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# Integral input-to-state stable time-delay systems in cascade <sup>★</sup>

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## Abstract

We consider the cascade interconnection of two nonlinear time-delay systems, each of which being integral input-to-state stable (iISS). We provide an explicit growth condition on the dissipation rate of the driving system and the input rate of the driven system under which the overall cascade is itself iISS. Building upon recent iISS characterizations of time-delay systems, the method allows to consider Lyapunov-Krasovskii functionals (LKF) that dissipate in a point-wise manner along solutions, which simplifies its applicability as compared to dissipation rates involving the whole LKF itself. Another key feature of our approach is that the growth condition is imposed only on the driving state variables that actually appear in the LKF analysis of the driven subsystem: this feature happens to be new also in a delay-free context and helps reducing the conservatism of existing approaches.

*Key words:* integral input-to-state stability, time-delay systems, cascades.

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## 1 Introduction

Cascades are a class of interconnected subsystems in which the influence of one subsystem on the other is unidirectional: the output of the driving subsystem impacts the driven subsystem, but the latter has no influence on the former. Cascades arise in many real-world applications, either due to their physical structure or as a result of the chosen feedback law (cascade-based control).

For finite-dimensional systems, local asymptotic stability is naturally preserved by cascade interconnection [34], but this not the case for global asymptotic stability [33, 22]. Additional requirements are needed, which may take the form of solutions' boundedness [27, 28] or growth rate conditions [22, 21, 6].

Beyond stability of autonomous cascades, practical applications often require to guarantee its robustness to

exogenous disturbances. This can be efficiently achieved through the framework of input-to-state stability (ISS). This notion was introduced in [29] and imposes that, in the absence of disturbances, the dynamical system evolves properly (namely, its equilibrium is globally asymptotically stable: 0-GAS) and that, in the presence of perturbations, this nominal behavior is preserved up to a steady-state error proportional to the magnitude of the applied disturbance. ISS is known to be preserved by cascade interconnection for finite-dimensional systems [31, 9], but sometimes constitutes a too demanding requirement.

A weaker robustness property is integral input-to-state stability (iISS, [30]): instead of measuring the impact of the input *magnitude* on the steady-state behavior of the system, it rather focuses on its *energy*. Unlike ISS, iISS is in general not preserved by cascade interconnection [6, 7]. For finite dimensional systems, iISS of a cascade of iISS subsystems can be ensured either under a growth rate condition [7] or when the dissipation rate of the driven subsystem is non-vanishing at infinity [15, 9].

All of those results are developed for finite-dimensional systems. Due to their pervasiveness in control applications and their possible destabilizing impact, there is a need to provide stability analysis of cascades involving delays. iISS was extended to time-delay systems (TDS)

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in [25], in which a sufficient condition for iISS is given in terms of Lyapunov-Krasovskii functionals (LKF). This sufficient condition was recently slightly simplified in [20] in the context of integral input-to-output stability, which allows in particular to consider an iISS-like behavior only part of the state variables. A LKF characterization of iISS was recently obtained in [18, 10]. Building up on these results, we provide explicit growth-rate conditions under which the cascade of two iISS TDS is itself iISS. Like in the finite-dimensional case [7], these growth rate conditions require that the dissipation rate of the driving subsystem has greater growth around zero than the input rate of the driven subsystem.

When dealing with TDS, a specific care must be taken on the way the LKF dissipates along solutions as, unlike for finite-dimensional systems, the LKF cannot always be interchanged with the state norm [11]. A particularly appealing feature of the proposed results is that they rely on LKF that dissipate in a *point-wise* manner, meaning solely in terms of the current value of the solution's norm. This differs from more stringent LKF conditions for iISS [25, 18], which require a dissipation rate that involves the whole LKF itself. We even further relax this requirement, by imposing merely a  $\mathcal{KL}$  dissipation rate, thus allowing for LKF of the form  $\ln(1 + V)$ , which are of common use in the study of iISS.

Another interesting feature of the results presented here is that the growth restrictions are imposed only on the driving state variables that actually appear in the derivative of the LKF of the driven subsystem. As we show through an example, this allows to reduce the conservatism of the existing approaches even in a finite-dimensional context.

So far, the vast majority of the existing stability results about cascades of iISS systems are restricted to finite-dimensional systems. A notable exception is [26], who extended the results in [22] to TDS by using a Lyapunov-Razumikin approach. Cascades can be seen as a particular case of feedback interconnection, thus allowing the use iISS small-gain results for time-delay systems such as [16]. However, this approach often leads to overly conservative stability conditions in the context of cascades. We also stress that a preliminary version of this work was published in [13], but that paper considered only a single discrete delay between the two subsystems, relied on a growth condition on all driving state variables, and was unable to conclude iISS for the cascade.

After recalling some necessary concepts in Section 2, we provide our main results on cascades of TDS in Section 3. A focus on finite-dimensional cascades is made in Section 4 to highlight the results novelty also for delay-free systems. Two academic examples are provided in Section 5. All proofs are given in Section 6 and we provide some directions for future work in Section 7.

## 2 Preliminaries

### 2.1 Notation

A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be of class  $\mathcal{PD}$  if it is continuous, and satisfies  $\alpha(0) = 0$  and  $\alpha(s) > 0$  for all  $s > 0$ .  $\alpha \in \mathcal{K}$  if  $\alpha \in \mathcal{PD}$  and it is increasing.  $\alpha \in \mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is unbounded.  $\alpha \in \mathcal{L}$ , if it is continuous, non-increasing and tends to zero as its argument tends to infinity. A function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}$  for each  $t \geq 0$  and  $\beta(s, \cdot) \in \mathcal{L}$  for each  $s > 0$ . Given  $q_1, q_2 \in \mathcal{PD}$ , we say that  $q_1$  has greater growth than  $q_2$  in a neighborhood of zero (and we write  $q_2(s) = \mathcal{O}_{s \rightarrow 0^+}(q_1(s))$ ) if  $\limsup_{s \rightarrow 0^+} q_2(s)/q_1(s) < +\infty$ . Note that this is equivalent to requiring that  $\sup_{s \in [0, 1]} q_2(s)/q_1(s) < +\infty$ . Given  $x \in \mathbb{R}^n$ ,  $|x|$  denotes its Euclidean norm. Given  $\delta \geq 0$ ,  $\mathcal{X}$  denotes the set of all continuous functions  $\varphi : [-\delta; 0] \rightarrow \mathbb{R}$ . Given any  $\phi \in \mathcal{X}^n$ ,  $\|\phi\| := \sup_{\tau \in [-\delta, 0]} |\phi(\tau)|$ . We denote by  $\mathcal{U}$  the set of all measurable essentially bounded signals  $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ .  $\mathbb{Z}$  is the set of integers whereas  $\mathbb{N}$  is the set of non-negative integers. Given any  $a, b \in \mathbb{Z}$  with  $b \geq a$ ,  $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$ . A property is said to hold almost everywhere (*a.e.*), if it holds everywhere except possibly on a set of zero Lebesgue measure zero.

### 2.2 Definitions

Consider the nonlinear TDS defined as

$$\dot{x}(t) = f(x_t, u(t)), \quad \forall t \geq 0 \text{ a.e.} \quad (1)$$

where  $u \in \mathcal{U}^m$  is the input,  $x(t) \in \mathbb{R}^n$  is the current value of the solution and  $x_t \in \mathcal{X}^n$  is the state history defined as

$$x_t(s) := x(t + s), \quad \forall s \in [-\delta, 0], \quad (2)$$

where  $\delta \geq 0$  denotes any constant larger than or equal to the largest delay involved.  $f : \mathcal{X}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is assumed to be Lipschitz on bounded sets, which ensures local existence and uniqueness of solutions for any initial state history  $x_0 \in \mathcal{X}^n$  and any input  $u \in \mathcal{U}^m$  [14]. We also assume that  $f(0, 0) = 0$ .

We recall the definition of iISS, originally introduced in a delay-free context [30] and extended to TDS in [25].

**Definition 1 (iISS)** *The system (1) is said to be integral input-to-state stable (iISS) if there exists  $\beta \in \mathcal{KL}$  and  $\nu, \sigma \in \mathcal{K}_\infty$  such that, for all  $x_0 \in \mathcal{X}^n$  and all  $u \in \mathcal{U}^m$ , its solution satisfies*

$$|x(t)| \leq \beta(\|x_0\|, t) + \nu \left( \int_0^t \sigma(|u(s)|) ds \right), \quad \forall t \geq 0. \quad (3)$$

A first consequence of iISS is that, in the absence of inputs, the system is globally asymptotically stable.

**Definition 2 (0-GAS)** *The TDS (1) is said to be globally asymptotically stable for zero input (0-GAS) if there exists  $\beta \in \mathcal{KL}$  such that, for all  $x_0 \in \mathcal{X}^n$ , the solution of the input-free system  $\dot{x}(t) = f(x_t, 0)$  satisfies*

$$|x(t)| \leq \beta(\|x_0\|, t), \quad \forall t \geq 0.$$

Note that, the concept defined above is usually called 0-UGAS in infinite-dimensional ISS literature (see for instance the survey [19]), whereas 0-GAS is often defined as the combination of stability and global attractivity. For finite-dimensional systems both properties are equivalent, but this is not the case for infinite-dimensional systems. Nevertheless, we decide to stick to the 0-GAS acronym (without explicitly stressing that convergence is uniform in the initial state) for the sake of homogeneity with the finite-dimensional terminology.

iISS actually goes beyond the internal stability property of 0-GAS, by ensuring a robustness property known as *bounded-energy converging-state*. Namely, if solutions of (1) satisfy the iISS estimate (3), then it holds that

$$\int_0^\infty \sigma(|u(s)|) ds < \infty \quad \Rightarrow \quad \lim_{t \rightarrow \infty} |x(t)| = 0. \quad (4)$$

Another robustness property induced by iISS is the following.

**Definition 3 (UBEBS)** *The TDS (1) is said to satisfy the uniform bounded energy-bounded state property if there exists  $\underline{\alpha}, \bar{\alpha}, \zeta \in \mathcal{K}_\infty$  and a constant  $c \geq 0$  such that, for all  $x_0 \in \mathcal{X}^n$  and all  $u \in \mathcal{U}^m$ , its solution satisfies*

$$\underline{\alpha}(|x(t)|) \leq \bar{\alpha}(\|x_0\|) + \int_0^t \zeta(|u(s)|) ds + c, \quad \forall t \geq 0. \quad (5)$$

For finite-dimensional systems, it was shown in [5] that iISS is equivalent to 0-GAS plus UBEBS. This result was recently extended to TDS in [10], as recalled next.

**Proposition 1 (iISS  $\Leftrightarrow$  0-GAS + UBEBS, [10])**

*The TDS (1) is iISS if and only if it is 0-GAS and owns the UBEBS property.*

**Remark 1** *In a finite-dimensional context, iISS is actually equivalent to 0-GAS plus the weaker requirement:*

$$\int_0^\infty \zeta(|u(s)|) ds < \infty \quad \Rightarrow \quad \liminf_{t \rightarrow \infty} |x(t)| < \infty,$$

*as proved in [2]. This property was at the core of the stability analysis of iISS finite-dimensional cascades conducted in [7]. No extension of this fact to TDS is yet available: the proof techniques presented below rather rely on Proposition 1.*

### 2.3 iISS characterization

Another strength of iISS is that it can be characterized using Lyapunov-Krasovskii functionals. We call a *Lyapunov-Krasovskii functional (LKF) candidate* any functional  $V : \mathcal{X}^n \rightarrow \mathbb{R}_{\geq 0}$ , Lipschitz on bounded sets, for which there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \bar{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}^n. \quad (6)$$

A LKF candidate is *coercive* if (6) is replaced by

$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \bar{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}^n.$$

The use of non-coercive LKF turns out to be useful in practice, as will be illustrated through an example in Section 5.1. Nevertheless, this comes at a price: unlike for finite-dimensional systems, it is often no longer possible to interchange dissipation rates in terms of  $|x(t)|$  and  $V(x_t)$ . This issue is discussed in more details in [11].

The *upper-right Dini derivative* of a LKF candidate  $V$  along the solutions of (1) is defined, wherever it exists, as

$$D_{(1)}^+ V(t) := \limsup_{h \rightarrow 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}. \quad (7)$$

It is worth stressing that this Dini derivative can be computed without knowing the solutions of (1), for instance using Driver's derivative: see [23].

Just like in finite dimension [4], iISS is characterized by the existence of a LKF whose dissipation rate is a  $\mathcal{PD}$  function of the LKF [18, 25]. In the recent paper [10], it was shown that a  $\mathcal{KL}$  dissipation rate involving the solution's norm  $|x(t)|$  and the supremum norm of the state history  $\|x_t\|$  is actually sufficient to establish iISS. More precisely, we have the following characterization of iISS for TDS.

**Theorem 1 (iISS LKF, [10, 18])** *The following statements are equivalent:*

- i) *there exist a coercive LKF candidate  $V$ ,  $\alpha \in \mathcal{PD}$  and  $\gamma \in \mathcal{K}_\infty$  such that, given any  $x_0 \in \mathcal{X}^n$  and any  $u \in \mathcal{U}^m$ , the solution of (1) satisfies*

$$D_{(1)}^+ V(t) \leq -\alpha(\|x_t\|) + \gamma(|u(t)|), \quad \forall t \geq 0 \text{ a.e.}$$

- ii) *there exists a LKF candidate  $V$ ,  $\alpha \in \mathcal{PD}$  and  $\gamma \in \mathcal{K}_\infty$  such that, given any  $x_0 \in \mathcal{X}^n$  and any  $u \in \mathcal{U}^m$ , the*

solution of (1) satisfies

$$D_{(1)}^+ V(t) \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \quad \forall t \geq 0 \text{ a.e.}$$

iii) there exists a LKF candidate  $V$ ,  $\sigma \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_\infty$  such that, given any  $x_0 \in \mathcal{X}^n$  and any  $u \in \mathcal{U}^m$ , the solution of (1) satisfies

$$D_{(1)}^+ V(t) \leq -\sigma(|x(t)|, \|x_t\|) + \gamma(|u(t)|), \quad \forall t \geq 0 \text{ a.e.}$$

iv) (1) is iISS.

Item iii), referred to as a  $\mathcal{KL}$  dissipation, is often the most handy way to establish iISS. As will become clear with Lemma 7 below, it holds readily under a *point-wise* dissipation:

$$D_{(1)}^+ V(t) \leq -\alpha(|x(t)|) + \gamma(|u(t)|), \quad \forall t \geq 0 \text{ a.e.,}$$

for some  $\alpha \in \mathcal{PD}$ , which is more in line with the classical iISS characterization of finite-dimensional systems [4]. The  $\mathcal{KL}$  nature of the dissipation rate allows in particular to use a LKF of the form  $V = \ln(1 + W)$ , where  $W$  denotes another LKF candidate. This common trick to show iISS turns out to be useful in a range of applications, as will be illustrated in Section 5.

On the other hand Item i), referred to as a *history-wise* dissipation, is probably the most interesting LKF feature one can derive from the iISS property, as the generated LKF is not only coercive (thus lower and upper bounded by  $\mathcal{K}_\infty$  functions of  $\|x_t\|$ ) but also dissipates in terms of the supremum norm of the whole state history.

We stress that, since the considered LKFs are Lipschitz on bounded sets, [24, Theorem 5] guarantees that  $t \mapsto V(x_t)$  is locally absolutely continuous on its domain of existence, for any continuously differentiable  $x_0 : [-\delta, 0] \rightarrow \mathbb{R}^n$  and any  $u \in \mathcal{U}^m$ . This regularity is a key requirement to derive qualitative information on the system's solutions based on the Dini derivative of its LKF. On the other hand, proceeding as in [24, Proposition 3], it can be seen that establishing iISS for all continuously differentiable initial states  $x_0$  is equivalent to establishing iISS for all  $x_0 \in \mathcal{X}^n$ . Based on this observation, we will directly work with  $x_0 \in \mathcal{X}^n$ , without explicitly invoking this reasoning.

All the above characterizations make use of a single dissipation rate, encompassing all non-positive terms appearing in the computation of  $D_{(1)}^+ V(t)$ . It is sometimes useful, especially in the context of cascades as will become clear in the next section, to allow a different dissipation rate for each state variables. This possibility is offered by the next result.

**Proposition 2 (iISS through multiple  $\mathcal{KL}$  rates)**  
The system (1) is iISS if and only if there exist a LKF

candidate  $V$ ,  $\sigma_i \in \mathcal{KL}$ ,  $i \in \llbracket 1, n \rrbracket$ , and  $\gamma_j \in \mathcal{K}_\infty$ ,  $j \in \llbracket 1, m \rrbracket$ , such that, given any  $x_0 \in \mathcal{X}^n$  and any  $u \in \mathcal{U}^m$ , it holds along its solution that

$$D_{(1)}^+ V(t) \leq -\sum_{i=1}^n \sigma_i(|x_i(t)|, V(x_t)) + \sum_{j=1}^m \gamma_j(|u_j(t)|), \quad (8)$$

for almost all  $t \geq 0$  in its maximal interval of existence.

Although this result can be obtained using rather standard manipulations, we provide its proof in Section 6.4 for the sake of completeness.

We stress that if (1) is iISS, then it is forward complete for any  $u \in \mathcal{U}^m$ , meaning that (8) holds for almost all  $t \geq 0$ .

### 3 Cascade of TDS

#### 3.1 With inputs

Consider two nonlinear TDS in cascade:

$$\dot{x}(t) = f_1(x_t, z_t, u(t)) \quad (9a)$$

$$\dot{z}(t) = f_2(z_t, u(t)), \quad (9b)$$

where  $x(t) \in \mathbb{R}^{n_1}$  and  $z(t) \in \mathbb{R}^{n_2}$  are the current values of the states of the driven and driving systems respectively,  $x_t \in \mathcal{X}^{n_1}$  and  $z_t \in \mathcal{X}^{n_2}$  are the corresponding state histories, and  $u \in \mathcal{U}^m$  denotes an exogenous input. The functions  $f_1 : \mathcal{X}^{n_1} \times \mathcal{X}^{n_2} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_1}$  and  $f_2 : \mathcal{X}^{n_2} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_2}$  are assumed to be Lipschitz on bounded sets and to satisfy  $f_1(0, 0, 0) = 0$  and  $f_2(0, 0) = 0$ .

Note that a feedthrough term is allowed in (9), meaning that the exogenous input may affect both the driving and the driven subsystems. The same input  $u \in \mathcal{U}^m$  is used in both subsystems: different inputs  $u_1 \in \mathcal{U}^{m_1}$  and  $u_2 \in \mathcal{U}^{m_2}$  can be accounted for by simply letting  $u := (u_1^T, u_2^T)^T \in \mathcal{U}^m$ , with  $m := m_1 + m_2$ .

The main purpose of this study is to investigate under which condition iISS of the cascade (9) can be ensured based on the assumption that each of the subsystems (9a) and (9b) is iISS. Our main result, proved in Section 6.5, is the following.

**Theorem 2 (Cascade of iISS systems)** Let  $p \in \mathbb{N}$  and  $\mathcal{I} \subset \llbracket 1, n_2 \rrbracket$ . Assume there exist two LKF candidates  $V_1 : \mathcal{X}^{n_1} \rightarrow \mathbb{R}_{\geq 0}$  and  $V_2 : \mathcal{X}^{n_2} \rightarrow \mathbb{R}_{\geq 0}$ ,  $\sigma \in \mathcal{KL}$ ,  $\gamma \in \mathcal{K}_\infty$  and, for each  $i \in \mathcal{I}$  and each  $j \in \llbracket 1, p \rrbracket$ ,  $\sigma_i \in \mathcal{KL}$ ,  $\gamma_{ij} \in \mathcal{K}_\infty$  and  $\delta_{ij} \in [0, \delta]$ , such that, almost everywhere

in the maximal interval of existence of any solution of the driven subsystem,

$$D_{(9a)}^+ V_1(t) \leq -\sigma(|x(t)|, V_1(x_t)) \quad (10)$$

$$+ \sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij}(|z_i(t - \delta_{ij})|) + \gamma(|u(t)|)$$

and, almost everywhere in the maximal interval of existence of any solution of the driving subsystem,

$$D_{(9b)}^+ V_2(t) \leq -\sum_{i=1}^{n_2} \sigma_i(|z_i(t)|, V_2(z_t)) + \gamma(|u(t)|). \quad (11)$$

Assume further that the following growth condition holds:

$$\gamma_{ij}(s) = \mathcal{O}_{s \rightarrow 0^+}(\sigma_i(s, 0)), \quad \forall j \in \llbracket 1, p \rrbracket, i \in \mathcal{I}. \quad (12)$$

Then the cascade (9) is iISS.

In view of Proposition 2, condition (11) is equivalent to requiring that the driving subsystem is iISS. Similarly, condition (10) is equivalent to imposing that the driven subsystem (9a) is iISS with respect to its input variables  $u$  and  $z_t$ . Note that not all state variables  $z_i$  of the driving system (9b) may appear in (10), as allowed by the fact that  $\mathcal{I}$  is not necessarily  $\llbracket 1, n_2 \rrbracket$ : a typical situation for this is when the interconnection between the two subsystem involves only the terms  $z_i(t - \delta_{ij})$  with  $i \in \mathcal{I}$  and  $j \in \llbracket 1, p \rrbracket$ , but this feature may also cover the case when some variables  $z_i$ ,  $i \notin \mathcal{I}$ , happen to have a stabilizing effect on the driven subsystem (in which case the corresponding input rate can be picked null).

It is worth stressing that the growth rate condition (12) involves only the dissipation rates  $\sigma_i$  corresponding to a state variable  $z_i$  appearing explicitly in the dissipation inequality of the driving subsystem. In other words, if some variable  $z_i$  does not appear in (10) (meaning  $i \notin \mathcal{I}$ ), then the corresponding dissipation rate  $\sigma_i(\cdot, 0)$  is allowed to be arbitrarily flat around zero. This feature seems to be new even in a finite-dimensional context, as detailed in Section 4.

In the above statement, no distinction between state variables is made in the dissipation rate  $\sigma$  in (10). This choice is motivated by clarity and conciseness concerns and comes with no loss of generality. For the same reason,  $\gamma$  was taken identical in both (10) and (11).

It is worth mentioning that small-gain results for iISS TDS can also be used to study cascade interconnections, see [16, Remark 13]. However, that result makes use of Lyapunov-wise dissipation rates for each subsystem, meaning in the form of Item *ii*) of Theorem 1. More crucially, that result imposes that these dissipation rates are of class  $\mathcal{K}$ , meaning that both systems are required

to have an ISS-like behavior for small inputs (this property is referred to Strong iISS in finite-dimensional literature [8]). In particular, the results in [16] cannot be used for the examples of Section 5.

The proof of Theorem 2 relies on the following change of dissipation rate result for input-free systems, whose proof is provided in Section 6.1.

**Lemma 3 (Changing rates for input-free TDS)**  
Consider an autonomous TDS

$$\dot{x}(t) = f(x_t), \quad \forall t \geq 0 \text{ a.e.}, \quad (13)$$

where  $f : \mathcal{X}^n \rightarrow \mathbb{R}^n$  is Lipschitz on bounded sets and  $f(0) = 0$ . Let  $V : \mathcal{X}^n \rightarrow \mathbb{R}_{\geq 0}$  be a LKF candidate satisfying, along any solution of (13),

$$D_{(13)}^+ V(t) \leq -\frac{\sum_{i=1}^n \alpha_i(|x_i(t)|)}{1 + \eta(V(x_t))}, \quad \forall t \geq 0 \text{ a.e.}, \quad (14)$$

for some  $\alpha_i \in \mathcal{PD}$ ,  $i \in \llbracket 1, n \rrbracket$ , and some  $\eta \in \mathcal{K}_\infty$ . Let  $\mathcal{I} \subset \llbracket 1, n \rrbracket$  and  $\tilde{\alpha}_i$  be any  $\mathcal{PD}$  functions satisfying

$$\tilde{\alpha}_i(s) = \mathcal{O}_{s \rightarrow 0^+}(\alpha_i(s)), \quad \forall i \in \mathcal{I}. \quad (15)$$

Then there exists a continuously differentiable function  $\rho \in \mathcal{K}_\infty$  such that the LKF candidate  $\tilde{V} := \rho \circ V$  satisfies, for almost all  $t \geq 0$ ,

$$D_{(13)}^+ \tilde{V}(t) \leq -\sum_{i \in \mathcal{I}} \tilde{\alpha}_i(|x_i(t)|) - \sum_{i \in \llbracket 1, n \rrbracket \setminus \mathcal{I}} \alpha_i(|x_i(t)|).$$

The above result makes it possible to transform a  $\mathcal{KL}$  dissipation rate as in (14) into a point-wise dissipation rate. It also shows that arbitrary dissipation rates  $\tilde{\alpha}_i$ ,  $i \in \mathcal{I}$ , can be assigned provided that they satisfy the growth condition (15). Unlike similar existing results such as [31, 7], the assignment of these new dissipation rates can be made only on part of the state variables  $x_i$ ,  $i \in \mathcal{I}$ , thus avoiding to impose any growth restrictions on the dissipation rates that do not need to be modified.

The proof of Theorem 2 also relies on the following technical lemma, whose proof is provided in Section 6.2.

**Lemma 4 (Lower bound on  $\mathcal{KL}$  function)** Given any  $\sigma \in \mathcal{KL}$ , there exist  $\alpha \in \mathcal{K}$ ,  $\eta \in \mathcal{K}_\infty$  and a constant  $c \in (0, 1]$  such that

$$\sigma(s, t) \geq \frac{\alpha(s)}{1 + \eta(t)}, \quad \forall s, t \geq 0,$$

$$\alpha(s) = c\sigma(s, 0), \quad \forall s \in [0, 1].$$

It is worth stressing that a similar  $\mathcal{KL}$  decomposition was already presented in [32, Lemma A.2]. Lemma 4 simply

underlines the fact that  $\alpha$  can be picked proportional to  $\sigma(\cdot, 0)$  on  $[0, 1]$ .

### 3.2 Without inputs

A straightforward consequence of Theorem 2 is that, under the above growth-rate condition, the cascade composed of an iISS subsystem driven by a globally asymptotically stable (GAS) one is itself GAS. To state this, consider the following input-free cascade:

$$\dot{x}(t) = f_1(x_t, z_t) \quad (16a)$$

$$\dot{z}(t) = f_2(z_t), \quad (16b)$$

under similar regularity assumptions as the ones on (9). Then we have the following.

**Corollary 1 (Cascade GAS + iISS)** *Let  $p \in \mathbb{N}$  and  $\mathcal{I} \subset \llbracket 1, n_2 \rrbracket$ . Assume there exist two LKF candidates  $V_1 : \mathcal{X}^{n_1} \rightarrow \mathbb{R}_{\geq 0}$  and  $V_2 : \mathcal{X}^{n_2} \rightarrow \mathbb{R}_{\geq 0}$ ,  $\sigma \in \mathcal{KL}$  and, for each  $i \in \mathcal{I}$  and each  $j \in \llbracket 1, p \rrbracket$ ,  $\sigma_i \in \mathcal{KL}$ ,  $\gamma_{ij} \in \mathcal{K}_\infty$  and  $\delta_{ij} \in [0, \delta]$ , such that, almost everywhere along any solution of the driven subsystem (16a),*

$$D_{(16a)}^+ V_1(t) \leq -\sigma(|x(t)|, V_1(x_t)) + \sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij}(z_i(t - \delta_{ij})) \quad (17)$$

and, almost everywhere along any solution of the driving subsystem (16b),

$$D_{(16b)}^+ V_2(t) \leq -\sum_{i=1}^{n_2} \sigma_i(|z_i(t)|, V_2(z_t)). \quad (18)$$

Assume further that the following growth conditions hold:

$$\gamma_{ij}(s) = \mathcal{O}_{s \rightarrow 0^+}(\sigma_i(s, 0)), \quad \forall j \in \llbracket 1, p \rrbracket, i \in \mathcal{I}. \quad (19)$$

Then the cascade (9) is GAS.

This result complements existing delay-free results [22, 6, 1, 7] as well as the Razumikhin approach developed in [26] for TDS.

## 4 Delay-free systems

The possibility to impose a growth rate condition only on the state variables that enter the driven subsystem appears to be new also for delay-free systems. The following statement focuses on this class of systems, namely:

$$\dot{x}(t) = f_1(x(t), z(t), u(t)) \quad (20a)$$

$$\dot{z}(t) = f_2(z(t), u(t)), \quad (20b)$$

where  $x(t) \in \mathbb{R}^{n_1}$ ,  $z(t) \in \mathbb{R}^{n_2}$ ,  $u \in \mathcal{U}^m$ , and  $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_1}$  and  $f_2 : \mathbb{R}^{n_2} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_2}$  are locally Lipschitz vector fields satisfying  $f_1(0, 0, 0) = 0$  and  $f_2(0, 0) = 0$ . For such finite-dimensional cascades, a consequence of Theorem 1 is the following.

### Theorem 5 (Cascade of iISS delay-free systems)

*Let  $\mathcal{I} \subset \llbracket 1, n_2 \rrbracket$ . Consider two continuous positive definite and radially unbounded functions  $V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}_{\geq 0}$  and  $V_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}_{\geq 0}$ , continuously differentiable on  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2} \setminus \{0\}$  respectively. Assume there exist  $\alpha \in \mathcal{PD}$ ,  $\gamma \in \mathcal{K}_\infty$  and, for each  $i \in \mathcal{I}$ ,  $\sigma_i \in \mathcal{KL}$  and  $\gamma_i \in \mathcal{K}_\infty$  such that, for all  $x \in \mathbb{R}^{n_1}$ , all  $z \in \mathbb{R}^{n_2}$  and all  $u \in \mathbb{R}^m$ ,*

$$\frac{\partial V_1}{\partial x}(x) f_1(x, z, u) \leq -\alpha(|x|) + \sum_{i \in \mathcal{I}} \gamma_i(|z_i|) + \gamma(|u|) \quad (21)$$

$$z \neq 0 \Rightarrow \frac{\partial V_2}{\partial z}(z) f_2(z, u) \leq -\sum_{i=1}^{n_2} \sigma_i(|z_i|, |z|) + \gamma(|u|). \quad (22)$$

Assume further that

$$\gamma_i(s) = \mathcal{O}_{s \rightarrow 0^+}(\sigma_i(s, 0)), \quad \forall i \in \mathcal{I}. \quad (23)$$

Then the delay-free cascade (20) is iISS.

The main difference between this statement and Theorem 2 is that the derivative of the Lyapunov functions  $V_1$  and  $V_2$  can be computed using the chain rule, which allows to get rid of the Dini derivative and gives rise to a purely algebraic condition. For the sake of generality,  $V_2$  is not requested to be differentiable at 0, which sometimes proves useful to derive a less conservative dissipation rate (see [7] for an illustration). Also, the second argument of the  $\mathcal{KL}$  functions  $\sigma_i$  is no longer  $V_2$ , but directly  $|z|$ , which may help simplifying computations. For the same reason, the function  $\sigma \in \mathcal{KL}$  has been replaced here by a  $\mathcal{PD}$  function  $\alpha$  in (21). The proof of this result is provided in Section 6.6.

The main result in [7] also studied delay-free cascades of iISS systems. In order to highlight the similarity and novelty of the present approach with respect to [7], we state the following consequence of Theorem 5, which does not make use of  $\mathcal{KL}$  dissipation rates.

### Corollary 6 (Cascade of iISS delay-free systems)

*Let  $\mathcal{I} \subset \llbracket 1, n_2 \rrbracket$ . Consider two continuous positive definite and radially unbounded functions  $V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}_{\geq 0}$  and  $V_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}_{\geq 0}$  continuously differentiable on  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2} \setminus \{0\}$  respectively. Assume there exist  $\alpha \in \mathcal{PD}$  and  $\gamma \in \mathcal{K}_\infty$  and, for each  $i \in \mathcal{I}$ ,  $\alpha_i \in \mathcal{PD}$  and  $\gamma_i \in \mathcal{K}_\infty$*

such that, for all  $x \in \mathbb{R}^{n_1}$ , all  $z \in \mathbb{R}^{n_2}$  and all  $u \in \mathbb{R}^m$ ,

$$\frac{\partial V_1}{\partial x}(x) f_1(x, z, u) \leq -\alpha(|x|) + \sum_{i \in \mathcal{I}} \gamma_i(|z_i|) + \gamma(|u|) \quad (24)$$

$$z \neq 0 \quad \Rightarrow \quad \frac{\partial V_2}{\partial z}(z) f_2(z, u) \leq -\sum_{i=1}^{n_2} \alpha_i(|z_i|) + \gamma(|u|). \quad (25)$$

Assume further that

$$\gamma_i(s) = \mathcal{O}_{s \rightarrow 0^+}(\alpha_i(s)), \quad \forall i \in \mathcal{I}. \quad (26)$$

Then the delay-free cascade (20) is *iISS*.

Here again,  $V_2$  is not requested to be differentiable at zero: this feature was already present in [7]. The key difference with the main result in [7] is that the growth rate condition (26) is imposed only on the variables  $z_i$  that are present in the dissipation inequality (24) (meaning for  $i \in \mathcal{I}$ ). We will show through an Example in Section 5.2 that this allows to cover a wider class of cascades.

The proof of Corollary 6 is provided in Section 6.7 and relies on the following technical lemma.

**Lemma 7 (Lower bound on  $\mathcal{PD}$  function)** *Given any  $\alpha \in \mathcal{PD}$ , there exist  $\mu \in \mathcal{K}_\infty$  and  $\nu \in \mathcal{L}$  such that*

$$\alpha(s) \geq \mu(s)\nu(s), \quad \forall s \geq 0 \quad (27)$$

$$\lim_{s \rightarrow 0^+} \frac{\mu(s)}{\alpha(s)} = 1. \quad (28)$$

The lower bound (27) was already established in [4]. The novelty here is to show that  $\mu$  can be picked in such a way that it behaves similarly to  $\alpha$  in a neighborhood of 0. The proof of this lemma is provided in Section 6.3.

## 5 Illustrative examples

The two academic examples presented below illustrate the applicability of the tools presented here and highlight their novelty with respect to existing works.

### 5.1 Time-delay system

Consider the following cascade TDS involving both discrete and distributed delays:

$$\dot{x}(t) = -\text{sat}(x(t)) + \frac{1}{4}\text{sat}(x(t-1)) + x(t)z(t-2)^2 \quad (29a)$$

$$\dot{z}(t) = -\frac{3}{2}z(t) + z(t-1) + u(t) \int_{t-1}^t z(\tau) d\tau, \quad (29b)$$

where  $\text{sat}(s) := \text{sign}(s) \min\{|s|, 1\}$  for all  $s \in \mathbb{R}$ . This system is in the form (9) with  $n_1 = n_2 = 1$ ,  $m = 1$ , and  $\delta = 2$ . Consider the LKF candidates defined, for all  $\phi \in \mathcal{X}$ , as

$$V_1(\phi) := \ln \left( 1 + \phi(0)^2 + \frac{1}{2} \int_{-1}^0 \phi(\tau) \text{sat}(\phi(\tau)) d\tau \right) \quad (30a)$$

$$V_2(\phi) := \ln \left( 1 + \phi(0)^2 + \int_{-1}^0 \phi(\tau)^2 d\tau \right). \quad (30b)$$

The Dini derivative of  $V_2$  along the solutions of the driving subsystem reads, for almost all  $t \geq 0$ ,

$$\begin{aligned} D_{(29b)}^+ V_2(t) &\leq \frac{1}{1 + z(t)^2 + \int_{-1}^0 z(t+\tau)^2 d\tau} \\ &\times \left[ 2z(t) \left( -\frac{3}{2}z(t) + z(t-1) + u(t) \int_{t-1}^t z(\tau) d\tau \right) \right. \\ &\quad \left. + z(t)^2 - z(t-1)^2 \right]. \end{aligned}$$

Observe that  $2z(t)z(t-1) \leq z(t)^2 + z(t-1)^2$  and that

$$\begin{aligned} z(t) \int_{t-1}^t z(\tau) d\tau &\leq \frac{1}{2} \left( z(t)^2 + \left( \int_{t-1}^t z(\tau) d\tau \right)^2 \right) \\ &\leq \frac{1}{2} \left( z(t)^2 + \int_{t-1}^t z(\tau)^2 d\tau \right). \end{aligned}$$

Defining  $\eta_2(s) := e^s - 1$ , we get that

$$\begin{aligned} D_{(29b)}^+ V_2(t) &\leq \frac{-z(t)^2 + u(t) \left( z(t)^2 + \int_{-1}^0 z(t+\tau)^2 d\tau \right)}{1 + z(t)^2 + \int_{-1}^0 z(t+\tau)^2 d\tau} \\ &\leq -\frac{z(t)^2}{1 + \eta_2(V_2(z_t))} + |u(t)|, \quad \forall t \geq 0 \text{ a.e.}, \end{aligned}$$

thus making (11) fulfilled with  $\sigma_1(s, t) = \frac{s^2}{1 + \eta_2(t)}$  and  $\gamma(s) = s$ , for all  $s, t \geq 0$ .

On the other hand, the derivative of  $V_1$  along the solutions of the driven subsystem reads almost everywhere

$$\begin{aligned} D_{(29a)}^+ V_1(t) &= \frac{1}{1 + x(t)^2 + \frac{1}{2} \int_{t-1}^t x(\tau) \text{sat}(x(\tau)) d\tau} \\ &\times \left[ 2x(t) \left( -\text{sat}(x(t)) + \frac{1}{4}\text{sat}(x(t-1)) + x(t)z(t-2)^2 \right) \right. \\ &\quad \left. + \frac{1}{2} \left( x(t) \text{sat}(x(t)) - x(t-1) \text{sat}(x(t-1)) \right) \right]. \end{aligned}$$

Observing that  $x(t) \text{sat}(x(t-1)) \leq x(t) \text{sat}(x(t)) + x(t-1) \text{sat}(x(t-1))$  (as can be seen by considering separately



the cases when  $|x(t)| \leq |x(t-1)|$  and  $|x(t)| \geq |x(t-1)|$  and defining  $\eta_1(s) := e^s - 1$ , we obtain that

$$D^+V_1(t) \leq -\frac{x(t)\text{sat}(x(t))}{1 + \eta_1(V_1(x_t))} + 2z(t-2)^2, \quad \forall t \geq 0 \text{ a.e.},$$

thus fulfilling (10) with  $\sigma(s, t) = \frac{s\text{sat}(s)}{1 + \eta_2(t)}$ ,  $\mathcal{I} = \{1\}$ ,  $\delta_{11} = 2$  and  $\gamma_{11}(s) = 2s^2$ . Observing that  $\sigma_1(s, 0) = s^2 = \gamma_{11}(s)/2$ , the growth condition (12) is satisfied and we conclude with Theorem 2 that the cascade (29) is iISS.

It is worth mentioning that this conclusion cannot be derived with the results in [16] as no class  $\mathcal{K}$  dissipation rate can be obtained for the driven subsystem (29a). It can also be seen that no linear combination of  $V_1$  and  $V_2$  results in an iISS LKF for the overall cascade, thus illustrating the benefits of the cascade analysis in terms of analysis complexity.

### 5.2 Delay-free system

Consider the following delay-free cascade system:

$$\dot{x}(t) = -\text{sat}(x(t)) + x(t)(z_1(t)^2 + u_1(t)) \quad (31a)$$

$$\dot{z}_1(t) = -z_1(t) + u_2(t) \quad (31b)$$

$$\dot{z}_2(t) = -z_2(t)^3, \quad (31c)$$

where  $x(t) \in \mathbb{R}$  is the state of the driven subsystem,  $z(t) := (z_1(t), z_2(t))^T \in \mathbb{R}^2$  is the state of the driving subsystem and  $u := (u_1, u_2)^T \in \mathcal{U}^2$  is the input. We let  $f(x, z, u) = (f_1(x, z_1, u_1), f_2(z, u_2))^T$  denote the right-hand side of (31). Let  $V_1(x) := \frac{1}{2} \ln(1 + x^2)$ . Then, for all  $x, z_1, u_1 \in \mathbb{R}$ , it holds that

$$\begin{aligned} \frac{\partial V_1}{\partial x}(x)f_1(x, z_1, u_1) &\leq \frac{x(-\text{sat}(x) + x(z_1^2 + u_1))}{1 + x^2} \\ &\leq -\frac{|x|\text{sat}(|x|)}{1 + x^2} + |z_1|^2 + |u_1|, \end{aligned}$$

meaning that (24) is satisfied with  $\alpha(s) = \frac{s\text{sat}(s)}{1+s^2}$ ,  $\mathcal{I} = \{1\}$ ,  $\gamma_1(s) = s^2$  and  $\gamma(s) = s$ . Moreover, letting  $V_2(z) := \frac{1}{2}(z_1^2 + z_2^2)$ , it holds that, for all  $z \in \mathbb{R}^2$  and all  $u_2 \in \mathbb{R}$ ,

$$\begin{aligned} \frac{\partial V_2}{\partial z}(z)f_2(z, u_2) &= -z_1^2 + z_1u_2 - z_2^4 \\ &\leq -\frac{1}{2}z_1^2 - z_2^4 + \frac{1}{2}u_2^2. \end{aligned} \quad (32)$$

This makes (25) satisfied with  $\alpha_1(s) = \frac{s^2}{2}$  and  $\alpha_2(s) = s^4$ . For these particular functions, it clearly holds that  $\gamma_1(s) = \mathcal{O}_{s \rightarrow 0^+}(\alpha_1(s))$ , thus making the growth condition (26) fulfilled. iISS of (31) follows from Corollary 6.

It is worth mentioning that this academic example cannot be treated by the results in [7] (at least not with

the proposed Lyapunov functions) since they impose a growth condition on a  $\mathcal{PD}$  dissipation rate that involves the whole state  $z$  of the driving subsystem. In our case, (32) indicates that such a dissipation rate would necessarily be smaller than a function in  $s^4$ , which does not dominate  $\gamma_1(s) = s^2$  when  $s \rightarrow 0^+$ . We thus believe that this example illustrates the benefits of imposing a growth conditions only on the variables  $z_i$  actually appearing in the computation of the derivative of  $V_1$ .

## 6 Proofs

### 6.1 Proof of Lemma 3

First recall that, since  $V$  is a LKF candidate, there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  such that, for all  $\phi \in \mathcal{X}^n$ ,

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \bar{\alpha}(\|\phi\|). \quad (33)$$

We will show that there exists a continuous non-decreasing function  $q : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  satisfying  $q(s) > 0$  for all  $s > 0$  such that the sought function  $\rho$  reads

$$\rho(s) = \int_0^s q(r)dr, \quad \forall s \geq 0.$$

Note that, with this choice,  $\rho$  is a continuously differentiable  $\mathcal{K}_\infty$  function and the function  $\tilde{V} = \rho \circ V$  is then Lipschitz on bounded sets. Moreover, in view of (33), it holds that

$$\rho \circ \underline{\alpha}(|\phi(0)|) \leq \tilde{V}(\phi) \leq \rho \circ \bar{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}^n,$$

meaning that  $\tilde{V}$  is a LKF candidate. Furthermore, its Dini derivative along the solutions of (13) reads, in view of (14) and [16, Lemma 7],

$$\begin{aligned} D_{(13)}^+ \tilde{V}(t) &\leq q(V(x_t))D_{(13)}^+ V(t) \\ &\leq -q(V(x_t)) \frac{\sum_{i=1}^n \alpha_i(|x_i(t)|)}{1 + \eta(V(x_t))}, \end{aligned}$$

for almost all  $t \geq 0$ . So we need to find a continuous non-decreasing function  $q$  satisfying  $q(s) > 0$  for all  $s > 0$  such that

$$q(V) \frac{\sum_{i=1}^n \alpha_i(|x_i|)}{1 + \eta(V)} \geq \sum_{i \in \mathcal{I}} \tilde{\alpha}_i(|x_i|) + \sum_{i \in \llbracket 1, n \rrbracket \setminus \mathcal{I}} \alpha_i(|x_i|), \quad (34)$$

where we omitted some arguments for brevity. Let  $\mu : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be defined for each  $s > 0$  as

$$\mu(s) := 1 + \sup_{r \in [0, s]} \left( \max_{i \in \mathcal{I}} \frac{\tilde{\alpha}_i(r)}{\alpha_i(r)} \right). \quad (35)$$

For each  $i \in \mathcal{I}$ , the function  $s \mapsto \tilde{\alpha}_i(s)/\alpha_i(s)$  is continuous on  $\mathbb{R}_{>0}$ . In addition, condition (15) ensures that it is bounded on any interval of the form  $[0, s]$ ,  $s \geq 0$ .  $\mu$  being clearly non-decreasing, it follows that it admits a limit at zero. Hence,  $\mu$  can be extended at zero to be continuous and non-decreasing on  $\mathbb{R}_{\geq 0}$ . We claim that a possible choice of  $q$  to fulfill (34) is then

$$q(s) := \mu \circ \underline{\alpha}^{-1}(s)(1 + \eta(s)), \quad \forall s \geq 0.$$

This function  $q$  is indeed a continuous positive non-decreasing function. With this choice, and invoking (33) and (35), we have that, for all  $i \in \mathcal{I}$  and almost all  $t \geq 0$ ,

$$\begin{aligned} q(V(x_t)) \frac{\alpha_i(|x_i(t)|)}{1 + \eta(V(x_t))} &\geq \mu \circ \underline{\alpha}^{-1}(V(x_t)) \alpha_i(|x_i(t)|) \\ &\geq \mu(|x(t)|) \alpha_i(|x_i(t)|) \\ &\geq \tilde{\alpha}_i(|x_i(t)|). \end{aligned} \quad (36)$$

For all  $i \in \llbracket 1, n \rrbracket \setminus \mathcal{I}$ , since  $\mu(s) \geq 1$ , we get that

$$\begin{aligned} q(V(x_t)) \frac{\alpha_i(|x_i(t)|)}{1 + \eta(V(x_t))} &\geq \mu(|x(t)|) \alpha_i(|x_i(t)|) \\ &\geq \alpha_i(|x_i(t)|). \end{aligned} \quad (37)$$

Summing (36)-(37) over  $i \in \llbracket 1, n \rrbracket$  thus establishes (34).

## 6.2 Proof of Lemma 4

Given any  $\sigma \in \mathcal{KL}$ , let

$$\tilde{\sigma}(s, t) := \frac{\sigma(s, t)}{1 + t}, \quad \forall s, t \geq 0.$$

Then  $\tilde{\sigma} \in \mathcal{KL}$ ,

$$\sigma(s, t) \geq \tilde{\sigma}(s, t), \quad \forall s, t \geq 0, \quad (38)$$

and, possibly unlike  $\sigma$ ,  $\tilde{\sigma}$  is decreasing in its second argument for each fixed positive value of its first argument. Let  $\bar{\eta}_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  be defined as

$$\bar{\eta}_1(t) := \max_{s \in [0, 1]} \frac{\tilde{\sigma}(s, 0)}{\tilde{\sigma}(s, t)} - 1, \quad \forall t \geq 0. \quad (39)$$

Then  $\bar{\eta}_1(t)$  is finite for all  $t \geq 0$  as the maximum of a continuous function over a closed interval,  $\bar{\eta}_1(0) = 0$ , and  $\bar{\eta}_1(t) > 0$  for all  $t > 0$  since  $t \mapsto \tilde{\sigma}(s, t)$  is decreasing for any fixed  $s > 0$ . In addition,  $\bar{\eta}_1$  is continuous at zero. To see this, observe that given  $\varepsilon > 0$ , there exists  $h > 0$  such that  $\frac{\tilde{\sigma}(s, 0)}{\tilde{\sigma}(s, h)} - 1 \leq \varepsilon$  for all  $s \in [0, 1]$ , as results from the continuity of  $\tilde{\sigma}$ . It follows that

$$\max_{s \in [0, 1]} \frac{\tilde{\sigma}(s, 0)}{\tilde{\sigma}(s, t)} - 1 \leq \varepsilon, \quad \forall t \in [0, h],$$

meaning that  $\bar{\eta}_1$  is indeed continuous at zero. Observing that  $\lim_{t \rightarrow \infty} \bar{\eta}_1(t) = \infty$ , we conclude with [12, Lemma 2.5] that there exists  $\eta_1 \in \mathcal{K}_\infty$  such that  $\bar{\eta}_1(t) \geq \eta_1(t)$  for all  $t \geq 0$  and it holds from (38) and (39) that

$$\frac{\tilde{\sigma}(s, 0)}{1 + \eta_1(t)} \leq \frac{\tilde{\sigma}(s, 0)}{1 + \bar{\eta}_1(t)} \leq \sigma(s, t), \quad \forall s \in [0, 1]. \quad (40)$$

Furthermore, we know by [32, Lemma A.2] that there exists  $\alpha_2 \in \mathcal{K}$  and  $\eta_2 \in \mathcal{K}_\infty$  such that

$$\sigma(s, t) \geq \frac{\alpha_2(s)}{1 + \eta_2(t)}, \quad \forall s, t \geq 0. \quad (41)$$

Note that, in this expression,  $\alpha_2$  can be picked in such a way that  $\alpha_2(s) \leq \sigma(s, 0)$  for all  $s \geq 0$  (if not, just replace it by  $\min\{\alpha_2(\cdot); \sigma(\cdot, 0)\}$ ). In particular, letting  $c := \frac{\alpha_2(1)}{\sigma(1, 0)}$ , it holds that  $c \in (0, 1]$ .

Let  $\eta \in \mathcal{K}_\infty$  be defined as  $\eta(t) := \max\{\eta_1(t), \eta_2(t)\}$  for all  $t \geq 0$  and

$$\alpha(s) := \begin{cases} c\sigma(s, 0) & \text{if } s \in [0, 1] \\ \alpha_2(s) & \text{if } s > 1. \end{cases}$$

It can easily be checked that  $\alpha \in \mathcal{K}$  and it holds from (40)-(41) that

$$\sigma(s, t) \geq \frac{\alpha(s)}{1 + \eta(t)}, \quad \forall s, t \geq 0.$$

## 6.3 Proof of Lemma 7

If  $\alpha \in \mathcal{K}$ , then the lemma trivially holds by picking  $\mu(s) = \alpha(s)$  and  $\nu(s) = \frac{1}{1+s}$  for all  $s \geq 0$ . So we now consider that  $\alpha \in \mathcal{PD} \setminus \mathcal{K}$ . In that case,  $\alpha$  reaches its maximum  $\alpha_M$  on  $\mathbb{R}_{>0}$ . Let  $s_M > 0$  be any number such that  $\alpha(s_M) = \alpha_M$  and define  $\hat{\mu}, \hat{\nu} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  as

$$\begin{aligned} \hat{\mu}(s) &:= \begin{cases} \min_{r \in [s, s_M]} \alpha(r), & \text{if } s \in [0, s_M] \\ \alpha_M, & \text{if } s > s_M \end{cases} \\ \hat{\nu}(s) &:= \begin{cases} \alpha_M, & \text{if } s \in [0, s_M] \\ \min_{r \in [s_M, s]} \alpha(r), & \text{if } s > s_M. \end{cases} \end{aligned}$$

These functions are continuous and satisfy

$$\hat{\mu}(s)\hat{\nu}(s) \leq \alpha_M \alpha(s), \quad \forall s \geq 0. \quad (42)$$

Moreover,  $\hat{\mu}$  is positive definite and non-decreasing, whereas  $\hat{\nu}$  is non-increasing and tends to zero as its

argument tends to infinity. Now let

$$\begin{aligned}\mu(s) &:= (1+s)\hat{\mu}(s) \\ \nu(s) &:= \frac{1}{\alpha_M(1+s)}\hat{\nu}(s),\end{aligned}$$

for all  $s \geq 0$ . Then  $\mu \in \mathcal{K}_\infty$  and  $\nu \in \mathcal{L}$ . Furthermore,  $\mu$  satisfies

$$\begin{aligned}\lim_{s \rightarrow 0^+} \frac{\mu(s)}{\alpha(s)} &= \lim_{s \rightarrow 0^+} \frac{(1+s)\hat{\mu}(s)}{\alpha(s)} \\ &= \lim_{s \rightarrow 0^+} \frac{\min_{r \in [s, s_M]} \alpha(r)}{\alpha(s)} \\ &= 1,\end{aligned}$$

thus establishing (28). Finally, using (42), we get

$$\alpha(s) \geq \frac{\hat{\mu}(s)\hat{\nu}(s)}{\alpha_M} = \frac{\frac{\mu(s)}{1+s}\alpha_M(1+s)\nu(s)}{\alpha_M} = \mu(s)\nu(s),$$

which establishes (27).

#### 6.4 Proof of Proposition 2

We start by establishing the sufficiency part of this statement. To that aim, assume that (8) holds almost everywhere on the maximal interval of existence of any solution of (1). If this maximal interval of existence  $[0, T)$  was finite for some initial state  $x_0 \in \mathcal{X}^n$  and some  $u \in \mathcal{U}^m$ , then we would have  $\limsup_{t \rightarrow T^-} |x(t)| = +\infty$  (see for instance [14, Theorem 3.2, p.43]), which contradicts the fact that

$$V(x_t) \leq V(x_0) + \sum_{j=1}^m \int_0^t \gamma_j(|u_j(\tau)|) d\tau, \quad \forall t \in [0, T),$$

as obtained by the integration of (8). Hence, the system is forward complete and (8) holds for almost all  $t \geq 0$ .

We first invoke Lemma 4, which asserts that there exist  $\alpha_i \in \mathcal{K}$  and  $\eta_i \in \mathcal{K}_\infty$ ,  $i \in \llbracket 1, n \rrbracket$ , such that

$$D_{(1)}^+ V(t) \leq - \sum_{i=1}^n \frac{\alpha_i(|x_i(t)|)}{1 + \eta_i(V(x_t))} + \sum_{j=1}^m \gamma_j(|u_j(t)|),$$

for almost all  $t \geq 0$ . Let  $\eta \in \mathcal{K}_\infty$  be defined as  $\eta(s) := \max_{i \in \llbracket 1, n \rrbracket} \eta_i(s)$ . Then it holds that

$$D_{(1)}^+ V(t) \leq - \sum_{i=1}^n \frac{\alpha_i(|x_i(t)|)}{1 + \eta(V(x_t))} + \sum_{j=1}^m \gamma_j(|u_j(t)|).$$

The function  $a : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$  defined as  $a(r) := \sum_{i=1}^n \alpha_i(r_i)$  is continuous, positive definite, and satisfies

$\lim_{|r| \rightarrow \infty} a(r) > 0$ . It follows that there exist a function  $\alpha \in \mathcal{K}$  such that  $a(r) \geq \alpha(|r|)$ . Letting  $\gamma \in \mathcal{K}_\infty$  be defined as  $\gamma(s) := \sum_{j=1}^m \gamma_j(s)$  for all  $s \geq 0$ , it follows that

$$D_{(1)}^+ V(t) \leq - \frac{\alpha(|x(t)|)}{1 + \eta(V(x_t))} + \gamma(|u(t)|), \quad \forall t \geq 0 \text{ a.e.},$$

and iISS follows from Item *iii*) of Theorem 1 by observing that  $(s, t) \mapsto \frac{\alpha(s)}{1 + \eta(t)}$  is of class  $\mathcal{KL}$ .

Conversely, assume that (1) is iISS. Then Theorem 1 ensures that there exist a LKF  $V$ ,  $\sigma \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_\infty$  such that

$$D_{(1)}^+ V(t) \leq -\sigma(|x(t)|, V(x_t)) + \gamma(|u(t)|), \quad \forall t \geq 0 \text{ a.e.}$$

Observing that  $\gamma\left(\sum_{j=1}^m r_j\right) \leq \sum_{j=1}^m \gamma(2^m r_j)$  for all  $r_1, \dots, r_m \geq 0$  (as can be seen by iteratively using the fact that  $\gamma(a+b) \leq \gamma(2a) + \gamma(2b)$  for all  $a, b \geq 0$ ), it follows that, for almost all  $t \geq 0$ ,

$$\begin{aligned}D_{(1)}^+ V(t) &\leq -\frac{1}{n} \sum_{i=1}^n \sigma(|x_i(t)|, V(x_t)) + \gamma\left(\sum_{j=1}^m |u_j(t)|\right) \\ &\leq -\frac{1}{n} \sum_{i=1}^n \sigma(|x_i(t)|, V(x_t)) + \sum_{j=1}^m \gamma(2^m |u_j(t)|).\end{aligned}$$

The conclusion follows by letting  $\sigma_i := \frac{1}{n} \sigma \in \mathcal{KL}$  for all  $i \in \llbracket 1, n \rrbracket$  and  $\gamma_j(s) := \gamma(2^m s)$  for all  $s \geq 0$  and all  $j \in \llbracket 1, m \rrbracket$ .

#### 6.5 Proof of Theorem 2

The proof consists in showing first that the cascade is forward complete, and then in showing that it is 0-GAS and owns the UBEBS property. iISS then follows from Proposition 1.

To that aim, first observe that Lemma 4 ensures that, for each  $i \in \llbracket 1, n_2 \rrbracket$ , there exist  $\eta_i \in \mathcal{K}_\infty$

$$\begin{aligned}\sigma_i(s, t) &\geq \frac{\alpha_i(s)}{1 + \eta_i(t)}, \quad \forall s, t \geq 0 \\ \alpha_i(s) &= c_i \sigma_i(s, 0), \quad \forall s \in [0, 1].\end{aligned}\tag{43}$$

Let  $\eta \in \mathcal{K}_\infty$  be defined as

$$\eta(s) := \max_{i \in \llbracket 1, n_2 \rrbracket} \eta_i(s), \quad \forall s \geq 0.\tag{44}$$

In view of (11), it follows that

$$D_{(9b)}^+ V_2(t) \leq - \frac{\sum_{i=1}^{n_2} \alpha_i(|z_i(t)|)}{1 + \eta(V_2(z_t))} + \gamma(|u(t)|)\tag{45}$$

for almost all  $t \geq 0$  in the maximal interval of existence of  $z(\cdot)$ . And, in view of (12) and (43), the functions  $\alpha_i$  satisfy the growth rate condition

$$\gamma_{ij}(s) = \mathcal{O}_{s \rightarrow 0^+}(\alpha_i(s)), \quad \forall i \in \mathcal{I}. \quad (46)$$

### 6.5.1 Forward completeness

We start by showing that solutions of the cascade (9) exist at all positive times. This is rather straightforward for the driving subsystem as the assumptions of Theorem 2 readily ensure that (9b) is iISS by Theorem 1 (hence, forward complete). This in turn implies that  $t \mapsto z_t$  is continuous on  $\mathbb{R}_{\geq 0}$  as the solution of a forward complete TDS. In particular, it is measurable and locally bounded on  $\mathbb{R}_{\geq 0}$ , meaning that it belongs to  $\mathcal{U}^{n_2}$ . Observe that Proposition 2 also ensures that the driven subsystem is iISS with  $u$  and  $t \mapsto z_t$  seen as inputs. It follows that (9a) is forward complete. In particular, (10) and (45) hold for almost all  $t \geq 0$ .

### 6.5.2 0-GAS

We next show that the cascade (9) is 0-GAS, meaning that the input-free system

$$\dot{x}(t) = f_1(x_t, z_t, 0) \quad (47a)$$

$$\dot{z}(t) = f_2(z_t, 0) \quad (47b)$$

is globally asymptotically stable. Note that, in view of (45), the LKF  $V_2$  satisfies

$$D_{(47b)}^+ V_2(t) \leq -\frac{\sum_{i=1}^{n_2} \alpha_i(|z_i(t)|)}{1 + \eta_2(V_2(z_t))}, \quad \forall t \geq 0 \text{ a.e.}$$

Observe that the growth rate condition (46) ensures that

$$2 \sum_{j=1}^p \gamma_{ij}(s) = \mathcal{O}_{s \rightarrow 0^+}(\alpha_i(s)), \quad \forall i \in \mathcal{I}.$$

It follows from Lemma 3 that there exists a continuously differentiable function  $\rho \in \mathcal{K}_\infty$  such that the functional

$$\tilde{V}_2 := \rho \circ V_2 \quad (48)$$

satisfies

$$D_{(47b)}^+ \tilde{V}_2(t) \leq -2 \sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij}(|z_i(t)|) - \sum_{i \in \llbracket 1, n_2 \rrbracket \setminus \mathcal{I}} \alpha_i(|z_i(t)|), \quad (49)$$

for almost all  $t \geq 0$ . We next modify the LKF  $\tilde{V}_2$  in such a way that its dissipation rate involves not only the current value of the driving state variables  $z_i(t)$  but also

its delayed values  $z_i(t - \delta_{ij})$ ,  $i \in \mathcal{I}$ ,  $j \in \llbracket 1, p \rrbracket$ . To that aim, consider the LKF defined for all  $\varphi \in \mathcal{X}^{n_2}$  as

$$\mathcal{V}_2(\varphi) := \tilde{V}_2(\varphi) + \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_{-\delta_{ij}}^0 \gamma_{ij}(|\varphi_i(\tau)|) d\tau.$$

Since  $V_2$  is a LKF candidate, there exist  $\underline{\alpha}_2, \bar{\alpha}_2 \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}_2(|\varphi(0)|) \leq V_2(\varphi) \leq \bar{\alpha}_2(\|\varphi\|), \quad \forall \varphi \in \mathcal{X}^{n_2}. \quad (50)$$

In view of this and (48), it holds that

$$\tilde{\alpha}_2(|\varphi(0)|) \leq \mathcal{V}_2(\varphi) \leq \tilde{\bar{\alpha}}_2(\|\varphi\|), \quad (51)$$

where  $\tilde{\alpha}_2 := \rho \circ \underline{\alpha}_2 \in \mathcal{K}_\infty$  and  $\tilde{\bar{\alpha}}_2 := \rho \circ \bar{\alpha}_2 + \delta \sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij} \in \mathcal{K}_\infty$ . Moreover, along the solutions of (47b), it holds that

$$\mathcal{V}_2(z_t) = \tilde{V}_2(z_t) + \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_{t-\delta_{ij}}^t \gamma_{ij}(|z_i(\tau)|) d\tau.$$

In view of (49), its Dini derivative therefore reads

$$\begin{aligned} D_{(47b)}^+ \mathcal{V}_2(t) &= D_{(47b)}^+ \tilde{V}_2(t) \\ &\quad + \sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij}(|z_i(t)|) - \gamma_{ij}(|z_i(t - \delta_{ij})|) \\ &\leq - \sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij}(|z_i(t)|) + \gamma_{ij}(|z(t - \delta_{ij})|) \\ &\quad - \sum_{i \in \llbracket 1, n_2 \rrbracket \setminus \mathcal{I}} \alpha_i(|z_i(t)|), \quad \forall t \geq 0 \text{ a.e.} \end{aligned}$$

For all  $\Phi := (\phi^T, \varphi^T)^T \in \mathcal{X}^{n_1+n_2}$ , let

$$\mathcal{V}(\Phi) := V_1(\phi) + \mathcal{V}_2(\varphi). \quad (52)$$

Then it can easily be checked from (51) that  $\mathcal{V}$  is a LKF candidate for the cascade (9) and, summing the above inequality with (10), we get that, for almost all  $t \geq 0$ ,

$$\begin{aligned} D_{(47a)}^+ \mathcal{V}(t) &\leq -\sigma(|x(t)|, V_1(x_t)) - \sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij}(|z_i(t)|) \\ &\quad - \sum_{i \in \llbracket 1, n_2 \rrbracket \setminus \mathcal{I}} \alpha_i(|z_i(t)|). \end{aligned}$$

Since  $\sigma \in \mathcal{KL}$ , Lemma 4 ensures that there exist  $\alpha \in \mathcal{K}$  and  $\bar{\eta} \in \mathcal{K}_\infty$  such that  $\sigma(s, t) \geq \frac{\alpha(s)}{1 + \bar{\eta}(t)}$  for all  $s, t \geq 0$ . It

follows that

$$D_{(47a)}^+ \mathcal{V}(t) \leq -\frac{1}{1 + \tilde{\eta}(\mathcal{V}(X_t))} \left( \alpha(|x(t)|) + \sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij}(|z_i(t)|) + \sum_{i \in [1, n_2] \setminus \mathcal{I}} \alpha_i(|z_i(t)|) \right),$$

where  $X_t := (x_t^T, z_t^T)^T \in \mathcal{X}^{n_1+n_2}$  denotes the state history of the whole cascade. Let  $\tilde{\gamma} \in \mathcal{K}$  be defined as

$$\tilde{\gamma}(s) := \min \left\{ \min_{i \in \mathcal{I}, j \in [1, p]} \gamma_{ij}(s); \min_{i \in [1, n_2] \setminus \mathcal{I}} \alpha_i(s) \right\},$$

for all  $s \geq 0$ . Then it holds that

$$D_{(47a)}^+ \mathcal{V}(t) \leq -\frac{1}{1 + \tilde{\eta}(\mathcal{V}(X_t))} \left( \alpha(|x(t)|) + \sum_{i=1}^{n_2} \tilde{\gamma}(|z_i(t)|) \right)$$

for almost all  $t \geq 0$ . The function  $X = (x^T, z^T)^T \mapsto \alpha(|x|) + \sum_{i=1}^{n_2} \tilde{\gamma}(|z_i|)$  being continuous, positive definite on  $\mathbb{R}^{n_1+n_2}$ , and non-vanishing as  $|X| \rightarrow \infty$ , there exists  $\tilde{\alpha} \in \mathcal{K}$  such that  $\alpha(|x|) + \sum_{i=1}^{n_2} \tilde{\gamma}(|z_i|) \geq \tilde{\alpha}(|X|)$  for all  $X \in \mathbb{R}^{n_1+n_2}$ : see [17, Lemma 4.3]. Therefore

$$D_{(9)}^+ \mathcal{V}(t) \leq -\frac{\tilde{\alpha}(|X(t)|)}{1 + \tilde{\eta}(\mathcal{V}(X_t))}, \quad \forall t \geq 0 \text{ a.e.} \quad (53)$$

GAS of (47) (hence, 0-GAS of (9)) then follows by invoking Item *iii*) of Theorem 1 with  $u \equiv 0$  (or [10, Proposition 1]).

### 6.5.3 UBEBS

We finally proceed to establishing the UBEBS property. To that end, first observe that (46) ensures the existence of a positive constant  $k$  such that

$$\alpha_i(s) \geq k\gamma_{ij}(s), \quad \forall s \in [0, 1], \quad (54)$$

for all  $i \in \mathcal{I}$  and all  $j \in [1, p]$ . Consider the function  $\underline{\alpha}_2, \eta \in \mathcal{K}_\infty$  in (44) and (50) and let  $\tilde{\eta}$  be defined as

$$\tilde{\eta}(s) := s + \eta(s) + \max_{i \in \mathcal{I}, j \in [1, p]} \frac{k}{\alpha_i(1)} \gamma_{ij} \circ \underline{\alpha}_2^{-1}(s) (1 + \eta(s)),$$

for all  $s \geq 0$ . It can easily be checked that  $\tilde{\eta} \in \mathcal{K}_\infty$ . We claim that this function satisfies, for all  $\varphi \in \mathcal{X}^{n_2}$ ,

$$\frac{\alpha_i(|\varphi_i(0)|)}{1 + \eta(V_2(\varphi))} \geq k \frac{\gamma_{ij}(|\varphi_i(0)|)}{1 + \tilde{\eta}(V_2(\varphi))}, \quad \forall i \in \mathcal{I}, j \in [1, p]. \quad (55)$$

To see this, pick any  $i \in \mathcal{I}$  and first consider the case when  $|\varphi_i(0)| \leq 1$ . Since  $\tilde{\eta}(s) \geq \eta(s)$ , (55) readily follows

from (54). On the other hand, if  $|\varphi_i(0)| \geq 1$ , then using (50) it holds that

$$\begin{aligned} 1 + \tilde{\eta}(V_2(\varphi)) &\geq \frac{k}{\alpha_i(1)} \gamma_{ij} \circ \underline{\alpha}_2^{-1}(V_2(\varphi)) (1 + \eta(V_2(\varphi))) \\ &\geq \frac{k}{\alpha_i(|\varphi_i(0)|)} \gamma_{ij}(|\varphi(0)|) (1 + \eta(V_2(\varphi))), \end{aligned}$$

and (55) follows. Combining (55) with (45), the driving system satisfies the following dissipation inequality:

$$D_{(9b)}^+ V_2(t) \leq -k \sum_{i \in \mathcal{I}} \sum_{j=1}^p \frac{\gamma_{ij}(|z_i(t)|)}{1 + \tilde{\eta}(V_2(z_t))} + \gamma(|u(t)|), \quad (56)$$

for almost all  $t \geq 0$ . This ensures in particular that  $D_{(9b)}^+ V_2(t) \leq \gamma(|u(t)|)$  for almost all  $t \geq 0$ , which in turn implies that

$$V_2(z_t) \leq V_2(z_0) + \int_0^t \gamma(|u(\tau)|) d\tau, \quad \forall t \geq 0. \quad (57)$$

For notation conciseness, let

$$E_u^\gamma(t) := \int_0^t \gamma(|u(\tau)|) d\tau, \quad \forall t \geq 0.$$

Observe that  $E_u^\gamma$  is a non-decreasing continuous function. Integrating (56) and using (57), it follows that

$$\begin{aligned} V_2(z_t) - V_2(z_0) &\leq -k \int_0^t \frac{\sum_{i \in \mathcal{I}} \sum_{j=1}^p \gamma_{ij}(|z_i(\tau)|)}{1 + \tilde{\eta}(V_2(z_\tau))} d\tau + E_u^\gamma(t) \\ &\leq -\frac{k \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_0^t \gamma_{ij}(|z_i(\tau)|) d\tau}{1 + \tilde{\eta}(V_2(z_0) + E_u^\gamma(t))} + E_u^\gamma(t). \end{aligned}$$

This ensures that

$$\begin{aligned} k \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_0^t \gamma_{ij}(|z_i(\tau)|) d\tau &\leq \left( 1 + \tilde{\eta}(V_2(z_0) + E_u^\gamma(t)) \right) (V_2(z_0) + E_u^\gamma(t)) \\ &\leq \left( 1 + \tilde{\eta}(V_2(z_0) + E_u^\gamma(t)) \right) \tilde{\eta}(V_2(z_0) + E_u^\gamma(t)), \end{aligned}$$

where we used the fact that  $\tilde{\eta}_2(s) \geq s$  for all  $s \geq 0$ . Recalling that  $(a+b)^2 \leq 2(a^2 + b^2)$  and using  $\tilde{\eta}(a+b) \leq$

$\tilde{\eta}(2a) + \tilde{\eta}(2b)$  for all  $a, b \geq 0$ , we get that

$$\begin{aligned} k \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_0^t \gamma_{ij}(|z_i(\tau)|) d\tau &\leq \tilde{\eta}(2V_2(z_0)) + \tilde{\eta}(2E_u^\gamma(t)) \\ &+ \left( \tilde{\eta}(2V_2(z_0)) + \tilde{\eta}(2E_u^\gamma(t)) \right)^2 \\ &\leq \tilde{\eta}(2V_2(z_0)) + 2\tilde{\eta}(2V_2(z_0))^2 \\ &+ \tilde{\eta}(2E_u^\gamma(t)) + 2\tilde{\eta}(2E_u^\gamma(t))^2 \\ &\leq \tilde{\eta} \circ 2\bar{\alpha}_2(\|z_0\|) + 2\tilde{\eta}^2 \circ 2\bar{\alpha}_2(\|z_0\|) \\ &+ \tilde{\eta}(2E_u^\gamma(t)) + 2\tilde{\eta}(2E_u^\gamma(t))^2, \end{aligned}$$

where  $\bar{\alpha}_2 \in \mathcal{K}_\infty$  comes from (50). Let  $\xi \in \mathcal{K}_\infty$  be defined as  $\xi(s) := \frac{1}{k}(\tilde{\eta} \circ 2\bar{\alpha}_2(s) + 2\tilde{\eta}^2 \circ 2\bar{\alpha}_2(s))$  for all  $s \geq 0$ . Then  $\xi \in \mathcal{K}_\infty$  and it holds that

$$\sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_0^t \gamma_{ij}(|z_i(\tau)|) d\tau \leq \xi(\|z_0\|) + \xi(E_u^\gamma(t)), \quad (58)$$

for all  $t \geq 0$ . Since  $V_1$  is a LKF candidate for the driven subsystem, there exist  $\underline{\alpha}_1, \bar{\alpha}_1 \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}_1(|\phi(0)|) \leq V_1(\phi) \leq \bar{\alpha}_1(\|\phi\|), \quad \forall \phi \in \mathcal{X}^{n_1}. \quad (59)$$

Based on this, integrating (10) yields

$$\begin{aligned} \underline{\alpha}_1(|x(t)|) &\leq \bar{\alpha}_1(\|x_0\|) - \int_0^t \sigma(|x(\tau)|, V(x_t)) d\tau \\ &+ \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_0^t \gamma_{ij}(|z_i(\tau - \delta_{ij})|) d\tau + E_u^\gamma(t) \\ &\leq \bar{\alpha}_1(\|x_{10}\|) + \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_{-\delta_{ij}}^{t-\delta_{ij}} \gamma_{ij}(|z_i(\tau)|) d\tau + E_u^\gamma(t) \\ &\leq \bar{\alpha}_1(\|x_0\|) + \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_{-\delta_{ij}}^t \gamma_{ij}(|z_i(\tau)|) d\tau + E_u^\gamma(t) \\ &\leq \bar{\alpha}_1(\|x_0\|) + n_2 p \delta \bar{\gamma}(\|z_0\|) + \sum_{i \in \mathcal{I}} \sum_{j=1}^p \int_0^t \gamma_{ij}(|z_i(\tau)|) d\tau \\ &+ E_u^\gamma(t), \end{aligned}$$

where  $\bar{\gamma} \in \mathcal{K}_\infty$  is defined as  $\bar{\gamma}(s) := \max_{i \in \mathcal{I}, j \in [1, p]} \gamma_{ij}(s)$  for all  $s \geq 0$ . Using (58) and replacing  $E_u^\gamma$  by its expression, we finally obtain that

$$\begin{aligned} \underline{\alpha}_1(|x(t)|) &\leq \bar{\alpha}_1(\|x_0\|) + n_2 p \delta \bar{\gamma}(\|z_0\|) + \xi(\|z_0\|) \quad (60) \\ &+ \xi \left( \int_0^t \gamma(|u(\tau)|) d\tau \right) + \int_0^t \gamma(|u(\tau)|) d\tau. \end{aligned}$$

Finally, observe that (11) and (50) readily ensure that

$$\underline{\alpha}_2(|z(t)|) \leq \bar{\alpha}_2(\|z_0\|) + \int_0^t \gamma(|u(\tau)|) d\tau. \quad (61)$$

Combining (60) and (61), we conclude that the cascade (9) owns the UBEBS property. Since we have already shown that it is 0-GAS, iISS follows from Proposition 1.

## 6.6 Proof of Theorem 5

Proceeding as in the proof of Theorem 1, the cascade can be shown to be forward complete (see also [3]). We start by recalling the technical result [7, Proposition 12]: if a locally Lipschitz function  $y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  satisfies, for all  $t \geq 0$ ,

$$\begin{aligned} y(t) > 0 &\Rightarrow \dot{y}(t) \leq w(t) \\ y(t) = 0 &\Rightarrow w(t) \geq 0, \end{aligned}$$

for some  $w \in \mathcal{U}$ , then  $\dot{y}(t) \leq w(t)$  for almost all  $t \geq 0$ . Considering any  $x_0 \in \mathbb{R}^{n_1}$ , any  $z_0 \in \mathbb{R}^{n_2}$  and any  $u \in \mathcal{U}^m$ , observe that  $\sum_{i=1}^{n_2} \sigma_i(|z_i|, |z|) = 0$  whenever  $V_2(z(t)) = 0$  (due to the fact that  $V_2$  is positive definite). Picking  $w(t) = -\sum_{i=1}^{n_2} \sigma_i(|z_i(t)|, |z(t)|) + \gamma(|u(t)|)$  and  $y(t) = V_2(z(t))$ , the two above implications hold, and we conclude from (22) that

$$\dot{V}_2(z(t)) \leq -\sum_{i=1}^{n_2} \sigma_i(|z_i(t)|, |z(t)|) + \gamma(|u(t)|), \quad \forall t \geq 0 \text{ a.e.}$$

Since  $V_2$  is a continuous positive definite radially unbounded function, there exist  $\underline{\alpha}_2, \bar{\alpha}_2 \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}_2(|z|) \leq V_2(z) \leq \bar{\alpha}_2(|z|), \quad \forall z \in \mathbb{R}^{n_2}.$$

It follows that

$$\dot{V}_2(z(t)) \leq -\sum_{i=1}^{n_2} \tilde{\sigma}_i(|z_i(t)|, V_2(z(t))) + \gamma(|u(t)|), \quad (62)$$

for almost all  $t \geq 0$ , where  $\tilde{\sigma}_i \in \mathcal{KL}$  is defined as  $\tilde{\sigma}_i(s, t) := \sigma_i(s, \underline{\alpha}_2^{-1}(t))$  for all  $s, t \geq 0$ . Notice that, in view of the growth condition (23), it holds that

$$\gamma_i(s) = \mathcal{O}_{s \rightarrow 0^+}(\tilde{\sigma}_i(s, 0)), \quad \forall i \in \mathcal{I}. \quad (63)$$

Finally, (21) readily ensures that

$$\dot{V}_1(x(t)) \leq -\frac{\alpha(|x(t)|)}{1 + V_1(x(t))} + \sum_{i \in \mathcal{I}} \gamma_i(|z_i(t)|) + \gamma(|u(t)|) \quad (64)$$

for all  $t \geq 0$ . With (62), (63) and (64), all the assumptions of Theorem 1 are fulfilled with  $\delta = 0$  and we conclude that the cascade (20) is iISS.

## 6.7 Proof of Corollary 6

Invoking Lemma 7, there exist  $\mu_i \in \mathcal{K}_\infty$  and  $\nu_i \in \mathcal{L}$ ,  $i \in \llbracket 1, n_2 \rrbracket$ , such that

$$\begin{aligned} \alpha_i(s) &\geq \mu_i(s)\nu_i(s), \quad \forall s \geq 0 \\ \lim_{s \rightarrow 0^+} \frac{\mu_i(s)}{\alpha_i(s)} &= 1. \end{aligned} \quad (65)$$

It follows in particular that, for all  $x \in \mathbb{R}^{n_1}$ , all  $z \in \mathbb{R}^{n_2} \setminus \{0\}$  and all  $u \in \mathbb{R}^m$ ,

$$\frac{\partial V_2}{\partial z}(z) f_2(x, z, u) \leq - \sum_{i=1}^{n_2} \mu_i(|z_i|) \nu_i(|z_i|) + \gamma(|u|).$$

Since  $V_2$  is a continuous positive finite radially unbounded function, there exist  $\underline{\alpha}_2, \bar{\alpha}_2 \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}_2(|z|) \leq V_2(z) \leq \bar{\alpha}_2(|z|), \quad \forall z \in \mathbb{R}^{n_2}.$$

Thus,

$$\frac{\partial V_2}{\partial z} f_2(x, z, u) \leq - \sum_{i=1}^{n_2} \mu_i(|z_i|) \nu_i \circ \underline{\alpha}_2^{-1}(V_2(z)) + \gamma(|u|).$$

For each  $i \in \llbracket 1, n_2 \rrbracket$ , define  $\sigma_i : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  as

$$\sigma_i(s, t) := \mu_i(s) \nu_i \circ \underline{\alpha}_2(t), \quad \forall s, t \geq 0.$$

Then each  $\sigma_i$  is a  $\mathcal{KL}$  function and it holds that

$$\frac{\partial V_2}{\partial z}(z) f_2(x, z, u) \leq - \sum_{i=1}^{n_2} \sigma_i(|z_i|, V_2(z)) + \gamma(|u|). \quad (66)$$

Furthermore, we have from (26) and (65) that

$$\gamma_i(s) = \mathcal{O}_{s \rightarrow 0^+}(\sigma_i(s, 0)), \quad \forall i \in \mathcal{I}. \quad (67)$$

Thus, we get from (24), (66) and (67) that all the assumptions of Theorem 5 are fulfilled and iISS of the cascade (20) follows.

## 7 Conclusion

We have provided a testable condition under which the cascade of two iISS time-delay systems is iISS. Like in the finite-dimensional case, this condition imposes that the input rate of the driven subsystem be dominated by the dissipation rate of the driving one in a neighborhood of zero. Beyond its applicability to time-delay systems, a key novelty of the results presented here is that this growth condition is imposed only on the driving state variables that actually appear in the Lyapunov analysis of the driven subsystem. This feature allows to cover

a wider class of cascades, as illustrated by an example involving delay-free dynamics.

A limitation of our main result lies in the way the two subsystems are interconnected: although discrete delays between the two subsystems are allowed in our analysis, condition (10) is unlikely to be fulfilled for more generic interconnections, such as distributed delays or other processes that would generate non-discrete delayed terms of  $z_i$  in the dynamics of the driven subsystem. Further work is needed in that direction, possibly relying on the  $M_a$  functions introduced in [25]. Notice however that, if a class  $\mathcal{K}$  LKF-wise dissipation rate can be obtained for the driven subsystem, then this class of cascade TDS can be treated by [16, Remark 13]. Nevertheless, some further work is needed in that direction to allow for merely  $\mathcal{PD}$  (or  $\mathcal{KL}$ ) dissipation rates.

Finally, an interesting compromise between iISS and ISS is the Strong iISS property [8], which is known to be preserved by cascade interconnection for finite-dimensional systems [9]. Further investigations are needed to extend this result to time-delay systems.

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