cdot 10^{-5}$ )
$v = \infty$	$2.05 \cdot 10^{-3}$ ( $2.93 \cdot 10^{-5}$ )	$2.16 \cdot 10^{-3}$ ( $4.23 \cdot 10^{-5}$ )	$2.09 \cdot 10^{-3}$ ( $3.39 \cdot 10^{-5}$ )	$2.04 \cdot 10^{-3}$ ( $2.88 \cdot 10^{-5}$ )	$5.06 \cdot 10^{-2}$ ( $1.09 \cdot 10^{-3}$ )	$2.51 \cdot 10^{-3}$ ( $8.36 \cdot 10^{-5}$ )
$v \in \{1/2, \dots, \infty\}$	$5.91 \cdot 10^{-4}$ ( $3.59 \cdot 10^{-6}$ )	$6.03 \cdot 10^{-4}$ ( $1.08 \cdot 10^{-5}$ )	$5.53 \cdot 10^{-4}$ ( $9.45 \cdot 10^{-6}$ )	$5.32 \cdot 10^{-4}$ ( $7.38 \cdot 10^{-6}$ )	$5.06 \cdot 10^{-2}$ ( $1.09 \cdot 10^{-3}$ )	$6.68 \cdot 10^{-4}$ ( $1.59 \cdot 10^{-5}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$5.97 \cdot 10^{-4}$ ( $7.58 \cdot 10^{-6}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.56 \cdot 10^{-1}$ ( $5.87 \cdot 10^{-4}$ )	$3.60 \cdot 10^{-1}$ ( $5.24 \cdot 10^{-3}$ )	$3.46 \cdot 10^{-1}$ ( $4.81 \cdot 10^{-3}$ )	$4.11 \cdot 10^{-1}$ ( $6.66 \cdot 10^{-3}$ )	$1.86 \cdot 10^0$ ( $2.10 \cdot 10^{-2}$ )	$3.41 \cdot 10^{-1}$ ( $4.70 \cdot 10^{-3}$ )
$v = 3/2$	$1.47 \cdot 10^{-2}$ ( $6.93 \cdot 10^{-5}$ )	$1.24 \cdot 10^{-2}$ ( $2.40 \cdot 10^{-4}$ )	$1.22 \cdot 10^{-2}$ ( $2.35 \cdot 10^{-4}$ )	$1.23 \cdot 10^{-2}$ ( $2.33 \cdot 10^{-4}$ )	$1.44 \cdot 10^0$ ( $2.27 \cdot 10^{-2}$ )	$1.24 \cdot 10^{-2}$ ( $2.33 \cdot 10^{-4}$ )
$v = 5/2$	$7.72 \cdot 10^{-3}$ ( $8.08 \cdot 10^{-5}$ )	$8.50 \cdot 10^{-3}$ ( $1.81 \cdot 10^{-4}$ )	$8.12 \cdot 10^{-3}$ ( $2.58 \cdot 10^{-4}$ )	$7.91 \cdot 10^{-3}$ ( $2.02 \cdot 10^{-4}$ )	$1.23 \cdot 10^0$ ( $2.22 \cdot 10^{-2}$ )	$9.83 \cdot 10^{-3}$ ( $3.65 \cdot 10^{-4}$ )
$v = 7/2$	$9.89 \cdot 10^{-3}$ ( $1.24 \cdot 10^{-4}$ )	$1.06 \cdot 10^{-2}$ ( $2.04 \cdot 10^{-4}$ )	$1.01 \cdot 10^{-2}$ ( $2.67 \cdot 10^{-4}$ )	$1.00 \cdot 10^{-2}$ ( $2.20 \cdot 10^{-4}$ )	$1.06 \cdot 10^0$ ( $2.27 \cdot 10^{-2}$ )	$1.31 \cdot 10^{-2}$ ( $4.44 \cdot 10^{-4}$ )
$v = 9/2$	$1.19 \cdot 10^{-2}$ ( $1.62 \cdot 10^{-4}$ )	$1.26 \cdot 10^{-2}$ ( $2.52 \cdot 10^{-4}$ )	$1.19 \cdot 10^{-2}$ ( $2.52 \cdot 10^{-4}$ )	$1.18 \cdot 10^{-2}$ ( $2.05 \cdot 10^{-4}$ )	$9.93 \cdot 10^{-1}$ ( $2.28 \cdot 10^{-2}$ )	$1.56 \cdot 10^{-2}$ ( $5.20 \cdot 10^{-4}$ )
$v = 13/2$	$1.52 \cdot 10^{-2}$ ( $2.50 \cdot 10^{-4}$ )	$1.51 \cdot 10^{-2}$ ( $3.17 \cdot 10^{-4}$ )	$1.45 \cdot 10^{-2}$ ( $2.56 \cdot 10^{-4}$ )	$1.43 \cdot 10^{-2}$ ( $2.40 \cdot 10^{-4}$ )	$9.09 \cdot 10^{-1}$ ( $2.22 \cdot 10^{-2}$ )	$1.82 \cdot 10^{-2}$ ( $5.95 \cdot 10^{-4}$ )
$v = 25/2$	$1.98 \cdot 10^{-2}$ ( $3.77 \cdot 10^{-4}$ )	$1.80 \cdot 10^{-2}$ ( $4.07 \cdot 10^{-4}$ )	$1.71 \cdot 10^{-2}$ ( $3.14 \cdot 10^{-4}$ )	$1.70 \cdot 10^{-2}$ ( $3.15 \cdot 10^{-4}$ )	$8.02 \cdot 10^{-1}$ ( $2.07 \cdot 10^{-2}$ )	$2.11 \cdot 10^{-2}$ ( $8.40 \cdot 10^{-4}$ )
$v = \infty$	$2.46 \cdot 10^{-2}$ ( $5.11 \cdot 10^{-4}$ )	$2.04 \cdot 10^{-2}$ ( $4.28 \cdot 10^{-4}$ )	$1.95 \cdot 10^{-2}$ ( $3.60 \cdot 10^{-4}$ )	$1.96 \cdot 10^{-2}$ ( $3.49 \cdot 10^{-4}$ )	$6.70 \cdot 10^{-1}$ ( $1.84 \cdot 10^{-2}$ )	$2.40 \cdot 10^{-2}$ ( $8.64 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$7.72 \cdot 10^{-3}$ ( $8.08 \cdot 10^{-5}$ )	$8.59 \cdot 10^{-3}$ ( $1.83 \cdot 10^{-4}$ )	$8.12 \cdot 10^{-3}$ ( $2.58 \cdot 10^{-4}$ )	$7.91 \cdot 10^{-3}$ ( $2.02 \cdot 10^{-4}$ )	$6.70 \cdot 10^{-1}$ ( $1.84 \cdot 10^{-2}$ )	$9.90 \cdot 10^{-3}$ ( $3.65 \cdot 10^{-4}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$7.78 \cdot 10^{-3}$ ( $1.06 \cdot 10^{-4}$ )

### SM-2.2.40 $\text{Re}(S_{6\pi}), n = 60, d = 6$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.94 \cdot 10^{-2}$ ( $1.71 \cdot 10^{-3}$ )	$5.11 \cdot 10^{-2}$ ( $1.61 \cdot 10^{-3}$ )	$4.84 \cdot 10^{-2}$ ( $1.63 \cdot 10^{-3}$ )	$4.84 \cdot 10^{-2}$ ( $1.68 \cdot 10^{-3}$ )	$4.29 \cdot 10^{-1}$ ( $8.01 \cdot 10^{-3}$ )	$4.73 \cdot 10^{-2}$ ( $1.64 \cdot 10^{-3}$ )
$v = 3/2$	$3.85 \cdot 10^{-3}$ ( $1.40 \cdot 10^{-4}$ )	$4.24 \cdot 10^{-3}$ ( $1.95 \cdot 10^{-4}$ )	$3.52 \cdot 10^{-3}$ ( $1.31 \cdot 10^{-4}$ )	$3.59 \cdot 10^{-3}$ ( $1.34 \cdot 10^{-4}$ )	$3.82 \cdot 10^{-1}$ ( $1.07 \cdot 10^{-2}$ )	$3.45 \cdot 10^{-3}$ ( $1.26 \cdot 10^{-4}$ )
$v = 5/2$	$4.02 \cdot 10^{-4}$ ( $1.62 \cdot 10^{-5}$ )	$5.11 \cdot 10^{-4}$ ( $2.92 \cdot 10^{-5}$ )	$4.18 \cdot 10^{-4}$ ( $1.42 \cdot 10^{-5}$ )	$4.31 \cdot 10^{-4}$ ( $1.63 \cdot 10^{-5}$ )	$3.71 \cdot 10^{-1}$ ( $1.18 \cdot 10^{-2}$ )	$4.93 \cdot 10^{-4}$ ( $2.33 \cdot 10^{-5}$ )
$v = 7/2$	$2.88 \cdot 10^{-4}$ ( $1.03 \cdot 10^{-5}$ )	$4.33 \cdot 10^{-4}$ ( $2.25 \cdot 10^{-5}$ )	$3.73 \cdot 10^{-4}$ ( $1.60 \cdot 10^{-5}$ )	$3.86 \cdot 10^{-4}$ ( $1.72 \cdot 10^{-5}$ )	$3.54 \cdot 10^{-1}$ ( $1.11 \cdot 10^{-2}$ )	$4.75 \cdot 10^{-4}$ ( $3.44 \cdot 10^{-5}$ )
$v = 9/2$	$2.96 \cdot 10^{-4}$ ( $1.02 \cdot 10^{-5}$ )	$4.39 \cdot 10^{-4}$ ( $1.93 \cdot 10^{-5}$ )	$4.22 \cdot 10^{-4}$ ( $2.56 \cdot 10^{-5}$ )	$3.95 \cdot 10^{-4}$ ( $1.57 \cdot 10^{-5}$ )	$3.42 \cdot 10^{-1}$ ( $1.10 \cdot 10^{-2}$ )	$5.44 \cdot 10^{-4}$ ( $4.85 \cdot 10^{-5}$ )
$v = 13/2$	$3.15 \cdot 10^{-4}$ ( $1.05 \cdot 10^{-5}$ )	$4.80 \cdot 10^{-4}$ ( $2.25 \cdot 10^{-5}$ )	$4.48 \cdot 10^{-4}$ ( $3.07 \cdot 10^{-5}$ )	$4.25 \cdot 10^{-4}$ ( $1.73 \cdot 10^{-5}$ )	$3.29 \cdot 10^{-1}$ ( $1.11 \cdot 10^{-2}$ )	$6.43 \cdot 10^{-4}$ ( $6.64 \cdot 10^{-5}$ )
$v = 25/2$	$3.46 \cdot 10^{-4}$ ( $1.12 \cdot 10^{-5}$ )	$5.29 \cdot 10^{-4}$ ( $3.00 \cdot 10^{-5}$ )	$4.92 \cdot 10^{-4}$ ( $3.61 \cdot 10^{-5}$ )	$4.61 \cdot 10^{-4}$ ( $1.84 \cdot 10^{-5}$ )	$3.14 \cdot 10^{-1}$ ( $1.14 \cdot 10^{-2}$ )	$7.43 \cdot 10^{-4}$ ( $8.57 \cdot 10^{-5}$ )
$v = \infty$	$3.80 \cdot 10^{-4}$ ( $1.21 \cdot 10^{-5}$ )	$6.34 \cdot 10^{-4}$ ( $4.87 \cdot 10^{-5}$ )	$5.38 \cdot 10^{-4}$ ( $4.06 \cdot 10^{-5}$ )	$5.38 \cdot 10^{-4}$ ( $4.18 \cdot 10^{-5}$ )	$2.94 \cdot 10^{-1}$ ( $1.13 \cdot 10^{-2}$ )	$8.75 \cdot 10^{-4}$ ( $1.32 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$3.07 \cdot 10^{-4}$ ( $1.09 \cdot 10^{-5}$ )	$4.90 \cdot 10^{-4}$ ( $2.59 \cdot 10^{-5}$ )	$4.64 \cdot 10^{-4}$ ( $4.07 \cdot 10^{-5}$ )	$4.31 \cdot 10^{-4}$ ( $1.87 \cdot 10^{-5}$ )	$2.98 \cdot 10^{-1}$ ( $1.12 \cdot 10^{-2}$ )	$6.89 \cdot 10^{-4}$ ( $8.81 \cdot 10^{-5}$ )

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$4.44 \cdot 10^{-2}$ ( $1.62 \cdot 10^{-3}$ )
$v = 3/2$	$2.97 \cdot 10^{-3}$ ( $1.13 \cdot 10^{-4}$ )
$v = 5/2$	$3.21 \cdot 10^{-4}$ ( $1.04 \cdot 10^{-5}$ )
$v = 7/2$	$2.26 \cdot 10^{-4}$ ( $7.54 \cdot 10^{-6}$ )
$v = 9/2$	$2.26 \cdot 10^{-4}$ ( $7.46 \cdot 10^{-6}$ )
$v = 13/2$	$2.32 \cdot 10^{-4}$ ( $7.49 \cdot 10^{-6}$ )
$v = 25/2$	$2.45 \cdot 10^{-4}$ ( $7.54 \cdot 10^{-6}$ )
$v = \infty$	$2.60 \cdot 10^{-4}$ ( $7.57 \cdot 10^{-6}$ )
$v \in \{1/2, \dots, \infty\}$	$2.13 \cdot 10^{-4}$ ( $6.61 \cdot 10^{-6}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$3.26 \cdot 10^{-4}$ ( $1.15 \cdot 10^{-5}$ )
linear	$5.10 \cdot 10^{-1}$ ( $3.00 \cdot 10^{-3}$ )
1NN	$4.56 \cdot 10^{-1}$ ( $2.93 \cdot 10^{-3}$ )
2NN	$3.62 \cdot 10^{-1}$ ( $4.15 \cdot 10^{-3}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.09 \cdot 10^{-1}$ ( $8.65 \cdot 10^{-4}$ )	$9.69 \cdot 10^{-2}$ ( $1.23 \cdot 10^{-3}$ )	$9.39 \cdot 10^{-2}$ ( $1.24 \cdot 10^{-3}$ )	$9.16 \cdot 10^{-2}$ ( $1.47 \cdot 10^{-3}$ )	$3.41 \cdot 10^{-1}$ ( $3.74 \cdot 10^{-3}$ )	$9.29 \cdot 10^{-2}$ ( $1.24 \cdot 10^{-3}$ )
$v = 3/2$	$2.59 \cdot 10^{-2}$ ( $3.48 \cdot 10^{-4}$ )	$2.57 \cdot 10^{-2}$ ( $5.51 \cdot 10^{-4}$ )	$2.31 \cdot 10^{-2}$ ( $4.22 \cdot 10^{-4}$ )	$2.34 \cdot 10^{-2}$ ( $4.38 \cdot 10^{-4}$ )	$3.12 \cdot 10^{-1}$ ( $5.36 \cdot 10^{-3}$ )	$2.26 \cdot 10^{-2}$ ( $3.93 \cdot 10^{-4}$ )
$v = 5/2$	$8.26 \cdot 10^{-3}$ ( $9.79 \cdot 10^{-5}$ )	$8.57 \cdot 10^{-3}$ ( $2.28 \cdot 10^{-4}$ )	$7.74 \cdot 10^{-3}$ ( $1.23 \cdot 10^{-4}$ )	$7.93 \cdot 10^{-3}$ ( $1.49 \cdot 10^{-4}$ )	$3.05 \cdot 10^{-1}$ ( $6.18 \cdot 10^{-3}$ )	$8.37 \cdot 10^{-3}$ ( $1.82 \cdot 10^{-4}$ )
$v = 7/2$	$6.86 \cdot 10^{-3}$ ( $8.36 \cdot 10^{-5}$ )	$8.12 \cdot 10^{-3}$ ( $1.84 \cdot 10^{-4}$ )	$7.58 \cdot 10^{-3}$ ( $1.55 \cdot 10^{-4}$ )	$7.75 \cdot 10^{-3}$ ( $1.61 \cdot 10^{-4}$ )	$2.96 \cdot 10^{-1}$ ( $5.87 \cdot 10^{-3}$ )	$8.30 \cdot 10^{-3}$ ( $2.02 \cdot 10^{-4}$ )
$v = 9/2$	$6.95 \cdot 10^{-3}$ ( $9.07 \cdot 10^{-5}$ )	$8.32 \cdot 10^{-3}$ ( $1.72 \cdot 10^{-4}$ )	$8.07 \cdot 10^{-3}$ ( $1.78 \cdot 10^{-4}$ )	$8.02 \cdot 10^{-3}$ ( $1.58 \cdot 10^{-4}$ )	$2.88 \cdot 10^{-1}$ ( $5.73 \cdot 10^{-3}$ )	$8.86 \cdot 10^{-3}$ ( $2.38 \cdot 10^{-4}$ )
$v = 13/2$	$7.20 \cdot 10^{-3}$ ( $1.00 \cdot 10^{-4}$ )	$8.82 \cdot 10^{-3}$ ( $1.96 \cdot 10^{-4}$ )	$8.45 \cdot 10^{-3}$ ( $1.82 \cdot 10^{-4}$ )	$8.46 \cdot 10^{-3}$ ( $1.74 \cdot 10^{-4}$ )	$2.79 \cdot 10^{-1}$ ( $5.77 \cdot 10^{-3}$ )	$9.76 \cdot 10^{-3}$ ( $3.03 \cdot 10^{-4}$ )
$v = 25/2$	$7.58 \cdot 10^{-3}$ ( $1.10 \cdot 10^{-4}$ )	$9.38 \cdot 10^{-3}$ ( $2.47 \cdot 10^{-4}$ )	$8.96 \cdot 10^{-3}$ ( $1.88 \cdot 10^{-4}$ )	$8.96 \cdot 10^{-3}$ ( $1.83 \cdot 10^{-4}$ )	$2.68 \cdot 10^{-1}$ ( $5.99 \cdot 10^{-3}$ )	$1.05 \cdot 10^{-2}$ ( $3.35 \cdot 10^{-4}$ )
$v = \infty$	$7.99 \cdot 10^{-3}$ ( $1.19 \cdot 10^{-4}$ )	$1.01 \cdot 10^{-2}$ ( $2.90 \cdot 10^{-4}$ )	$9.42 \cdot 10^{-3}$ ( $2.00 \cdot 10^{-4}$ )	$9.48 \cdot 10^{-3}$ ( $2.13 \cdot 10^{-4}$ )	$2.55 \cdot 10^{-1}$ ( $6.04 \cdot 10^{-3}$ )	$1.13 \cdot 10^{-2}$ ( $4.47 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$7.08 \cdot 10^{-3}$ ( $9.98 \cdot 10^{-5}$ )	$8.80 \cdot 10^{-3}$ ( $2.09 \cdot 10^{-4}$ )	$8.48 \cdot 10^{-3}$ ( $2.14 \cdot 10^{-4}$ )	$8.40 \cdot 10^{-3}$ ( $1.79 \cdot 10^{-4}$ )	$2.58 \cdot 10^{-1}$ ( $6.14 \cdot 10^{-3}$ )	$9.88 \cdot 10^{-3}$ ( $3.52 \cdot 10^{-4}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$7.28 \cdot 10^{-3}$ ( $1.02 \cdot 10^{-4}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.44 \cdot 10^0 (1.35 \cdot 10^{-2})$	$1.38 \cdot 10^0 (3.54 \cdot 10^{-2})$	$1.34 \cdot 10^0 (3.56 \cdot 10^{-2})$	$1.48 \cdot 10^0 (4.47 \cdot 10^{-2})$	$3.69 \cdot 10^0 (5.16 \cdot 10^{-2})$	$1.32 \cdot 10^0 (3.53 \cdot 10^{-2})$
$v = 3/2$	$2.80 \cdot 10^{-1} (3.52 \cdot 10^{-3})$	$3.05 \cdot 10^{-1} (9.78 \cdot 10^{-3})$	$2.75 \cdot 10^{-1} (8.16 \cdot 10^{-3})$	$2.88 \cdot 10^{-1} (9.36 \cdot 10^{-3})$	$3.43 \cdot 10^0 (6.18 \cdot 10^{-2})$	$2.69 \cdot 10^{-1} (7.48 \cdot 10^{-3})$
$v = 5/2$	$9.30 \cdot 10^{-2} (6.54 \cdot 10^{-4})$	$9.42 \cdot 10^{-2} (3.19 \cdot 10^{-3})$	$8.55 \cdot 10^{-2} (2.32 \cdot 10^{-3})$	$9.11 \cdot 10^{-2} (2.88 \cdot 10^{-3})$	$3.36 \cdot 10^0 (6.71 \cdot 10^{-2})$	$9.18 \cdot 10^{-2} (2.71 \cdot 10^{-3})$
$v = 7/2$	$6.82 \cdot 10^{-2} (5.82 \cdot 10^{-4})$	$8.82 \cdot 10^{-2} (3.20 \cdot 10^{-3})$	$8.42 \cdot 10^{-2} (3.05 \cdot 10^{-3})$	$9.10 \cdot 10^{-2} (3.74 \cdot 10^{-3})$	$3.23 \cdot 10^0 (6.24 \cdot 10^{-2})$	$9.17 \cdot 10^{-2} (3.49 \cdot 10^{-3})$
$v = 9/2$	$6.50 \cdot 10^{-2} (6.50 \cdot 10^{-4})$	$9.08 \cdot 10^{-2} (3.28 \cdot 10^{-3})$	$9.30 \cdot 10^{-2} (3.75 \cdot 10^{-3})$	$9.53 \cdot 10^{-2} (3.88 \cdot 10^{-3})$	$3.14 \cdot 10^0 (6.16 \cdot 10^{-2})$	$1.00 \cdot 10^{-1} (3.99 \cdot 10^{-3})$
$v = 13/2$	$6.48 \cdot 10^{-2} (8.76 \cdot 10^{-4})$	$9.95 \cdot 10^{-2} (4.05 \cdot 10^{-3})$	$9.99 \cdot 10^{-2} (4.31 \cdot 10^{-3})$	$1.03 \cdot 10^{-1} (4.27 \cdot 10^{-3})$	$3.02 \cdot 10^0 (6.27 \cdot 10^{-2})$	$1.16 \cdot 10^{-1} (5.44 \cdot 10^{-3})$
$v = 25/2$	$6.77 \cdot 10^{-2} (1.21 \cdot 10^{-3})$	$1.10 \cdot 10^{-1} (5.70 \cdot 10^{-3})$	$1.08 \cdot 10^{-1} (4.73 \cdot 10^{-3})$	$1.12 \cdot 10^{-1} (4.91 \cdot 10^{-3})$	$2.90 \cdot 10^0 (6.52 \cdot 10^{-2})$	$1.29 \cdot 10^{-1} (6.36 \cdot 10^{-3})$
$v = \infty$	$7.23 \cdot 10^{-2} (1.57 \cdot 10^{-3})$	$1.23 \cdot 10^{-1} (6.70 \cdot 10^{-3})$	$1.16 \cdot 10^{-1} (5.20 \cdot 10^{-3})$	$1.23 \cdot 10^{-1} (6.18 \cdot 10^{-3})$	$2.74 \cdot 10^0 (6.73 \cdot 10^{-2})$	$1.44 \cdot 10^{-1} (8.33 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$6.70 \cdot 10^{-2} (8.65 \cdot 10^{-4})$	$1.04 \cdot 10^{-1} (4.50 \cdot 10^{-3})$	$1.06 \cdot 10^{-1} (5.29 \cdot 10^{-3})$	$1.08 \cdot 10^{-1} (4.92 \cdot 10^{-3})$	$2.78 \cdot 10^0 (6.97 \cdot 10^{-2})$	$1.23 \cdot 10^{-1} (6.81 \cdot 10^{-3})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$7.06 \cdot 10^{-2} (1.24 \cdot 10^{-3})$

**SM-2.2.41**  $\text{Re}(S_{6\pi}), n = 120, d = 6$

**Averages of  $R(\theta; S^{\text{SPE}})$**

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.42 \cdot 10^{-2} (7.58 \cdot 10^{-4})$	$2.55 \cdot 10^{-2} (7.81 \cdot 10^{-4})$	$2.43 \cdot 10^{-2} (7.49 \cdot 10^{-4})$	$2.42 \cdot 10^{-2} (7.63 \cdot 10^{-4})$	$2.68 \cdot 10^{-1} (6.22 \cdot 10^{-3})$	$2.38 \cdot 10^{-2} (7.38 \cdot 10^{-4})$
$v = 3/2$	$1.24 \cdot 10^{-3} (3.82 \cdot 10^{-5})$	$1.38 \cdot 10^{-3} (4.73 \cdot 10^{-5})$	$1.24 \cdot 10^{-3} (3.85 \cdot 10^{-5})$	$1.27 \cdot 10^{-3} (4.03 \cdot 10^{-5})$	$2.13 \cdot 10^{-1} (5.64 \cdot 10^{-3})$	$1.23 \cdot 10^{-3} (3.94 \cdot 10^{-5})$
$v = 5/2$	$1.48 \cdot 10^{-4} (4.43 \cdot 10^{-6})$	$1.73 \cdot 10^{-4} (5.55 \cdot 10^{-6})$	$1.59 \cdot 10^{-4} (5.02 \cdot 10^{-6})$	$1.62 \cdot 10^{-4} (5.01 \cdot 10^{-6})$	$1.97 \cdot 10^{-1} (5.70 \cdot 10^{-3})$	$1.68 \cdot 10^{-4} (5.49 \cdot 10^{-6})$
$v = 7/2$	$9.31 \cdot 10^{-5} (2.51 \cdot 10^{-6})$	$1.05 \cdot 10^{-4} (3.76 \cdot 10^{-6})$	$1.02 \cdot 10^{-4} (3.38 \cdot 10^{-6})$	$1.00 \cdot 10^{-4} (3.37 \cdot 10^{-6})$	$1.77 \cdot 10^{-1} (4.76 \cdot 10^{-3})$	$1.20 \cdot 10^{-4} (4.55 \cdot 10^{-6})$
$v = 9/2$	$8.68 \cdot 10^{-5} (2.23 \cdot 10^{-6})$	$1.01 \cdot 10^{-4} (3.78 \cdot 10^{-6})$	$9.75 \cdot 10^{-5} (3.31 \cdot 10^{-6})$	$9.53 \cdot 10^{-5} (3.19 \cdot 10^{-6})$	$1.65 \cdot 10^{-1} (4.39 \cdot 10^{-3})$	$1.27 \cdot 10^{-4} (6.13 \cdot 10^{-6})$
$v = 13/2$	$9.08 \cdot 10^{-5} (2.28 \cdot 10^{-6})$	$1.06 \cdot 10^{-4} (3.59 \cdot 10^{-6})$	$1.04 \cdot 10^{-4} (3.46 \cdot 10^{-6})$	$1.02 \cdot 10^{-4} (3.28 \cdot 10^{-6})$	$1.53 \cdot 10^{-1} (4.21 \cdot 10^{-3})$	$1.44 \cdot 10^{-4} (9.01 \cdot 10^{-6})$
$v = 25/2$	$9.20 \cdot 10^{-5} (2.27 \cdot 10^{-6})$	$1.06 \cdot 10^{-4} (3.48 \cdot 10^{-6})$	$1.05 \cdot 10^{-4} (3.31 \cdot 10^{-6})$	$1.02 \cdot 10^{-4} (3.16 \cdot 10^{-6})$	$1.41 \cdot 10^{-1} (4.11 \cdot 10^{-3})$	$1.55 \cdot 10^{-4} (1.14 \cdot 10^{-5})$
$v = \infty$	$1.02 \cdot 10^{-4} (2.51 \cdot 10^{-6})$	$1.15 \cdot 10^{-4} (3.81 \cdot 10^{-6})$	$1.12 \cdot 10^{-4} (3.50 \cdot 10^{-6})$	$1.11 \cdot 10^{-4} (3.29 \cdot 10^{-6})$	$1.28 \cdot 10^{-1} (4.00 \cdot 10^{-3})$	$1.66 \cdot 10^{-4} (1.30 \cdot 10^{-5})$
$v \in \{1/2, \dots, \infty\}$	$9.29 \cdot 10^{-5} (2.50 \cdot 10^{-6})$	$1.07 \cdot 10^{-4} (3.90 \cdot 10^{-6})$	$1.01 \cdot 10^{-4} (3.33 \cdot 10^{-6})$	$1.01 \cdot 10^{-4} (3.41 \cdot 10^{-6})$	$1.29 \cdot 10^{-1} (4.20 \cdot 10^{-3})$	$1.28 \cdot 10^{-4} (6.02 \cdot 10^{-6})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$2.26 \cdot 10^{-2} (7.24 \cdot 10^{-4})$
$v = 3/2$	$1.15 \cdot 10^{-3} (3.77 \cdot 10^{-5})$
$v = 5/2$	$1.30 \cdot 10^{-4} (3.77 \cdot 10^{-6})$
$v = 7/2$	$7.74 \cdot 10^{-5} (1.97 \cdot 10^{-6})$
$v = 9/2$	$7.15 \cdot 10^{-5} (1.71 \cdot 10^{-6})$
$v = 13/2$	$7.54 \cdot 10^{-5} (1.77 \cdot 10^{-6})$
$v = 25/2$	$7.65 \cdot 10^{-5} (1.79 \cdot 10^{-6})$
$v = \infty$	$8.56 \cdot 10^{-5} (1.98 \cdot 10^{-6})$
$v \in \{1/2, \dots, \infty\}$	$6.95 \cdot 10^{-5} (1.64 \cdot 10^{-6})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$9.60 \cdot 10^{-5} (2.97 \cdot 10^{-6})$
linear	$4.83 \cdot 10^{-1} (1.40 \cdot 10^{-3})$
INN	$3.66 \cdot 10^{-1} (1.73 \cdot 10^{-3})$
2NN	$2.84 \cdot 10^{-1} (2.23 \cdot 10^{-3})$

**Averages of  $R(\theta; S^{\text{CRPS}})$**

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$7.36 \cdot 10^{-2}$ ( $4.45 \cdot 10^{-4}$ )	$6.24 \cdot 10^{-2}$ ( $7.51 \cdot 10^{-4}$ )	$6.08 \cdot 10^{-2}$ ( $6.99 \cdot 10^{-4}$ )	$5.88 \cdot 10^{-2}$ ( $8.35 \cdot 10^{-4}$ )	$2.55 \cdot 10^{-1}$ ( $3.65 \cdot 10^{-3}$ )	$6.02 \cdot 10^{-2}$ ( $6.80 \cdot 10^{-4}$ )
$v = 3/2$	$1.32 \cdot 10^{-2}$ ( $1.16 \cdot 10^{-4}$ )	$1.26 \cdot 10^{-2}$ ( $1.97 \cdot 10^{-4}$ )	$1.18 \cdot 10^{-2}$ ( $1.62 \cdot 10^{-4}$ )	$1.19 \cdot 10^{-2}$ ( $1.71 \cdot 10^{-4}$ )	$2.14 \cdot 10^{-1}$ ( $3.98 \cdot 10^{-3}$ )	$1.17 \cdot 10^{-2}$ ( $1.64 \cdot 10^{-4}$ )
$v = 5/2$	$4.20 \cdot 10^{-3}$ ( $3.21 \cdot 10^{-5}$ )	$4.22 \cdot 10^{-3}$ ( $5.83 \cdot 10^{-5}$ )	$3.93 \cdot 10^{-3}$ ( $4.68 \cdot 10^{-5}$ )	$4.01 \cdot 10^{-3}$ ( $4.86 \cdot 10^{-5}$ )	$2.01 \cdot 10^{-1}$ ( $4.35 \cdot 10^{-3}$ )	$4.01 \cdot 10^{-3}$ ( $4.71 \cdot 10^{-5}$ )
$v = 7/2$	$3.31 \cdot 10^{-3}$ ( $3.06 \cdot 10^{-5}$ )	$3.44 \cdot 10^{-3}$ ( $4.77 \cdot 10^{-5}$ )	$3.29 \cdot 10^{-3}$ ( $4.11 \cdot 10^{-5}$ )	$3.31 \cdot 10^{-3}$ ( $4.33 \cdot 10^{-5}$ )	$1.84 \cdot 10^{-1}$ ( $3.44 \cdot 10^{-3}$ )	$3.51 \cdot 10^{-3}$ ( $5.39 \cdot 10^{-5}$ )
$v = 9/2$	$3.25 \cdot 10^{-3}$ ( $3.38 \cdot 10^{-5}$ )	$3.48 \cdot 10^{-3}$ ( $4.92 \cdot 10^{-5}$ )	$3.35 \cdot 10^{-3}$ ( $4.40 \cdot 10^{-5}$ )	$3.36 \cdot 10^{-3}$ ( $4.35 \cdot 10^{-5}$ )	$1.72 \cdot 10^{-1}$ ( $2.95 \cdot 10^{-3}$ )	$3.71 \cdot 10^{-3}$ ( $7.17 \cdot 10^{-5}$ )
$v = 13/2$	$3.39 \cdot 10^{-3}$ ( $3.79 \cdot 10^{-5}$ )	$3.69 \cdot 10^{-3}$ ( $5.68 \cdot 10^{-5}$ )	$3.60 \cdot 10^{-3}$ ( $5.29 \cdot 10^{-5}$ )	$3.60 \cdot 10^{-3}$ ( $5.26 \cdot 10^{-5}$ )	$1.60 \cdot 10^{-1}$ ( $2.67 \cdot 10^{-3}$ )	$4.02 \cdot 10^{-3}$ ( $8.84 \cdot 10^{-5}$ )
$v = 25/2$	$3.48 \cdot 10^{-3}$ ( $4.05 \cdot 10^{-5}$ )	$3.77 \cdot 10^{-3}$ ( $5.93 \cdot 10^{-5}$ )	$3.71 \cdot 10^{-3}$ ( $5.48 \cdot 10^{-5}$ )	$3.71 \cdot 10^{-3}$ ( $5.73 \cdot 10^{-5}$ )	$1.49 \cdot 10^{-1}$ ( $2.57 \cdot 10^{-3}$ )	$4.25 \cdot 10^{-3}$ ( $1.10 \cdot 10^{-4}$ )
$v = \infty$	$3.70 \cdot 10^{-3}$ ( $4.31 \cdot 10^{-5}$ )	$3.94 \cdot 10^{-3}$ ( $6.28 \cdot 10^{-5}$ )	$3.89 \cdot 10^{-3}$ ( $5.71 \cdot 10^{-5}$ )	$3.90 \cdot 10^{-3}$ ( $5.80 \cdot 10^{-5}$ )	$1.38 \cdot 10^{-1}$ ( $2.51 \cdot 10^{-3}$ )	$4.44 \cdot 10^{-3}$ ( $1.14 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$3.31 \cdot 10^{-3}$ ( $3.07 \cdot 10^{-5}$ )	$3.55 \cdot 10^{-3}$ ( $5.02 \cdot 10^{-5}$ )	$3.38 \cdot 10^{-3}$ ( $4.55 \cdot 10^{-5}$ )	$3.41 \cdot 10^{-3}$ ( $4.74 \cdot 10^{-5}$ )	$1.39 \cdot 10^{-1}$ ( $2.72 \cdot 10^{-3}$ )	$3.74 \cdot 10^{-3}$ ( $7.08 \cdot 10^{-5}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$3.39 \cdot 10^{-3}$ ( $4.16 \cdot 10^{-5}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.04 \cdot 10^0$ ( $8.08 \cdot 10^{-3}$ )	$9.64 \cdot 10^{-1}$ ( $2.22 \cdot 10^{-2}$ )	$9.37 \cdot 10^{-1}$ ( $2.14 \cdot 10^{-2}$ )	$1.05 \cdot 10^0$ ( $2.57 \cdot 10^{-2}$ )	$2.98 \cdot 10^0$ ( $3.58 \cdot 10^{-2}$ )	$9.23 \cdot 10^{-1}$ ( $2.08 \cdot 10^{-2}$ )
$v = 3/2$	$1.59 \cdot 10^{-1}$ ( $1.45 \cdot 10^{-3}$ )	$1.70 \cdot 10^{-1}$ ( $4.47 \cdot 10^{-3}$ )	$1.61 \cdot 10^{-1}$ ( $4.05 \cdot 10^{-3}$ )	$1.67 \cdot 10^{-1}$ ( $4.35 \cdot 10^{-3}$ )	$2.58 \cdot 10^0$ ( $3.97 \cdot 10^{-2}$ )	$1.60 \cdot 10^{-1}$ ( $4.02 \cdot 10^{-3}$ )
$v = 5/2$	$4.90 \cdot 10^{-2}$ ( $3.47 \cdot 10^{-4}$ )	$5.18 \cdot 10^{-2}$ ( $1.06 \cdot 10^{-3}$ )	$4.90 \cdot 10^{-2}$ ( $1.08 \cdot 10^{-3}$ )	$5.08 \cdot 10^{-2}$ ( $1.09 \cdot 10^{-3}$ )	$2.42 \cdot 10^0$ ( $4.17 \cdot 10^{-2}$ )	$4.95 \cdot 10^{-2}$ ( $1.09 \cdot 10^{-3}$ )
$v = 7/2$	$3.41 \cdot 10^{-2}$ ( $2.92 \cdot 10^{-4}$ )	$3.85 \cdot 10^{-2}$ ( $8.26 \cdot 10^{-4}$ )	$3.79 \cdot 10^{-2}$ ( $8.62 \cdot 10^{-4}$ )	$3.85 \cdot 10^{-2}$ ( $8.83 \cdot 10^{-4}$ )	$2.23 \cdot 10^0$ ( $4.10 \cdot 10^{-2}$ )	$4.00 \cdot 10^{-2}$ ( $9.96 \cdot 10^{-4}$ )
$v = 9/2$	$3.15 \cdot 10^{-2}$ ( $3.38 \cdot 10^{-4}$ )	$3.71 \cdot 10^{-2}$ ( $8.20 \cdot 10^{-4}$ )	$3.68 \cdot 10^{-2}$ ( $8.90 \cdot 10^{-4}$ )	$3.73 \cdot 10^{-2}$ ( $9.00 \cdot 10^{-4}$ )	$2.12 \cdot 10^0$ ( $4.23 \cdot 10^{-2}$ )	$4.09 \cdot 10^{-2}$ ( $1.25 \cdot 10^{-3}$ )
$v = 13/2$	$3.19 \cdot 10^{-2}$ ( $4.15 \cdot 10^{-4}$ )	$3.85 \cdot 10^{-2}$ ( $1.02 \cdot 10^{-3}$ )	$3.87 \cdot 10^{-2}$ ( $1.11 \cdot 10^{-3}$ )	$3.90 \cdot 10^{-2}$ ( $1.12 \cdot 10^{-3}$ )	$2.01 \cdot 10^0$ ( $4.37 \cdot 10^{-2}$ )	$4.34 \cdot 10^{-2}$ ( $1.55 \cdot 10^{-3}$ )
$v = 25/2$	$3.27 \cdot 10^{-2}$ ( $5.00 \cdot 10^{-4}$ )	$3.86 \cdot 10^{-2}$ ( $1.08 \cdot 10^{-3}$ )	$3.92 \cdot 10^{-2}$ ( $1.16 \cdot 10^{-3}$ )	$3.97 \cdot 10^{-2}$ ( $1.17 \cdot 10^{-3}$ )	$1.88 \cdot 10^0$ ( $4.44 \cdot 10^{-2}$ )	$4.50 \cdot 10^{-2}$ ( $1.70 \cdot 10^{-3}$ )
$v = \infty$	$3.49 \cdot 10^{-2}$ ( $5.69 \cdot 10^{-4}$ )	$3.95 \cdot 10^{-2}$ ( $1.09 \cdot 10^{-3}$ )	$4.01 \cdot 10^{-2}$ ( $1.15 \cdot 10^{-3}$ )	$4.07 \cdot 10^{-2}$ ( $1.16 \cdot 10^{-3}$ )	$1.72 \cdot 10^0$ ( $4.40 \cdot 10^{-2}$ )	$4.60 \cdot 10^{-2}$ ( $1.81 \cdot 10^{-3}$ )
$v \in \{1/2, \dots, \infty\}$	$3.40 \cdot 10^{-2}$ ( $2.98 \cdot 10^{-4}$ )	$3.93 \cdot 10^{-2}$ ( $8.88 \cdot 10^{-4}$ )	$3.84 \cdot 10^{-2}$ ( $9.64 \cdot 10^{-4}$ )	$3.96 \cdot 10^{-2}$ ( $9.73 \cdot 10^{-4}$ )	$1.73 \cdot 10^0$ ( $4.50 \cdot 10^{-2}$ )	$4.21 \cdot 10^{-2}$ ( $1.26 \cdot 10^{-3}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$3.43 \cdot 10^{-2}$ ( $5.61 \cdot 10^{-4}$ )

### SM-2.2.42 $\text{Re}(S_{6\pi}), n = 300, d = 6$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$9.04 \cdot 10^{-3}$ ( $2.64 \cdot 10^{-4}$ )	$9.46 \cdot 10^{-3}$ ( $2.65 \cdot 10^{-4}$ )	$9.16 \cdot 10^{-3}$ ( $2.63 \cdot 10^{-4}$ )	$9.07 \cdot 10^{-3}$ ( $2.70 \cdot 10^{-4}$ )	$1.31 \cdot 10^{-1}$ ( $3.27 \cdot 10^{-3}$ )	$9.01 \cdot 10^{-3}$ ( $2.64 \cdot 10^{-4}$ )
$v = 3/2$	$3.42 \cdot 10^{-4}$ ( $9.44 \cdot 10^{-6}$ )	$3.76 \cdot 10^{-4}$ ( $1.08 \cdot 10^{-5}$ )	$3.51 \cdot 10^{-4}$ ( $9.83 \cdot 10^{-6}$ )	$3.52 \cdot 10^{-4}$ ( $9.93 \cdot 10^{-6}$ )	$9.49 \cdot 10^{-2}$ ( $2.56 \cdot 10^{-3}$ )	$3.46 \cdot 10^{-4}$ ( $9.76 \cdot 10^{-6}$ )
$v = 5/2$	$3.75 \cdot 10^{-5}$ ( $9.97 \cdot 10^{-7}$ )	$4.32 \cdot 10^{-5}$ ( $2.36 \cdot 10^{-6}$ )	$3.93 \cdot 10^{-5}$ ( $1.11 \cdot 10^{-6}$ )	$3.88 \cdot 10^{-5}$ ( $1.14 \cdot 10^{-6}$ )	$8.33 \cdot 10^{-2}$ ( $2.44 \cdot 10^{-3}$ )	$4.34 \cdot 10^{-5}$ ( $1.39 \cdot 10^{-6}$ )
$v = 7/2$	$2.11 \cdot 10^{-5}$ ( $4.94 \cdot 10^{-7}$ )	$2.44 \cdot 10^{-5}$ ( $1.53 \cdot 10^{-6}$ )	$2.51 \cdot 10^{-5}$ ( $6.94 \cdot 10^{-7}$ )	$2.22 \cdot 10^{-5}$ ( $5.83 \cdot 10^{-7}$ )	$7.30 \cdot 10^{-2}$ ( $1.97 \cdot 10^{-3}$ )	$3.10 \cdot 10^{-5}$ ( $1.42 \cdot 10^{-6}$ )
$v = 9/2$	$2.10 \cdot 10^{-5}$ ( $4.89 \cdot 10^{-7}$ )	$2.33 \cdot 10^{-5}$ ( $7.97 \cdot 10^{-7}$ )	$2.62 \cdot 10^{-5}$ ( $8.27 \cdot 10^{-7}$ )	$2.24 \cdot 10^{-5}$ ( $6.10 \cdot 10^{-7}$ )	$6.75 \cdot 10^{-2}$ ( $1.85 \cdot 10^{-3}$ )	$3.34 \cdot 10^{-5}$ ( $2.10 \cdot 10^{-6}$ )
$v = 13/2$	$2.57 \cdot 10^{-5}$ ( $6.49 \cdot 10^{-7}$ )	$3.01 \cdot 10^{-5}$ ( $1.05 \cdot 10^{-6}$ )	$3.28 \cdot 10^{-5}$ ( $1.23 \cdot 10^{-6}$ )	$2.90 \cdot 10^{-5}$ ( $8.44 \cdot 10^{-7}$ )	$6.16 \cdot 10^{-2}$ ( $1.76 \cdot 10^{-3}$ )	$4.04 \cdot 10^{-5}$ ( $1.97 \cdot 10^{-6}$ )
$v = 25/2$	$3.01 \cdot 10^{-5}$ ( $8.24 \cdot 10^{-7}$ )	$3.76 \cdot 10^{-5}$ ( $1.45 \cdot 10^{-6}$ )	$3.80 \cdot 10^{-5}$ ( $1.46 \cdot 10^{-6}$ )	$3.54 \cdot 10^{-5}$ ( $1.15 \cdot 10^{-6}$ )	$5.50 \cdot 10^{-2}$ ( $1.65 \cdot 10^{-3}$ )	$4.81 \cdot 10^{-5}$ ( $2.61 \cdot 10^{-6}$ )
$v = \infty$	$3.47 \cdot 10^{-5}$ ( $9.43 \cdot 10^{-7}$ )	$4.29 \cdot 10^{-5}$ ( $1.65 \cdot 10^{-6}$ )	$4.27 \cdot 10^{-5}$ ( $1.72 \cdot 10^{-6}$ )	$3.91 \cdot 10^{-5}$ ( $1.27 \cdot 10^{-6}$ )	$4.72 \cdot 10^{-2}$ ( $1.49 \cdot 10^{-3}$ )	$5.56 \cdot 10^{-5}$ ( $3.78 \cdot 10^{-6}$ )
$v \in \{1/2, \dots, \infty\}$	$2.11 \cdot 10^{-5}$ ( $4.94 \cdot 10^{-7}$ )	$2.26 \cdot 10^{-5}$ ( $6.35 \cdot 10^{-7}$ )	$2.57 \cdot 10^{-5}$ ( $7.73 \cdot 10^{-7}$ )	$2.20 \cdot 10^{-5}$ ( $6.06 \cdot 10^{-7}$ )	$4.72 \cdot 10^{-2}$ ( $1.49 \cdot 10^{-3}$ )	$3.40 \cdot 10^{-5}$ ( $1.66 \cdot 10^{-6}$ )



$R^*(S^{\text{SPE}})$	
$v = 1/2$	$8.64 \cdot 10^{-3}$ ( $2.61 \cdot 10^{-4}$ )
$v = 3/2$	$3.26 \cdot 10^{-4}$ ( $9.34 \cdot 10^{-6}$ )
$v = 5/2$	$3.23 \cdot 10^{-5}$ ( $7.68 \cdot 10^{-7}$ )
$v = 7/2$	$1.71 \cdot 10^{-5}$ ( $3.30 \cdot 10^{-7}$ )
$v = 9/2$	$1.76 \cdot 10^{-5}$ ( $3.67 \cdot 10^{-7}$ )
$v = 13/2$	$2.38 \cdot 10^{-5}$ ( $5.72 \cdot 10^{-7}$ )
$v = 25/2$	$2.93 \cdot 10^{-5}$ ( $7.54 \cdot 10^{-7}$ )
$v = \infty$	$3.42 \cdot 10^{-5}$ ( $8.78 \cdot 10^{-7}$ )
$v \in \{1/2, \dots, \infty\}$	$1.66 \cdot 10^{-5}$ ( $3.25 \cdot 10^{-7}$ )

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$2.15 \cdot 10^{-5}$ ( $6.08 \cdot 10^{-7}$ )
linear	$4.68 \cdot 10^{-1}$ ( $5.47 \cdot 10^{-4}$ )
1NN	$2.73 \cdot 10^{-1}$ ( $8.82 \cdot 10^{-4}$ )
2NN	$2.03 \cdot 10^{-1}$ ( $1.23 \cdot 10^{-3}$ )

### Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.41 \cdot 10^{-2}$ ( $1.88 \cdot 10^{-4}$ )	$3.42 \cdot 10^{-2}$ ( $3.18 \cdot 10^{-4}$ )	$3.36 \cdot 10^{-2}$ ( $3.24 \cdot 10^{-4}$ )	$3.14 \cdot 10^{-2}$ ( $3.82 \cdot 10^{-4}$ )	$1.63 \cdot 10^{-1}$ ( $2.60 \cdot 10^{-3}$ )	$3.34 \cdot 10^{-2}$ ( $3.22 \cdot 10^{-4}$ )
$v = 3/2$	$6.07 \cdot 10^{-3}$ ( $3.84 \cdot 10^{-5}$ )	$5.47 \cdot 10^{-3}$ ( $7.11 \cdot 10^{-5}$ )	$5.14 \cdot 10^{-3}$ ( $5.82 \cdot 10^{-5}$ )	$5.16 \cdot 10^{-3}$ ( $6.02 \cdot 10^{-5}$ )	$1.23 \cdot 10^{-1}$ ( $2.49 \cdot 10^{-3}$ )	$5.06 \cdot 10^{-3}$ ( $5.74 \cdot 10^{-5}$ )
$v = 5/2$	$1.73 \cdot 10^{-3}$ ( $8.56 \cdot 10^{-6}$ )	$1.75 \cdot 10^{-3}$ ( $2.83 \cdot 10^{-5}$ )	$1.60 \cdot 10^{-3}$ ( $1.28 \cdot 10^{-5}$ )	$1.62 \cdot 10^{-3}$ ( $1.31 \cdot 10^{-5}$ )	$1.10 \cdot 10^{-1}$ ( $2.62 \cdot 10^{-3}$ )	$1.65 \cdot 10^{-3}$ ( $1.43 \cdot 10^{-5}$ )
$v = 7/2$	$1.31 \cdot 10^{-3}$ ( $7.41 \cdot 10^{-6}$ )	$1.38 \cdot 10^{-3}$ ( $2.19 \cdot 10^{-5}$ )	$1.28 \cdot 10^{-3}$ ( $1.09 \cdot 10^{-5}$ )	$1.27 \cdot 10^{-3}$ ( $9.92 \cdot 10^{-6}$ )	$9.66 \cdot 10^{-2}$ ( $1.65 \cdot 10^{-3}$ )	$1.36 \cdot 10^{-3}$ ( $1.41 \cdot 10^{-5}$ )
$v = 9/2$	$1.32 \cdot 10^{-3}$ ( $1.06 \cdot 10^{-5}$ )	$1.37 \cdot 10^{-3}$ ( $1.81 \cdot 10^{-5}$ )	$1.32 \cdot 10^{-3}$ ( $1.35 \cdot 10^{-5}$ )	$1.30 \cdot 10^{-3}$ ( $1.16 \cdot 10^{-5}$ )	$8.93 \cdot 10^{-2}$ ( $1.41 \cdot 10^{-3}$ )	$1.41 \cdot 10^{-3}$ ( $1.90 \cdot 10^{-5}$ )
$v = 13/2$	$1.49 \cdot 10^{-3}$ ( $1.74 \cdot 10^{-5}$ )	$1.54 \cdot 10^{-3}$ ( $2.22 \cdot 10^{-5}$ )	$1.52 \cdot 10^{-3}$ ( $1.94 \cdot 10^{-5}$ )	$1.49 \cdot 10^{-3}$ ( $1.73 \cdot 10^{-5}$ )	$8.20 \cdot 10^{-2}$ ( $1.34 \cdot 10^{-3}$ )	$1.62 \cdot 10^{-3}$ ( $2.60 \cdot 10^{-5}$ )
$v = 25/2$	$1.65 \cdot 10^{-3}$ ( $2.31 \cdot 10^{-5}$ )	$1.75 \cdot 10^{-3}$ ( $2.94 \cdot 10^{-5}$ )	$1.71 \cdot 10^{-3}$ ( $2.47 \cdot 10^{-5}$ )	$1.69 \cdot 10^{-3}$ ( $2.23 \cdot 10^{-5}$ )	$7.44 \cdot 10^{-2}$ ( $1.29 \cdot 10^{-3}$ )	$1.84 \cdot 10^{-3}$ ( $3.48 \cdot 10^{-5}$ )
$v = \infty$	$1.79 \cdot 10^{-3}$ ( $2.64 \cdot 10^{-5}$ )	$1.91 \cdot 10^{-3}$ ( $3.29 \cdot 10^{-5}$ )	$1.88 \cdot 10^{-3}$ ( $2.87 \cdot 10^{-5}$ )	$1.84 \cdot 10^{-3}$ ( $2.57 \cdot 10^{-5}$ )	$6.62 \cdot 10^{-2}$ ( $1.22 \cdot 10^{-3}$ )	$2.03 \cdot 10^{-3}$ ( $4.43 \cdot 10^{-5}$ )
$v \in \{1/2, \dots, \infty\}$	$1.31 \cdot 10^{-3}$ ( $7.41 \cdot 10^{-6}$ )	$1.35 \cdot 10^{-3}$ ( $1.41 \cdot 10^{-5}$ )	$1.31 \cdot 10^{-3}$ ( $1.24 \cdot 10^{-5}$ )	$1.27 \cdot 10^{-3}$ ( $1.05 \cdot 10^{-5}$ )	$6.62 \cdot 10^{-2}$ ( $1.22 \cdot 10^{-3}$ )	$1.44 \cdot 10^{-3}$ ( $2.28 \cdot 10^{-5}$ )

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$1.32 \cdot 10^{-3}$ ( $1.08 \cdot 10^{-5}$ )

### Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$6.56 \cdot 10^{-1}$ ( $3.82 \cdot 10^{-3}$ )	$5.54 \cdot 10^{-1}$ ( $9.76 \cdot 10^{-3}$ )	$5.43 \cdot 10^{-1}$ ( $9.67 \cdot 10^{-3}$ )	$6.22 \cdot 10^{-1}$ ( $1.18 \cdot 10^{-2}$ )	$2.17 \cdot 10^0$ ( $2.50 \cdot 10^{-2}$ )	$5.37 \cdot 10^{-1}$ ( $9.64 \cdot 10^{-3}$ )
$v = 3/2$	$7.98 \cdot 10^{-2}$ ( $5.89 \cdot 10^{-4}$ )	$8.36 \cdot 10^{-2}$ ( $1.78 \cdot 10^{-3}$ )	$7.98 \cdot 10^{-2}$ ( $1.62 \cdot 10^{-3}$ )	$8.47 \cdot 10^{-2}$ ( $1.71 \cdot 10^{-3}$ )	$1.73 \cdot 10^0$ ( $2.66 \cdot 10^{-2}$ )	$7.82 \cdot 10^{-2}$ ( $1.61 \cdot 10^{-3}$ )
$v = 5/2$	$2.05 \cdot 10^{-2}$ ( $1.53 \cdot 10^{-4}$ )	$2.39 \cdot 10^{-2}$ ( $5.46 \cdot 10^{-4}$ )	$2.22 \cdot 10^{-2}$ ( $3.71 \cdot 10^{-4}$ )	$2.22 \cdot 10^{-2}$ ( $3.77 \cdot 10^{-4}$ )	$1.53 \cdot 10^0$ ( $2.73 \cdot 10^{-2}$ )	$2.27 \cdot 10^{-2}$ ( $3.93 \cdot 10^{-4}$ )
$v = 7/2$	$1.45 \cdot 10^{-2}$ ( $1.12 \cdot 10^{-4}$ )	$1.66 \cdot 10^{-2}$ ( $3.35 \cdot 10^{-4}$ )	$1.65 \cdot 10^{-2}$ ( $2.59 \cdot 10^{-4}$ )	$1.61 \cdot 10^{-2}$ ( $2.42 \cdot 10^{-4}$ )	$1.37 \cdot 10^0$ ( $2.69 \cdot 10^{-2}$ )	$1.75 \cdot 10^{-2}$ ( $3.12 \cdot 10^{-4}$ )
$v = 9/2$	$1.43 \cdot 10^{-2}$ ( $1.29 \cdot 10^{-4}$ )	$1.55 \cdot 10^{-2}$ ( $2.38 \cdot 10^{-4}$ )	$1.58 \cdot 10^{-2}$ ( $2.40 \cdot 10^{-4}$ )	$1.55 \cdot 10^{-2}$ ( $2.13 \cdot 10^{-4}$ )	$1.28 \cdot 10^0$ ( $2.68 \cdot 10^{-2}$ )	$1.70 \cdot 10^{-2}$ ( $3.21 \cdot 10^{-4}$ )
$v = 13/2$	$1.56 \cdot 10^{-2}$ ( $1.93 \cdot 10^{-4}$ )	$1.64 \cdot 10^{-2}$ ( $2.59 \cdot 10^{-4}$ )	$1.67 \cdot 10^{-2}$ ( $2.69 \cdot 10^{-4}$ )	$1.66 \cdot 10^{-2}$ ( $2.45 \cdot 10^{-4}$ )	$1.19 \cdot 10^0$ ( $2.63 \cdot 10^{-2}$ )	$1.81 \cdot 10^{-2}$ ( $3.83 \cdot 10^{-4}$ )
$v = 25/2$	$1.72 \cdot 10^{-2}$ ( $2.82 \cdot 10^{-4}$ )	$1.79 \cdot 10^{-2}$ ( $3.15 \cdot 10^{-4}$ )	$1.77 \cdot 10^{-2}$ ( $2.96 \cdot 10^{-4}$ )	$1.79 \cdot 10^{-2}$ ( $2.87 \cdot 10^{-4}$ )	$1.06 \cdot 10^0$ ( $2.51 \cdot 10^{-2}$ )	$1.94 \cdot 10^{-2}$ ( $4.64 \cdot 10^{-4}$ )
$v = \infty$	$1.89 \cdot 10^{-2}$ ( $3.60 \cdot 10^{-4}$ )	$1.89 \cdot 10^{-2}$ ( $3.50 \cdot 10^{-4}$ )	$1.86 \cdot 10^{-2}$ ( $3.27 \cdot 10^{-4}$ )	$1.86 \cdot 10^{-2}$ ( $3.04 \cdot 10^{-4}$ )	$9.14 \cdot 10^{-1}$ ( $2.28 \cdot 10^{-2}$ )	$2.06 \cdot 10^{-2}$ ( $5.65 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$1.45 \cdot 10^{-2}$ ( $1.12 \cdot 10^{-4}$ )	$1.60 \cdot 10^{-2}$ ( $2.30 \cdot 10^{-4}$ )	$1.61 \cdot 10^{-2}$ ( $2.61 \cdot 10^{-4}$ )	$1.59 \cdot 10^{-2}$ ( $2.40 \cdot 10^{-4}$ )	$9.14 \cdot 10^{-1}$ ( $2.28 \cdot 10^{-2}$ )	$1.74 \cdot 10^{-2}$ ( $3.39 \cdot 10^{-4}$ )

$R(\theta; S_{0.95}^{\text{IS}})$	
NLL/SPE	$1.46 \cdot 10^{-2}$ ( $1.55 \cdot 10^{-4}$ )

### SM-2.2.43 $\text{Re}(S_{10\pi})$ , $n = 60$ , $d = 6$

### Averages of $R(\theta; S^{\text{SPE}})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$5.76 \cdot 10^{-2} (1.62 \cdot 10^{-3})$	$4.72 \cdot 10^{-2} (1.43 \cdot 10^{-3})$	$4.66 \cdot 10^{-2} (1.50 \cdot 10^{-3})$	$4.18 \cdot 10^{-2} (1.39 \cdot 10^{-3})$	$1.75 \cdot 10^{-1} (5.05 \cdot 10^{-3})$	$4.44 \cdot 10^{-2} (1.37 \cdot 10^{-3})$
$v = 3/2$	$1.63 \cdot 10^{-2} (8.27 \cdot 10^{-4})$	$1.88 \cdot 10^{-2} (1.32 \cdot 10^{-3})$	$1.55 \cdot 10^{-2} (9.79 \cdot 10^{-4})$	$1.43 \cdot 10^{-2} (7.84 \cdot 10^{-4})$	$1.63 \cdot 10^{-1} (5.91 \cdot 10^{-3})$	$1.55 \cdot 10^{-2} (1.14 \cdot 10^{-3})$
$v = 5/2$	$1.21 \cdot 10^{-2} (8.97 \cdot 10^{-4})$	$1.36 \cdot 10^{-2} (1.10 \cdot 10^{-3})$	$1.22 \cdot 10^{-2} (9.02 \cdot 10^{-4})$	$1.15 \cdot 10^{-2} (7.73 \cdot 10^{-4})$	$1.56 \cdot 10^{-1} (5.70 \cdot 10^{-3})$	$1.18 \cdot 10^{-2} (6.24 \cdot 10^{-4})$
$v = 7/2$	$1.18 \cdot 10^{-2} (9.86 \cdot 10^{-4})$	$1.34 \cdot 10^{-2} (1.01 \cdot 10^{-3})$	$1.15 \cdot 10^{-2} (7.52 \cdot 10^{-4})$	$1.14 \cdot 10^{-2} (6.82 \cdot 10^{-4})$	$1.40 \cdot 10^{-1} (5.66 \cdot 10^{-3})$	$1.21 \cdot 10^{-2} (8.02 \cdot 10^{-4})$
$v = 9/2$	$1.18 \cdot 10^{-2} (9.03 \cdot 10^{-4})$	$1.47 \cdot 10^{-2} (2.33 \cdot 10^{-3})$	$1.11 \cdot 10^{-2} (6.43 \cdot 10^{-4})$	$1.20 \cdot 10^{-2} (9.69 \cdot 10^{-4})$	$1.30 \cdot 10^{-1} (5.63 \cdot 10^{-3})$	$1.25 \cdot 10^{-2} (7.49 \cdot 10^{-4})$
$v = 13/2$	$1.23 \cdot 10^{-2} (1.01 \cdot 10^{-3})$	$1.60 \cdot 10^{-2} (2.68 \cdot 10^{-3})$	$1.20 \cdot 10^{-2} (7.35 \cdot 10^{-4})$	$1.28 \cdot 10^{-2} (9.60 \cdot 10^{-4})$	$1.23 \cdot 10^{-1} (5.64 \cdot 10^{-3})$	$1.31 \cdot 10^{-2} (7.76 \cdot 10^{-4})$
$v = 25/2$	$1.23 \cdot 10^{-2} (9.80 \cdot 10^{-4})$	$1.50 \cdot 10^{-2} (1.23 \cdot 10^{-3})$	$1.31 \cdot 10^{-2} (8.72 \cdot 10^{-4})$	$1.34 \cdot 10^{-2} (1.06 \cdot 10^{-3})$	$1.15 \cdot 10^{-1} (5.63 \cdot 10^{-3})$	$1.38 \cdot 10^{-2} (8.60 \cdot 10^{-4})$
$v = \infty$	$1.28 \cdot 10^{-2} (1.09 \cdot 10^{-3})$	$1.60 \cdot 10^{-2} (1.43 \cdot 10^{-3})$	$1.34 \cdot 10^{-2} (9.61 \cdot 10^{-4})$	$1.41 \cdot 10^{-2} (1.17 \cdot 10^{-3})$	$1.10 \cdot 10^{-1} (6.73 \cdot 10^{-3})$	$1.57 \cdot 10^{-2} (9.70 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$1.20 \cdot 10^{-2} (8.94 \cdot 10^{-4})$	$1.45 \cdot 10^{-2} (1.28 \cdot 10^{-3})$	$1.32 \cdot 10^{-2} (9.68 \cdot 10^{-4})$	$1.36 \cdot 10^{-2} (1.19 \cdot 10^{-3})$	$1.12 \cdot 10^{-1} (5.35 \cdot 10^{-3})$	$1.45 \cdot 10^{-2} (9.25 \cdot 10^{-4})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$3.62 \cdot 10^{-2} (1.27 \cdot 10^{-3})$
$v = 3/2$	$9.75 \cdot 10^{-3} (2.74 \cdot 10^{-4})$
$v = 5/2$	$6.77 \cdot 10^{-3} (1.44 \cdot 10^{-4})$
$v = 7/2$	$6.42 \cdot 10^{-3} (1.25 \cdot 10^{-4})$
$v = 9/2$	$6.40 \cdot 10^{-3} (1.24 \cdot 10^{-4})$
$v = 13/2$	$6.45 \cdot 10^{-3} (1.25 \cdot 10^{-4})$
$v = 25/2$	$6.59 \cdot 10^{-3} (1.43 \cdot 10^{-4})$
$v = \infty$	$6.72 \cdot 10^{-3} (1.58 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$6.17 \cdot 10^{-3} (1.21 \cdot 10^{-4})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.28 \cdot 10^{-2} (1.25 \cdot 10^{-3})$
linear	$3.48 \cdot 10^{-1} (2.59 \cdot 10^{-3})$
1NN	$5.04 \cdot 10^{-1} (3.07 \cdot 10^{-3})$
2NN	$3.96 \cdot 10^{-1} (4.17 \cdot 10^{-3})$

## Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.05 \cdot 10^{-1} (1.24 \cdot 10^{-3})$	$9.22 \cdot 10^{-2} (1.35 \cdot 10^{-3})$	$9.01 \cdot 10^{-2} (1.37 \cdot 10^{-3})$	$8.27 \cdot 10^{-2} (1.19 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (2.58 \cdot 10^{-3})$	$8.80 \cdot 10^{-2} (1.19 \cdot 10^{-3})$
$v = 3/2$	$4.36 \cdot 10^{-2} (9.74 \cdot 10^{-4})$	$5.12 \cdot 10^{-2} (2.01 \cdot 10^{-3})$	$4.28 \cdot 10^{-2} (1.13 \cdot 10^{-3})$	$4.22 \cdot 10^{-2} (1.11 \cdot 10^{-3})$	$1.73 \cdot 10^{-1} (3.86 \cdot 10^{-3})$	$4.25 \cdot 10^{-2} (1.32 \cdot 10^{-3})$
$v = 5/2$	$3.40 \cdot 10^{-2} (1.00 \cdot 10^{-3})$	$4.02 \cdot 10^{-2} (1.62 \cdot 10^{-3})$	$3.53 \cdot 10^{-2} (1.07 \cdot 10^{-3})$	$3.58 \cdot 10^{-2} (1.10 \cdot 10^{-3})$	$1.64 \cdot 10^{-1} (3.39 \cdot 10^{-3})$	$3.52 \cdot 10^{-2} (8.48 \cdot 10^{-4})$
$v = 7/2$	$3.33 \cdot 10^{-2} (1.05 \cdot 10^{-3})$	$3.91 \cdot 10^{-2} (1.43 \cdot 10^{-3})$	$3.42 \cdot 10^{-2} (8.76 \cdot 10^{-4})$	$3.62 \cdot 10^{-2} (1.11 \cdot 10^{-3})$	$1.44 \cdot 10^{-1} (3.52 \cdot 10^{-3})$	$3.54 \cdot 10^{-2} (9.21 \cdot 10^{-4})$
$v = 9/2$	$3.35 \cdot 10^{-2} (8.66 \cdot 10^{-4})$	$3.93 \cdot 10^{-2} (1.91 \cdot 10^{-3})$	$3.40 \cdot 10^{-2} (7.57 \cdot 10^{-4})$	$3.65 \cdot 10^{-2} (1.18 \cdot 10^{-3})$	$1.34 \cdot 10^{-1} (3.56 \cdot 10^{-3})$	$3.65 \cdot 10^{-2} (9.47 \cdot 10^{-4})$
$v = 13/2$	$3.44 \cdot 10^{-2} (9.03 \cdot 10^{-4})$	$4.06 \cdot 10^{-2} (1.95 \cdot 10^{-3})$	$3.55 \cdot 10^{-2} (8.35 \cdot 10^{-4})$	$3.80 \cdot 10^{-2} (1.14 \cdot 10^{-3})$	$1.25 \cdot 10^{-1} (3.59 \cdot 10^{-3})$	$3.76 \cdot 10^{-2} (1.01 \cdot 10^{-3})$
$v = 25/2$	$3.50 \cdot 10^{-2} (8.46 \cdot 10^{-4})$	$4.04 \cdot 10^{-2} (1.44 \cdot 10^{-3})$	$3.71 \cdot 10^{-2} (8.99 \cdot 10^{-4})$	$3.92 \cdot 10^{-2} (1.15 \cdot 10^{-3})$	$1.18 \cdot 10^{-1} (3.61 \cdot 10^{-3})$	$3.85 \cdot 10^{-2} (1.04 \cdot 10^{-3})$
$v = \infty$	$3.60 \cdot 10^{-2} (8.87 \cdot 10^{-4})$	$4.18 \cdot 10^{-2} (1.56 \cdot 10^{-3})$	$3.81 \cdot 10^{-2} (9.68 \cdot 10^{-4})$	$4.00 \cdot 10^{-2} (1.18 \cdot 10^{-3})$	$1.12 \cdot 10^{-1} (4.16 \cdot 10^{-3})$	$4.11 \cdot 10^{-2} (1.15 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$3.41 \cdot 10^{-2} (8.96 \cdot 10^{-4})$	$4.03 \cdot 10^{-2} (1.43 \cdot 10^{-3})$	$3.76 \cdot 10^{-2} (9.78 \cdot 10^{-4})$	$3.91 \cdot 10^{-2} (1.22 \cdot 10^{-3})$	$1.18 \cdot 10^{-1} (4.26 \cdot 10^{-3})$	$3.97 \cdot 10^{-2} (1.14 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$3.57 \cdot 10^{-2} (1.25 \cdot 10^{-3})$

## Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.37 \cdot 10^0 (2.45 \cdot 10^{-2})$	$1.36 \cdot 10^0 (3.07 \cdot 10^{-2})$	$1.35 \cdot 10^0 (3.05 \cdot 10^{-2})$	$1.50 \cdot 10^0 (3.79 \cdot 10^{-2})$	$2.60 \cdot 10^0 (5.00 \cdot 10^{-2})$	$1.31 \cdot 10^0 (2.92 \cdot 10^{-2})$
$v = 3/2$	$6.23 \cdot 10^{-1} (2.17 \cdot 10^{-2})$	$7.42 \cdot 10^{-1} (2.78 \cdot 10^{-2})$	$6.79 \cdot 10^{-1} (2.59 \cdot 10^{-2})$	$6.73 \cdot 10^{-1} (2.47 \cdot 10^{-2})$	$2.20 \cdot 10^0 (4.85 \cdot 10^{-2})$	$6.49 \cdot 10^{-1} (2.49 \cdot 10^{-2})$
$v = 5/2$	$5.16 \cdot 10^{-1} (2.28 \cdot 10^{-2})$	$5.86 \cdot 10^{-1} (2.41 \cdot 10^{-2})$	$5.70 \cdot 10^{-1} (2.41 \cdot 10^{-2})$	$6.09 \cdot 10^{-1} (2.58 \cdot 10^{-2})$	$1.97 \cdot 10^0 (4.16 \cdot 10^{-2})$	$5.30 \cdot 10^{-1} (1.69 \cdot 10^{-2})$
$v = 7/2$	$5.05 \cdot 10^{-1} (2.31 \cdot 10^{-2})$	$5.78 \cdot 10^{-1} (2.30 \cdot 10^{-2})$	$5.37 \cdot 10^{-1} (1.87 \cdot 10^{-2})$	$6.25 \cdot 10^{-1} (2.51 \cdot 10^{-2})$	$1.72 \cdot 10^0 (4.41 \cdot 10^{-2})$	$5.30 \cdot 10^{-1} (1.73 \cdot 10^{-2})$
$v = 9/2$	$5.05 \cdot 10^{-1} (1.67 \cdot 10^{-2})$	$5.72 \cdot 10^{-1} (2.60 \cdot 10^{-2})$	$5.32 \cdot 10^{-1} (1.41 \cdot 10^{-2})$	$6.21 \cdot 10^{-1} (2.63 \cdot 10^{-2})$	$1.60 \cdot 10^0 (4.57 \cdot 10^{-2})$	$5.46 \cdot 10^{-1} (1.78 \cdot 10^{-2})$
$v = 13/2$	$5.25 \cdot 10^{-1} (1.73 \cdot 10^{-2})$	$5.90 \cdot 10^{-1} (2.57 \cdot 10^{-2})$	$5.64 \cdot 10^{-1} (1.70 \cdot 10^{-2})$	$6.59 \cdot 10^{-1} (2.47 \cdot 10^{-2})$	$1.50 \cdot 10^0 (4.62 \cdot 10^{-2})$	$5.71 \cdot 10^{-1} (2.03 \cdot 10^{-2})$
$v = 25/2$	$5.40 \cdot 10^{-1} (1.56 \cdot 10^{-2})$	$5.90 \cdot 10^{-1} (2.25 \cdot 10^{-2})$	$5.99 \cdot 10^{-1} (1.85 \cdot 10^{-2})$	$6.95 \cdot 10^{-1} (2.58 \cdot 10^{-2})$	$1.40 \cdot 10^0 (4.60 \cdot 10^{-2})$	$5.92 \cdot 10^{-1} (2.06 \cdot 10^{-2})$
$v = \infty$	$5.64 \cdot 10^{-1} (1.60 \cdot 10^{-2})$	$6.20 \cdot 10^{-1} (2.59 \cdot 10^{-2})$	$6.30 \cdot 10^{-1} (2.02 \cdot 10^{-2})$	$7.19 \cdot 10^{-1} (2.77 \cdot 10^{-2})$	$1.31 \cdot 10^0 (4.87 \cdot 10^{-2})$	$6.51 \cdot 10^{-1} (2.44 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$5.28 \cdot 10^{-1} (1.74 \cdot 10^{-2})$	$6.05 \cdot 10^{-1} (2.27 \cdot 10^{-2})$	$6.23 \cdot 10^{-1} (2.00 \cdot 10^{-2})$	$7.14 \cdot 10^{-1} (2.80 \cdot 10^{-2})$	$1.41 \cdot 10^0 (5.88 \cdot 10^{-2})$	$6.30 \cdot 10^{-1} (2.42 \cdot 10^{-2})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$5.54 \cdot 10^{-1} (2.07 \cdot 10^{-2})$

**SM-2.2.44**  $\text{Re}(S_{10\pi}), n = 120, d = 6$

**Averages of  $R(\theta; S^{\text{SPE}})$**

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.78 \cdot 10^{-2} (8.04 \cdot 10^{-4})$	$2.42 \cdot 10^{-2} (6.84 \cdot 10^{-4})$	$2.29 \cdot 10^{-2} (6.18 \cdot 10^{-4})$	$2.22 \cdot 10^{-2} (6.42 \cdot 10^{-4})$	$1.04 \cdot 10^{-1} (3.37 \cdot 10^{-3})$	$2.26 \cdot 10^{-2} (6.28 \cdot 10^{-4})$
$v = 3/2$	$8.67 \cdot 10^{-3} (2.99 \cdot 10^{-4})$	$9.98 \cdot 10^{-3} (9.50 \cdot 10^{-4})$	$8.01 \cdot 10^{-3} (2.85 \cdot 10^{-4})$	$8.12 \cdot 10^{-3} (5.52 \cdot 10^{-4})$	$9.76 \cdot 10^{-2} (4.06 \cdot 10^{-3})$	$8.16 \cdot 10^{-3} (3.77 \cdot 10^{-4})$
$v = 5/2$	$6.84 \cdot 10^{-3} (3.43 \cdot 10^{-4})$	$7.78 \cdot 10^{-3} (5.51 \cdot 10^{-4})$	$7.28 \cdot 10^{-3} (6.84 \cdot 10^{-4})$	$6.54 \cdot 10^{-3} (2.58 \cdot 10^{-4})$	$7.69 \cdot 10^{-2} (3.29 \cdot 10^{-3})$	$8.36 \cdot 10^{-3} (1.20 \cdot 10^{-3})$
$v = 7/2$	$6.35 \cdot 10^{-3} (3.90 \cdot 10^{-4})$	$7.05 \cdot 10^{-3} (4.24 \cdot 10^{-4})$	$6.40 \cdot 10^{-3} (2.84 \cdot 10^{-4})$	$6.20 \cdot 10^{-3} (2.92 \cdot 10^{-4})$	$6.00 \cdot 10^{-2} (2.88 \cdot 10^{-3})$	$7.39 \cdot 10^{-3} (7.05 \cdot 10^{-4})$
$v = 9/2$	$6.29 \cdot 10^{-3} (4.39 \cdot 10^{-4})$	$7.08 \cdot 10^{-3} (4.33 \cdot 10^{-4})$	$6.19 \cdot 10^{-3} (2.85 \cdot 10^{-4})$	$6.26 \cdot 10^{-3} (3.16 \cdot 10^{-4})$	$5.47 \cdot 10^{-2} (2.79 \cdot 10^{-3})$	$7.42 \cdot 10^{-3} (1.05 \cdot 10^{-3})$
$v = 13/2$	$6.37 \cdot 10^{-3} (4.85 \cdot 10^{-4})$	$7.18 \cdot 10^{-3} (4.14 \cdot 10^{-4})$	$6.36 \cdot 10^{-3} (3.23 \cdot 10^{-4})$	$6.28 \cdot 10^{-3} (3.24 \cdot 10^{-4})$	$5.06 \cdot 10^{-2} (2.69 \cdot 10^{-3})$	$7.57 \cdot 10^{-3} (7.46 \cdot 10^{-4})$
$v = 25/2$	$6.56 \cdot 10^{-3} (5.38 \cdot 10^{-4})$	$7.65 \cdot 10^{-3} (4.94 \cdot 10^{-4})$	$6.45 \cdot 10^{-3} (2.86 \cdot 10^{-4})$	$6.66 \cdot 10^{-3} (4.04 \cdot 10^{-4})$	$4.70 \cdot 10^{-2} (2.59 \cdot 10^{-3})$	$8.40 \cdot 10^{-3} (1.01 \cdot 10^{-3})$
$v = \infty$	$6.75 \cdot 10^{-3} (5.53 \cdot 10^{-4})$	$7.80 \cdot 10^{-3} (5.29 \cdot 10^{-4})$	$7.28 \cdot 10^{-3} (5.79 \cdot 10^{-4})$	$7.00 \cdot 10^{-3} (4.66 \cdot 10^{-4})$	$4.29 \cdot 10^{-2} (2.45 \cdot 10^{-3})$	$8.75 \cdot 10^{-3} (1.11 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$6.46 \cdot 10^{-3} (4.82 \cdot 10^{-4})$	$7.70 \cdot 10^{-3} (6.28 \cdot 10^{-4})$	$6.43 \cdot 10^{-3} (2.86 \cdot 10^{-4})$	$6.62 \cdot 10^{-3} (4.62 \cdot 10^{-4})$	$4.35 \cdot 10^{-2} (2.47 \cdot 10^{-3})$	$8.44 \cdot 10^{-3} (1.10 \cdot 10^{-3})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.99 \cdot 10^{-2} (5.62 \cdot 10^{-4})$
$v = 3/2$	$5.90 \cdot 10^{-3} (1.02 \cdot 10^{-4})$
$v = 5/2$	$4.46 \cdot 10^{-3} (6.52 \cdot 10^{-5})$
$v = 7/2$	$4.18 \cdot 10^{-3} (6.07 \cdot 10^{-5})$
$v = 9/2$	$4.12 \cdot 10^{-3} (6.11 \cdot 10^{-5})$
$v = 13/2$	$4.13 \cdot 10^{-3} (6.34 \cdot 10^{-5})$
$v = 25/2$	$4.21 \cdot 10^{-3} (6.68 \cdot 10^{-5})$
$v = \infty$	$4.35 \cdot 10^{-3} (7.21 \cdot 10^{-5})$
$v \in \{1/2, \dots, \infty\}$	$4.04 \cdot 10^{-3} (5.93 \cdot 10^{-5})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$6.70 \cdot 10^{-3} (5.46 \cdot 10^{-4})$
linear	$3.30 \cdot 10^{-1} (1.02 \cdot 10^{-3})$
INN	$4.03 \cdot 10^{-1} (1.76 \cdot 10^{-3})$
2NN	$3.07 \cdot 10^{-1} (2.26 \cdot 10^{-3})$

**Averages of  $R(\theta; S^{\text{CRPS}})$**

$R(\theta; \mathcal{S}^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$6.99 \cdot 10^{-2}$ ( $6.49 \cdot 10^{-4}$ )	$6.11 \cdot 10^{-2}$ ( $7.90 \cdot 10^{-4}$ )	$5.87 \cdot 10^{-2}$ ( $5.98 \cdot 10^{-4}$ )	$5.45 \cdot 10^{-2}$ ( $6.74 \cdot 10^{-4}$ )	$1.35 \cdot 10^{-1}$ ( $1.92 \cdot 10^{-3}$ )	$5.80 \cdot 10^{-2}$ ( $5.81 \cdot 10^{-4}$ )
$v = 3/2$	$2.75 \cdot 10^{-2}$ ( $4.72 \cdot 10^{-4}$ )	$3.10 \cdot 10^{-2}$ ( $1.16 \cdot 10^{-3}$ )	$2.67 \cdot 10^{-2}$ ( $5.37 \cdot 10^{-4}$ )	$2.65 \cdot 10^{-2}$ ( $5.87 \cdot 10^{-4}$ )	$1.30 \cdot 10^{-1}$ ( $3.79 \cdot 10^{-3}$ )	$2.64 \cdot 10^{-2}$ ( $5.04 \cdot 10^{-4}$ )
$v = 5/2$	$2.07 \cdot 10^{-2}$ ( $4.42 \cdot 10^{-4}$ )	$2.53 \cdot 10^{-2}$ ( $9.45 \cdot 10^{-4}$ )	$2.14 \cdot 10^{-2}$ ( $6.57 \cdot 10^{-4}$ )	$2.15 \cdot 10^{-2}$ ( $5.07 \cdot 10^{-4}$ )	$1.00 \cdot 10^{-1}$ ( $2.48 \cdot 10^{-3}$ )	$2.21 \cdot 10^{-2}$ ( $7.05 \cdot 10^{-4}$ )
$v = 7/2$	$1.93 \cdot 10^{-2}$ ( $4.35 \cdot 10^{-4}$ )	$2.34 \cdot 10^{-2}$ ( $8.94 \cdot 10^{-4}$ )	$2.01 \cdot 10^{-2}$ ( $4.81 \cdot 10^{-4}$ )	$2.05 \cdot 10^{-2}$ ( $5.16 \cdot 10^{-4}$ )	$7.57 \cdot 10^{-2}$ ( $1.92 \cdot 10^{-3}$ )	$2.10 \cdot 10^{-2}$ ( $6.07 \cdot 10^{-4}$ )
$v = 9/2$	$1.92 \cdot 10^{-2}$ ( $4.52 \cdot 10^{-4}$ )	$2.32 \cdot 10^{-2}$ ( $8.81 \cdot 10^{-4}$ )	$1.98 \cdot 10^{-2}$ ( $4.55 \cdot 10^{-4}$ )	$2.06 \cdot 10^{-2}$ ( $5.35 \cdot 10^{-4}$ )	$6.77 \cdot 10^{-2}$ ( $1.79 \cdot 10^{-3}$ )	$2.11 \cdot 10^{-2}$ ( $6.37 \cdot 10^{-4}$ )
$v = 13/2$	$1.95 \cdot 10^{-2}$ ( $4.68 \cdot 10^{-4}$ )	$2.36 \cdot 10^{-2}$ ( $8.67 \cdot 10^{-4}$ )	$2.05 \cdot 10^{-2}$ ( $5.21 \cdot 10^{-4}$ )	$2.08 \cdot 10^{-2}$ ( $5.08 \cdot 10^{-4}$ )	$6.20 \cdot 10^{-2}$ ( $1.67 \cdot 10^{-3}$ )	$2.18 \cdot 10^{-2}$ ( $6.33 \cdot 10^{-4}$ )
$v = 25/2$	$2.00 \cdot 10^{-2}$ ( $4.85 \cdot 10^{-4}$ )	$2.45 \cdot 10^{-2}$ ( $9.56 \cdot 10^{-4}$ )	$2.10 \cdot 10^{-2}$ ( $4.70 \cdot 10^{-4}$ )	$2.17 \cdot 10^{-2}$ ( $5.81 \cdot 10^{-4}$ )	$5.77 \cdot 10^{-2}$ ( $1.60 \cdot 10^{-3}$ )	$2.32 \cdot 10^{-2}$ ( $7.30 \cdot 10^{-4}$ )
$v = \infty$	$2.06 \cdot 10^{-2}$ ( $4.97 \cdot 10^{-4}$ )	$2.47 \cdot 10^{-2}$ ( $8.93 \cdot 10^{-4}$ )	$2.19 \cdot 10^{-2}$ ( $5.57 \cdot 10^{-4}$ )	$2.26 \cdot 10^{-2}$ ( $7.04 \cdot 10^{-4}$ )	$5.34 \cdot 10^{-2}$ ( $1.49 \cdot 10^{-3}$ )	$2.40 \cdot 10^{-2}$ ( $7.42 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$1.94 \cdot 10^{-2}$ ( $4.61 \cdot 10^{-4}$ )	$2.38 \cdot 10^{-2}$ ( $8.80 \cdot 10^{-4}$ )	$2.09 \cdot 10^{-2}$ ( $4.78 \cdot 10^{-4}$ )	$2.11 \cdot 10^{-2}$ ( $5.99 \cdot 10^{-4}$ )	$5.45 \cdot 10^{-2}$ ( $1.81 \cdot 10^{-3}$ )	$2.33 \cdot 10^{-2}$ ( $7.35 \cdot 10^{-4}$ )

$R(\theta; \mathcal{S}^{\text{CRPS}})$	
NLL/SPE	$2.04 \cdot 10^{-2}$ ( $6.79 \cdot 10^{-4}$ )

### Averages of $R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$

$R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$9.74 \cdot 10^{-1}$ ( $1.22 \cdot 10^{-2}$ )	$9.59 \cdot 10^{-1}$ ( $1.72 \cdot 10^{-2}$ )	$9.33 \cdot 10^{-1}$ ( $1.62 \cdot 10^{-2}$ )	$1.11 \cdot 10^0$ ( $2.24 \cdot 10^{-2}$ )	$1.94 \cdot 10^0$ ( $4.06 \cdot 10^{-2}$ )	$9.24 \cdot 10^{-1}$ ( $1.60 \cdot 10^{-2}$ )
$v = 3/2$	$4.32 \cdot 10^{-1}$ ( $9.95 \cdot 10^{-3}$ )	$4.96 \cdot 10^{-1}$ ( $2.06 \cdot 10^{-2}$ )	$4.46 \cdot 10^{-1}$ ( $1.02 \cdot 10^{-2}$ )	$4.89 \cdot 10^{-1}$ ( $1.41 \cdot 10^{-2}$ )	$1.63 \cdot 10^0$ ( $3.93 \cdot 10^{-2}$ )	$4.40 \cdot 10^{-1}$ ( $9.97 \cdot 10^{-3}$ )
$v = 5/2$	$3.44 \cdot 10^{-1}$ ( $9.53 \cdot 10^{-3}$ )	$3.81 \cdot 10^{-1}$ ( $1.28 \cdot 10^{-2}$ )	$3.58 \cdot 10^{-1}$ ( $1.29 \cdot 10^{-2}$ )	$3.85 \cdot 10^{-1}$ ( $1.17 \cdot 10^{-2}$ )	$1.26 \cdot 10^0$ ( $2.70 \cdot 10^{-2}$ )	$3.65 \cdot 10^{-1}$ ( $1.33 \cdot 10^{-2}$ )
$v = 7/2$	$3.20 \cdot 10^{-1}$ ( $9.62 \cdot 10^{-3}$ )	$3.42 \cdot 10^{-1}$ ( $1.16 \cdot 10^{-2}$ )	$3.27 \cdot 10^{-1}$ ( $9.35 \cdot 10^{-3}$ )	$3.54 \cdot 10^{-1}$ ( $1.19 \cdot 10^{-2}$ )	$1.04 \cdot 10^0$ ( $2.96 \cdot 10^{-2}$ )	$3.38 \cdot 10^{-1}$ ( $1.22 \cdot 10^{-2}$ )
$v = 9/2$	$3.15 \cdot 10^{-1}$ ( $1.02 \cdot 10^{-2}$ )	$3.31 \cdot 10^{-1}$ ( $1.14 \cdot 10^{-2}$ )	$3.13 \cdot 10^{-1}$ ( $8.70 \cdot 10^{-3}$ )	$3.48 \cdot 10^{-1}$ ( $1.18 \cdot 10^{-2}$ )	$9.59 \cdot 10^{-1}$ ( $3.06 \cdot 10^{-2}$ )	$3.30 \cdot 10^{-1}$ ( $1.31 \cdot 10^{-2}$ )
$v = 13/2$	$3.14 \cdot 10^{-1}$ ( $1.06 \cdot 10^{-2}$ )	$3.29 \cdot 10^{-1}$ ( $1.10 \cdot 10^{-2}$ )	$3.15 \cdot 10^{-1}$ ( $9.29 \cdot 10^{-3}$ )	$3.42 \cdot 10^{-1}$ ( $1.08 \cdot 10^{-2}$ )	$8.87 \cdot 10^{-1}$ ( $3.01 \cdot 10^{-2}$ )	$3.34 \cdot 10^{-1}$ ( $1.20 \cdot 10^{-2}$ )
$v = 25/2$	$3.19 \cdot 10^{-1}$ ( $1.09 \cdot 10^{-2}$ )	$3.34 \cdot 10^{-1}$ ( $1.27 \cdot 10^{-2}$ )	$3.16 \cdot 10^{-1}$ ( $8.44 \cdot 10^{-3}$ )	$3.49 \cdot 10^{-1}$ ( $1.27 \cdot 10^{-2}$ )	$8.14 \cdot 10^{-1}$ ( $2.87 \cdot 10^{-2}$ )	$3.48 \cdot 10^{-1}$ ( $1.35 \cdot 10^{-2}$ )
$v = \infty$	$3.24 \cdot 10^{-1}$ ( $1.10 \cdot 10^{-2}$ )	$3.37 \cdot 10^{-1}$ ( $1.29 \cdot 10^{-2}$ )	$3.29 \cdot 10^{-1}$ ( $1.08 \cdot 10^{-2}$ )	$3.56 \cdot 10^{-1}$ ( $1.36 \cdot 10^{-2}$ )	$7.39 \cdot 10^{-1}$ ( $2.71 \cdot 10^{-2}$ )	$3.54 \cdot 10^{-1}$ ( $1.45 \cdot 10^{-2}$ )
$v \in \{1/2, \dots, \infty\}$	$3.22 \cdot 10^{-1}$ ( $1.04 \cdot 10^{-2}$ )	$3.42 \cdot 10^{-1}$ ( $1.31 \cdot 10^{-2}$ )	$3.22 \cdot 10^{-1}$ ( $8.39 \cdot 10^{-3}$ )	$3.61 \cdot 10^{-1}$ ( $1.33 \cdot 10^{-2}$ )	$7.51 \cdot 10^{-1}$ ( $2.89 \cdot 10^{-2}$ )	$3.51 \cdot 10^{-1}$ ( $1.41 \cdot 10^{-2}$ )

$R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$	
NLL/SPE	$3.33 \cdot 10^{-1}$ ( $1.20 \cdot 10^{-2}$ )

### SM-2.2.45 $\text{Re}(S_{10\pi})$ , $n = 300$ , $d = 6$

### Averages of $R(\theta; \mathcal{S}^{\text{SPE}})$

$R(\theta; \mathcal{S}^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.22 \cdot 10^{-2}$ ( $2.90 \cdot 10^{-4}$ )	$1.11 \cdot 10^{-2}$ ( $2.69 \cdot 10^{-4}$ )	$1.06 \cdot 10^{-2}$ ( $2.61 \cdot 10^{-4}$ )	$1.03 \cdot 10^{-2}$ ( $2.54 \cdot 10^{-4}$ )	$5.10 \cdot 10^{-2}$ ( $1.57 \cdot 10^{-3}$ )	$1.05 \cdot 10^{-2}$ ( $2.50 \cdot 10^{-4}$ )
$v = 3/2$	$4.87 \cdot 10^{-3}$ ( $2.03 \cdot 10^{-4}$ )	$4.89 \cdot 10^{-3}$ ( $1.81 \cdot 10^{-4}$ )	$4.72 \cdot 10^{-3}$ ( $1.97 \cdot 10^{-4}$ )	$4.29 \cdot 10^{-3}$ ( $1.11 \cdot 10^{-4}$ )	$5.40 \cdot 10^{-2}$ ( $3.00 \cdot 10^{-3}$ )	$4.71 \cdot 10^{-3}$ ( $1.97 \cdot 10^{-4}$ )
$v = 5/2$	$4.01 \cdot 10^{-3}$ ( $2.15 \cdot 10^{-4}$ )	$4.27 \cdot 10^{-3}$ ( $1.97 \cdot 10^{-4}$ )	$4.10 \cdot 10^{-3}$ ( $2.31 \cdot 10^{-4}$ )	$3.75 \cdot 10^{-3}$ ( $1.09 \cdot 10^{-4}$ )	$3.25 \cdot 10^{-2}$ ( $1.41 \cdot 10^{-3}$ )	$4.21 \cdot 10^{-3}$ ( $3.38 \cdot 10^{-4}$ )
$v = 7/2$	$3.43 \cdot 10^{-3}$ ( $1.63 \cdot 10^{-4}$ )	$3.70 \cdot 10^{-3}$ ( $1.34 \cdot 10^{-4}$ )	$3.54 \cdot 10^{-3}$ ( $1.90 \cdot 10^{-4}$ )	$3.31 \cdot 10^{-3}$ ( $9.34 \cdot 10^{-5}$ )	$2.49 \cdot 10^{-2}$ ( $1.13 \cdot 10^{-3}$ )	$3.56 \cdot 10^{-3}$ ( $1.19 \cdot 10^{-4}$ )
$v = 9/2$	$3.17 \cdot 10^{-3}$ ( $1.24 \cdot 10^{-4}$ )	$3.55 \cdot 10^{-3}$ ( $1.33 \cdot 10^{-4}$ )	$3.22 \cdot 10^{-3}$ ( $1.15 \cdot 10^{-4}$ )	$3.12 \cdot 10^{-3}$ ( $8.04 \cdot 10^{-5}$ )	$2.19 \cdot 10^{-2}$ ( $9.76 \cdot 10^{-4}$ )	$3.42 \cdot 10^{-3}$ ( $1.19 \cdot 10^{-4}$ )
$v = 13/2$	$3.04 \cdot 10^{-3}$ ( $1.01 \cdot 10^{-4}$ )	$3.53 \cdot 10^{-3}$ ( $1.52 \cdot 10^{-4}$ )	$3.11 \cdot 10^{-3}$ ( $9.66 \cdot 10^{-5}$ )	$3.00 \cdot 10^{-3}$ ( $7.05 \cdot 10^{-5}$ )	$1.95 \cdot 10^{-2}$ ( $8.40 \cdot 10^{-4}$ )	$3.41 \cdot 10^{-3}$ ( $9.80 \cdot 10^{-5}$ )
$v = 25/2$	$3.06 \cdot 10^{-3}$ ( $9.93 \cdot 10^{-5}$ )	$3.45 \cdot 10^{-3}$ ( $1.23 \cdot 10^{-4}$ )	$3.21 \cdot 10^{-3}$ ( $1.02 \cdot 10^{-4}$ )	$3.09 \cdot 10^{-3}$ ( $8.41 \cdot 10^{-5}$ )	$1.75 \cdot 10^{-2}$ ( $7.66 \cdot 10^{-4}$ )	$3.89 \cdot 10^{-3}$ ( $2.02 \cdot 10^{-4}$ )
$v = \infty$	$3.20 \cdot 10^{-3}$ ( $1.07 \cdot 10^{-4}$ )	$3.75 \cdot 10^{-3}$ ( $1.66 \cdot 10^{-4}$ )	$3.51 \cdot 10^{-3}$ ( $1.32 \cdot 10^{-4}$ )	$3.28 \cdot 10^{-3}$ ( $1.02 \cdot 10^{-4}$ )	$1.54 \cdot 10^{-2}$ ( $6.68 \cdot 10^{-4}$ )	$4.44 \cdot 10^{-3}$ ( $2.73 \cdot 10^{-4}$ )
$v \in \{1/2, \dots, \infty\}$	$3.25 \cdot 10^{-3}$ ( $1.24 \cdot 10^{-4}$ )	$3.73 \cdot 10^{-3}$ ( $2.02 \cdot 10^{-4}$ )	$3.26 \cdot 10^{-3}$ ( $1.10 \cdot 10^{-4}$ )	$3.13 \cdot 10^{-3}$ ( $8.08 \cdot 10^{-5}$ )	$1.55 \cdot 10^{-2}$ ( $6.70 \cdot 10^{-4}$ )	$3.81 \cdot 10^{-3}$ ( $1.59 \cdot 10^{-4}$ )

$R^*(S^{SPE})$	
$v = 1/2$	$9.44 \cdot 10^{-3} (2.09 \cdot 10^{-4})$
$v = 3/2$	$3.46 \cdot 10^{-3} (5.06 \cdot 10^{-5})$
$v = 5/2$	$2.77 \cdot 10^{-3} (4.56 \cdot 10^{-5})$
$v = 7/2$	$2.50 \cdot 10^{-3} (4.59 \cdot 10^{-5})$
$v = 9/2$	$2.39 \cdot 10^{-3} (4.50 \cdot 10^{-5})$
$v = 13/2$	$2.37 \cdot 10^{-3} (4.23 \cdot 10^{-5})$
$v = 25/2$	$2.40 \cdot 10^{-3} (3.84 \cdot 10^{-5})$
$v = \infty$	$2.48 \cdot 10^{-3} (3.52 \cdot 10^{-5})$
$v \in \{1/2, \dots, \infty\}$	$2.33 \cdot 10^{-3} (4.18 \cdot 10^{-5})$

$R(\theta; S^{SPE})$	
NLL/SPE	$3.34 \cdot 10^{-3} (1.56 \cdot 10^{-4})$
linear	$3.19 \cdot 10^{-1} (3.59 \cdot 10^{-4})$
1NN	$3.05 \cdot 10^{-1} (9.04 \cdot 10^{-4})$
2NN	$2.27 \cdot 10^{-1} (1.24 \cdot 10^{-3})$

### Averages of $R(\theta; S^{CRPS})$

$R(\theta; S^{CRPS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$4.36 \cdot 10^{-2} (2.71 \cdot 10^{-4})$	$3.75 \cdot 10^{-2} (4.08 \cdot 10^{-4})$	$3.61 \cdot 10^{-2} (3.58 \cdot 10^{-4})$	$3.20 \cdot 10^{-2} (3.60 \cdot 10^{-4})$	$8.62 \cdot 10^{-2} (1.59 \cdot 10^{-3})$	$3.58 \cdot 10^{-2} (3.29 \cdot 10^{-4})$
$v = 3/2$	$1.67 \cdot 10^{-2} (2.85 \cdot 10^{-4})$	$1.83 \cdot 10^{-2} (4.77 \cdot 10^{-4})$	$1.63 \cdot 10^{-2} (3.39 \cdot 10^{-4})$	$1.54 \cdot 10^{-2} (2.51 \cdot 10^{-4})$	$9.32 \cdot 10^{-2} (3.96 \cdot 10^{-3})$	$1.60 \cdot 10^{-2} (3.12 \cdot 10^{-4})$
$v = 5/2$	$1.17 \cdot 10^{-2} (2.72 \cdot 10^{-4})$	$1.42 \cdot 10^{-2} (5.33 \cdot 10^{-4})$	$1.19 \cdot 10^{-2} (2.93 \cdot 10^{-4})$	$1.18 \cdot 10^{-2} (2.64 \cdot 10^{-4})$	$5.32 \cdot 10^{-2} (1.48 \cdot 10^{-3})$	$1.20 \cdot 10^{-2} (2.93 \cdot 10^{-4})$
$v = 7/2$	$1.01 \cdot 10^{-2} (2.32 \cdot 10^{-4})$	$1.22 \cdot 10^{-2} (4.35 \cdot 10^{-4})$	$1.03 \cdot 10^{-2} (2.49 \cdot 10^{-4})$	$1.03 \cdot 10^{-2} (2.38 \cdot 10^{-4})$	$3.91 \cdot 10^{-2} (1.19 \cdot 10^{-3})$	$1.07 \cdot 10^{-2} (2.46 \cdot 10^{-4})$
$v = 9/2$	$9.72 \cdot 10^{-3} (2.12 \cdot 10^{-4})$	$1.17 \cdot 10^{-2} (4.05 \cdot 10^{-4})$	$9.98 \cdot 10^{-3} (2.15 \cdot 10^{-4})$	$9.99 \cdot 10^{-3} (2.19 \cdot 10^{-4})$	$3.39 \cdot 10^{-2} (9.63 \cdot 10^{-4})$	$1.07 \cdot 10^{-2} (2.56 \cdot 10^{-4})$
$v = 13/2$	$9.87 \cdot 10^{-3} (2.03 \cdot 10^{-4})$	$1.20 \cdot 10^{-2} (4.53 \cdot 10^{-4})$	$1.02 \cdot 10^{-2} (2.09 \cdot 10^{-4})$	$1.02 \cdot 10^{-2} (2.07 \cdot 10^{-4})$	$3.00 \cdot 10^{-2} (7.61 \cdot 10^{-4})$	$1.10 \cdot 10^{-2} (2.44 \cdot 10^{-4})$
$v = 25/2$	$1.03 \cdot 10^{-2} (2.11 \cdot 10^{-4})$	$1.22 \cdot 10^{-2} (3.57 \cdot 10^{-4})$	$1.08 \cdot 10^{-2} (2.22 \cdot 10^{-4})$	$1.08 \cdot 10^{-2} (2.11 \cdot 10^{-4})$	$2.71 \cdot 10^{-2} (6.37 \cdot 10^{-4})$	$1.19 \cdot 10^{-2} (3.13 \cdot 10^{-4})$
$v = \infty$	$1.09 \cdot 10^{-2} (2.30 \cdot 10^{-4})$	$1.30 \cdot 10^{-2} (4.34 \cdot 10^{-4})$	$1.16 \cdot 10^{-2} (2.65 \cdot 10^{-4})$	$1.15 \cdot 10^{-2} (2.32 \cdot 10^{-4})$	$2.46 \cdot 10^{-2} (5.52 \cdot 10^{-4})$	$1.29 \cdot 10^{-2} (3.64 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$9.83 \cdot 10^{-3} (2.09 \cdot 10^{-4})$	$1.22 \cdot 10^{-2} (5.01 \cdot 10^{-4})$	$1.04 \cdot 10^{-2} (2.42 \cdot 10^{-4})$	$1.01 \cdot 10^{-2} (2.24 \cdot 10^{-4})$	$2.47 \cdot 10^{-2} (5.50 \cdot 10^{-4})$	$1.18 \cdot 10^{-2} (2.93 \cdot 10^{-4})$

$R(\theta; S^{CRPS})$	
NLL/SPE	$1.07 \cdot 10^{-2} (5.15 \cdot 10^{-4})$

### Averages of $R(\theta; S_{0.95}^{IS})$

$R(\theta; S_{0.95}^{IS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$6.70 \cdot 10^{-1} (6.06 \cdot 10^{-3})$	$6.32 \cdot 10^{-1} (9.34 \cdot 10^{-3})$	$6.19 \cdot 10^{-1} (8.88 \cdot 10^{-3})$	$7.57 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$1.32 \cdot 10^0 (1.96 \cdot 10^{-2})$	$6.15 \cdot 10^{-1} (8.24 \cdot 10^{-3})$
$v = 3/2$	$2.92 \cdot 10^{-1} (6.92 \cdot 10^{-3})$	$3.19 \cdot 10^{-1} (7.71 \cdot 10^{-3})$	$2.99 \cdot 10^{-1} (7.01 \cdot 10^{-3})$	$3.32 \cdot 10^{-1} (6.88 \cdot 10^{-3})$	$1.18 \cdot 10^0 (4.01 \cdot 10^{-2})$	$2.94 \cdot 10^{-1} (6.80 \cdot 10^{-3})$
$v = 5/2$	$2.19 \cdot 10^{-1} (6.80 \cdot 10^{-3})$	$2.43 \cdot 10^{-1} (7.67 \cdot 10^{-3})$	$2.25 \cdot 10^{-1} (6.80 \cdot 10^{-3})$	$2.45 \cdot 10^{-1} (6.05 \cdot 10^{-3})$	$7.39 \cdot 10^{-1} (1.89 \cdot 10^{-2})$	$2.22 \cdot 10^{-1} (6.43 \cdot 10^{-3})$
$v = 7/2$	$1.90 \cdot 10^{-1} (5.61 \cdot 10^{-3})$	$2.05 \cdot 10^{-1} (6.18 \cdot 10^{-3})$	$1.92 \cdot 10^{-1} (5.62 \cdot 10^{-3})$	$2.10 \cdot 10^{-1} (5.52 \cdot 10^{-3})$	$5.84 \cdot 10^{-1} (1.67 \cdot 10^{-2})$	$1.93 \cdot 10^{-1} (4.70 \cdot 10^{-3})$
$v = 9/2$	$1.79 \cdot 10^{-1} (4.93 \cdot 10^{-3})$	$1.93 \cdot 10^{-1} (5.84 \cdot 10^{-3})$	$1.78 \cdot 10^{-1} (4.46 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (4.90 \cdot 10^{-3})$	$5.21 \cdot 10^{-1} (1.37 \cdot 10^{-2})$	$1.85 \cdot 10^{-1} (4.53 \cdot 10^{-3})$
$v = 13/2$	$1.74 \cdot 10^{-1} (4.47 \cdot 10^{-3})$	$1.88 \cdot 10^{-1} (5.92 \cdot 10^{-3})$	$1.73 \cdot 10^{-1} (4.02 \cdot 10^{-3})$	$1.86 \cdot 10^{-1} (4.19 \cdot 10^{-3})$	$4.60 \cdot 10^{-1} (1.11 \cdot 10^{-2})$	$1.82 \cdot 10^{-1} (3.81 \cdot 10^{-3})$
$v = 25/2$	$1.73 \cdot 10^{-1} (4.31 \cdot 10^{-3})$	$1.81 \cdot 10^{-1} (4.79 \cdot 10^{-3})$	$1.73 \cdot 10^{-1} (4.00 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (3.85 \cdot 10^{-3})$	$4.01 \cdot 10^{-1} (9.62 \cdot 10^{-3})$	$1.89 \cdot 10^{-1} (5.10 \cdot 10^{-3})$
$v = \infty$	$1.77 \cdot 10^{-1} (4.31 \cdot 10^{-3})$	$1.85 \cdot 10^{-1} (5.06 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (4.52 \cdot 10^{-3})$	$1.83 \cdot 10^{-1} (3.71 \cdot 10^{-3})$	$3.44 \cdot 10^{-1} (8.12 \cdot 10^{-3})$	$1.97 \cdot 10^{-1} (5.77 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$1.82 \cdot 10^{-1} (4.85 \cdot 10^{-3})$	$1.98 \cdot 10^{-1} (7.12 \cdot 10^{-3})$	$1.77 \cdot 10^{-1} (4.33 \cdot 10^{-3})$	$1.96 \cdot 10^{-1} (4.80 \cdot 10^{-3})$	$3.45 \cdot 10^{-1} (8.06 \cdot 10^{-3})$	$1.91 \cdot 10^{-1} (4.87 \cdot 10^{-3})$

$R(\theta; S_{0.95}^{IS})$	
NLL/SPE	$1.88 \cdot 10^{-1} (8.12 \cdot 10^{-3})$

**SM-2.2.46**  $\text{Re}(S_{14\pi}), n = 60, d = 6$

### Averages of $R(\theta; S^{SPE})$

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.40 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$2.09 \cdot 10^{-1} (7.27 \cdot 10^{-3})$	$2.32 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$2.13 \cdot 10^{-1} (8.34 \cdot 10^{-3})$	$4.72 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$2.40 \cdot 10^{-1} (1.27 \cdot 10^{-2})$
$v = 3/2$	$1.24 \cdot 10^{-1} (1.07 \cdot 10^{-2})$	$1.48 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$1.26 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$1.13 \cdot 10^{-1} (7.54 \cdot 10^{-3})$	$4.81 \cdot 10^{-1} (1.72 \cdot 10^{-2})$	$1.49 \cdot 10^{-1} (1.46 \cdot 10^{-2})$
$v = 5/2$	$9.95 \cdot 10^{-2} (5.04 \cdot 10^{-3})$	$1.23 \cdot 10^{-1} (1.62 \cdot 10^{-2})$	$1.04 \cdot 10^{-1} (9.78 \cdot 10^{-3})$	$9.08 \cdot 10^{-2} (5.48 \cdot 10^{-3})$	$4.84 \cdot 10^{-1} (1.85 \cdot 10^{-2})$	$1.20 \cdot 10^{-1} (7.82 \cdot 10^{-3})$
$v = 7/2$	$1.06 \cdot 10^{-1} (5.11 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (9.24 \cdot 10^{-3})$	$9.63 \cdot 10^{-2} (6.20 \cdot 10^{-3})$	$8.82 \cdot 10^{-2} (5.44 \cdot 10^{-3})$	$4.74 \cdot 10^{-1} (1.88 \cdot 10^{-2})$	$1.37 \cdot 10^{-1} (8.17 \cdot 10^{-3})$
$v = 9/2$	$1.12 \cdot 10^{-1} (4.90 \cdot 10^{-3})$	$1.08 \cdot 10^{-1} (9.66 \cdot 10^{-3})$	$9.91 \cdot 10^{-2} (5.74 \cdot 10^{-3})$	$9.24 \cdot 10^{-2} (6.11 \cdot 10^{-3})$	$4.66 \cdot 10^{-1} (1.86 \cdot 10^{-2})$	$1.49 \cdot 10^{-1} (8.47 \cdot 10^{-3})$
$v = 13/2$	$1.20 \cdot 10^{-1} (5.26 \cdot 10^{-3})$	$1.06 \cdot 10^{-1} (8.88 \cdot 10^{-3})$	$1.04 \cdot 10^{-1} (6.04 \cdot 10^{-3})$	$9.36 \cdot 10^{-2} (6.15 \cdot 10^{-3})$	$4.63 \cdot 10^{-1} (1.96 \cdot 10^{-2})$	$1.61 \cdot 10^{-1} (9.73 \cdot 10^{-3})$
$v = 25/2$	$1.29 \cdot 10^{-1} (5.71 \cdot 10^{-3})$	$1.38 \cdot 10^{-1} (3.35 \cdot 10^{-2})$	$1.13 \cdot 10^{-1} (6.83 \cdot 10^{-3})$	$1.01 \cdot 10^{-1} (6.59 \cdot 10^{-3})$	$4.62 \cdot 10^{-1} (2.08 \cdot 10^{-2})$	$1.73 \cdot 10^{-1} (9.81 \cdot 10^{-3})$
$v = \infty$	$1.38 \cdot 10^{-1} (6.25 \cdot 10^{-3})$	$1.09 \cdot 10^{-1} (7.38 \cdot 10^{-3})$	$1.18 \cdot 10^{-1} (7.53 \cdot 10^{-3})$	$1.05 \cdot 10^{-1} (6.61 \cdot 10^{-3})$	$4.47 \cdot 10^{-1} (1.95 \cdot 10^{-2})$	$1.93 \cdot 10^{-1} (1.17 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$1.04 \cdot 10^{-1} (5.74 \cdot 10^{-3})$	$1.50 \cdot 10^{-1} (3.04 \cdot 10^{-2})$	$1.06 \cdot 10^{-1} (6.48 \cdot 10^{-3})$	$9.72 \cdot 10^{-2} (6.21 \cdot 10^{-3})$	$4.48 \cdot 10^{-1} (1.72 \cdot 10^{-2})$	$1.76 \cdot 10^{-1} (1.21 \cdot 10^{-2})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.73 \cdot 10^{-1} (6.96 \cdot 10^{-3})$
$v = 3/2$	$5.81 \cdot 10^{-2} (2.79 \cdot 10^{-3})$
$v = 5/2$	$3.89 \cdot 10^{-2} (1.92 \cdot 10^{-3})$
$v = 7/2$	$3.39 \cdot 10^{-2} (1.48 \cdot 10^{-3})$
$v = 9/2$	$3.31 \cdot 10^{-2} (1.35 \cdot 10^{-3})$
$v = 13/2$	$3.23 \cdot 10^{-2} (1.23 \cdot 10^{-3})$
$v = 25/2$	$3.22 \cdot 10^{-2} (1.15 \cdot 10^{-3})$
$v = \infty$	$3.20 \cdot 10^{-2} (1.11 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$3.03 \cdot 10^{-2} (1.14 \cdot 10^{-3})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$1.20 \cdot 10^{-1} (6.53 \cdot 10^{-3})$
linear	$7.39 \cdot 10^{-1} (1.18 \cdot 10^{-2})$
1NN	$7.44 \cdot 10^{-1} (5.33 \cdot 10^{-3})$
2NN	$6.53 \cdot 10^{-1} (6.13 \cdot 10^{-3})$

## Averages of $R(\theta; S^{\text{CRPS}})$

$R(\theta; S^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.28 \cdot 10^{-1} (3.04 \cdot 10^{-3})$	$1.20 \cdot 10^{-1} (2.68 \cdot 10^{-3})$	$1.22 \cdot 10^{-1} (3.22 \cdot 10^{-3})$	$9.84 \cdot 10^{-2} (2.17 \cdot 10^{-3})$	$2.31 \cdot 10^{-1} (9.08 \cdot 10^{-3})$	$1.27 \cdot 10^{-1} (4.26 \cdot 10^{-3})$
$v = 3/2$	$6.81 \cdot 10^{-2} (2.28 \cdot 10^{-3})$	$8.71 \cdot 10^{-2} (3.86 \cdot 10^{-3})$	$7.03 \cdot 10^{-2} (2.51 \cdot 10^{-3})$	$6.78 \cdot 10^{-2} (2.43 \cdot 10^{-3})$	$2.31 \cdot 10^{-1} (1.34 \cdot 10^{-2})$	$8.03 \cdot 10^{-2} (5.53 \cdot 10^{-3})$
$v = 5/2$	$5.58 \cdot 10^{-2} (1.54 \cdot 10^{-3})$	$7.30 \cdot 10^{-2} (5.01 \cdot 10^{-3})$	$5.87 \cdot 10^{-2} (2.19 \cdot 10^{-3})$	$5.84 \cdot 10^{-2} (1.79 \cdot 10^{-3})$	$2.29 \cdot 10^{-1} (1.40 \cdot 10^{-2})$	$6.32 \cdot 10^{-2} (2.04 \cdot 10^{-3})$
$v = 7/2$	$5.60 \cdot 10^{-2} (1.45 \cdot 10^{-3})$	$6.69 \cdot 10^{-2} (3.92 \cdot 10^{-3})$	$5.71 \cdot 10^{-2} (1.59 \cdot 10^{-3})$	$5.81 \cdot 10^{-2} (1.61 \cdot 10^{-3})$	$2.18 \cdot 10^{-1} (1.26 \cdot 10^{-2})$	$6.77 \cdot 10^{-2} (2.24 \cdot 10^{-3})$
$v = 9/2$	$5.74 \cdot 10^{-2} (1.37 \cdot 10^{-3})$	$7.03 \cdot 10^{-2} (4.07 \cdot 10^{-3})$	$5.92 \cdot 10^{-2} (1.53 \cdot 10^{-3})$	$5.99 \cdot 10^{-2} (1.78 \cdot 10^{-3})$	$2.09 \cdot 10^{-1} (1.14 \cdot 10^{-2})$	$6.95 \cdot 10^{-2} (2.12 \cdot 10^{-3})$
$v = 13/2$	$5.91 \cdot 10^{-2} (1.40 \cdot 10^{-3})$	$7.05 \cdot 10^{-2} (3.78 \cdot 10^{-3})$	$6.10 \cdot 10^{-2} (1.59 \cdot 10^{-3})$	$6.17 \cdot 10^{-2} (1.83 \cdot 10^{-3})$	$2.10 \cdot 10^{-1} (1.33 \cdot 10^{-2})$	$7.29 \cdot 10^{-2} (2.43 \cdot 10^{-3})$
$v = 25/2$	$6.15 \cdot 10^{-2} (1.46 \cdot 10^{-3})$	$7.55 \cdot 10^{-2} (5.96 \cdot 10^{-3})$	$6.41 \cdot 10^{-2} (1.82 \cdot 10^{-3})$	$6.44 \cdot 10^{-2} (1.71 \cdot 10^{-3})$	$2.03 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$7.59 \cdot 10^{-2} (2.41 \cdot 10^{-3})$
$v = \infty$	$6.36 \cdot 10^{-2} (1.54 \cdot 10^{-3})$	$7.37 \cdot 10^{-2} (3.18 \cdot 10^{-3})$	$6.58 \cdot 10^{-2} (1.91 \cdot 10^{-3})$	$6.68 \cdot 10^{-2} (1.97 \cdot 10^{-3})$	$1.94 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$8.02 \cdot 10^{-2} (2.91 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$5.69 \cdot 10^{-2} (1.68 \cdot 10^{-3})$	$8.17 \cdot 10^{-2} (6.09 \cdot 10^{-3})$	$6.20 \cdot 10^{-2} (1.66 \cdot 10^{-3})$	$6.41 \cdot 10^{-2} (2.16 \cdot 10^{-3})$	$2.15 \cdot 10^{-1} (1.39 \cdot 10^{-2})$	$7.83 \cdot 10^{-2} (4.12 \cdot 10^{-3})$

$R(\theta; S^{\text{CRPS}})$	
NLL/SPE	$6.08 \cdot 10^{-2} (1.86 \cdot 10^{-3})$

## Averages of $R(\theta; S_{0.95}^{\text{IS}})$

$R(\theta; S_{0.95}^S)$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.64 \cdot 10^0 (7.42 \cdot 10^{-2})$	$2.45 \cdot 10^0 (5.78 \cdot 10^{-2})$	$2.64 \cdot 10^0 (7.67 \cdot 10^{-2})$	$2.92 \cdot 10^0 (8.40 \cdot 10^{-2})$	$4.38 \cdot 10^0 (1.17 \cdot 10^{-1})$	$2.71 \cdot 10^0 (9.41 \cdot 10^{-2})$
$v = 3/2$	$1.59 \cdot 10^0 (7.57 \cdot 10^{-2})$	$1.84 \cdot 10^0 (7.24 \cdot 10^{-2})$	$1.71 \cdot 10^0 (8.34 \cdot 10^{-2})$	$1.86 \cdot 10^0 (8.49 \cdot 10^{-2})$	$4.18 \cdot 10^0 (1.81 \cdot 10^{-1})$	$1.76 \cdot 10^0 (9.40 \cdot 10^{-2})$
$v = 5/2$	$1.27 \cdot 10^0 (4.72 \cdot 10^{-2})$	$1.41 \cdot 10^0 (7.04 \cdot 10^{-2})$	$1.36 \cdot 10^0 (7.10 \cdot 10^{-2})$	$1.45 \cdot 10^0 (6.27 \cdot 10^{-2})$	$4.06 \cdot 10^0 (1.90 \cdot 10^{-1})$	$1.38 \cdot 10^0 (5.83 \cdot 10^{-2})$
$v = 7/2$	$1.26 \cdot 10^0 (4.43 \cdot 10^{-2})$	$1.24 \cdot 10^0 (5.78 \cdot 10^{-2})$	$1.27 \cdot 10^0 (5.35 \cdot 10^{-2})$	$1.35 \cdot 10^0 (5.91 \cdot 10^{-2})$	$3.89 \cdot 10^0 (1.63 \cdot 10^{-1})$	$1.48 \cdot 10^0 (6.38 \cdot 10^{-2})$
$v = 9/2$	$1.28 \cdot 10^0 (4.26 \cdot 10^{-2})$	$1.29 \cdot 10^0 (6.52 \cdot 10^{-2})$	$1.29 \cdot 10^0 (4.96 \cdot 10^{-2})$	$1.36 \cdot 10^0 (6.02 \cdot 10^{-2})$	$3.75 \cdot 10^0 (1.44 \cdot 10^{-1})$	$1.53 \cdot 10^0 (6.26 \cdot 10^{-2})$
$v = 13/2$	$1.32 \cdot 10^0 (4.42 \cdot 10^{-2})$	$1.27 \cdot 10^0 (5.95 \cdot 10^{-2})$	$1.32 \cdot 10^0 (5.02 \cdot 10^{-2})$	$1.36 \cdot 10^0 (6.06 \cdot 10^{-2})$	$3.73 \cdot 10^0 (1.76 \cdot 10^{-1})$	$1.59 \cdot 10^0 (7.08 \cdot 10^{-2})$
$v = 25/2$	$1.38 \cdot 10^0 (4.64 \cdot 10^{-2})$	$1.31 \cdot 10^0 (6.64 \cdot 10^{-2})$	$1.39 \cdot 10^0 (5.68 \cdot 10^{-2})$	$1.43 \cdot 10^0 (6.43 \cdot 10^{-2})$	$3.59 \cdot 10^0 (1.56 \cdot 10^{-1})$	$1.65 \cdot 10^0 (6.98 \cdot 10^{-2})$
$v = \infty$	$1.42 \cdot 10^0 (4.89 \cdot 10^{-2})$	$1.32 \cdot 10^0 (6.47 \cdot 10^{-2})$	$1.41 \cdot 10^0 (6.17 \cdot 10^{-2})$	$1.47 \cdot 10^0 (6.70 \cdot 10^{-2})$	$3.43 \cdot 10^0 (1.52 \cdot 10^{-1})$	$1.76 \cdot 10^0 (7.66 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$1.31 \cdot 10^0 (5.22 \cdot 10^{-2})$	$1.45 \cdot 10^0 (7.08 \cdot 10^{-2})$	$1.37 \cdot 10^0 (5.44 \cdot 10^{-2})$	$1.52 \cdot 10^0 (7.21 \cdot 10^{-2})$	$3.85 \cdot 10^0 (1.96 \cdot 10^{-1})$	$1.68 \cdot 10^0 (7.87 \cdot 10^{-2})$

$R(\theta; S_{0.95}^S)$	
NLL/SPE	$1.39 \cdot 10^0 (5.56 \cdot 10^{-2})$

**SM-2.2.47**  $\text{Re}(S_{14\pi}), n = 120, d = 6$

**Averages of  $R(\theta; S^{\text{SPE}})$**

$R(\theta; S^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.66 \cdot 10^{-1} (5.10 \cdot 10^{-3})$	$1.51 \cdot 10^{-1} (5.29 \cdot 10^{-3})$	$1.61 \cdot 10^{-1} (5.52 \cdot 10^{-3})$	$1.44 \cdot 10^{-1} (5.35 \cdot 10^{-3})$	$3.68 \cdot 10^{-1} (1.00 \cdot 10^{-2})$	$1.53 \cdot 10^{-1} (5.05 \cdot 10^{-3})$
$v = 3/2$	$6.20 \cdot 10^{-2} (2.63 \cdot 10^{-3})$	$8.39 \cdot 10^{-2} (9.93 \cdot 10^{-3})$	$6.41 \cdot 10^{-2} (2.71 \cdot 10^{-3})$	$6.01 \cdot 10^{-2} (2.99 \cdot 10^{-3})$	$3.76 \cdot 10^{-1} (1.80 \cdot 10^{-2})$	$6.17 \cdot 10^{-2} (2.78 \cdot 10^{-3})$
$v = 5/2$	$4.99 \cdot 10^{-2} (2.24 \cdot 10^{-3})$	$5.44 \cdot 10^{-2} (6.37 \cdot 10^{-3})$	$4.97 \cdot 10^{-2} (2.66 \cdot 10^{-3})$	$4.55 \cdot 10^{-2} (2.51 \cdot 10^{-3})$	$3.53 \cdot 10^{-1} (1.73 \cdot 10^{-2})$	$5.62 \cdot 10^{-2} (3.47 \cdot 10^{-3})$
$v = 7/2$	$5.54 \cdot 10^{-2} (2.69 \cdot 10^{-3})$	$4.75 \cdot 10^{-2} (3.24 \cdot 10^{-3})$	$4.85 \cdot 10^{-2} (2.73 \cdot 10^{-3})$	$4.77 \cdot 10^{-2} (3.22 \cdot 10^{-3})$	$3.28 \cdot 10^{-1} (1.68 \cdot 10^{-2})$	$6.77 \cdot 10^{-2} (3.96 \cdot 10^{-3})$
$v = 9/2$	$6.18 \cdot 10^{-2} (3.03 \cdot 10^{-3})$	$4.90 \cdot 10^{-2} (3.88 \cdot 10^{-3})$	$5.46 \cdot 10^{-2} (3.31 \cdot 10^{-3})$	$5.04 \cdot 10^{-2} (3.40 \cdot 10^{-3})$	$3.17 \cdot 10^{-1} (1.67 \cdot 10^{-2})$	$7.52 \cdot 10^{-2} (4.28 \cdot 10^{-3})$
$v = 13/2$	$7.03 \cdot 10^{-2} (3.38 \cdot 10^{-3})$	$5.28 \cdot 10^{-2} (4.09 \cdot 10^{-3})$	$6.37 \cdot 10^{-2} (3.88 \cdot 10^{-3})$	$5.52 \cdot 10^{-2} (3.64 \cdot 10^{-3})$	$3.09 \cdot 10^{-1} (1.81 \cdot 10^{-2})$	$9.50 \cdot 10^{-2} (5.36 \cdot 10^{-3})$
$v = 25/2$	$7.81 \cdot 10^{-2} (3.68 \cdot 10^{-3})$	$5.58 \cdot 10^{-2} (4.31 \cdot 10^{-3})$	$7.41 \cdot 10^{-2} (4.58 \cdot 10^{-3})$	$6.23 \cdot 10^{-2} (4.33 \cdot 10^{-3})$	$2.97 \cdot 10^{-1} (1.73 \cdot 10^{-2})$	$1.12 \cdot 10^{-1} (7.16 \cdot 10^{-3})$
$v = \infty$	$8.57 \cdot 10^{-2} (3.94 \cdot 10^{-3})$	$6.38 \cdot 10^{-2} (5.14 \cdot 10^{-3})$	$8.09 \cdot 10^{-2} (4.91 \cdot 10^{-3})$	$7.47 \cdot 10^{-2} (6.23 \cdot 10^{-3})$	$2.84 \cdot 10^{-1} (1.69 \cdot 10^{-2})$	$1.24 \cdot 10^{-1} (7.71 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$5.04 \cdot 10^{-2} (2.25 \cdot 10^{-3})$	$6.66 \cdot 10^{-2} (1.09 \cdot 10^{-2})$	$5.53 \cdot 10^{-2} (3.44 \cdot 10^{-3})$	$4.75 \cdot 10^{-2} (2.82 \cdot 10^{-3})$	$3.09 \cdot 10^{-1} (1.75 \cdot 10^{-2})$	$8.25 \cdot 10^{-2} (5.16 \cdot 10^{-3})$

$R^*(S^{\text{SPE}})$	
$v = 1/2$	$1.27 \cdot 10^{-1} (5.41 \cdot 10^{-3})$
$v = 3/2$	$3.81 \cdot 10^{-2} (1.96 \cdot 10^{-3})$
$v = 5/2$	$2.33 \cdot 10^{-2} (1.05 \cdot 10^{-3})$
$v = 7/2$	$2.01 \cdot 10^{-2} (8.10 \cdot 10^{-4})$
$v = 9/2$	$1.93 \cdot 10^{-2} (7.21 \cdot 10^{-4})$
$v = 13/2$	$1.94 \cdot 10^{-2} (6.63 \cdot 10^{-4})$
$v = 25/2$	$1.94 \cdot 10^{-2} (6.07 \cdot 10^{-4})$
$v = \infty$	$2.05 \cdot 10^{-2} (6.06 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$1.82 \cdot 10^{-2} (6.77 \cdot 10^{-4})$

$R(\theta; S^{\text{SPE}})$	
NLL/SPE	$5.84 \cdot 10^{-2} (2.80 \cdot 10^{-3})$
linear	$6.88 \cdot 10^{-1} (6.37 \cdot 10^{-3})$
1NN	$6.49 \cdot 10^{-1} (3.84 \cdot 10^{-3})$
2NN	$5.72 \cdot 10^{-1} (5.06 \cdot 10^{-3})$

**Averages of  $R(\theta; S^{\text{CRPS}})$**

$R(\theta; \mathcal{S}^{\text{CRPS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$9.75 \cdot 10^{-2} (1.96 \cdot 10^{-3})$	$9.05 \cdot 10^{-2} (2.38 \cdot 10^{-3})$	$9.04 \cdot 10^{-2} (2.28 \cdot 10^{-3})$	$6.70 \cdot 10^{-2} (1.18 \cdot 10^{-3})$	$1.82 \cdot 10^{-1} (7.16 \cdot 10^{-3})$	$8.78 \cdot 10^{-2} (1.87 \cdot 10^{-3})$
$v = 3/2$	$4.15 \cdot 10^{-2} (9.31 \cdot 10^{-4})$	$5.25 \cdot 10^{-2} (2.64 \cdot 10^{-3})$	$4.30 \cdot 10^{-2} (1.04 \cdot 10^{-3})$	$4.06 \cdot 10^{-2} (1.13 \cdot 10^{-3})$	$1.75 \cdot 10^{-1} (1.01 \cdot 10^{-2})$	$4.10 \cdot 10^{-2} (8.40 \cdot 10^{-4})$
$v = 5/2$	$3.26 \cdot 10^{-2} (5.81 \cdot 10^{-4})$	$4.13 \cdot 10^{-2} (3.11 \cdot 10^{-3})$	$3.38 \cdot 10^{-2} (7.68 \cdot 10^{-4})$	$3.29 \cdot 10^{-2} (6.47 \cdot 10^{-4})$	$1.59 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$3.46 \cdot 10^{-2} (8.35 \cdot 10^{-4})$
$v = 7/2$	$3.30 \cdot 10^{-2} (5.85 \cdot 10^{-4})$	$3.72 \cdot 10^{-2} (1.34 \cdot 10^{-3})$	$3.29 \cdot 10^{-2} (6.91 \cdot 10^{-4})$	$3.32 \cdot 10^{-2} (7.56 \cdot 10^{-4})$	$1.42 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$3.69 \cdot 10^{-2} (9.04 \cdot 10^{-4})$
$v = 9/2$	$3.43 \cdot 10^{-2} (6.27 \cdot 10^{-4})$	$3.80 \cdot 10^{-2} (1.35 \cdot 10^{-3})$	$3.49 \cdot 10^{-2} (7.35 \cdot 10^{-4})$	$3.45 \cdot 10^{-2} (7.31 \cdot 10^{-4})$	$1.34 \cdot 10^{-1} (1.19 \cdot 10^{-2})$	$3.89 \cdot 10^{-2} (8.57 \cdot 10^{-4})$
$v = 13/2$	$3.66 \cdot 10^{-2} (6.67 \cdot 10^{-4})$	$4.07 \cdot 10^{-2} (1.45 \cdot 10^{-3})$	$3.76 \cdot 10^{-2} (8.05 \cdot 10^{-4})$	$3.67 \cdot 10^{-2} (6.79 \cdot 10^{-4})$	$1.28 \cdot 10^{-1} (1.21 \cdot 10^{-2})$	$4.42 \cdot 10^{-2} (1.16 \cdot 10^{-3})$
$v = 25/2$	$3.86 \cdot 10^{-2} (6.93 \cdot 10^{-4})$	$4.19 \cdot 10^{-2} (1.27 \cdot 10^{-3})$	$4.02 \cdot 10^{-2} (8.72 \cdot 10^{-4})$	$3.92 \cdot 10^{-2} (7.57 \cdot 10^{-4})$	$1.22 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$4.80 \cdot 10^{-2} (1.44 \cdot 10^{-3})$
$v = \infty$	$4.10 \cdot 10^{-2} (7.12 \cdot 10^{-4})$	$4.50 \cdot 10^{-2} (1.57 \cdot 10^{-3})$	$4.29 \cdot 10^{-2} (8.83 \cdot 10^{-4})$	$4.31 \cdot 10^{-2} (1.18 \cdot 10^{-3})$	$1.16 \cdot 10^{-1} (1.18 \cdot 10^{-2})$	$5.11 \cdot 10^{-2} (1.53 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$3.27 \cdot 10^{-2} (5.84 \cdot 10^{-4})$	$4.54 \cdot 10^{-2} (3.63 \cdot 10^{-3})$	$3.53 \cdot 10^{-2} (8.30 \cdot 10^{-4})$	$3.40 \cdot 10^{-2} (7.06 \cdot 10^{-4})$	$1.41 \cdot 10^{-1} (1.26 \cdot 10^{-2})$	$4.08 \cdot 10^{-2} (1.20 \cdot 10^{-3})$

$R(\theta; \mathcal{S}^{\text{CRPS}})$	
NLL/SPE	$3.55 \cdot 10^{-2} (8.04 \cdot 10^{-4})$

### Averages of $R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$

$R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$2.02 \cdot 10^0 (4.37 \cdot 10^{-2})$	$1.89 \cdot 10^0 (4.32 \cdot 10^{-2})$	$1.99 \cdot 10^0 (4.86 \cdot 10^{-2})$	$2.06 \cdot 10^0 (5.07 \cdot 10^{-2})$	$3.53 \cdot 10^0 (9.63 \cdot 10^{-2})$	$1.92 \cdot 10^0 (4.19 \cdot 10^{-2})$
$v = 3/2$	$9.44 \cdot 10^{-1} (3.01 \cdot 10^{-2})$	$1.14 \cdot 10^0 (5.62 \cdot 10^{-2})$	$1.03 \cdot 10^0 (3.23 \cdot 10^{-2})$	$1.14 \cdot 10^0 (3.62 \cdot 10^{-2})$	$3.20 \cdot 10^0 (1.31 \cdot 10^{-1})$	$9.56 \cdot 10^{-1} (2.83 \cdot 10^{-2})$
$v = 5/2$	$7.30 \cdot 10^{-1} (2.06 \cdot 10^{-2})$	$8.27 \cdot 10^{-1} (4.71 \cdot 10^{-2})$	$7.85 \cdot 10^{-1} (2.66 \cdot 10^{-2})$	$8.53 \cdot 10^{-1} (2.57 \cdot 10^{-2})$	$2.92 \cdot 10^0 (1.69 \cdot 10^{-1})$	$7.79 \cdot 10^{-1} (2.72 \cdot 10^{-2})$
$v = 7/2$	$7.30 \cdot 10^{-1} (2.13 \cdot 10^{-2})$	$7.18 \cdot 10^{-1} (2.63 \cdot 10^{-2})$	$7.19 \cdot 10^{-1} (2.43 \cdot 10^{-2})$	$8.12 \cdot 10^{-1} (3.08 \cdot 10^{-2})$	$2.69 \cdot 10^0 (1.69 \cdot 10^{-1})$	$8.20 \cdot 10^{-1} (2.95 \cdot 10^{-2})$
$v = 9/2$	$7.56 \cdot 10^{-1} (2.27 \cdot 10^{-2})$	$6.95 \cdot 10^{-1} (2.76 \cdot 10^{-2})$	$7.46 \cdot 10^{-1} (2.58 \cdot 10^{-2})$	$7.98 \cdot 10^{-1} (3.12 \cdot 10^{-2})$	$2.56 \cdot 10^0 (1.69 \cdot 10^{-1})$	$8.46 \cdot 10^{-1} (2.99 \cdot 10^{-2})$
$v = 13/2$	$7.98 \cdot 10^{-1} (2.37 \cdot 10^{-2})$	$7.20 \cdot 10^{-1} (3.03 \cdot 10^{-2})$	$7.90 \cdot 10^{-1} (2.89 \cdot 10^{-2})$	$8.06 \cdot 10^{-1} (2.89 \cdot 10^{-2})$	$2.44 \cdot 10^0 (1.73 \cdot 10^{-1})$	$9.62 \cdot 10^{-1} (3.84 \cdot 10^{-2})$
$v = 25/2$	$8.34 \cdot 10^{-1} (2.43 \cdot 10^{-2})$	$7.19 \cdot 10^{-1} (2.92 \cdot 10^{-2})$	$8.32 \cdot 10^{-1} (3.06 \cdot 10^{-2})$	$8.30 \cdot 10^{-1} (3.17 \cdot 10^{-2})$	$2.30 \cdot 10^0 (1.72 \cdot 10^{-1})$	$1.05 \cdot 10^0 (4.99 \cdot 10^{-2})$
$v = \infty$	$8.78 \cdot 10^{-1} (2.48 \cdot 10^{-2})$	$7.64 \cdot 10^{-1} (3.21 \cdot 10^{-2})$	$8.68 \cdot 10^{-1} (3.11 \cdot 10^{-2})$	$9.11 \cdot 10^{-1} (4.76 \cdot 10^{-2})$	$2.14 \cdot 10^0 (1.69 \cdot 10^{-1})$	$1.10 \cdot 10^0 (5.25 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$7.34 \cdot 10^{-1} (2.08 \cdot 10^{-2})$	$8.32 \cdot 10^{-1} (4.85 \cdot 10^{-2})$	$7.84 \cdot 10^{-1} (2.80 \cdot 10^{-2})$	$8.54 \cdot 10^{-1} (2.78 \cdot 10^{-2})$	$2.59 \cdot 10^0 (1.83 \cdot 10^{-1})$	$9.07 \cdot 10^{-1} (3.71 \cdot 10^{-2})$

$R(\theta; \mathcal{S}_{0.95}^{\text{IS}})$	
NLL/SPE	$7.91 \cdot 10^{-1} (2.57 \cdot 10^{-2})$

### SM-2.2.48 $\text{Re}(S_{14\pi}), n = 300, d = 6$

### Averages of $R(\theta; \mathcal{S}^{\text{SPE}})$

$R(\theta; \mathcal{S}^{\text{SPE}})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$9.39 \cdot 10^{-2} (2.73 \cdot 10^{-3})$	$8.84 \cdot 10^{-2} (3.38 \cdot 10^{-3})$	$8.98 \cdot 10^{-2} (2.77 \cdot 10^{-3})$	$8.41 \cdot 10^{-2} (3.37 \cdot 10^{-3})$	$2.45 \cdot 10^{-1} (5.90 \cdot 10^{-3})$	$8.33 \cdot 10^{-2} (2.90 \cdot 10^{-3})$
$v = 3/2$	$2.47 \cdot 10^{-2} (1.26 \cdot 10^{-3})$	$3.44 \cdot 10^{-2} (4.32 \cdot 10^{-3})$	$2.49 \cdot 10^{-2} (1.26 \cdot 10^{-3})$	$2.48 \cdot 10^{-2} (1.32 \cdot 10^{-3})$	$2.49 \cdot 10^{-1} (1.04 \cdot 10^{-2})$	$2.51 \cdot 10^{-2} (1.37 \cdot 10^{-3})$
$v = 5/2$	$1.74 \cdot 10^{-2} (9.56 \cdot 10^{-4})$	$1.80 \cdot 10^{-2} (1.05 \cdot 10^{-3})$	$1.71 \cdot 10^{-2} (9.10 \cdot 10^{-4})$	$1.57 \cdot 10^{-2} (9.05 \cdot 10^{-4})$	$2.09 \cdot 10^{-1} (9.84 \cdot 10^{-3})$	$2.16 \cdot 10^{-2} (1.46 \cdot 10^{-3})$
$v = 7/2$	$1.88 \cdot 10^{-2} (1.09 \cdot 10^{-3})$	$1.55 \cdot 10^{-2} (1.24 \cdot 10^{-3})$	$1.66 \cdot 10^{-2} (9.33 \cdot 10^{-4})$	$1.44 \cdot 10^{-2} (9.15 \cdot 10^{-4})$	$1.68 \cdot 10^{-1} (5.73 \cdot 10^{-3})$	$2.96 \cdot 10^{-2} (2.31 \cdot 10^{-3})$
$v = 9/2$	$2.18 \cdot 10^{-2} (1.27 \cdot 10^{-3})$	$1.36 \cdot 10^{-2} (9.51 \cdot 10^{-4})$	$1.85 \cdot 10^{-2} (1.31 \cdot 10^{-3})$	$1.56 \cdot 10^{-2} (1.13 \cdot 10^{-3})$	$1.56 \cdot 10^{-1} (5.16 \cdot 10^{-3})$	$3.55 \cdot 10^{-2} (2.96 \cdot 10^{-3})$
$v = 13/2$	$2.78 \cdot 10^{-2} (1.57 \cdot 10^{-3})$	$1.48 \cdot 10^{-2} (1.12 \cdot 10^{-3})$	$2.28 \cdot 10^{-2} (1.67 \cdot 10^{-3})$	$1.82 \cdot 10^{-2} (1.51 \cdot 10^{-3})$	$1.47 \cdot 10^{-1} (4.87 \cdot 10^{-3})$	$4.48 \cdot 10^{-2} (3.53 \cdot 10^{-3})$
$v = 25/2$	$3.47 \cdot 10^{-2} (1.88 \cdot 10^{-3})$	$1.60 \cdot 10^{-2} (1.08 \cdot 10^{-3})$	$2.67 \cdot 10^{-2} (1.95 \cdot 10^{-3})$	$2.17 \cdot 10^{-2} (2.02 \cdot 10^{-3})$	$1.40 \cdot 10^{-1} (4.61 \cdot 10^{-3})$	$5.46 \cdot 10^{-2} (3.57 \cdot 10^{-3})$
$v = \infty$	$4.13 \cdot 10^{-2} (2.13 \cdot 10^{-3})$	$1.89 \cdot 10^{-2} (1.46 \cdot 10^{-3})$	$3.22 \cdot 10^{-2} (2.47 \cdot 10^{-3})$	$2.35 \cdot 10^{-2} (2.16 \cdot 10^{-3})$	$1.31 \cdot 10^{-1} (4.33 \cdot 10^{-3})$	$6.47 \cdot 10^{-2} (4.01 \cdot 10^{-3})$
$v \in \{1/2, \dots, \infty\}$	$1.80 \cdot 10^{-2} (9.98 \cdot 10^{-4})$	$1.54 \cdot 10^{-2} (1.18 \cdot 10^{-3})$	$1.76 \cdot 10^{-2} (1.21 \cdot 10^{-3})$	$1.48 \cdot 10^{-2} (9.05 \cdot 10^{-4})$	$1.56 \cdot 10^{-1} (5.64 \cdot 10^{-3})$	$3.53 \cdot 10^{-2} (2.70 \cdot 10^{-3})$



$R^*(S^{SPE})$	
$v = 1/2$	$7.22 \cdot 10^{-2} (3.22 \cdot 10^{-3})$
$v = 3/2$	$1.72 \cdot 10^{-2} (9.39 \cdot 10^{-4})$
$v = 5/2$	$9.66 \cdot 10^{-3} (4.33 \cdot 10^{-4})$
$v = 7/2$	$8.10 \cdot 10^{-3} (3.03 \cdot 10^{-4})$
$v = 9/2$	$7.75 \cdot 10^{-3} (2.56 \cdot 10^{-4})$
$v = 13/2$	$8.01 \cdot 10^{-3} (2.32 \cdot 10^{-4})$
$v = 25/2$	$8.36 \cdot 10^{-3} (2.12 \cdot 10^{-4})$
$v = \infty$	$8.93 \cdot 10^{-3} (2.05 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$7.48 \cdot 10^{-3} (2.48 \cdot 10^{-4})$

$R(\theta; S^{SPE})$	
NLL/SPE	$2.04 \cdot 10^{-2} (1.17 \cdot 10^{-3})$
linear	$6.58 \cdot 10^{-1} (2.28 \cdot 10^{-3})$
1NN	$5.39 \cdot 10^{-1} (2.85 \cdot 10^{-3})$
2NN	$4.67 \cdot 10^{-1} (3.86 \cdot 10^{-3})$

### Averages of $R(\theta; S^{CRPS})$

$R(\theta; S^{CRPS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$6.63 \cdot 10^{-2} (1.01 \cdot 10^{-3})$	$6.19 \cdot 10^{-2} (1.78 \cdot 10^{-3})$	$5.90 \cdot 10^{-2} (1.02 \cdot 10^{-3})$	$4.11 \cdot 10^{-2} (5.46 \cdot 10^{-4})$	$1.37 \cdot 10^{-1} (4.22 \cdot 10^{-3})$	$5.66 \cdot 10^{-2} (7.53 \cdot 10^{-4})$
$v = 3/2$	$2.23 \cdot 10^{-2} (2.98 \cdot 10^{-4})$	$2.90 \cdot 10^{-2} (1.62 \cdot 10^{-3})$	$2.21 \cdot 10^{-2} (3.61 \cdot 10^{-4})$	$2.07 \cdot 10^{-2} (3.75 \cdot 10^{-4})$	$1.28 \cdot 10^{-1} (5.40 \cdot 10^{-3})$	$2.18 \cdot 10^{-2} (3.05 \cdot 10^{-4})$
$v = 5/2$	$1.63 \cdot 10^{-2} (2.22 \cdot 10^{-4})$	$1.91 \cdot 10^{-2} (5.39 \cdot 10^{-4})$	$1.62 \cdot 10^{-2} (2.42 \cdot 10^{-4})$	$1.58 \cdot 10^{-2} (2.63 \cdot 10^{-4})$	$8.74 \cdot 10^{-2} (3.09 \cdot 10^{-3})$	$1.71 \cdot 10^{-2} (3.23 \cdot 10^{-4})$
$v = 7/2$	$1.62 \cdot 10^{-2} (2.23 \cdot 10^{-4})$	$1.77 \cdot 10^{-2} (4.24 \cdot 10^{-4})$	$1.58 \cdot 10^{-2} (2.12 \cdot 10^{-4})$	$1.53 \cdot 10^{-2} (2.35 \cdot 10^{-4})$	$6.47 \cdot 10^{-2} (1.87 \cdot 10^{-3})$	$1.87 \cdot 10^{-2} (4.41 \cdot 10^{-4})$
$v = 9/2$	$1.71 \cdot 10^{-2} (2.33 \cdot 10^{-4})$	$1.75 \cdot 10^{-2} (3.85 \cdot 10^{-4})$	$1.67 \cdot 10^{-2} (2.57 \cdot 10^{-4})$	$1.61 \cdot 10^{-2} (2.47 \cdot 10^{-4})$	$5.65 \cdot 10^{-2} (1.45 \cdot 10^{-3})$	$2.03 \cdot 10^{-2} (5.18 \cdot 10^{-4})$
$v = 13/2$	$1.90 \cdot 10^{-2} (2.61 \cdot 10^{-4})$	$1.88 \cdot 10^{-2} (3.03 \cdot 10^{-4})$	$1.86 \cdot 10^{-2} (2.99 \cdot 10^{-4})$	$1.79 \cdot 10^{-2} (2.86 \cdot 10^{-4})$	$5.05 \cdot 10^{-2} (1.14 \cdot 10^{-3})$	$2.30 \cdot 10^{-2} (5.99 \cdot 10^{-4})$
$v = 25/2$	$2.12 \cdot 10^{-2} (2.93 \cdot 10^{-4})$	$2.04 \cdot 10^{-2} (3.34 \cdot 10^{-4})$	$2.04 \cdot 10^{-2} (3.07 \cdot 10^{-4})$	$1.99 \cdot 10^{-2} (3.51 \cdot 10^{-4})$	$4.65 \cdot 10^{-2} (9.39 \cdot 10^{-4})$	$2.54 \cdot 10^{-2} (5.55 \cdot 10^{-4})$
$v = \infty$	$2.33 \cdot 10^{-2} (3.20 \cdot 10^{-4})$	$2.27 \cdot 10^{-2} (6.72 \cdot 10^{-4})$	$2.26 \cdot 10^{-2} (3.59 \cdot 10^{-4})$	$2.17 \cdot 10^{-2} (3.47 \cdot 10^{-4})$	$4.32 \cdot 10^{-2} (8.02 \cdot 10^{-4})$	$2.80 \cdot 10^{-2} (5.77 \cdot 10^{-4})$
$v \in \{1/2, \dots, \infty\}$	$1.62 \cdot 10^{-2} (2.18 \cdot 10^{-4})$	$1.87 \cdot 10^{-2} (6.74 \cdot 10^{-4})$	$1.63 \cdot 10^{-2} (2.60 \cdot 10^{-4})$	$1.56 \cdot 10^{-2} (2.36 \cdot 10^{-4})$	$6.55 \cdot 10^{-2} (5.49 \cdot 10^{-3})$	$2.04 \cdot 10^{-2} (4.61 \cdot 10^{-4})$

$R(\theta; S^{CRPS})$	
NLL/SPE	$1.72 \cdot 10^{-2} (2.81 \cdot 10^{-4})$

### Averages of $R(\theta; S_{0.95}^{IS})$

$R(\theta; S_{0.95}^{IS})$	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	KA	GCV
$v = 1/2$	$1.34 \cdot 10^0 (2.11 \cdot 10^{-2})$	$1.26 \cdot 10^0 (2.95 \cdot 10^{-2})$	$1.27 \cdot 10^0 (2.31 \cdot 10^{-2})$	$1.29 \cdot 10^0 (2.36 \cdot 10^{-2})$	$2.62 \cdot 10^0 (5.92 \cdot 10^{-2})$	$1.21 \cdot 10^0 (1.80 \cdot 10^{-2})$
$v = 3/2$	$4.84 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$6.22 \cdot 10^{-1} (3.33 \cdot 10^{-2})$	$5.10 \cdot 10^{-1} (1.38 \cdot 10^{-2})$	$6.00 \cdot 10^{-1} (1.54 \cdot 10^{-2})$	$2.23 \cdot 10^0 (7.20 \cdot 10^{-2})$	$4.96 \cdot 10^{-1} (1.28 \cdot 10^{-2})$
$v = 5/2$	$3.56 \cdot 10^{-1} (9.25 \cdot 10^{-3})$	$3.93 \cdot 10^{-1} (1.18 \cdot 10^{-2})$	$3.69 \cdot 10^{-1} (9.73 \cdot 10^{-3})$	$4.12 \cdot 10^{-1} (1.06 \cdot 10^{-2})$	$1.63 \cdot 10^0 (5.05 \cdot 10^{-2})$	$3.90 \cdot 10^{-1} (1.16 \cdot 10^{-2})$
$v = 7/2$	$3.50 \cdot 10^{-1} (9.38 \cdot 10^{-3})$	$3.37 \cdot 10^{-1} (9.45 \cdot 10^{-3})$	$3.42 \cdot 10^{-1} (9.05 \cdot 10^{-3})$	$3.70 \cdot 10^{-1} (1.03 \cdot 10^{-2})$	$1.27 \cdot 10^0 (3.49 \cdot 10^{-2})$	$4.21 \cdot 10^{-1} (1.54 \cdot 10^{-2})$
$v = 9/2$	$3.65 \cdot 10^{-1} (9.88 \cdot 10^{-3})$	$3.12 \cdot 10^{-1} (8.28 \cdot 10^{-3})$	$3.44 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$3.63 \cdot 10^{-1} (1.11 \cdot 10^{-2})$	$1.15 \cdot 10^0 (2.93 \cdot 10^{-2})$	$4.47 \cdot 10^{-1} (1.86 \cdot 10^{-2})$
$v = 13/2$	$4.01 \cdot 10^{-1} (1.09 \cdot 10^{-2})$	$3.13 \cdot 10^{-1} (7.99 \cdot 10^{-3})$	$3.63 \cdot 10^{-1} (1.14 \cdot 10^{-2})$	$3.70 \cdot 10^{-1} (1.24 \cdot 10^{-2})$	$1.05 \cdot 10^0 (2.57 \cdot 10^{-2})$	$4.89 \cdot 10^{-1} (2.08 \cdot 10^{-2})$
$v = 25/2$	$4.39 \cdot 10^{-1} (1.17 \cdot 10^{-2})$	$3.19 \cdot 10^{-1} (7.75 \cdot 10^{-3})$	$3.77 \cdot 10^{-1} (1.10 \cdot 10^{-2})$	$3.80 \cdot 10^{-1} (1.43 \cdot 10^{-2})$	$9.60 \cdot 10^{-1} (2.30 \cdot 10^{-2})$	$5.22 \cdot 10^{-1} (1.77 \cdot 10^{-2})$
$v = \infty$	$4.75 \cdot 10^{-1} (1.22 \cdot 10^{-2})$	$3.38 \cdot 10^{-1} (9.61 \cdot 10^{-3})$	$4.00 \cdot 10^{-1} (1.20 \cdot 10^{-2})$	$3.86 \cdot 10^{-1} (1.36 \cdot 10^{-2})$	$8.60 \cdot 10^{-1} (1.99 \cdot 10^{-2})$	$5.57 \cdot 10^{-1} (1.87 \cdot 10^{-2})$
$v \in \{1/2, \dots, \infty\}$	$3.55 \cdot 10^{-1} (9.12 \cdot 10^{-3})$	$3.46 \cdot 10^{-1} (1.31 \cdot 10^{-2})$	$3.45 \cdot 10^{-1} (9.90 \cdot 10^{-3})$	$3.85 \cdot 10^{-1} (1.02 \cdot 10^{-2})$	$1.24 \cdot 10^0 (9.04 \cdot 10^{-2})$	$4.43 \cdot 10^{-1} (1.61 \cdot 10^{-2})$

$R(\theta; S_{0.95}^{IS})$	
NLL/SPE	$3.72 \cdot 10^{-1} (9.34 \cdot 10^{-3})$

## SM-3 Additional analysis of the results

### SM-3.1 Some comments about the results

Detailed results are provided in Section [SM-2.2](#). Here we briefly outline the few problems where the choice of the selection criterion has a more noticeable impact.

Observe the Rosenbrock collection with  $d = 2$  and  $n = 20$  where LOO-SPE, LOO-CRPS and LOO-NLPD strongly outperform ML for more suitable values of  $v$ . This is caused by ML screening out one variable by choosing a very large range parameter along this dimension on some repetitions.

We count other situations where ML exhibits flaws like the Rosenbrock collection with  $d = 5$  and  $n = 100$ , where CV methods slightly outperform ML. There is also the Goldstein-Price function with  $n = 20$  where GCV slightly outperforms other methods.

On the other hand, ML appears to be part of the top performers for all GKLS problems, for the Mystery function with  $n = 100$ ,  $n = 40$  and  $n = 20$  and for the Rosenbrock collection with  $d = 5$  and  $n = 50$ . Note that for the last two cases, ML appears to give better results especially for weaker models, which is a property often associated with CV.

## SM-3.2 ANOVA analysis

In Section SM-3.2.1 and Section SM-3.2.2, we provide more details about the influence of  $\nu$  using one-factor ANOVA analysis as in the article. We consider the three metrics SPE, CRPS and IS.

Observe that the large variations are also mainly explained by  $\nu$  for CRPS and IS. Note that the Rosenbrock collection with  $(n, d) = (20, 2)$  mentioned in Section SM-3.1 appears as an singular exception on Figure SM2. Indeed, it presents significant fluctuations of  $\log_{10}(R)$  with a non-negligible part that is not uniquely explained by  $\nu$ .

Observe also that the fluctuations are typically larger on very smooth problems than on GKLS problems. This is consistent with the results of Tuo and Wang (2020), that suggest that underestimating the regularity of an underlying function has much more dramatic consequences than overestimating it. As a consequence the more regular the function is the more critical is the choice of  $\nu$ . The phenomenon is more strikeful as the design becomes larger since an infinitely differentiable function can seem irregular when the spacing between points is too large.

### SM-3.2.1 Raw ANOVA tables

We provide here the Sobol' indexes along with the total variances  $V_s$  for all problems.

#### ANOVA tables for the low-pass filter example

Table SM1: Sobol indices for experiments, SPE,  $n \in \{10d, 20d, 50d\}$

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	60	0.006	0.989	0.005	0.803
$\text{Re}(S_{6\pi})$	6	60	0.008	0.987	0.005	0.492
$\text{Re}(S_{10\pi})$	6	60	0.021	0.962	0.017	0.036
$\text{Re}(S_{14\pi})$	6	60	0.233	0.69	0.078	0.015

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	120	0.003	0.996	0.002	0.785
$\text{Re}(S_{6\pi})$	6	120	0.002	0.996	0.002	0.639
$\text{Re}(S_{10\pi})$	6	120	0.024	0.959	0.017	0.032
$\text{Re}(S_{14\pi})$	6	120	0.104	0.804	0.092	0.026

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	300	0.017	0.977	0.007	0.7
$\text{Re}(S_{6\pi})$	6	300	0.003	0.996	0.001	0.702
$\text{Re}(S_{10\pi})$	6	300	0.016	0.969	0.015	0.028
$\text{Re}(S_{14\pi})$	6	300	0.158	0.724	0.118	0.061

Table SM2: Sobol indices for experiments, CRPS,  $n \in \{10d, 20d, 50d\}$

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	60	0.002	0.994	0.004	0.23
$\text{Re}(S_{6\pi})$	6	60	0.004	0.991	0.006	0.132
$\text{Re}(S_{10\pi})$	6	60	0.021	0.96	0.02	0.017
$\text{Re}(S_{14\pi})$	6	60	0.123	0.838	0.039	0.009

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	120	0.002	0.996	0.003	0.234
$\text{Re}(S_{6\pi})$	6	120	0.001	0.997	0.002	0.176
$\text{Re}(S_{10\pi})$	6	120	0.023	0.961	0.016	0.022
$\text{Re}(S_{14\pi})$	6	120	0.041	0.916	0.043	0.016

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	300	0.004	0.99	0.006	0.218
$\text{Re}(S_{6\pi})$	6	300	0.001	0.997	0.002	0.211
$\text{Re}(S_{10\pi})$	6	300	0.019	0.968	0.013	0.031
$\text{Re}(S_{14\pi})$	6	300	0.033	0.931	0.036	0.028

Table SM3: Sobol indices for experiments, IS95%,  $n \in \{10d, 20d, 50d\}$

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	60	0.002	0.992	0.006	0.239
$\text{Re}(S_{6\pi})$	6	60	0.014	0.973	0.012	0.162
$\text{Re}(S_{10\pi})$	6	60	0.043	0.945	0.012	0.016
$\text{Re}(S_{14\pi})$	6	60	0.056	0.91	0.035	0.01

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	120	0.001	0.997	0.002	0.26
$\text{Re}(S_{6\pi})$	6	120	0.002	0.996	0.002	0.228
$\text{Re}(S_{10\pi})$	6	120	0.012	0.984	0.005	0.024
$\text{Re}(S_{14\pi})$	6	120	0.027	0.928	0.045	0.018

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
$\text{Re}(S_{2\pi})$	6	300	0.003	0.993	0.004	0.245
$\text{Re}(S_{6\pi})$	6	300	0.0	0.998	0.002	0.269
$\text{Re}(S_{10\pi})$	6	300	0.007	0.988	0.006	0.034
$\text{Re}(S_{14\pi})$	6	300	0.031	0.918	0.051	0.033

## **ANOVA tables for GKLS and other test functions**

Table SM4: Sobol indices for experiments, SPE,  $n \in \{10d, 20d, 50d\}$

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	80	0.004	0.994	0.002	0.276	GKLS $C^0$	2	20	0.367	0.333	0.299	0.062
Rosenbrock	2	20	0.188	0.621	0.191	0.329	GKLS $C^0$	5	50	0.236	0.691	0.073	0.017
Rosenbrock	5	50	0.08	0.812	0.109	0.006	GKLS $C^1$	2	20	0.411	0.406	0.182	0.036
Goldstein-Price	2	20	0.134	0.663	0.203	0.004	GKLS $C^1$	5	50	0.183	0.718	0.099	0.015
Mystery	2	20	0.284	0.565	0.151	0.033	GKLS $C^2$	2	20	0.407	0.447	0.147	0.029
Goldstein-Price (1D)	1	10	0.012	0.982	0.006	2.002	GKLS $C^2$	5	50	0.211	0.685	0.104	0.015
Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	160	0.0	0.999	0.0	0.426	GKLS $C^0$	2	40	0.453	0.229	0.318	0.032
Rosenbrock	2	40	0.0	0.999	0.001	0.549	GKLS $C^0$	5	100	0.22	0.734	0.047	0.022
Rosenbrock	5	100	0.18	0.735	0.086	0.014	GKLS $C^1$	2	40	0.441	0.253	0.306	0.048
Goldstein-Price	2	40	0.06	0.912	0.028	0.03	GKLS $C^1$	5	100	0.156	0.806	0.038	0.022
Mystery	2	40	0.414	0.259	0.327	0.032	GKLS $C^2$	2	40	0.429	0.284	0.287	0.046
Goldstein-Price (1D)	1	20	0.0	1.0	0.0	2.25	GKLS $C^2$	5	100	0.142	0.818	0.041	0.02
Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	400	0.0	1.0	0.0	0.581	GKLS $C^0$	2	100	0.343	0.35	0.307	0.017
Rosenbrock	2	100	0.0	0.995	0.004	0.639	GKLS $C^0$	5	250	0.166	0.797	0.037	0.016
Rosenbrock	5	250	0.001	0.995	0.004	0.348	GKLS $C^1$	2	100	0.349	0.348	0.303	0.015
Goldstein-Price	2	100	0.015	0.982	0.004	0.243	GKLS $C^1$	5	250	0.108	0.872	0.02	0.018
Mystery	2	100	0.064	0.898	0.038	0.036	GKLS $C^2$	2	100	0.424	0.222	0.353	0.013
Goldstein-Price (1D)	1	50	0.0	1.0	0.0	3.259	GKLS $C^2$	5	250	0.101	0.881	0.018	0.018



Table SM5: Sobol indices for experiments, CRPS,  $n \in \{10d, 20d, 50d\}$

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	80	0.002	0.985	0.013	0.068	GKLS $C^0$	2	20	0.475	0.35	0.176	0.006
Rosenbrock	2	20	0.14	0.7	0.16	0.101	GKLS $C^0$	5	50	0.205	0.708	0.086	0.005
Rosenbrock	5	50	0.124	0.771	0.106	0.003	GKLS $C^1$	2	20	0.385	0.509	0.106	0.006
Goldstein-Price	2	20	0.089	0.706	0.205	0.002	GKLS $C^1$	5	50	0.125	0.769	0.106	0.005
Mystery	2	20	0.366	0.529	0.106	0.004	GKLS $C^2$	2	20	0.377	0.541	0.082	0.005
Goldstein-Price (1D)	1	10	0.01	0.985	0.005	0.675	GKLS $C^2$	5	50	0.12	0.778	0.102	0.005
Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	160	0.002	0.991	0.006	0.105	GKLS $C^0$	2	40	0.482	0.207	0.311	0.003
Rosenbrock	2	40	0.001	0.998	0.002	0.203	GKLS $C^0$	5	100	0.156	0.8	0.045	0.006
Rosenbrock	5	100	0.142	0.785	0.073	0.004	GKLS $C^1$	2	40	0.473	0.29	0.237	0.003
Goldstein-Price	2	40	0.028	0.95	0.021	0.014	GKLS $C^1$	5	100	0.086	0.866	0.048	0.006
Mystery	2	40	0.423	0.215	0.362	0.003	GKLS $C^2$	2	40	0.487	0.314	0.198	0.003
Goldstein-Price (1D)	1	20	0.002	0.994	0.004	0.817	GKLS $C^2$	5	100	0.071	0.883	0.047	0.005
Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	400	0.005	0.988	0.007	0.15	GKLS $C^0$	2	100	0.235	0.543	0.222	0.003
Rosenbrock	2	100	0.007	0.984	0.009	0.25	GKLS $C^0$	5	250	0.1	0.864	0.035	0.005
Rosenbrock	5	250	0.001	0.996	0.003	0.104	GKLS $C^1$	2	100	0.215	0.595	0.19	0.004
Goldstein-Price	2	100	0.005	0.988	0.007	0.099	GKLS $C^1$	5	250	0.062	0.901	0.037	0.006
Mystery	2	100	0.004	0.993	0.003	0.039	GKLS $C^2$	2	100	0.323	0.412	0.265	0.003
Goldstein-Price (1D)	1	50	0.008	0.985	0.007	1.164	GKLS $C^2$	5	250	0.062	0.898	0.04	0.006

Table SM6: Sobol indices for experiments, IS95%,  $n \in \{10d, 20d, 50d\}$

Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	80	0.011	0.975	0.014	0.085	GKLS $C^0$	2	20	0.371	0.499	0.13	0.01
Rosenbrock	2	20	0.152	0.635	0.214	0.151	GKLS $C^0$	5	50	0.259	0.684	0.057	0.017
Rosenbrock	5	50	0.185	0.72	0.094	0.011	GKLS $C^1$	2	20	0.279	0.656	0.065	0.01
Goldstein-Price	2	20	0.32	0.28	0.4	0.001	GKLS $C^1$	5	50	0.251	0.671	0.078	0.015
Mystery	2	20	0.218	0.731	0.051	0.021	GKLS $C^2$	2	20	0.242	0.719	0.039	0.008
Goldstein-Price (1D)	1	10	0.022	0.966	0.013	0.72	GKLS $C^2$	5	50	0.254	0.662	0.084	0.013
Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	160	0.001	0.996	0.003	0.137	GKLS $C^0$	2	40	0.462	0.365	0.173	0.004
Rosenbrock	2	40	0.001	0.997	0.002	0.227	GKLS $C^0$	5	100	0.259	0.692	0.049	0.007
Rosenbrock	5	100	0.131	0.799	0.07	0.007	GKLS $C^1$	2	40	0.361	0.522	0.118	0.005
Goldstein-Price	2	40	0.06	0.899	0.041	0.018	GKLS $C^1$	5	100	0.189	0.763	0.048	0.006
Mystery	2	40	0.411	0.324	0.264	0.007	GKLS $C^2$	2	40	0.324	0.578	0.098	0.004
Goldstein-Price (1D)	1	20	0.001	0.996	0.003	0.946	GKLS $C^2$	5	100	0.181	0.769	0.05	0.006
Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance	Problem	$d$	$n$	$S_{\text{criterion}}$	$S_v$	$S_{\text{int}}$	total variance
Borehole	8	400	0.002	0.995	0.003	0.189	GKLS $C^0$	2	100	0.344	0.436	0.22	0.004
Rosenbrock	2	100	0.001	0.99	0.009	0.258	GKLS $C^0$	5	250	0.383	0.564	0.054	0.003
Rosenbrock	5	250	0.001	0.995	0.003	0.136	GKLS $C^1$	2	100	0.208	0.658	0.134	0.006
Goldstein-Price	2	100	0.01	0.985	0.005	0.133	GKLS $C^1$	5	250	0.24	0.708	0.052	0.003
Mystery	2	100	0.098	0.856	0.046	0.02	GKLS $C^2$	2	100	0.338	0.447	0.215	0.004
Goldstein-Price (1D)	1	50	0.0	0.993	0.007	1.156	GKLS $C^2$	5	250	0.222	0.73	0.048	0.003

### SM-3.2.2 Graphical summaries of the ANOVA tables

We provide here the same plot as in the article, but also for  $S^{\text{SPE}}$ ,  $S^{\text{CRPS}}$  and  $S_{0.95}^{\text{IS}}$ .

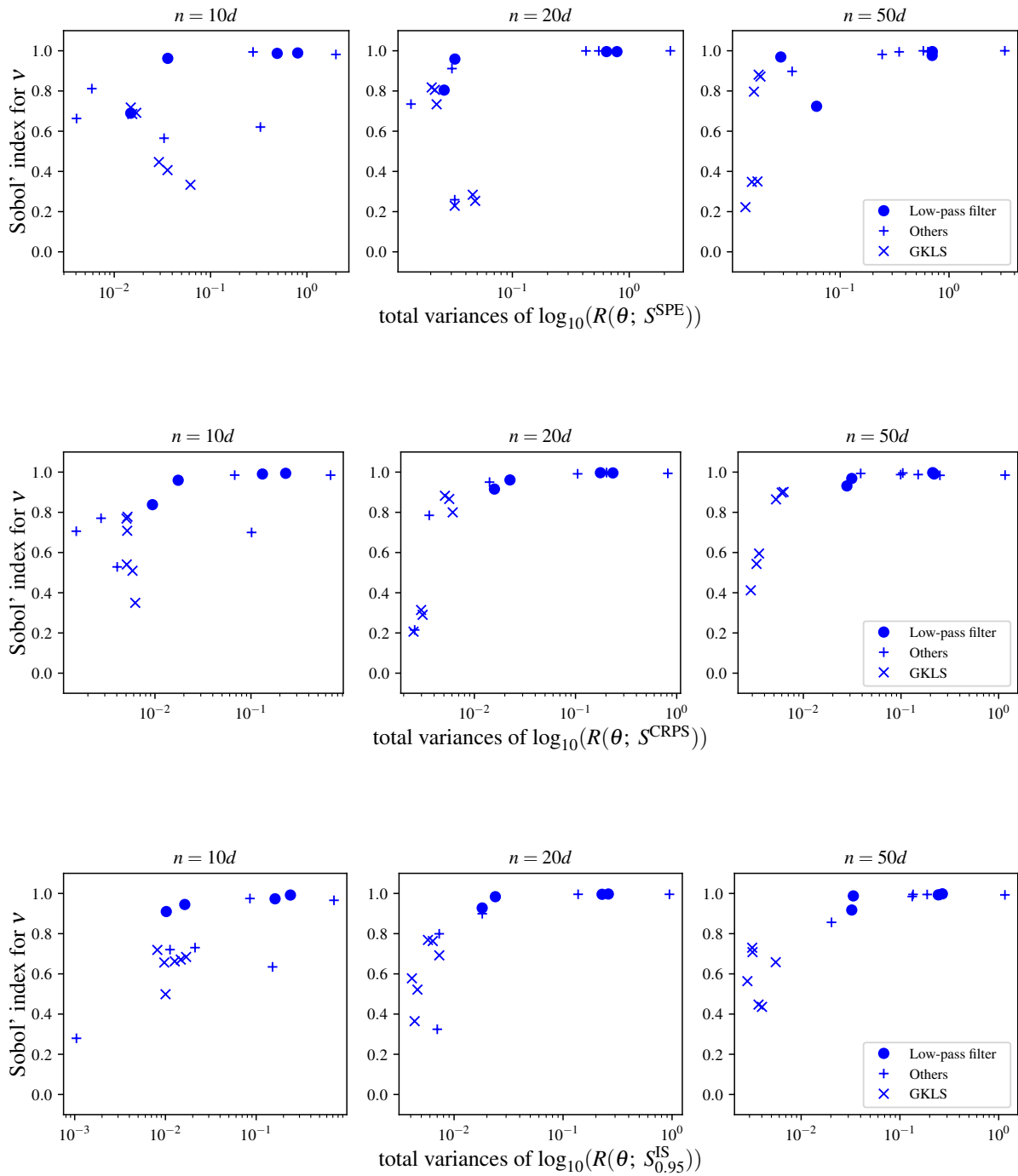


Figure SM2: One-factor ANOVA for SPE, CRPS and IS.

### SM-3.3 Additional criteria comparison

We provide here more criteria comparison. We also add percentages which show the fraction of problems where a given criterion average score is statistically indistinguishable from the best selection criterion under study. Dotted lines indicates a ratio 2, 4, 6, 8 and 10.

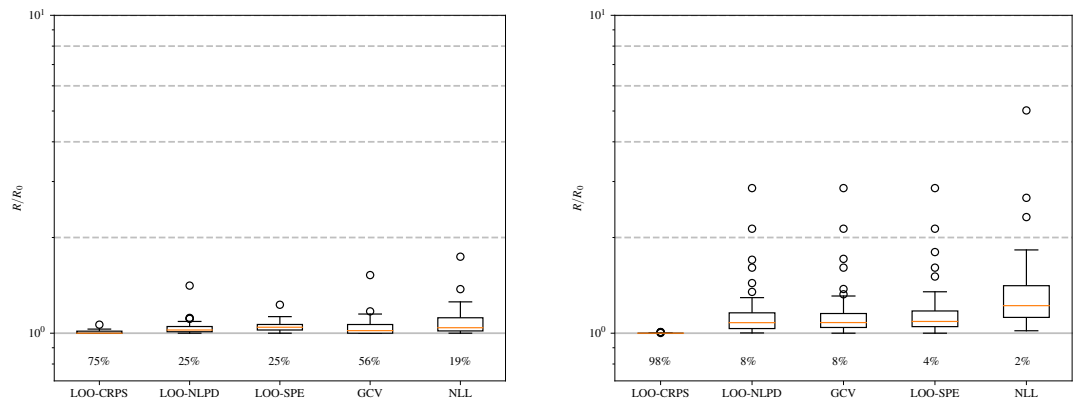
We observe that the NLL criterion appears to dominate the other criteria when  $\nu$  is automatically selected.

The situation is a bit different when  $\nu$  is enforced to a “weak” (as measured by  $R^*$ ) value. When  $\nu$  is underestimated, then the LOO criteria and GCV appear to dominate NLL. More precisely the LOO-CRPS criterion appears to be the most efficient in terms of CRPS and surprisingly of SPE and the LOO-SPE seems to be the most efficient according to IS in this situation.

Note also that the NLL selection criterion appears to dominate the others when  $\nu$  is overestimated.

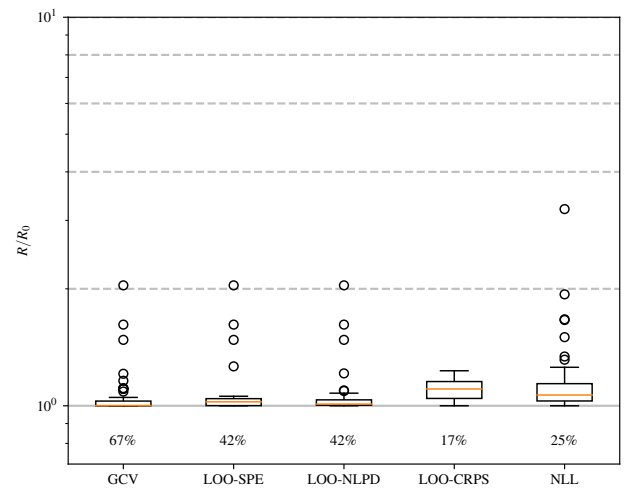
#### SM-3.3.1 Varying regularity

Here we present comparison for  $\nu = 1/2, 3/2, 5/2, \hat{\nu}$ , on all test instances and for all design sizes.



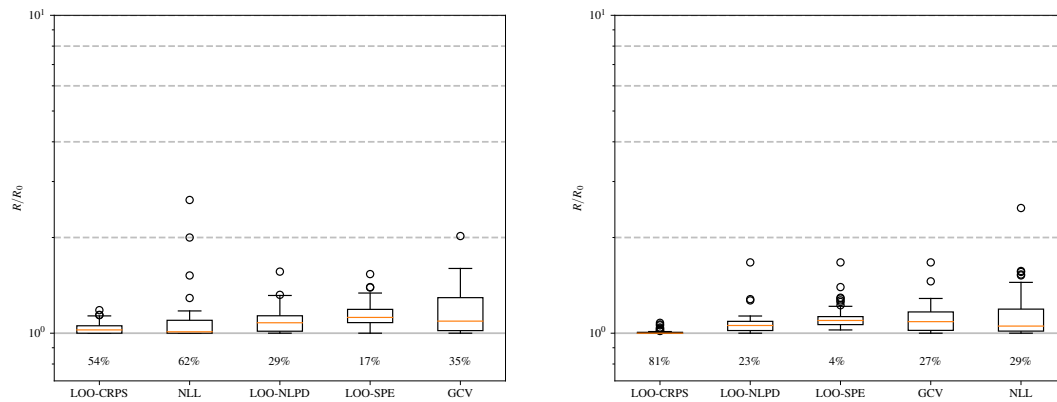
(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



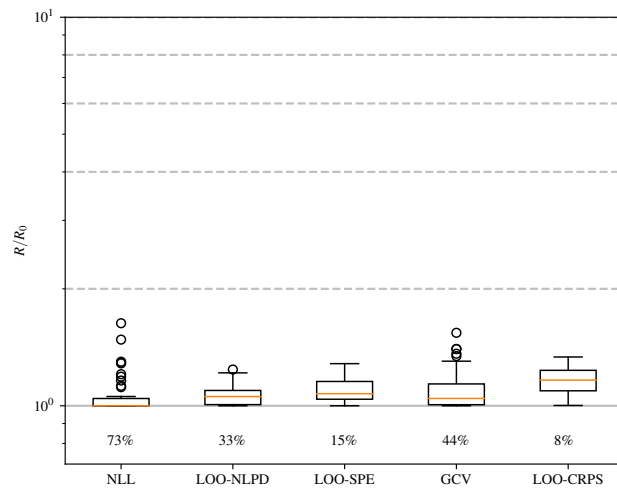
(c)  $S^{\text{IS}}_{0.95}$

Figure SM3:  $\nu = 1/2$



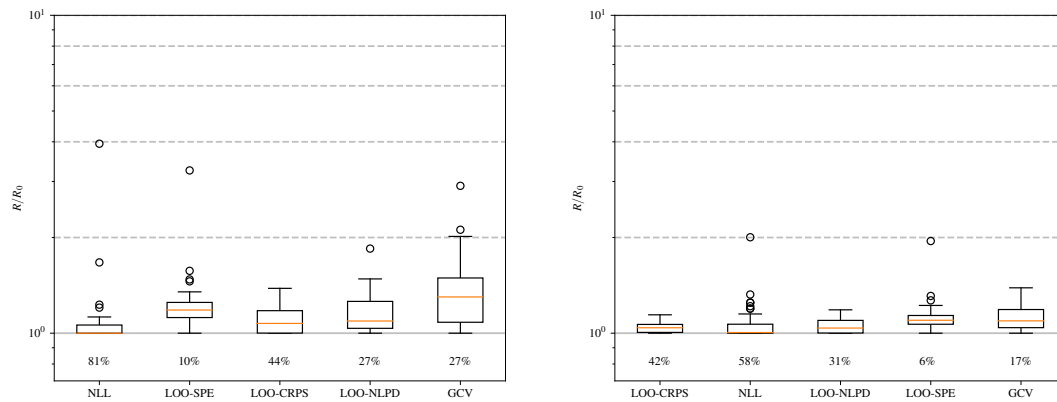
(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



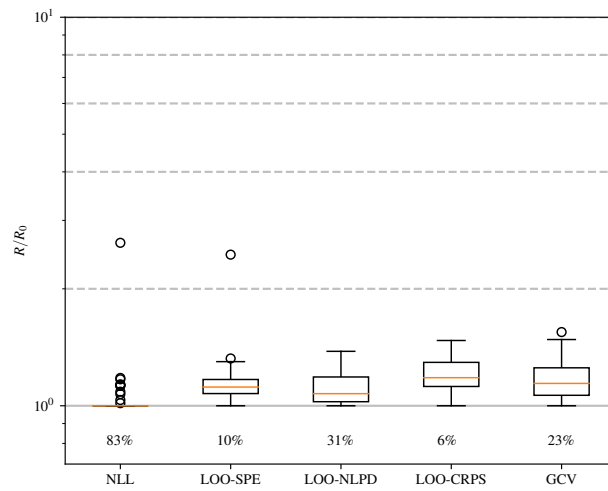
(c)  $S^{\text{IS}}_{0.95}$

Figure SM4:  $\nu = 3/2$



(a)  $S^{\text{SPE}}$

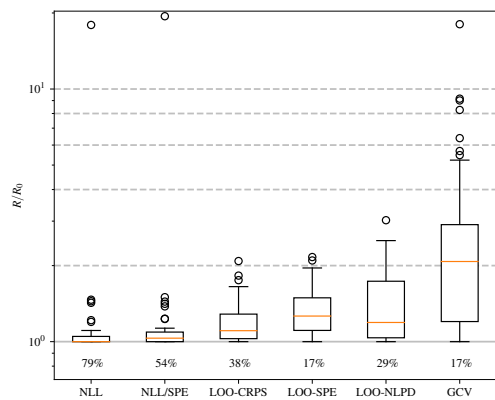
(b)  $S^{\text{CRPS}}$



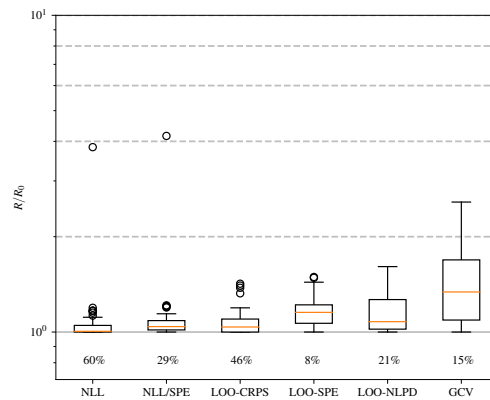
(c)  $S^{\text{IS}}_{0.95}$

Figure SM5:  $\nu = 5/2$

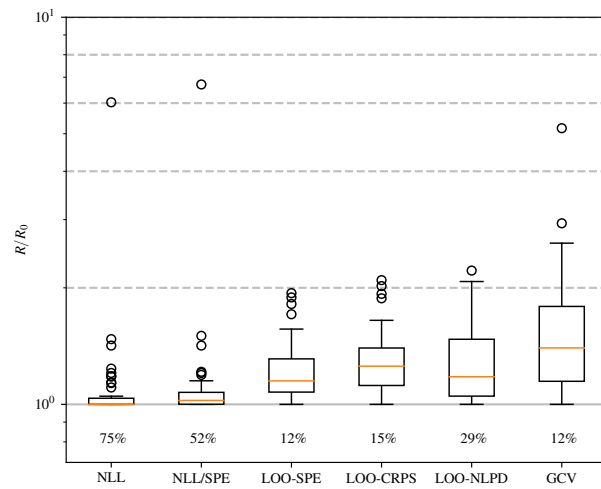




(a)  $S^{\text{SPE}}$



(b)  $S^{\text{CRPS}}$

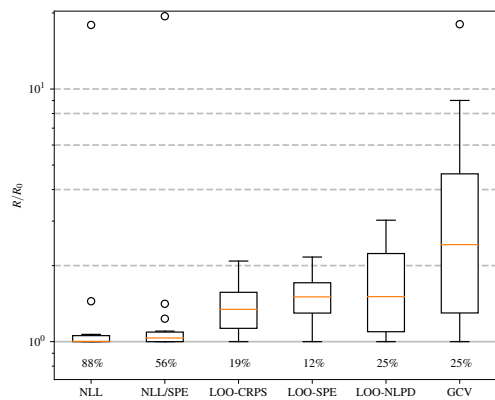


(c)  $S_{0.95}^{\text{IS}}$

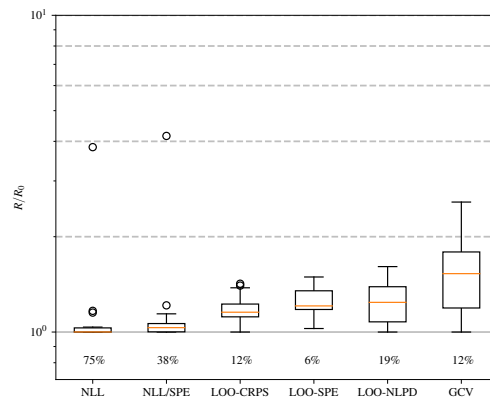
Figure SM6:  $v = \hat{v}$

### SM-3.3.2 Varying the design size

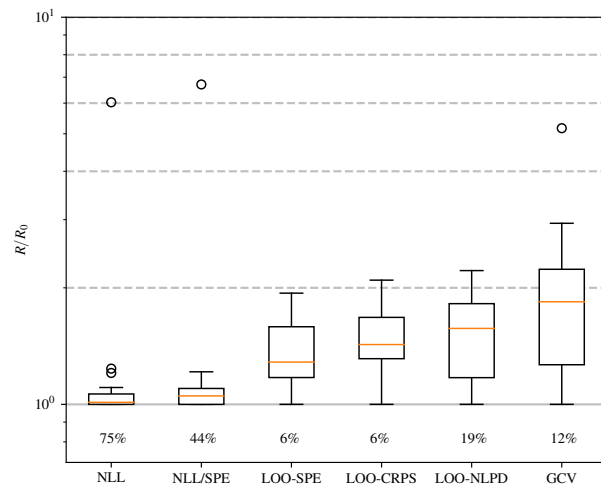
Here we present comparisons for  $n = 10d, 20d, 50d$  on all problems and for  $\nu = \hat{\nu}$ .



(a)  $S^{\text{SPE}}$

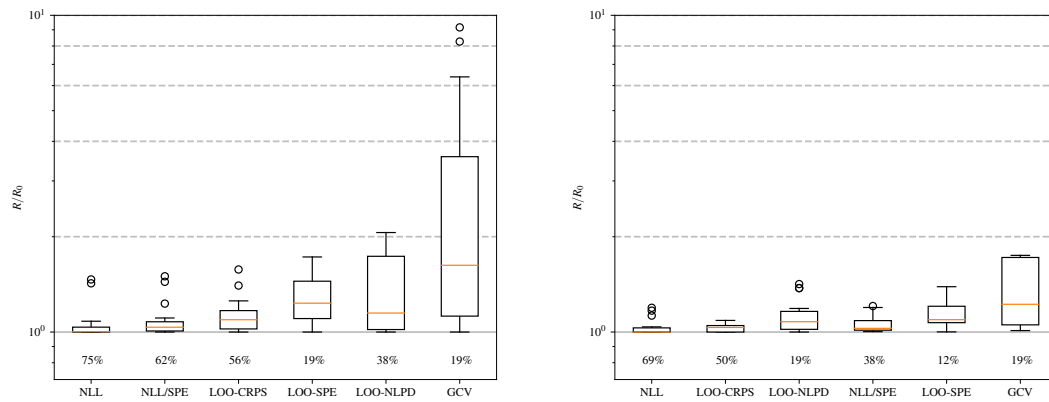


(b)  $S^{\text{CRPS}}$



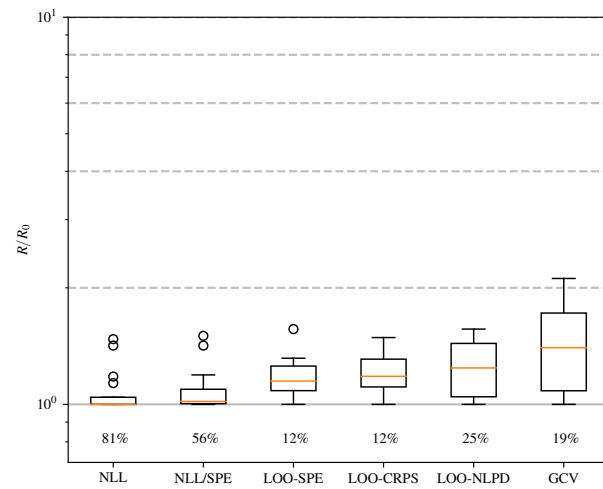
(c)  $S^{\text{IS}}_{0.95}$

Figure SM7:  $n = 10d$



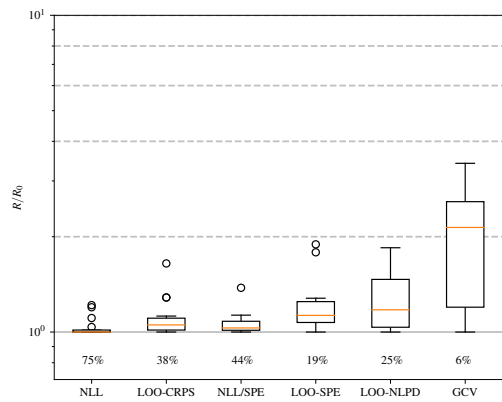
(a)  $S^{SPE}$

(b)  $S^{CRPS}$

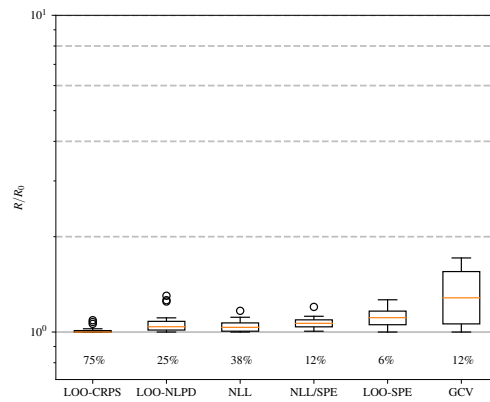


(c)  $S^{IS}_{0.95}$

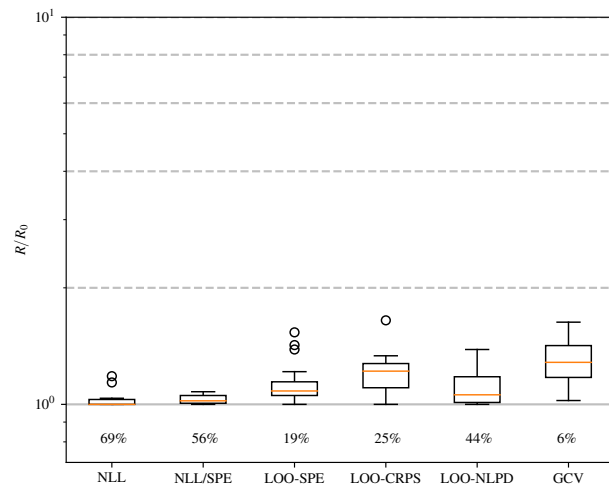
Figure SM8:  $n = 20d$



(a)  $S^{\text{SPE}}$



(b)  $S^{\text{CRPS}}$

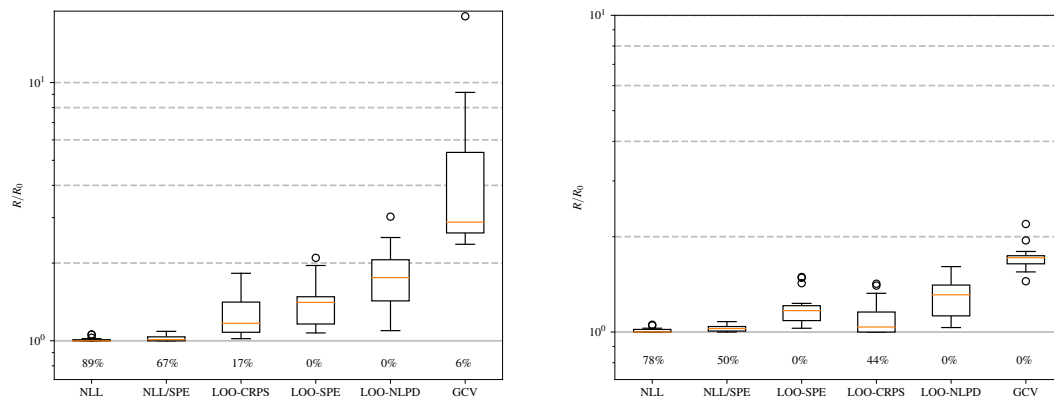


(c)  $S^{\text{IS}}_{0.95}$

Figure SM9:  $n = 50d$

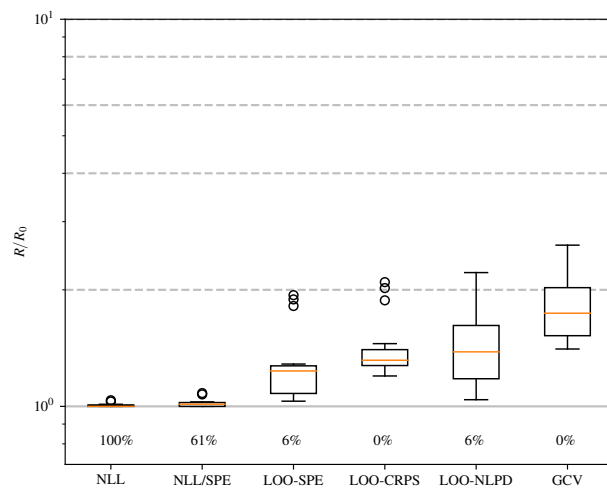
### **SM-3.3.3 Varying the type of problem**

Here we present comparisons on specific sets of problems for all design sizes and for  $\nu = \hat{\nu}$ .



(a)  $S^{SPE}$

(b)  $S^{CRPS}$



(c)  $S_{0.95}^{IS}$

Figure SM10: GKLS

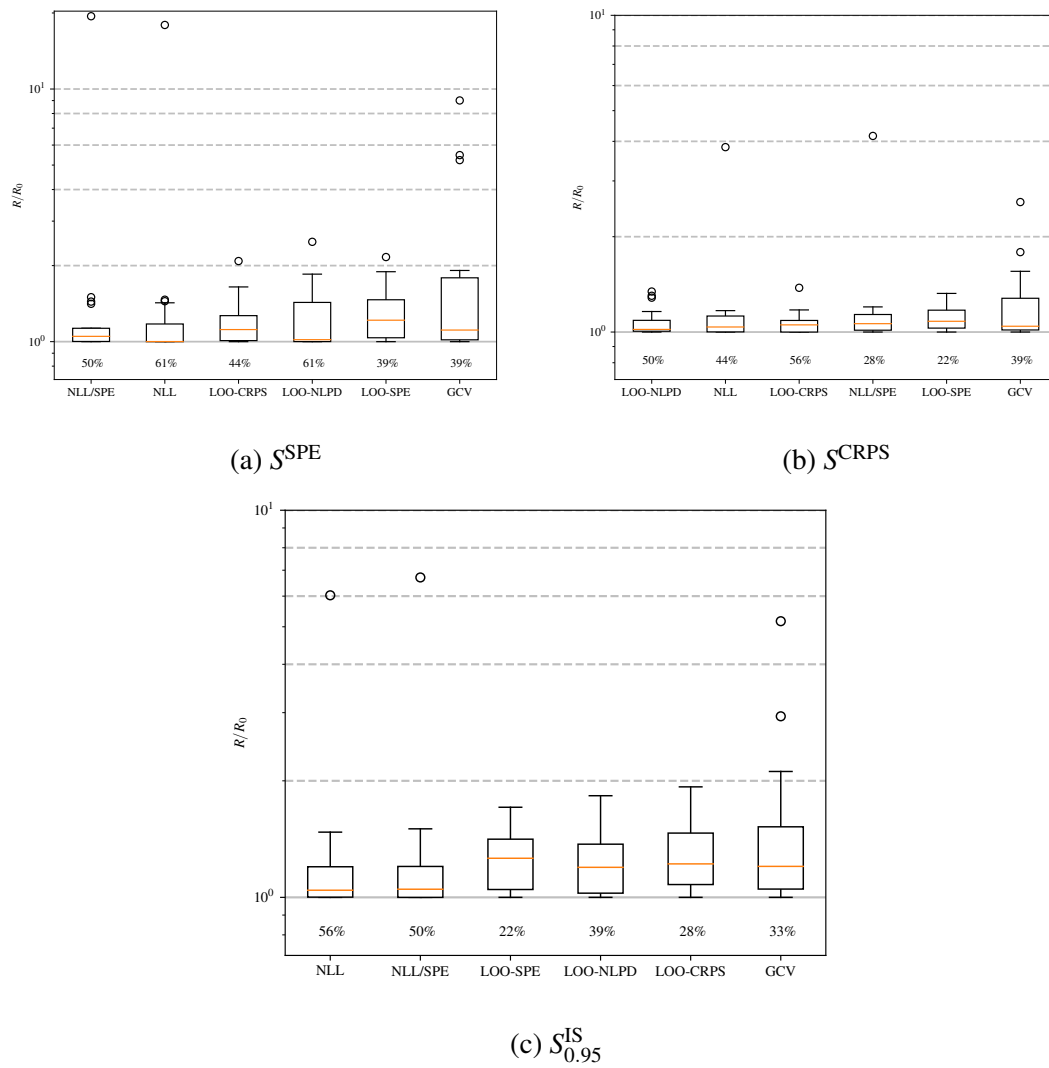
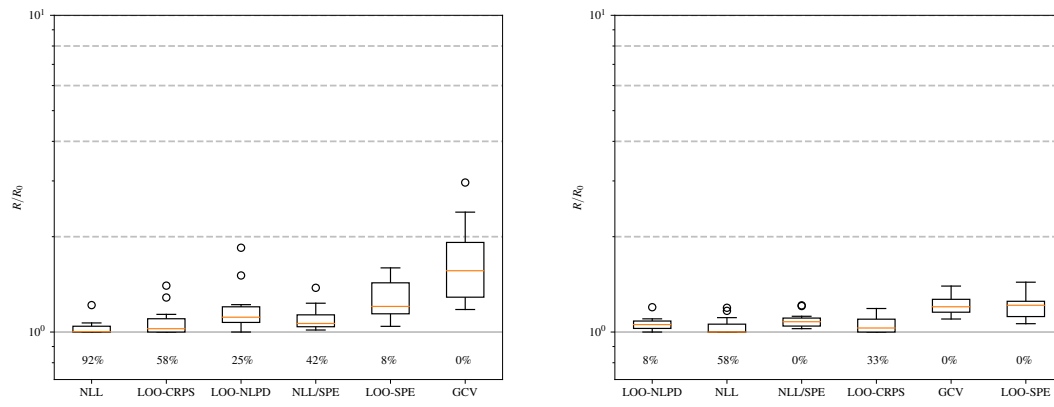


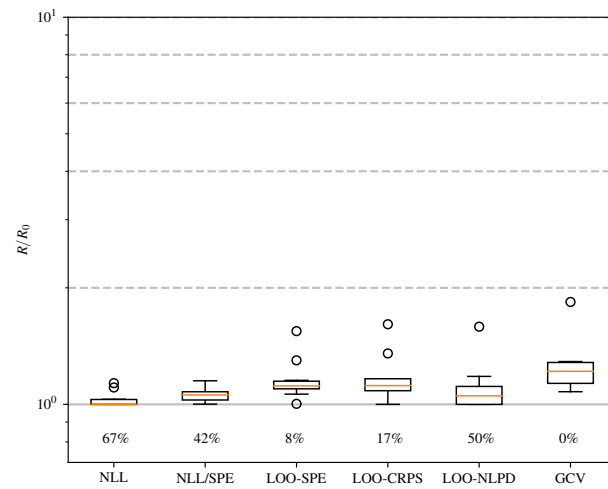
Figure SM11: Other test functions





(a)  $S^{\text{SPE}}$

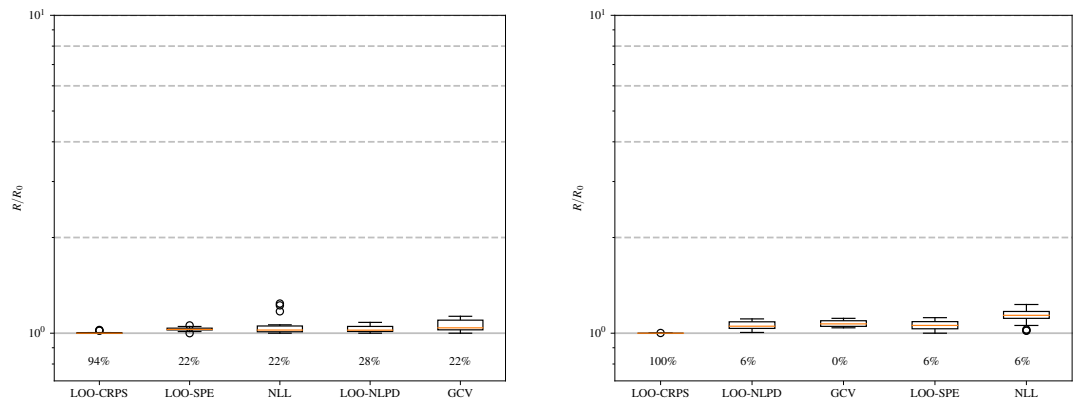
(b)  $S^{\text{CRPS}}$



(c)  $S_{0.95}^{\text{IS}}$

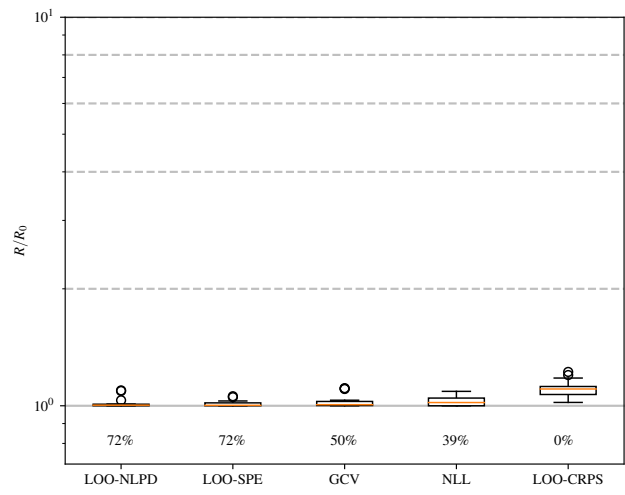
Figure SM12: lowpass filter

### **SM-3.3.4 Varying both the type of problem and the regularity**



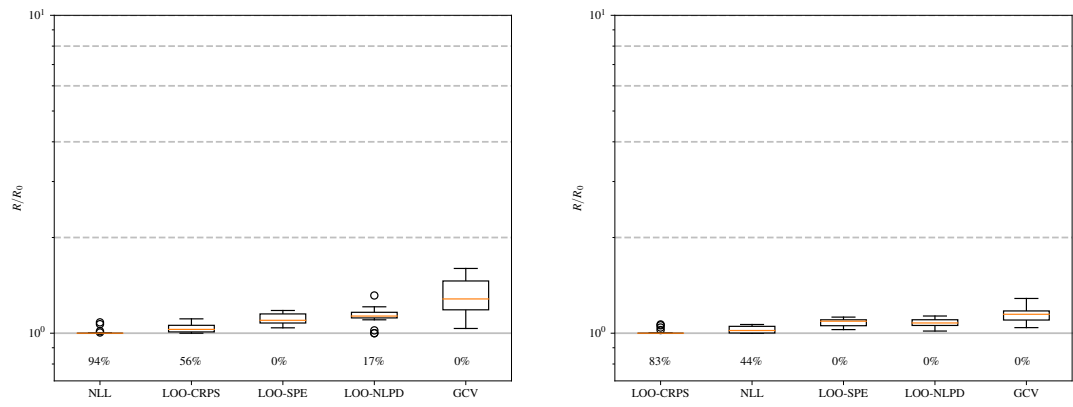
(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



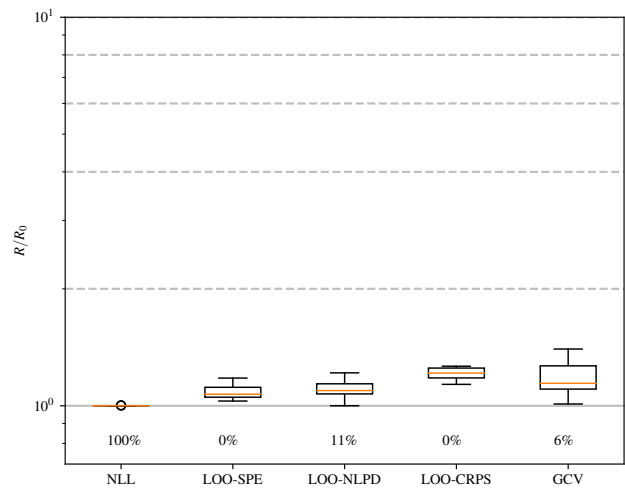
(c)  $S^{\text{IS}}_{0.95}$

Figure SM13: GKLS,  $\nu = 1/2$



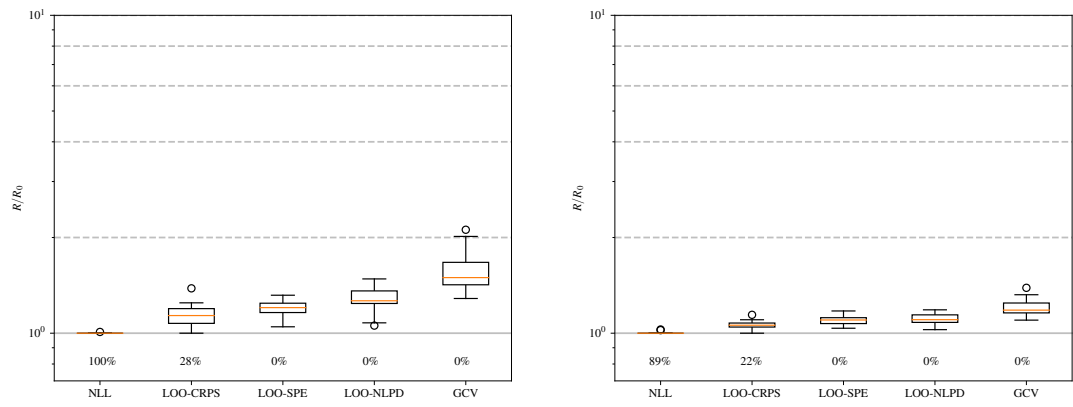
(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



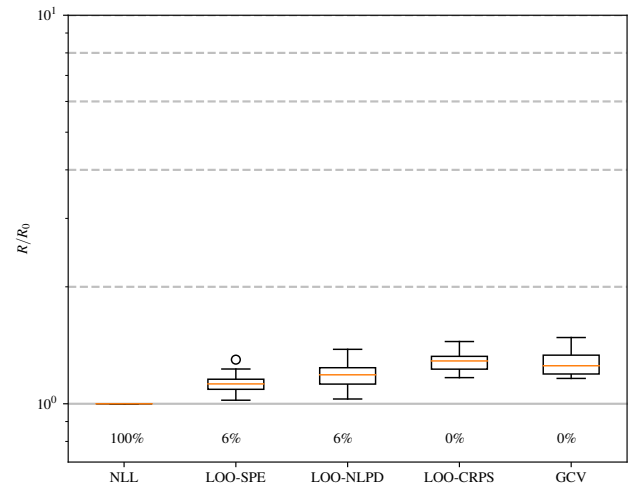
(c)  $S_{0.95}^{\text{IS}}$

Figure SM14: GKLS,  $\nu = 3/2$



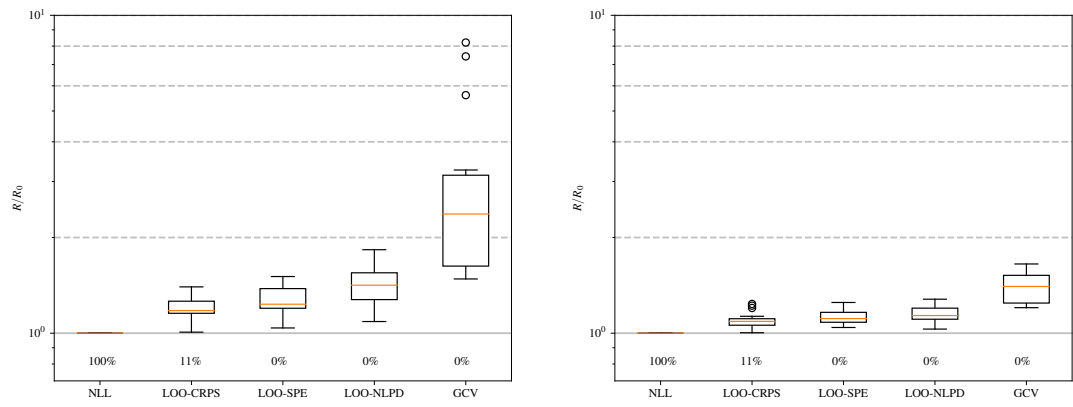
(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



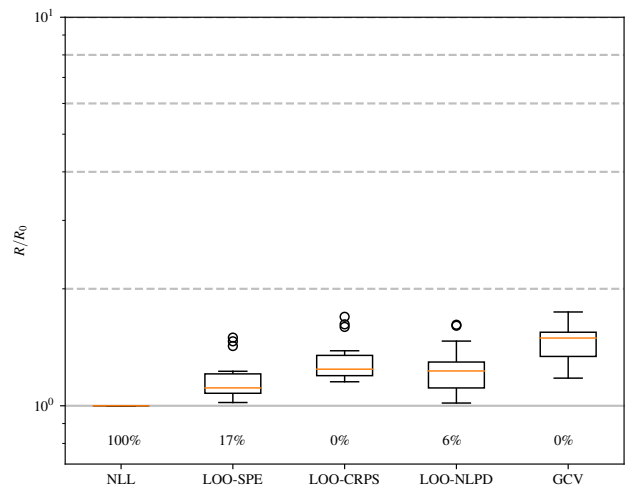
(c)  $S^{\text{IS}}_{0.95}$

Figure SM15: GKLS,  $\nu = 5/2$



(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



(c)  $S^{\text{IS}}_{0.95}$

Figure SM16: GKLS,  $\nu = 9/2$

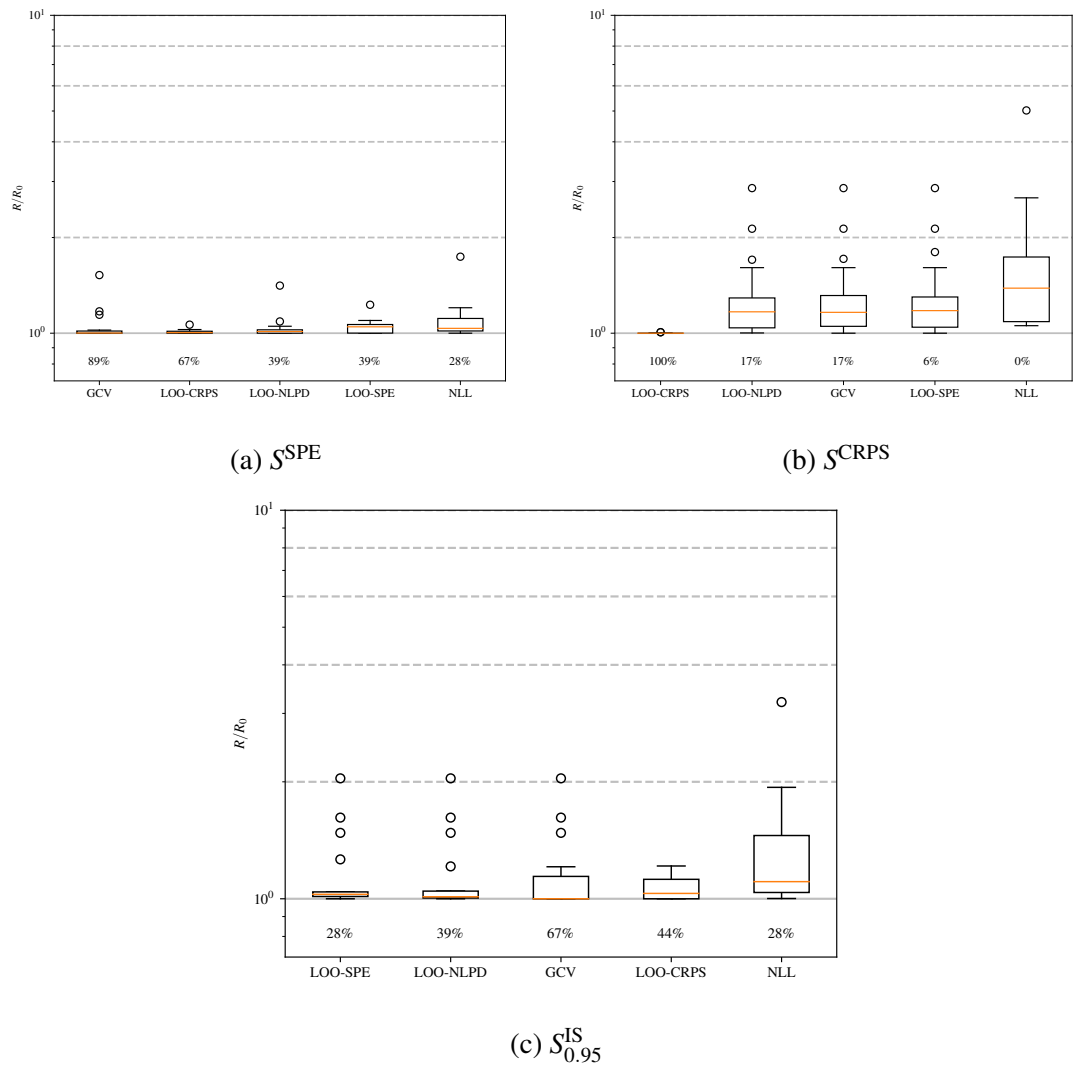


Figure SM17: Other test functions,  $\nu = 1/2$

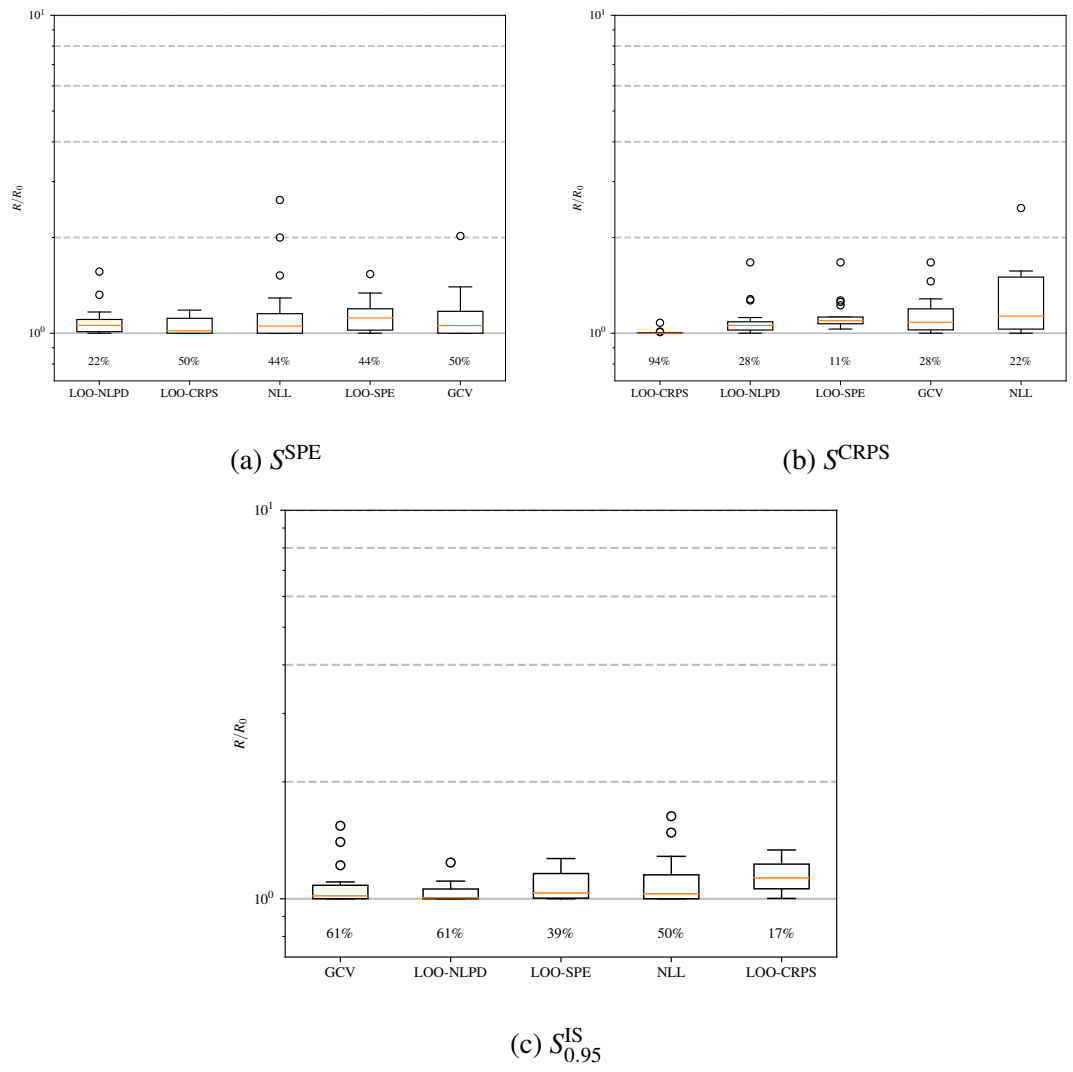
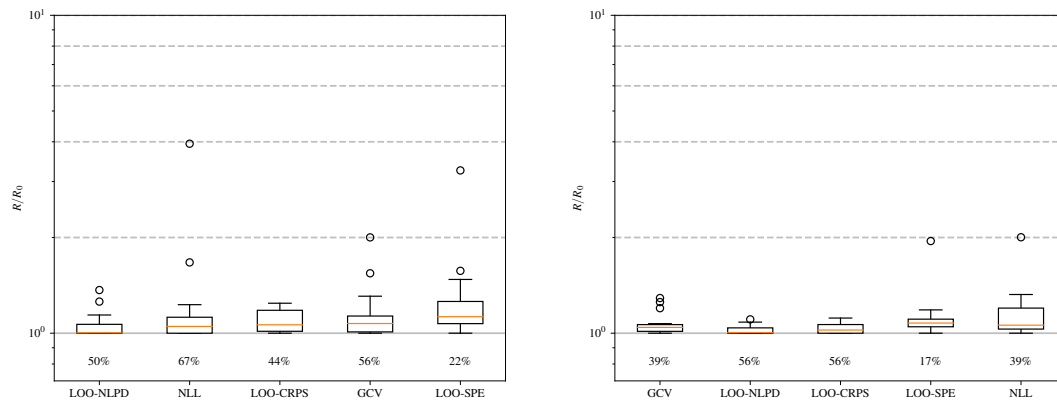


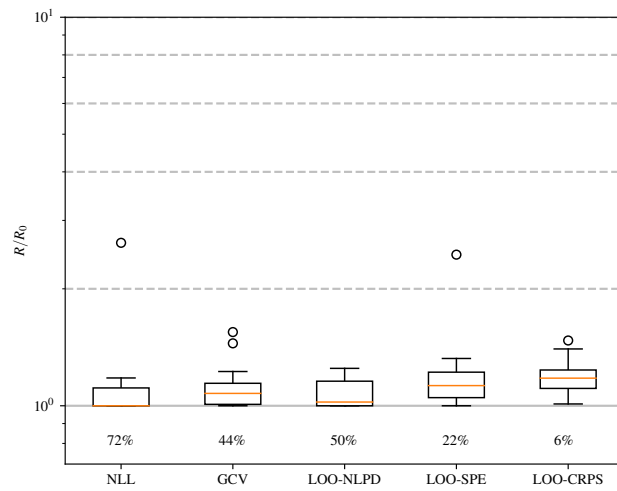
Figure SM18: Other test functions,  $\nu = 3/2$





(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



(c)  $S^{\text{IS}}_{0.95}$

Figure SM19: Other test functions,  $\nu = 5/2$

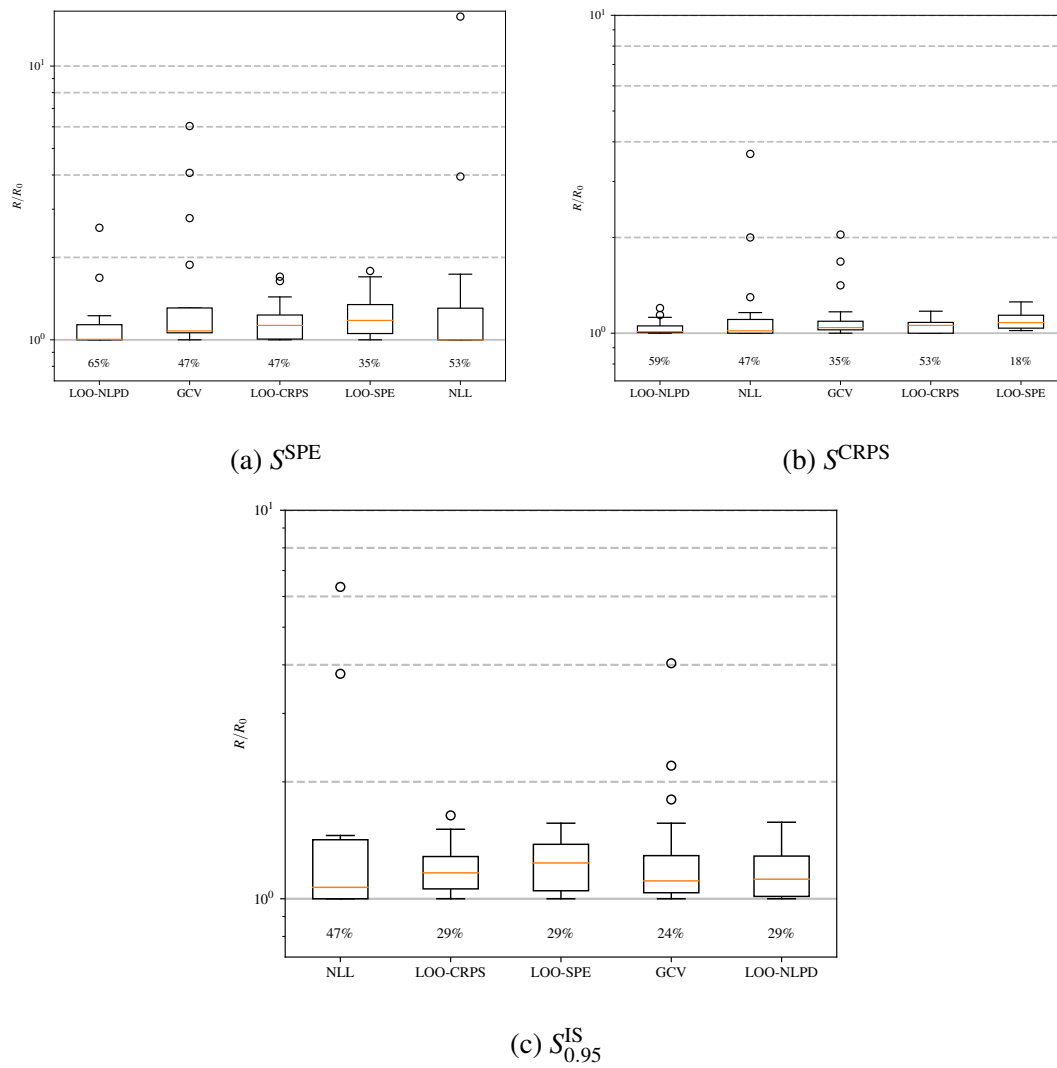
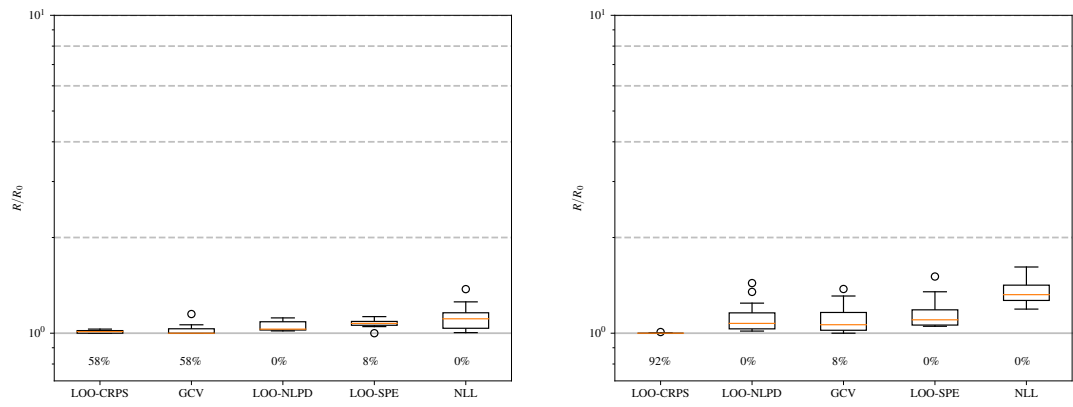
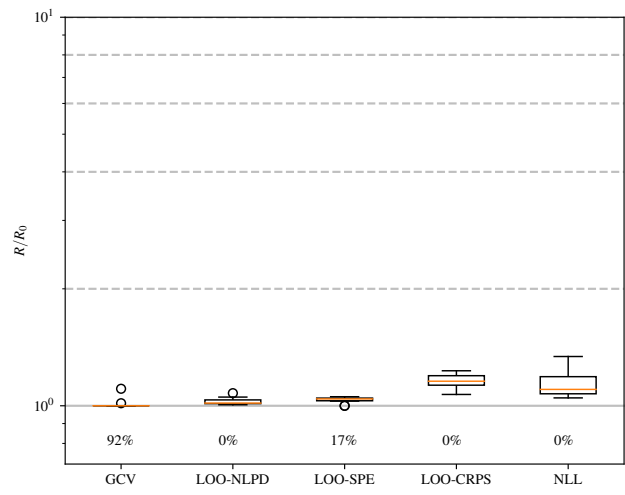


Figure SM20: Other test functions,  $\nu = 9/2$



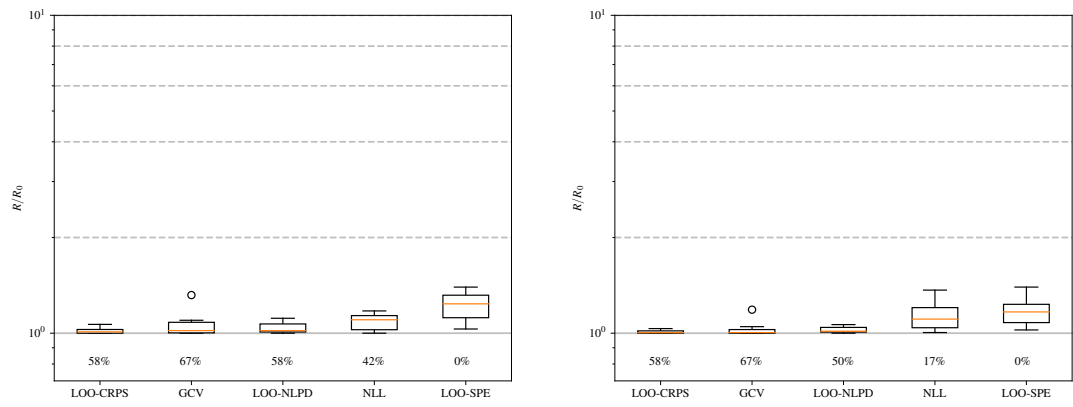
(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



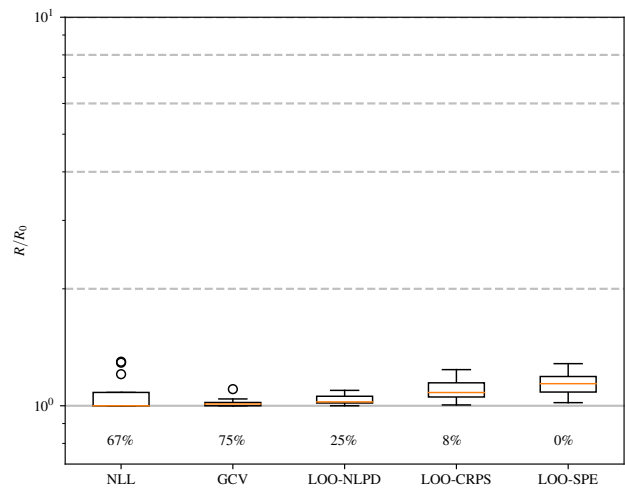
(c)  $S_{0.95}^{\text{IS}}$

Figure SM21: lowpass filter,  $\nu = 1/2$



(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



(c)  $S^{\text{IS}}_{0.95}$

Figure SM22: lowpass filter,  $\nu = 3/2$

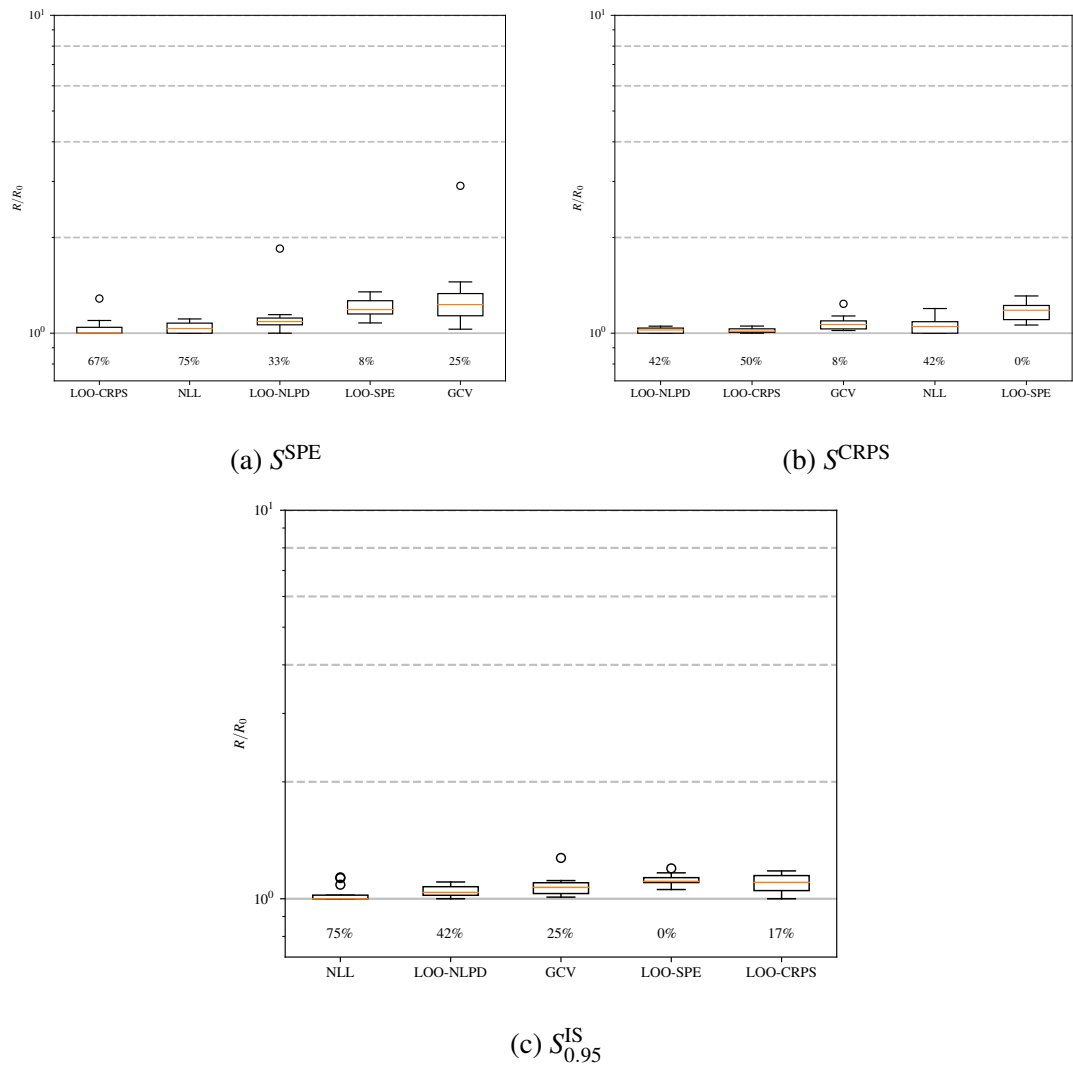
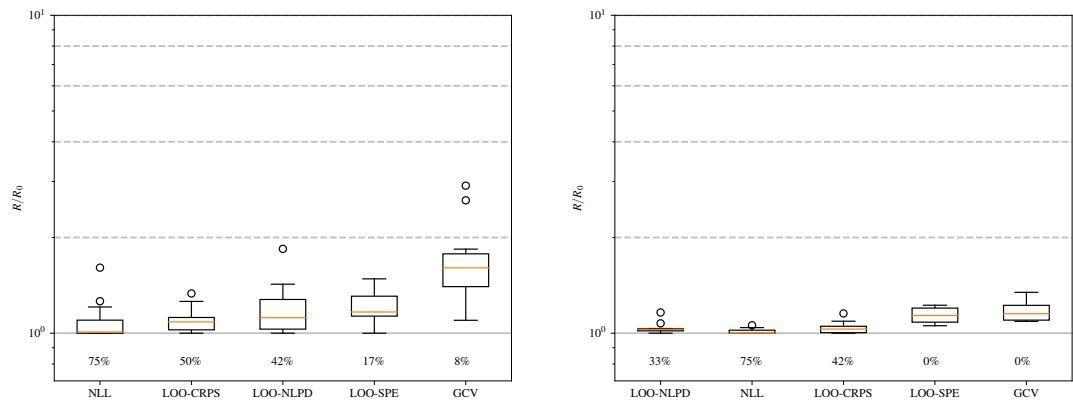
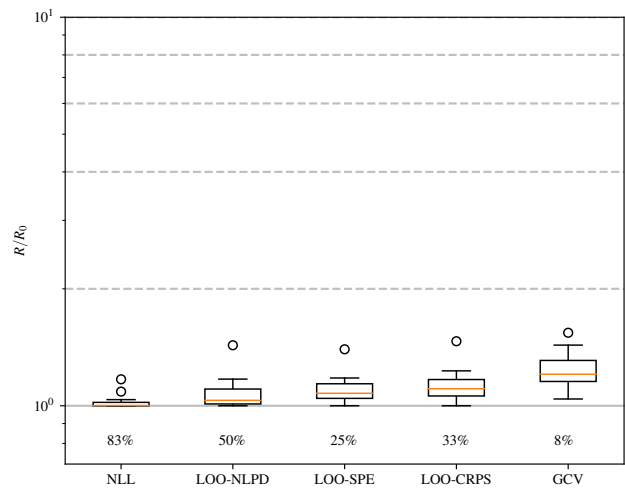


Figure SM23: lowpass filter,  $\nu = 5/2$



(a)  $S^{\text{SPE}}$

(b)  $S^{\text{CRPS}}$



(c)  $S_{0.95}^{\text{IS}}$

Figure SM24: lowpass filter,  $\nu = 9/2$

### SM-3.4 Additional model comparison

Here we present the same kind of analysis as in the article regarding the regularity parameter. Note here that we use the notation  $\chi = \nu - 1/2$ . A summary is given in Section [SM-3.4.6](#).

Observe that automatically selecting the model with NLL appears to be very efficient as it leads to a very small loss compared to the situation where the “best”  $\nu$  is known in advance.

The other criteria appear to be efficient to select  $\nu$  on smooth problems but not on GKLS problems, especially GCV which seems to be the least efficient to select  $\nu$ . Note that automatically selecting  $\nu$  with the LOO criteria or even GCV still exhibits practical benefits as the regularity parameter has a much larger impact on smooth problems.

### SM-3.4.1 Accuracy of NLL

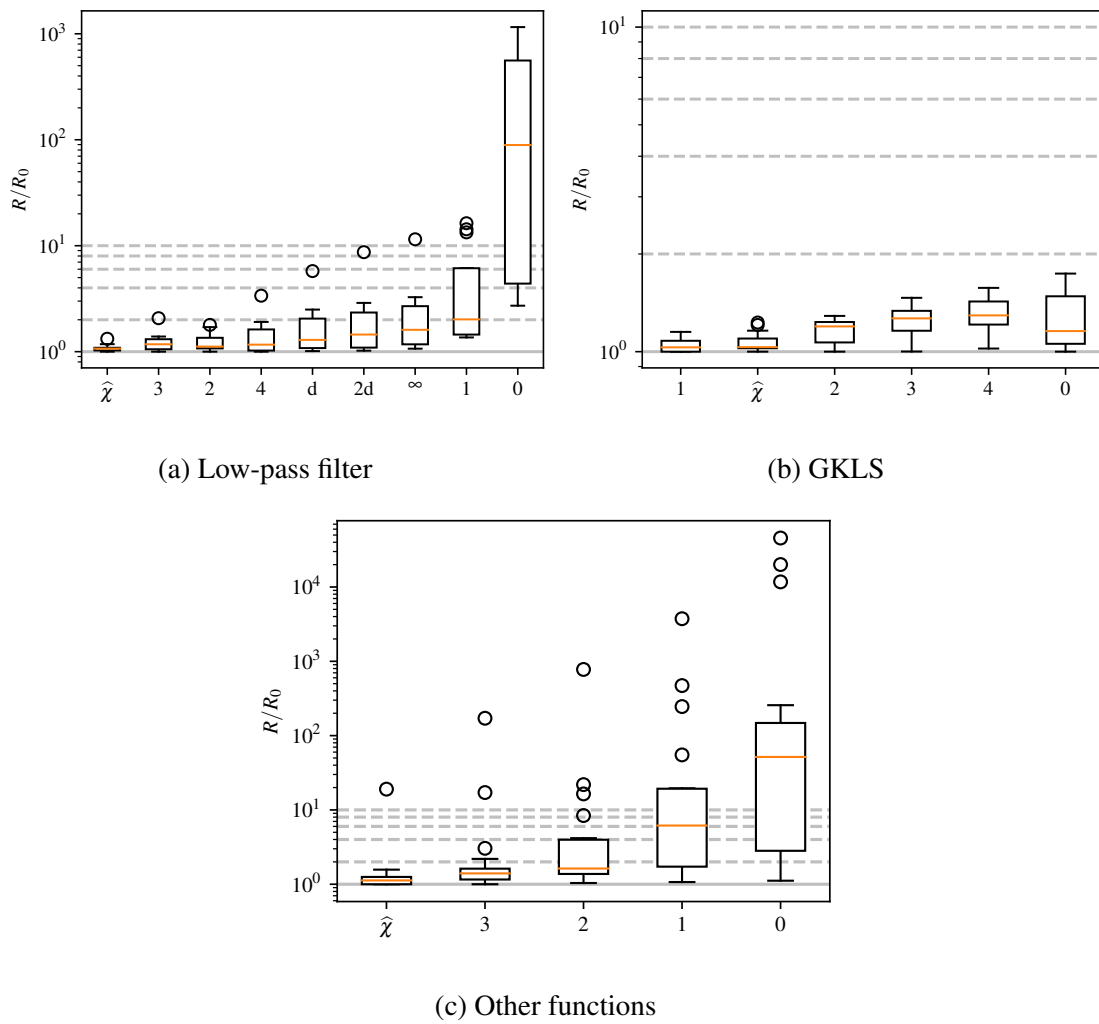
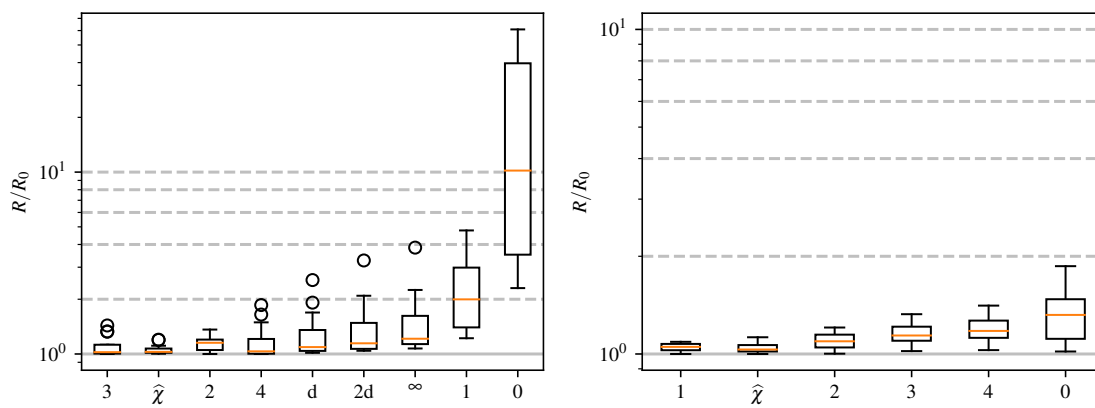


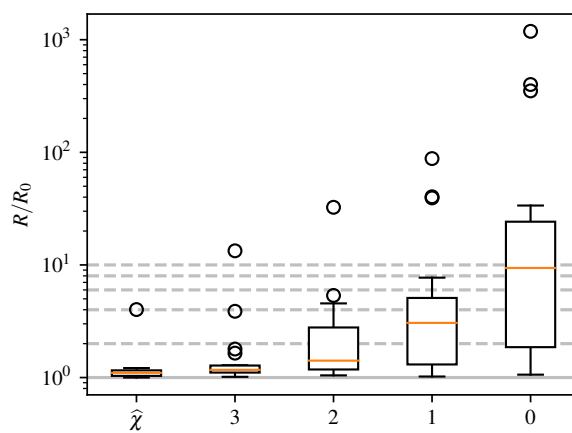
Figure SM25: Accuracy according to  $S^{\text{SPE}}$  for criterion NLL





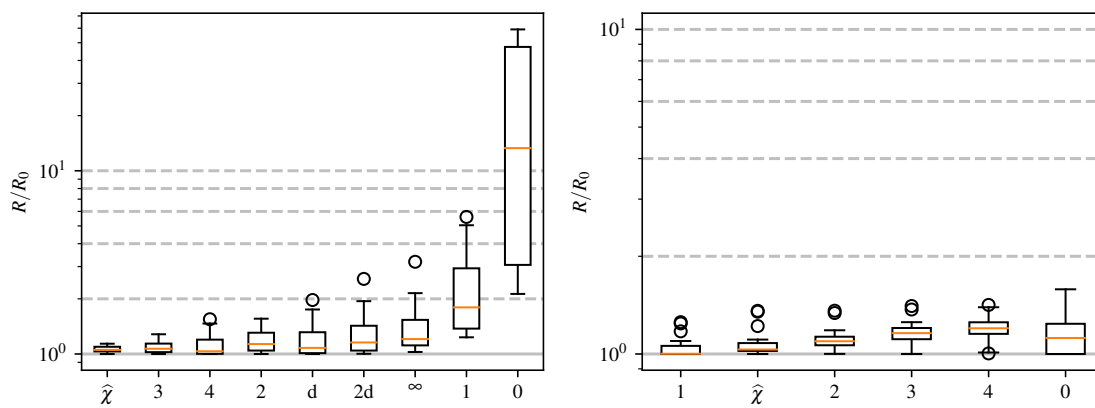
(a) Low-pass filter

(b) GKLS



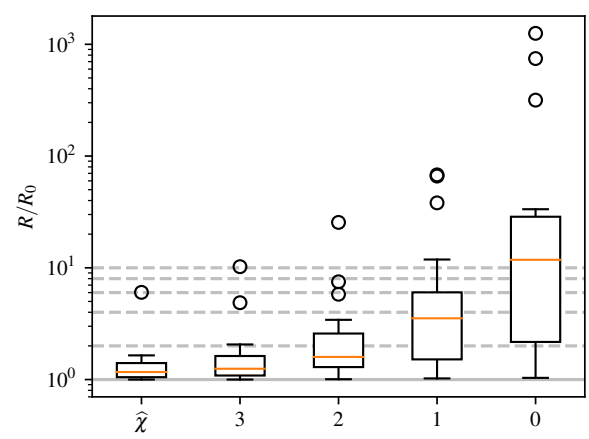
(c) Other functions

Figure SM26: Accuracy according to  $S^{CRPS}$  for criterion NLL



(a) Low-pass filter

(b) GKLS



(c) Other functions

Figure SM27: Accuracy according to  $S_{0.95}^{IS}$  for criterion NLL

### SM-3.4.2 Accuracy of LOO-CRPS

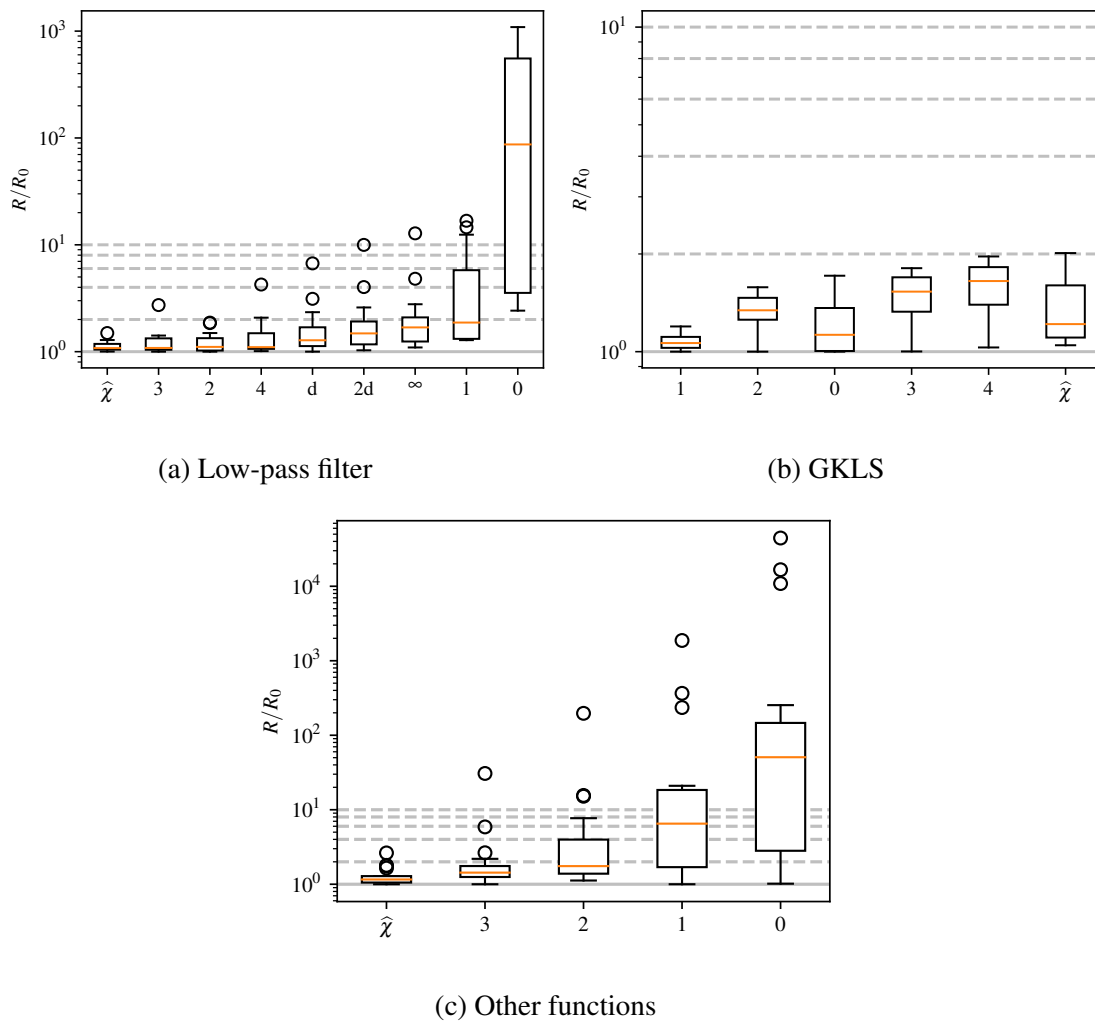
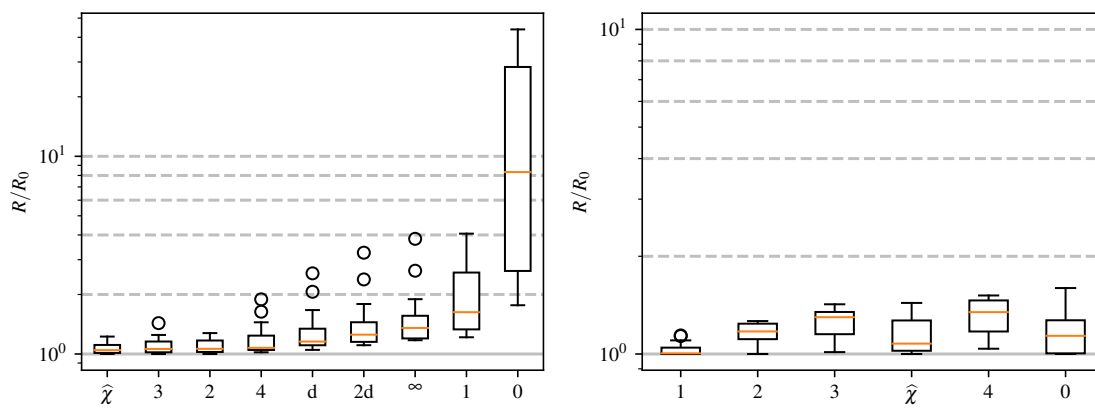
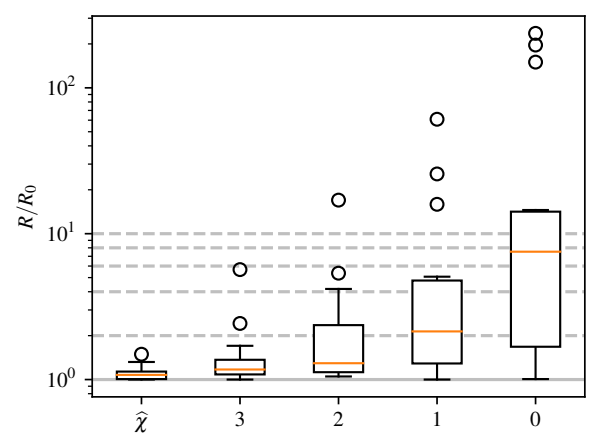


Figure SM28: Accuracy according to  $S^{\text{SPE}}$  for criterion LOO-CRPS



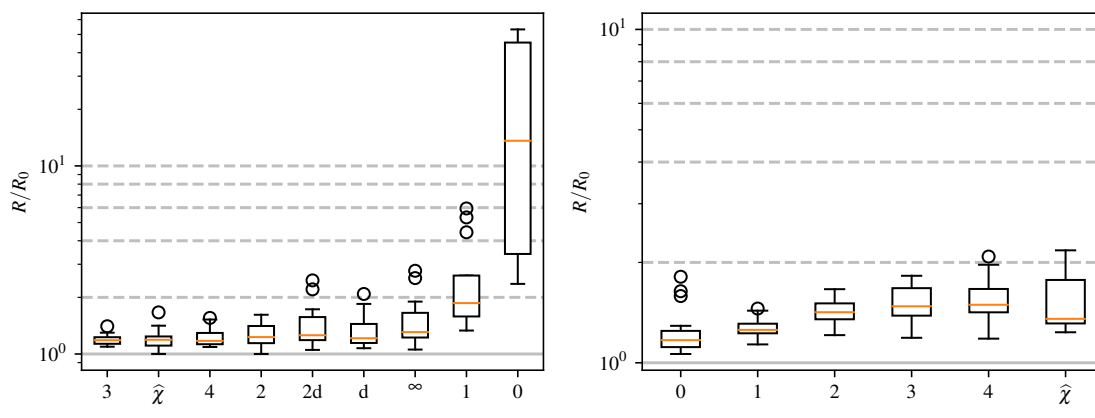
(a) Low-pass filter

(b) GKLS



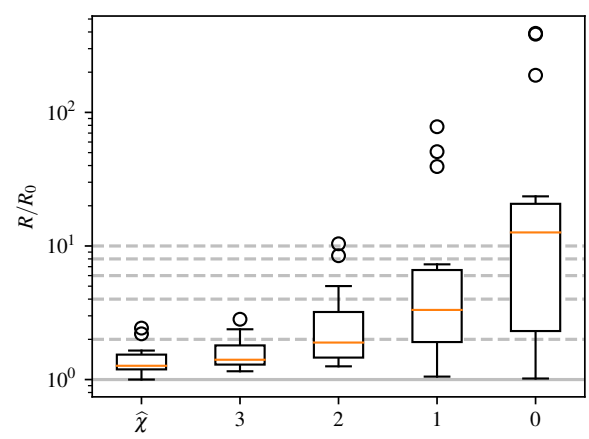
(c) Other functions

Figure SM29: Accuracy according to  $S^{CRPS}$  for criterion LOO-CRPS



(a) Low-pass filter

(b) GKLS



(c) Other functions

Figure SM30: Accuracy according to  $S_{0.95}^{clS}$  for criterion LOO-CRPS

### SM-3.4.3 Accuracy of LOO-SPE

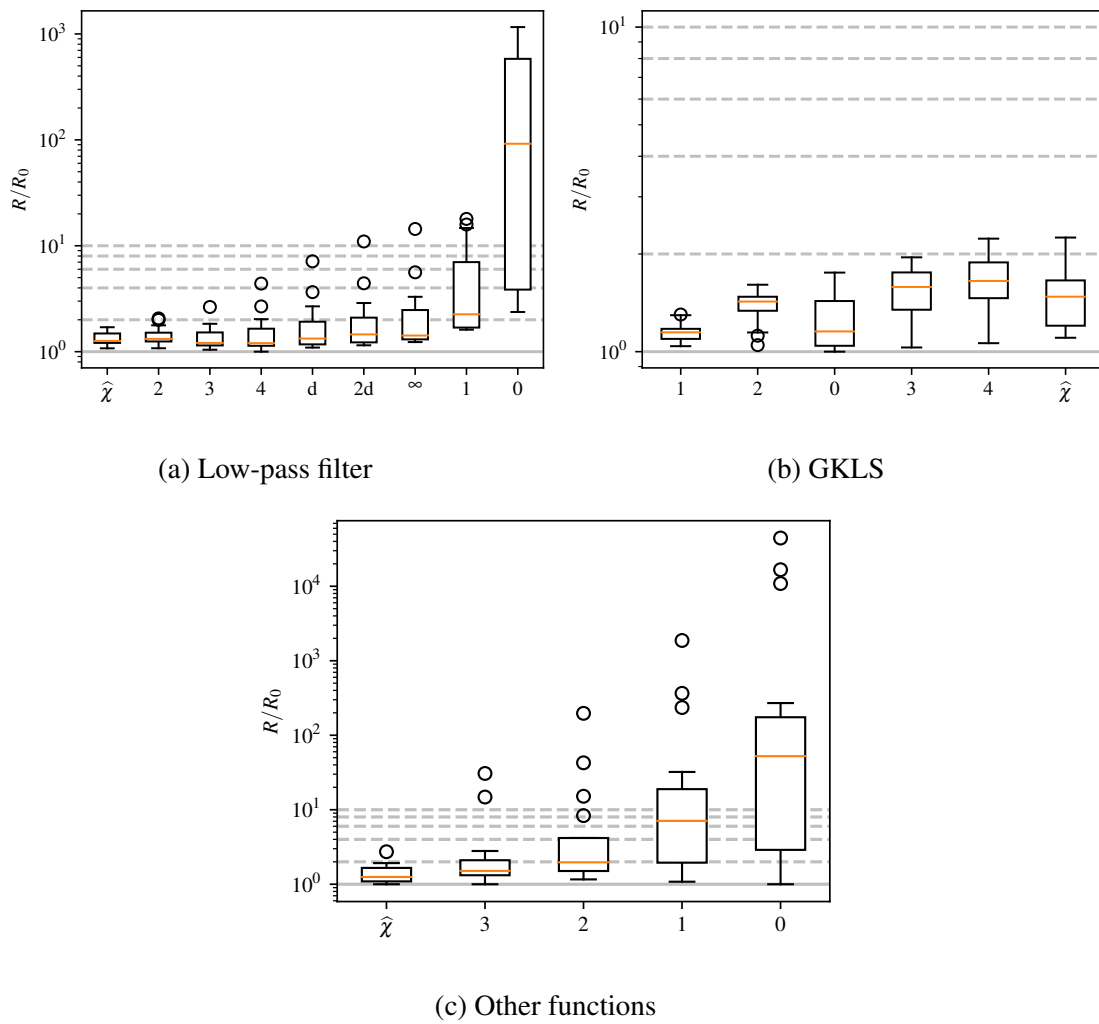
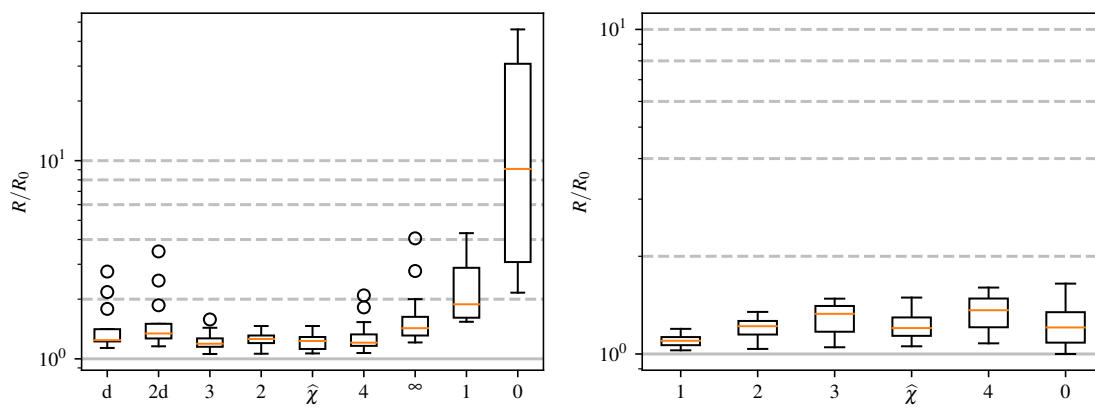
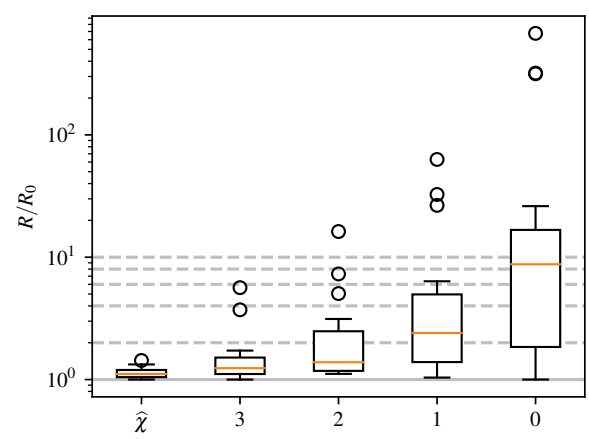


Figure SM31: Accuracy according to  $S^{\text{SPE}}$  for criterion LOO-SPE



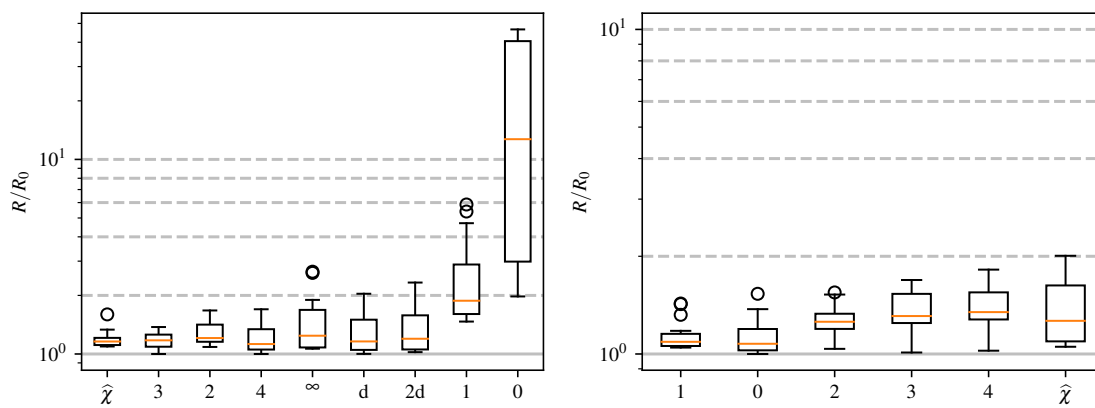
(a) Low-pass filter

(b) GKLS



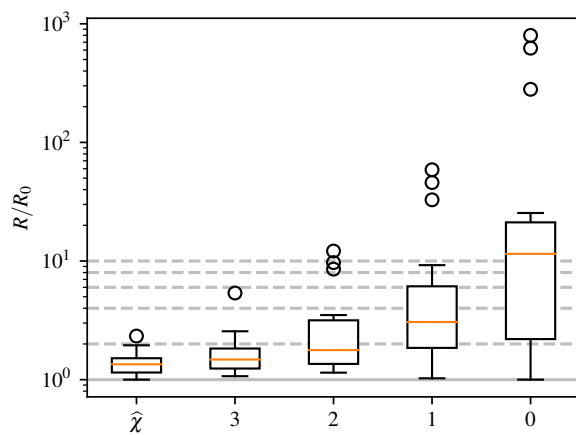
(c) Other functions

Figure SM32: Accuracy according to  $S^{CRPS}$  for criterion LOO-SPE



(a) Low-pass filter

(b) GKLS



(c) Other functions

Figure SM33: Accuracy according to  $S_{0.95}^{IS}$  for criterion LOO-SPE



### SM-3.4.4 Accuracy of LOO-NLPD

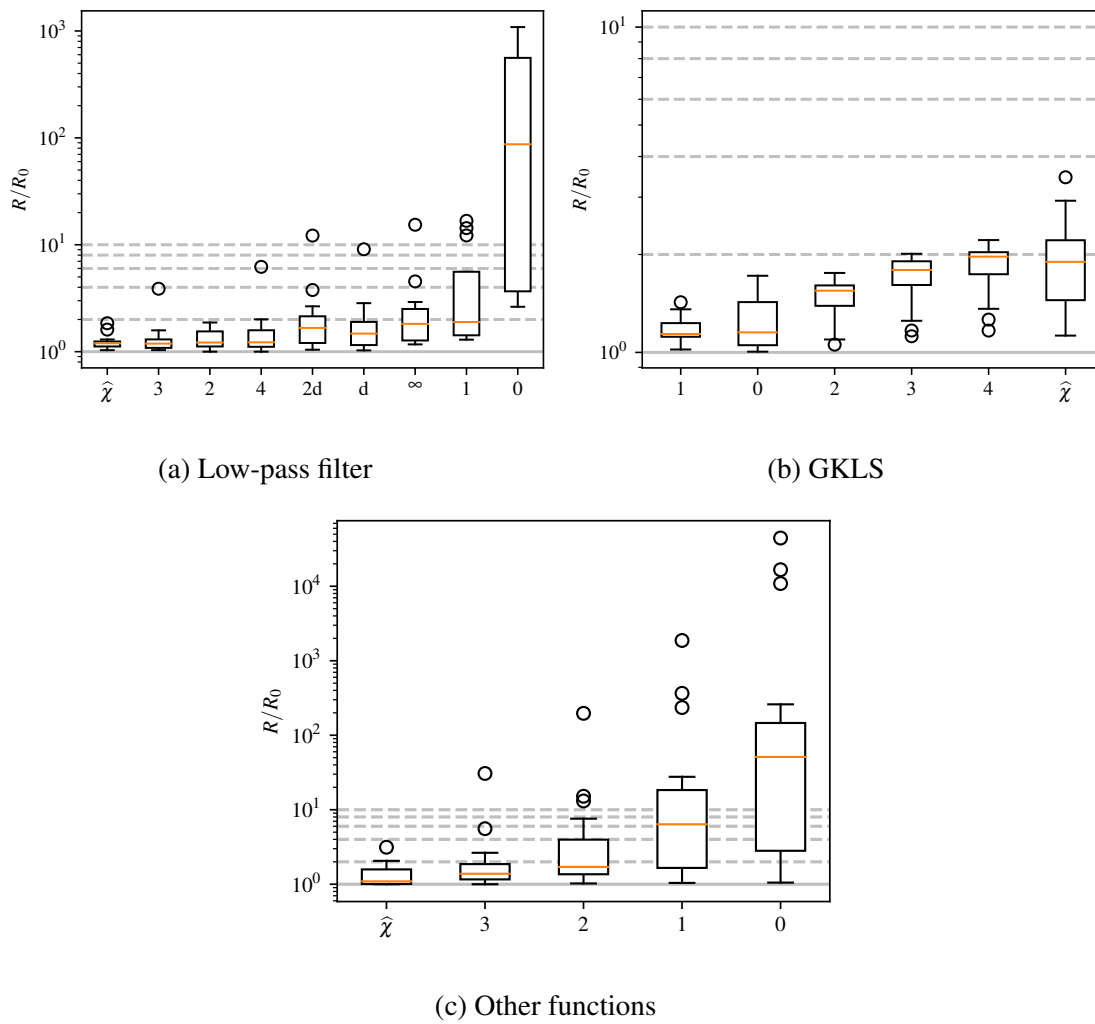
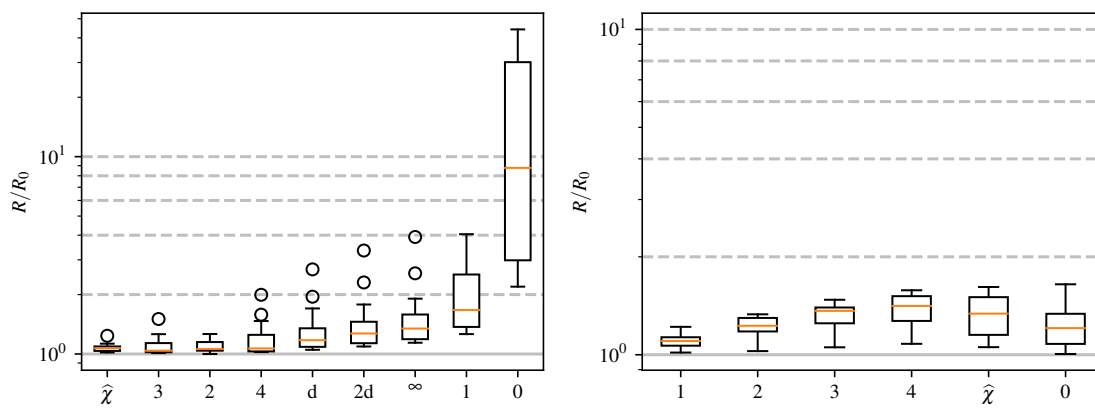
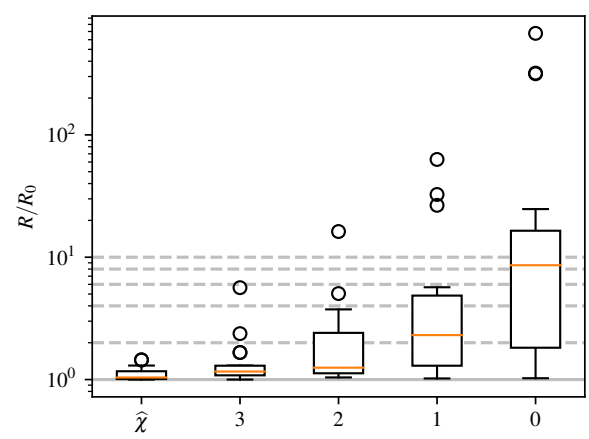


Figure SM34: Accuracy according to  $S^{\text{SPE}}$  for criterion LOO-NLPD



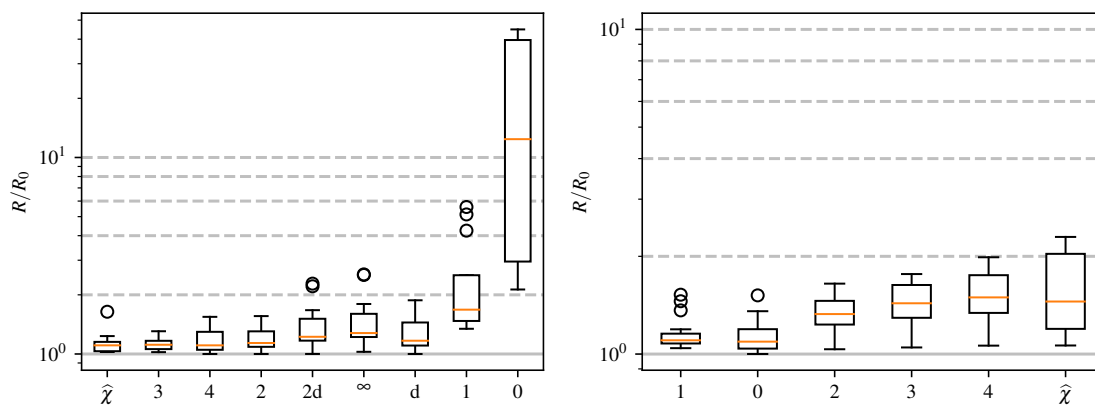
(a) Low-pass filter

(b) GKLS



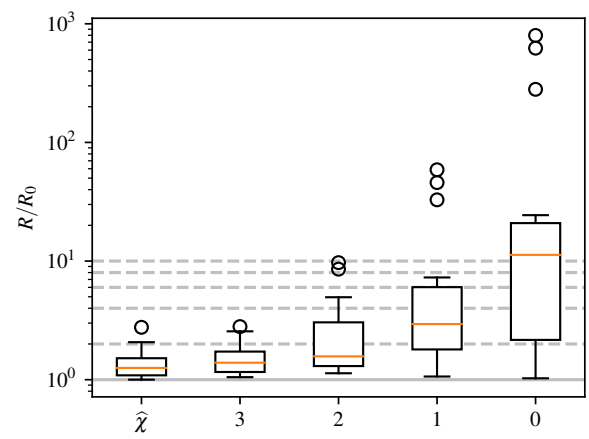
(c) Other functions

Figure SM35: Accuracy according to  $S^{CRPS}$  for criterion LOO-NLPD



(a) Low-pass filter

(b) GKLS



(c) Other functions

Figure SM36: Accuracy according to  $S_{0.95}^{IS}$  for criterion LOO-NLPD

### SM-3.4.5 Accuracy of GCV

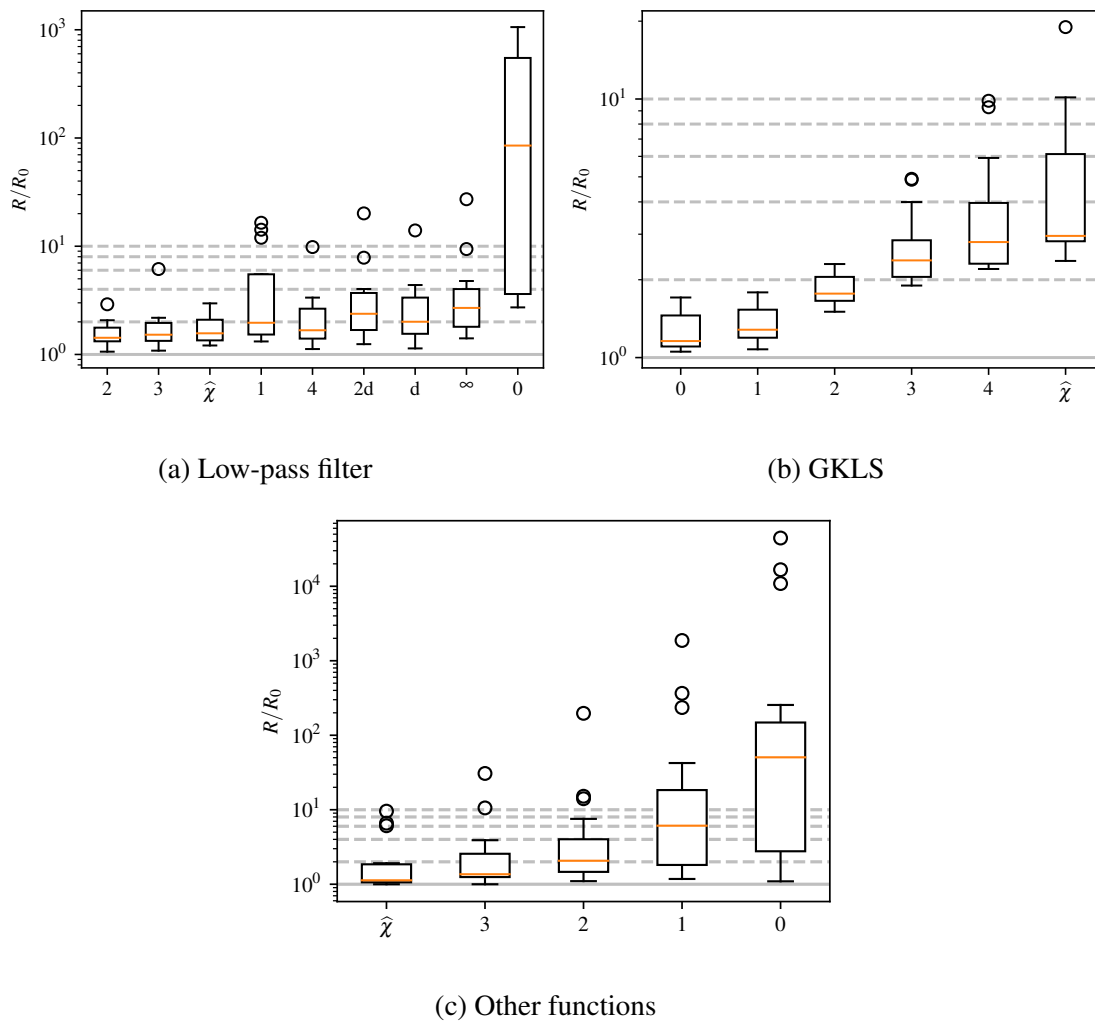
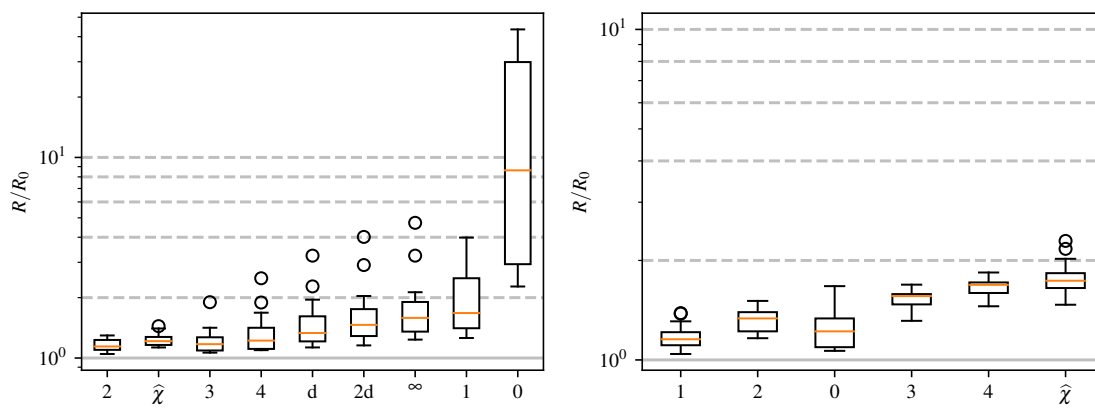
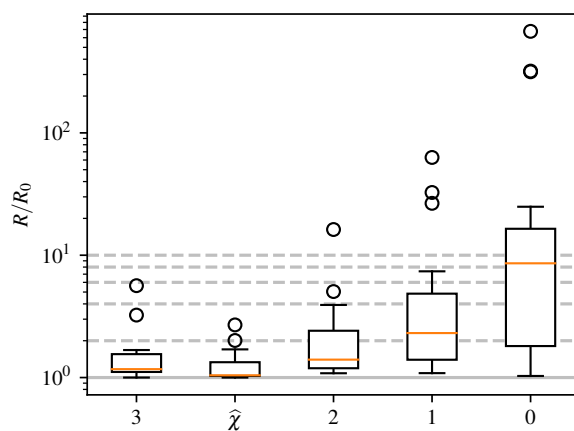


Figure SM37: Accuracy according to  $S^{\text{SPE}}$  for criterion GCV



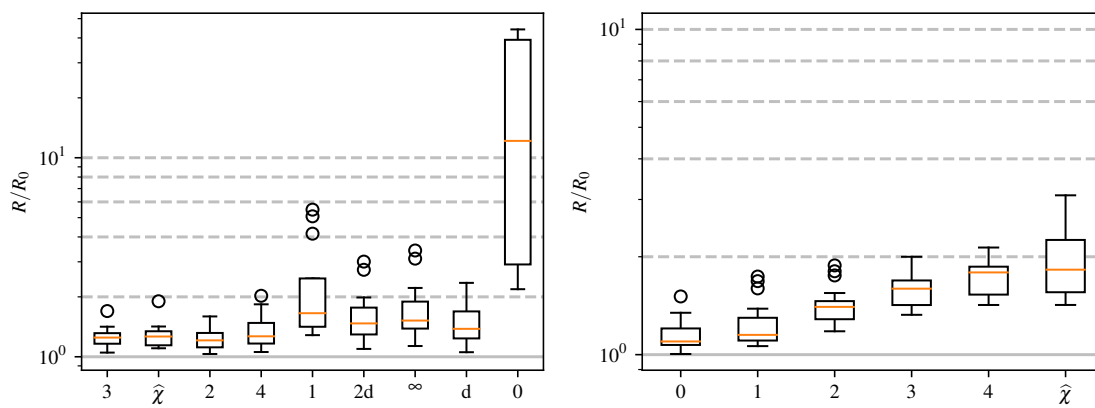
(a) Low-pass filter

(b) GKLS



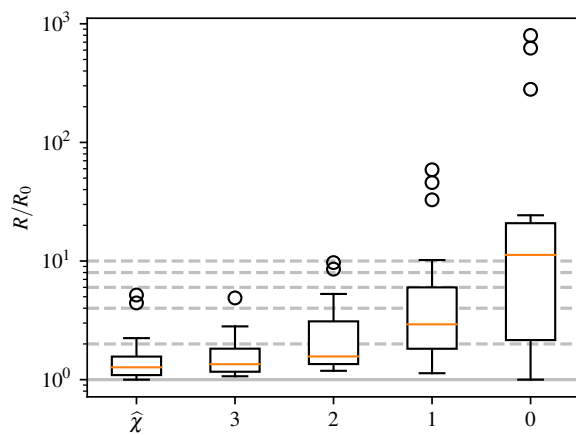
(c) Other functions

Figure SM38: Accuracy according to  $S^{CRPS}$  for criterion GCV



(a) Low-pass filter

(b) GKLS



(c) Other functions

Figure SM39: Accuracy according to  $S_{0.95}^{IS}$  for criterion GCV

### SM-3.4.6 Summary

We provide here a last comparison between models on Table SM7, Table SM8 and Table SM9 by taking all problems as a perimeter. These results clearly confirm that:

1. Under-smoothing has (much) more dramatic consequences than over-smoothing. This is true for all selection criteria.
2. Good results are achieved in quite a robust manner (since we are looking at the 95%-quantile) by selecting the value of  $\nu$  automatically, jointly with the other hyper-parameters. This is true for all selection criteria, with GCV as a notable exception. It is particularly true for NLL (for other criteria, we have seen that the performance drops a little on the non-smooth GKLS problems).

Table SM7: 95% quantiles for  $R/R_0$  with SPE where  $R_0$  is the best model for a given criteria and a given problem.

	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	GCV
$\nu = 1/2$	$8.0 \cdot 10^3$	$7.4 \cdot 10^3$	$7.4 \cdot 10^3$	$7.4 \cdot 10^3$	$7.4 \cdot 10^3$
$\nu = 3/2$	$1.7 \cdot 10^2$	$1.6 \cdot 10^2$	$1.6 \cdot 10^2$	$1.6 \cdot 10^2$	$1.6 \cdot 10^2$
$\nu = 5/2$	6.2	13	11	11	5.7
$\nu = 7/2$	2.1	2.5	2.5	2.5	4.2
$\nu = \hat{\nu}$	<b>1.1</b>	1.8	2.6	1.9	7.7

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Table SM8: 95% quantiles for  $R/R_0$  with CRPS where  $R_0$  is the best model for a given criteria and a given problem.

	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	GCV
$\nu = 1/2$	$2.4 \cdot 10^2$	$2.1 \cdot 10^2$	$2.1 \cdot 10^2$	$1.1 \cdot 10^2$	$2.1 \cdot 10^2$
$\nu = 3/2$	26	19	19	11	19
$\nu = 5/2$	2.9	4.3	3.5	3.4	2.7
$\nu = 7/2$	1.5	1.7	1.6	1.7	1.6
$\nu = \hat{\nu}$	<b>1.1</b>	1.3	1.5	1.4	1.9

Table SM9: 95% quantiles for  $R/R_0$  with IS where  $R_0$  is the best model for a given criteria and a given problem.

	NLL	LOO-SPE	LOO-NLPD	LOO-CRPS	GCV
$\nu = 1/2$	$1.5 \cdot 10^2$	$2.0 \cdot 10^2$	$2.0 \cdot 10^2$	$1.2 \cdot 10^2$	$2.0 \cdot 10^2$
$\nu = 3/2$	27	19	18	22	17
$\nu = 5/2$	3.0	4.7	3.9	4.0	2.8
$\nu = 7/2$	1.5	1.9	1.9	2.0	1.8
$\nu = \hat{\nu}$	<b>1.3</b>	1.9	2.1	2.0	2.5

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