



Technische
Universität
Braunschweig

Institut für  DS
Dynamik und Schwingungen



Gaussian Process Regression for Complex-valued Frequency Response Functions

with J. Bect³, U. Römer¹ and S. Schöps²; ECCOMAS YIC 2021

Niklas Georg^{1,2}, July 8, 2021

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² Centre of Computational Engineering and TEMF, Technische Universität Darmstadt, Germany

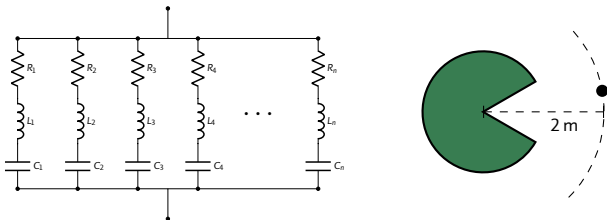
³ Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, France

Overview

- Introduction
- Approximation of Frequency Response Functions
- Numerical results
- Conclusion

Frequency Response Functions (FRFs)

- Response to harmonic (sinusoidal) excitation with a certain frequency f
 - FRF: Measure of magnitude and phase of the output, relative to the input



- Frequency sweep requires many simulation runs
- Goal:** Approximation of FRFs based on a few training points
- Approach:** Describe FRF with a complex Gaussian Process

Gaussian Process Regression

- Gaussian Process

$$f(x) \sim \text{GP}(m(x), k(x, x'))$$

- Mean function $m(x)$
- Covariance function $k(x, x')$

- Gaussian Process Regression (GPR):
infer unobserved values by conditioning
on training data

- Posterior mean: minimum norm
interpolant in the RKHS associated to k

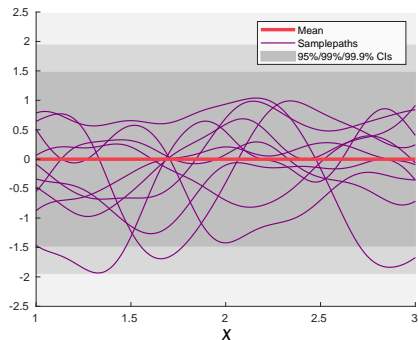
- Approximation of complex functions

1. Use separate Gaussian Process approximations for real and imaginary part

2. Use complex Gaussian Process $f(x) \sim \text{CGP}(m(x), k(x, x'), c(x, x'))$

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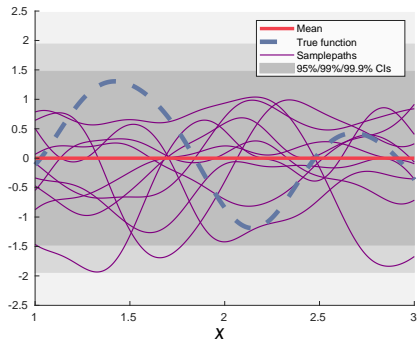
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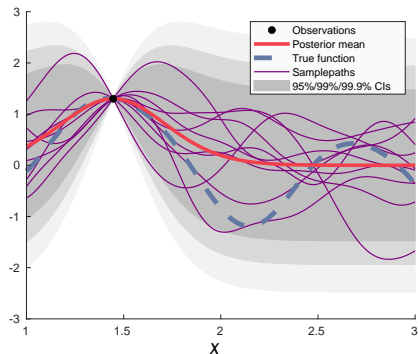
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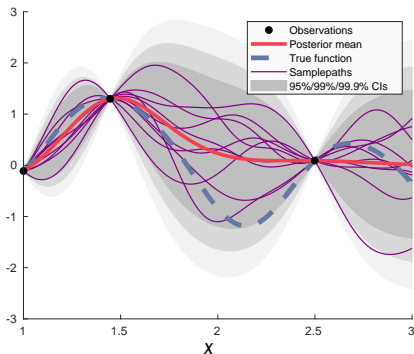
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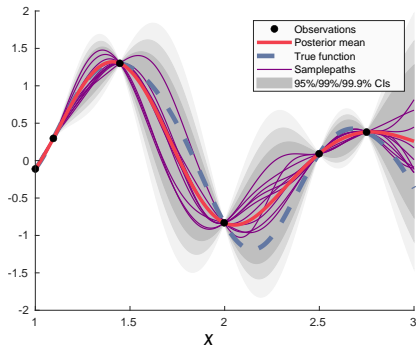
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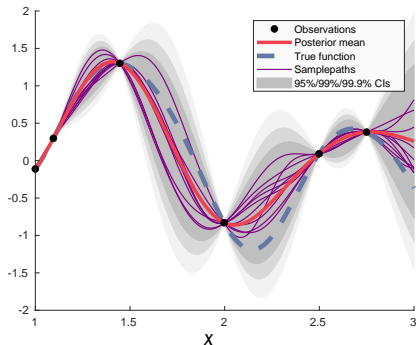
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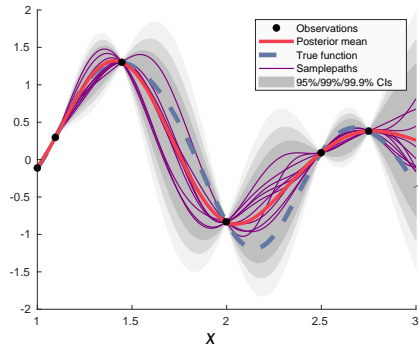
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Parametric Problem

- We address linear parametric problems in the form

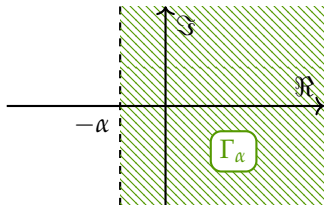
$$(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M})\mathbf{u} = \mathbf{g}$$

- Stiffness, damping and mass matrix $\mathbf{K}, \mathbf{D}, \mathbf{M}, \in \mathbb{R}^{n_h \times n_h}$
- Complex frequency variable $s = i\omega$, where $\omega \in \Omega \subset \mathbb{R}^+$
- Quantity of Interest (QoI)

$$F : \mathbb{C} \rightarrow \mathbb{C}, s \mapsto \mathcal{J}(\mathbf{u}(s))$$

- Assumption: $F \in H^2(\Gamma_\alpha)$, where

$$\Gamma_\alpha := \{s \in \mathbb{C} \mid \mathcal{R}[s] > -\alpha\}, \alpha > 0$$



- Hardy space $H^2(\Gamma_\alpha)$ is an RKHS with the Szegő kernel

$$k(s_k, s_l) = \frac{1}{2\pi(2\alpha + s_k + \bar{s}_l)}$$

Choice of Pseudo-Covariance

- FRF $F(i\omega)$ is the Laplace transform of a real-valued signal $\tilde{f}(t)$, hence

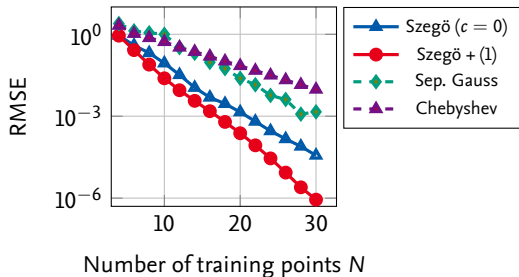
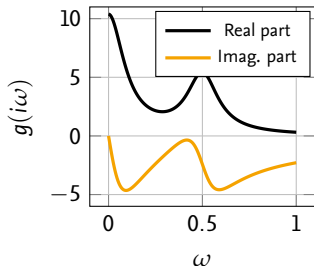
$$F \in H_{\text{sub}}^2(\Gamma_\alpha) := \{u \in H^2(\Gamma_\alpha) \mid \mathcal{L}^{-1}[u](t) \text{ real-valued}\}$$

- Information can be incorporated by choosing the pseudo-covariance¹

$$c(s_k, s_l) = \mathbb{E}[f(s_k)f(s_l)] = \mathbb{E}[f(s_k)\overline{f(\bar{s}_l)}] = k(s, \bar{s}_l) \quad (1)$$

of a mean-free complex GP, i.e., $f(s) \sim \text{CGP}(0, k, c)$

- Numerical example: third order rational function $g \in H_{\text{sub}}^2(\Gamma_\alpha)$, $\omega \in [0, 1]$



¹J. Lataire and T. Chen. "Transfer function and transient estimation by Gaussian process regression in the frequency domain." Automatica (2016)

Approach: GPR + Rational Basis (Sz.-Rat.)

- Problem: slow convergence of Szegő-GPR prediction $f_N(i\omega; \alpha)$ if F has a few dominant poles limiting Γ_α
- Idea: combine GPR with rational basis functions²

$$f_N^{(K)}(i\omega; \mathbf{p}, \alpha) = f_N(i\omega; \alpha) + \sum_{i=1}^K \frac{1}{i\omega - p_i} r_i + \frac{1}{i\omega - \bar{p}_i} \bar{r}_i,$$

- Approach: heuristic algorithm starting with K_{\max} initial pole pairs p_i, \bar{p}_i
 - Tune kernel hyper-parameter α and K pole pairs simultaneously (MLE)
 - Store model $f_N^{(K)}$
 - Remove pole pair with smallest contribution to the likelihood and repeat
- Output: set of approximations $\{f^{(K)}\}_{K=0}^{K_{\max}}$
- \Rightarrow Model selection using a novel criterion based on leave-on-out cross-validation with an additional instability penalty

²B. Gustavsen and A. Semlyen, "Rational approximation of frequency domain responses by vector fitting", IEEE Trans. Power Delivery (1999)

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Electric Circuit

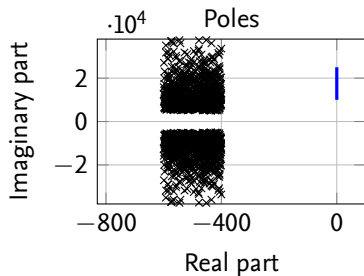
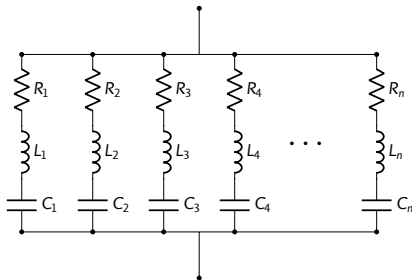
- Parallel connection of n underdamped series RLC circuits

$$Y(i\omega) = \sum_{i=1}^n \frac{i\omega}{(i\omega)^2 L_i + i\omega R_i + C_i^{-1}}, \quad \omega \in [10 \text{ kHz}, 25 \text{ kHz}]$$

- Setting 1: $n_1 = 1000$ random series RLC elements

$$C_i \sim \mathcal{U}(1, 20) \mu\text{F}, \quad L_i \sim \mathcal{U}(0.1, 2) \text{ mH}, \quad R_i = L_i(1 + \mathcal{U}(-0.2, 0.2)) \frac{\Omega}{\text{mH}}$$

- Setting 2: two additional circuit elements with very low damping



Electric Circuit

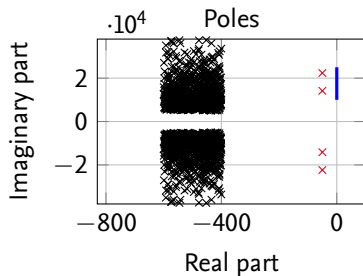
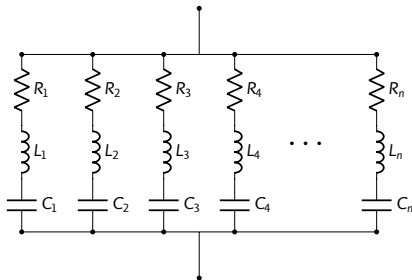
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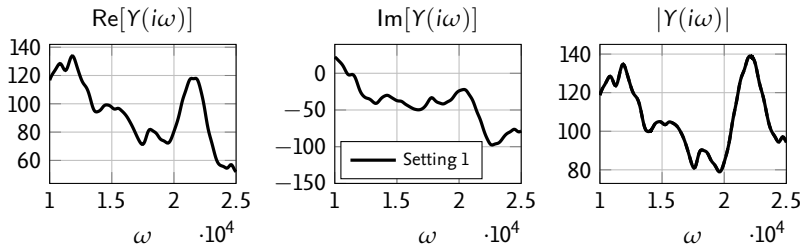
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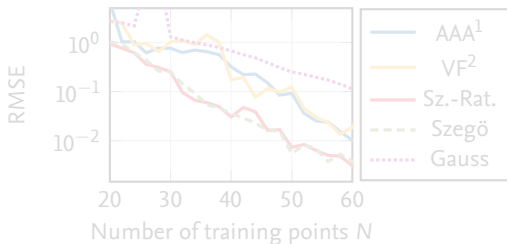
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Electric Circuit - Setting 1



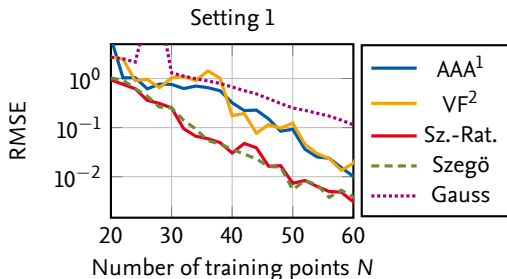
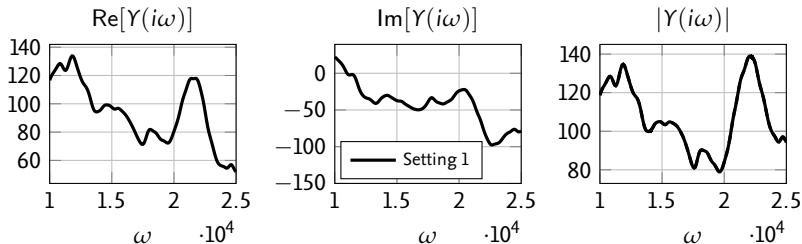
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¹Y. Nakatsukasa et al. "The AAA algorithm for rational approximation" (2018)

²B. Gustavsen et al. "Rational approximation of frequency domain responses by vector fitting" (1999)

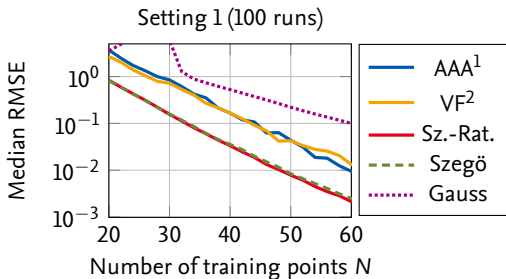
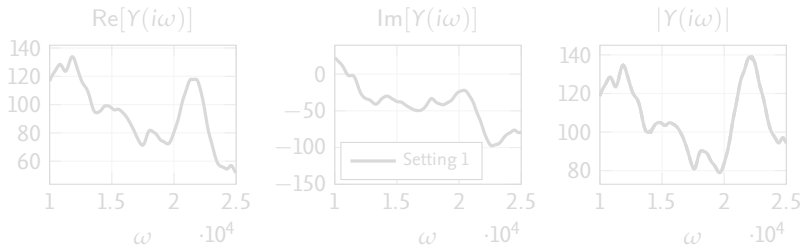
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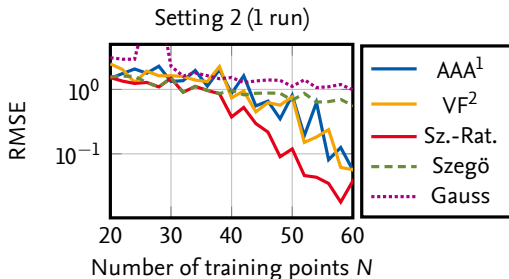
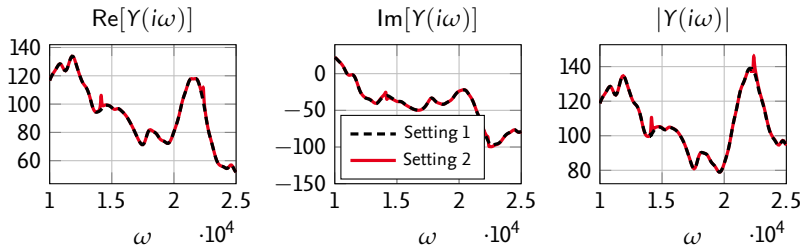
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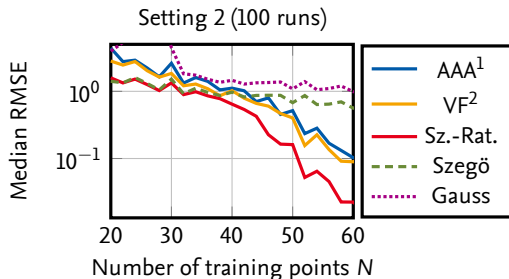
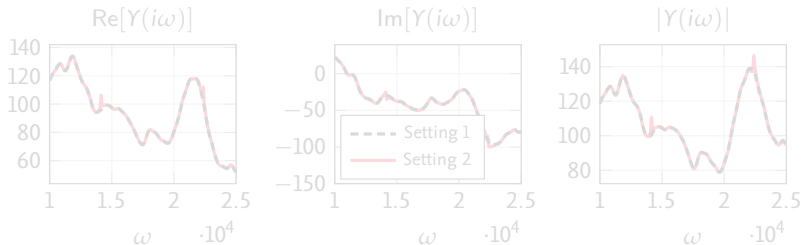
Electric Circuit - Setting 2



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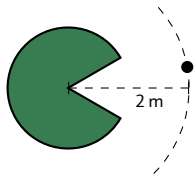
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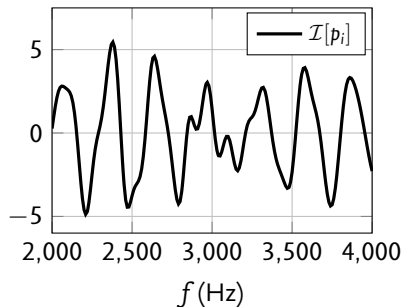
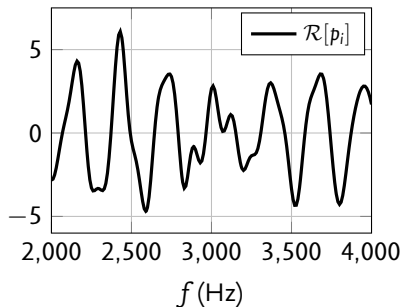
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Benchmark Problem: Acoustic Scattering³

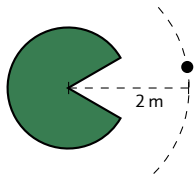


- PACMAN shape: vibrating surface (frequency $f = \frac{\omega}{2\pi}$)
- Acoustic pressure p_i at black dot
- Approximation using 28 training points (blue crosses)
- Method:

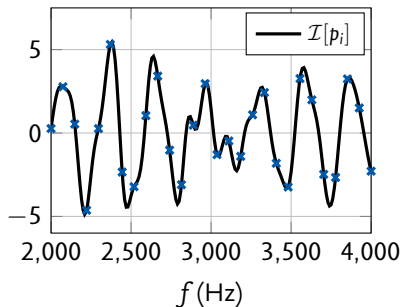
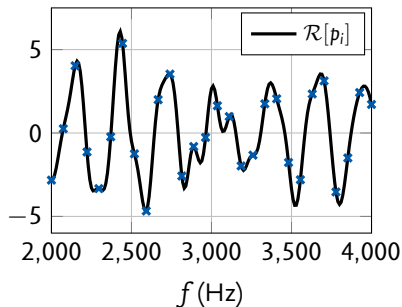


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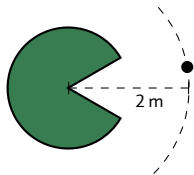


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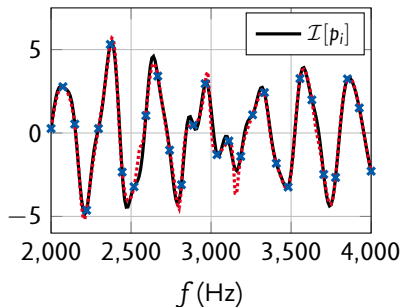
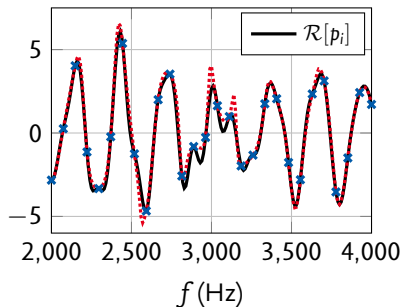


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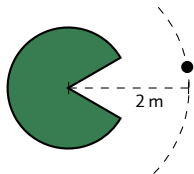


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- Method: **AAA (rational approximation)**

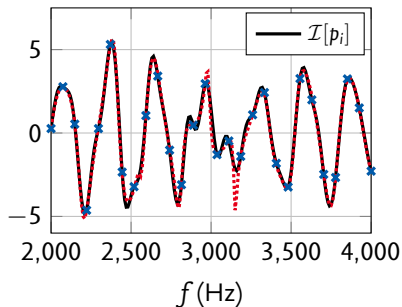
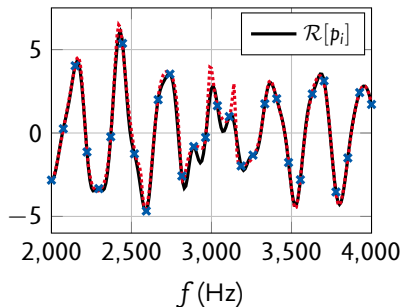


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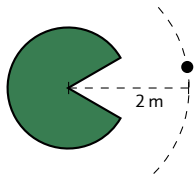


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- Method: **Vector Fitting (rational approximation)**

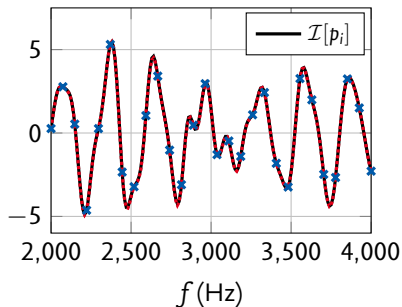
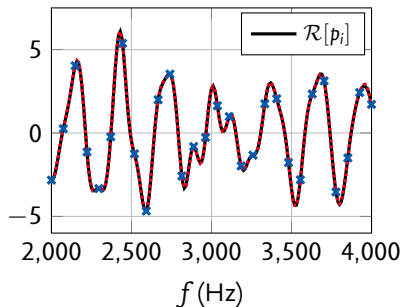


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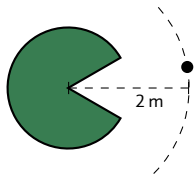


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- Method: **GPR (Szegö kernel) + rational basis**

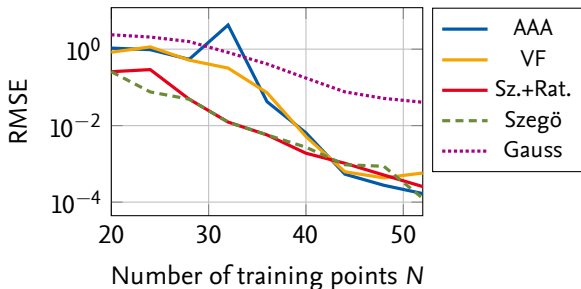


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Benchmark Problem: Acoustic Scattering³



- PACMAN shape: vibrating surface (frequency $f = \frac{\omega}{2\pi}$)
- Acoustic pressure p_i at black dot
- Approximation using N training points



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Conclusion⁵

- Complex GPR using a suitable (pseudo-)kernel for FRFs
- Additional rational basis functions to capture dominant poles
- Model selection based on LOO and instability penalty
- Improvements w.r.t. AAA/VF for a number of benchmark problems

Outlook:

- Generalization to multivariate case \Rightarrow UQ for parametric PDEs⁴

⁴M. Griebel and C. Rieger. "Reproducing kernel Hilbert spaces for parametric partial differential equations." SIAM-ASA J. Uncertain. (2017)

⁵N. Georg, J. Bect, U. Römer, S. Schöps. *Rational kernel-based interpolation for complex-valued frequency response functions*. Will appear on ARXIV soon.

Thank you for your attention!

Please ask questions now or contact

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<http://www.ids.tu-braunschweig.de>

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