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# Online estimation of Hilbert-Schmidt operators and application to kernel reconstruction of neural fields

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## Abstract

An adaptive observer is designed for online estimation of Hilbert-Schmidt operators from online measurement of the state for some class of nonlinear infinite-dimensional dynamical systems. Convergence is ensured under detectability and persistency of excitation assumptions. The class of systems considered is motivated by an application to kernel reconstruction of neural fields, commonly used to model spatiotemporal activity of neuronal populations. Numerical simulations confirm the relevance of the approach.

## 1 Introduction

The problem of online estimation of unknown parameters in dynamical systems from measured state variables is a major issue in many control systems. It can be addressed by means of adaptive observers, that are observers estimating the unmeasured part of the state and the unknown parameters simultaneously. The theory of adaptive observer design, well-known for linear finite-dimensional systems (see, e.g., [19]), is still an active area of research when it comes to nonlinear [3, 4, 18] and/or infinite-dimensional [9, 11, 12] systems.

In this paper, we design an adaptive observer for a class of nonlinear infinite-dimensional systems that allows the reconstruction of unknown linear operators appearing in the dynamics. These operators are estimated in the Hilbert-Schmidt topology. Therefore, not only the state of the system

is infinite-dimensional, but also the “parameters” (now, operators) to be estimated.

The specific class of systems we consider is motivated by an application to kernel reconstruction in neural fields. The offline estimation of these kernels is now a classical issue in inverse problems for neuroscience (see [1, 17] and references therein), that can be addressed for instance using a Tikhonov regularization. We instead rely on adaptive observer strategies to address the online estimation problem. The crucial additional constraint is that the reconstruction can only be based on past values of the measurements and estimates. The recent work [6] considers a similar problem but uses finite-dimensional conductance-based models (which differs from the infinite-dimensional Wilson- Cowan type equation considered here), and estimate finite-dimensional parameters (while we reconstruct linear operators on infinite- dimensional spaces).

**Organization of the paper** The class of infinite-dimensional nonlinear systems under consideration and a precise formulation of the estimation problem are given in Section 2. In Section 3, we show how this problem may be applied to the issue of online reconstruction of kernels in neural fields dynamics. An adaptive observer is proposed in Section 4, and its convergence is proved under a persistency of excitation condition. Finally, numerical simulations are provided in Section 5.

**Notation** Given a Hilbert space  $X$ , we denote by  $\langle \cdot, \cdot \rangle_X$  and  $\| \cdot \|_X$  its corresponding scalar product and norm. The identity operator over  $X$  is denoted by  $\text{Id}_X$ . If  $Y$  is a Hilbert space, we denote by  $\mathcal{L}(X, Y)$  the space of bounded linear operators from  $X$  to  $Y$ . For all  $B \in \mathcal{L}(X, Y)$ , we denote by  $B^* \in \mathcal{L}(Y, X)$  its adjoint. If  $B \in \mathcal{L}(X, X)$ , we denote by  $\text{Tr}(B)$  the trace of  $B$  if it exists. For any open interval  $I \subset \mathbb{R}$ ,  $L^p(I, X)$  and  $W^{m,p}(I, X)$  stand for the usual Lebesgue and Sobolev spaces, endowed with their canonical norms.

## 2 Problem statement

### 2.1 Functional setting

Let  $(X, \| \cdot \|_X)$  and  $(Y, \| \cdot \|_Y)$  be two separable Hilbert spaces. Consider an infinite-dimensional dynamical system of the following form:

$$\begin{cases} \dot{x} = A_1(x) + \psi(y) + u_1 \\ \dot{y} = A_2(y) + B_1\phi_1(x) + B_2\phi_2(y) + u_2 \end{cases} \quad (1)$$

where  $(x, y)$  is the state of the system lying in  $X \times Y$ ,  $u_1$  and  $u_2$  are inputs respectively lying in  $X$  and  $Y$ , and  $A_1 : \mathcal{D}(A_1) \rightarrow X$  and  $A_2 : \mathcal{D}(A_2) \rightarrow Y$  are

singled-valued  $m$ -dissipative operators (see [16] for a definition), respectively defined on dense subsets  $\mathcal{D}(A_1) \subset X$  and  $\mathcal{D}(A_2) \subset Y$ , such that  $A_1(0) = 0$  and  $A_2(0) = 0$ . The linear operators  $B_1 \in \mathcal{L}(X, Y)$  and  $B_2 \in \mathcal{L}(Y, Y)$  are bounded, and  $\psi : Y \rightarrow X$ ,  $\phi_1 : X \rightarrow X$  and  $\phi_2 : Y \rightarrow Y$  are Lipschitz continuous on any bounded set. According to [20, Chapter IV, Proposition 3.1],  $A_1$  and  $A_2$  are generators of nonlinear strongly continuous contraction semigroups over  $X$  and  $Y$  respectively.

It follows from [20, Chapter IV, Theorems 4.1 and 4.1A] that if  $u_1$  and  $u_2$  are absolutely continuous over  $\mathbb{R}_+$ , then for all  $(x_0, y_0) \in \mathcal{D}(A_1) \times \mathcal{D}(A_2)$ , there exists  $t_{\max} \in (0, +\infty]$  such that (1) admits a unique strong solution  $(x, y) : [0, t_{\max}) \rightarrow X \times Y$ , i.e., such that  $(x(0), y(0)) = (x_0, y_0)$ ,  $(x, y)$  is absolutely continuous, satisfies (1) almost everywhere and lies in  $\mathcal{D}(A_1) \times \mathcal{D}(A_2)$ . Moreover, if  $(x, y)$  is bounded in  $X \times Y$  over  $[0, t_{\max})$ , then  $t_{\max} = +\infty$ .

## 2.2 Problem formulation

In this paper, we consider the following online estimation problem.

**Problem 2.1** *From the knowledge of  $A_1$ ,  $A_2$ ,  $\psi$ ,  $\phi_1$ ,  $\phi_2$  and the online measurement of  $u_1$ ,  $u_2$  and  $y$ , estimate online  $x$  and the operators  $B_1$  and  $B_2$ .*

In addition to the hypotheses made to ensure the well-posedness of the system, we consider the following two main assumptions.

**Assumption 2.2 (Strong dissipativity)** *The nonlinear operator  $A_1$  is strongly dissipative, that is, there exists a positive constant  $\alpha$  such that for all  $(x_1, x_2) \in \mathcal{D}(A_1)^2$ ,*

$$\langle A_1(x_1) - A_1(x_2), x_1 - x_2 \rangle_X \leq -\alpha \|x_1 - x_2\|_X^2. \quad (2)$$

Since  $A_1$  is supposed to be  $m$ -dissipative, we already have that  $\langle A_1(x_1) - A_1(x_2), x_1 - x_2 \rangle_X \leq 0$ , so that Assumption 2.2 is indeed a stronger dissipativity assumption. Assumption 2.2 implies that every solutions of  $\dot{x} = A_1(x)$  are exponentially converging to one another in  $X$  at exponential rate  $\alpha$ . Since  $y$  is supposed to be known online while  $x$  is unknown, Assumption 2.2 can be interpreted as a detectability hypothesis: the unknown part of the state has a contracting dynamics. In the estimation strategy, this allows to estimate  $x$  online simply by simulating a particular trajectory of the  $x$ -subsystem (as all other solutions will eventually converge to it). Note that the unmeasured state  $x$  can also be assumed of null dimension, in which case the system (1) reduces to

$$\dot{y} = A_2(y) + B_2\phi_2(y) + u_2.$$

All the results of the paper are still valid in that easier case.

**Assumption 2.3 (Hilbert-Schmidt operators)** *The linear bounded operators  $B_1$  and  $B_2$  are Hilbert-Schmidt operators, that is, for any Hilbert basis  $(e_k^1)_{k \in \mathbb{N}}$  and  $(e_k^2)_{k \in \mathbb{N}}$  of  $X$  and  $Y$  respectively,*

$$\begin{aligned}\|B_1\|_{\mathcal{L}_2(X,Y)}^2 &:= \sum_{k \in \mathbb{N}} \|B_1 e_k^1\|_X^2 = \text{Tr}(B_1^* B_1) < +\infty, \\ \|B_2\|_{\mathcal{L}_2(Y,Y)}^2 &:= \sum_{k \in \mathbb{N}} \|B_2 e_k^2\|_X^2 = \text{Tr}(B_2^* B_2) < +\infty,\end{aligned}$$

where  $B_i^*$  denotes the adjoint operator of  $B_i$  for  $i \in \{1, 2\}$ .

We denote by  $(\mathcal{L}_2(X, Y), \|\cdot\|_{\mathcal{L}_2(X,Y)})$  and  $(\mathcal{L}_2(Y, Y), \|\cdot\|_{\mathcal{L}_2(Y,Y)})$  the Hilbert spaces of Hilbert-Schmidt operators from  $X$  to  $Y$  and  $Y$  to  $Y$ , respectively. Then, Problem 2.1 consists in finding  $\hat{B}_1(t)$  and  $\hat{B}_2(t)$  for all  $t \geq 0$  such that  $\|\hat{B}_1(t) - B_1\|_{\mathcal{L}_2(X,Y)} \rightarrow 0$  and  $\|\hat{B}_2(t) - B_2\|_{\mathcal{L}_2(Y,Y)} \rightarrow 0$  as  $t \rightarrow +\infty$ . Such estimators must only depend on the knowledge of  $A_1, A_2, \psi, \phi_1, \phi_2, u_1(s), u_2(s)$  and  $y(s)$  for  $s \in [0, t]$ . Note that the topology induced by the Hilbert-Schmidt norm is finer than the one induced by the operator norm since  $\|\cdot\|_{\mathcal{L}} \leq \|\cdot\|_{\mathcal{L}_2}$ .

### 3 Kernel reconstruction of neural fields

#### 3.1 Neural fields

Problem 2.1 is motivated by an application to kernel reconstruction of neural fields. Neural fields are nonlinear integro-differential equations modeling the spatiotemporal evolution of the activity of neuronal populations. They are based on the seminal works [2, 23] and surveys on their extensive use in mathematical neuroscience can be found in [5, 8]. Given a compact set  $\Omega \subset \mathbb{R}^q$  (where, typically,  $q \in \{1, 2, 3\}$ ) representing the physical support of the population, the evolution of neuronal activity  $z(t, r) \in \mathbb{R}^n$  at time  $t \in \mathbb{R}_+$  and position  $r \in \Omega$  is modeled as

$$\begin{aligned}\tau(r) \frac{\partial z}{\partial t}(t, r) &= -z(t, r) + \int_{r' \in \Omega} w(r, r') S(z(t, r')) dr' \\ &\quad + u(t, r),\end{aligned}\quad (3)$$

where  $\tau(r)$  is a positive diagonal matrix of size  $n \times n$  representing the time decay constant of neuronal activity at position  $r$ ,  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a nonlinear activation function (typically, a sigmoid),  $w(r, r') \in \mathbb{R}^{n \times n}$  defines a kernel describing the synaptic strength between location  $r$  and  $r'$  and  $u(t, r) \in \mathbb{R}^n$  is an input. We consider the problem of online reconstruction of the kernel  $w$  from the measurement of the neuronal activity  $z$ .

### 3.2 Application

Now we show how (3) fits into (1) and discuss the relevance of Assumptions 2.2 and 2.3 in this context. In order to ensure well-posedness, we make the following usual assumptions on  $S$  and  $w$ :

- $S$  is bounded, differentiable and has bounded derivative;
- $w$  is square-integrable over  $\Omega^2$ .

These assumptions are standard in neural fields analysis. In particular, the boundedness of  $S$  reflects the biological limitations of the maximal activity that can be reached by the population.

We assume that the neuronal population can be decomposed into  $z(t, r) = (z_1(t, r), z_2(t, r)) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$  where  $z_1$  corresponds to the unmeasured part of the state and  $z_2$  to the measured part. Such a decomposition is natural when the two considered populations are physically separated, as it happens in the brain structures involved in Parkinson's disease [7]. It can also be relevant for imagery techniques that discriminate among neuron types within a given population. Accordingly, we define  $\tau_i$ ,  $S_i$ ,  $w_{ij}$  and  $u_i$  of suitable dimensions for each population  $i, j \in \{1, 2\}$  so that

$$\tau_i(r) \frac{\partial z_i}{\partial t}(t, r) = -z_i(t, r) + \sum_{j=1}^2 \int_{r' \in \Omega} w_{ij}(r, r') S_j(z_j(t, r')) dr' + u_i(t, r). \quad (4)$$

Denote by  $m$  the dimension of the measured activity  $z_2(r, t)$ . In order to fit (4) in the form of (1), set  $X = L^2(\Omega; \mathbb{R}^{n-m})$ ,  $Y = L^2(\Omega; \mathbb{R}^m)$ ,  $x = \tau_1 z_1$ ,  $y = \tau_2 z_2$ ,  $W_{ij}(z_j) = \int_{r' \in \Omega} w_{ij}(\cdot, r') z_j(r') dr'$ ,  $A_1 = (-\text{Id}_X + W_{11} S_1)$ ,  $\mathcal{D}(A_1) = X$ ,  $A_2 = -\text{Id}_Y$ ,  $\mathcal{D}(A_2) = Y$ ,  $\psi = W_{12} S_2$ ,  $\phi_i = S_i$  and  $B_i = W_{2i}$ .

Since  $w$  is square-integrable,  $B_1$  and  $B_2$  are Hilbert-Schmidt integral operators with kernels  $w_{21}$  and  $w_{22}$ , hence Assumption 2.3 is satisfied. In order to satisfy the detectability Assumption 2.2, we need to assume that  $z_1$  has a strongly dissipative dynamics, namely, that  $A_1$  is strongly dissipative. Remark that due to the structure of  $A_1$ , this is the case if

$$\ell_1 \|W_{11}\|_{\mathcal{L}(X, X)} < 1 \quad (5)$$

where  $\ell_1$  is the Lipschitz constant of  $S_1$ . Indeed,

$$\begin{aligned} & \langle A_1(x_1) - A_1(x_2), x_1 - x_2 \rangle_X \\ &= -\|x_1 - x_2\|_X^2 + \langle W_{11}(S_1(x_1) - S_1(x_2)), x_1 - x_2 \rangle_X \\ &\leq -\alpha \|x_1 - x_2\|_X^2 \end{aligned}$$

for  $\alpha = 1 - \ell_1 \|W_{11}\|_{\mathcal{L}(X,X)}$ . We stress that condition (5) is commonly used in the stability analysis of neural fields [15] and ensures dissipativity even in the presence of axonal propagation delays [13].

We thus assume that each population is either measured online (taken into account in  $z_2$ ) or unmeasured but stable and with known kernels (taken into account in  $z_1$ ). Problem 2.1 is now equivalent to online reconstruction of  $w_{21}$  and  $w_{22}$  (in  $L^2(\Omega^2)$ ) from the online measurement of  $z_2$  and  $u_i$  and the knowledge of  $\tau_i, w_{1i}, S_i$  for all  $i \in \{1, 2\}$ . Note that if the full state  $z$  is measured (i.e.  $m = n$ ), then no dissipative part  $z_1$  of the system is required, hence the full kernel  $w$  is to be estimated.

## 4 Online estimation of hilbert-schmidt operators

### 4.1 Adaptive observer design

In order to solve Problem 2.1, we propose to consider  $B_1$  and  $B_2$  as additional constant variables to system (1), so that the resulting state space is the Hilbert space  $H := X \times Y \times \mathcal{L}_2(X, Y) \times \mathcal{L}_2(Y, Y)$ . Set also  $\mathcal{D} := \mathcal{D}(A_1) \times \mathcal{D}(A_2) \times \mathcal{L}_2(X, Y) \times \mathcal{L}_2(Y, Y) \subset H$ . Inspired by the estimator proposed in [3] for finite-dimensional nonlinear systems, we consider the following observer over  $H$ :

$$\begin{cases} \dot{\hat{x}} = A_1(\hat{x}) + \psi(y) + u_1 \\ \dot{\hat{y}} = A_2(y) + \hat{B}_1\phi_1(\hat{x}) + \hat{B}_2\phi_2(y) + u_2 - \beta(\hat{y} - y) \\ \dot{\hat{B}}_1 = -\gamma_1(\hat{y} - y)\phi_1(\hat{x})^* \\ \dot{\hat{B}}_2 = -\gamma_2(\hat{y} - y)\phi_2(y)^* \end{cases} \quad (6)$$

where  $\beta, \gamma_1,$  and  $\gamma_2$  are positive constants, called observer gains, that need to be appropriately tuned to guarantee the convergence of the observer state to the real state. Note that for any  $v$  in  $Y$  and any  $w$  in  $X$  (resp. in  $Y$ ),  $vw^*$  lies in  $\mathcal{L}_2(X, Y)$  (resp. in  $\mathcal{L}_2(Y, Y)$ ) and  $\|vw^*\|_{\mathcal{L}_2(X, Y)} = \|v\|_Y \|w\|_X$  (resp.  $\|vw^*\|_{\mathcal{L}_2(Y, Y)} = \|v\|_Y \|w\|_Y$ ). Reasoning as in Section 2.1, one can show that the cascade system (1)-(6) is well-posed.

### 4.2 Main result

Our main result, proved in Section 4.3, relies on the notion of persistence of excitation.

**Definition 4.1 (Persistence of excitation)** *A signal  $g : \mathbb{R}_+ \rightarrow V$  is persistently exciting over the Hilbert space  $V$  if there exists positive constants  $T$  and  $\kappa$  such that*

$$\int_t^{t+T} g(\tau)g^*(\tau)d\tau \geq \kappa \text{Id}_V, \quad \forall t \geq 0. \quad (7)$$





Then, we have the following, which immediately follows from Theorem 4.2 for this particular system.

**Corollary 4.3 (Neural fields estimation)** *Suppose that  $S_1$  (resp.  $S_2$ ) is bounded, differentiable, and that its derivative is bounded by some  $\ell_1 > 0$  (resp.  $\ell_2 > 0$ ), and that  $w$  is square-integrable over  $\Omega^2$ . Assuming that (5) is satisfied, pick the observer gains in such a way that  $\gamma_1, \gamma_2 > 0$  and*

$$4(1 - \ell_1 \|W_{11}\|_{\mathcal{L}(X,X)})\beta > \ell_1^2 \|B_1\|_{\mathcal{L}(X,Y)}^2. \quad (10)$$

*Consider any solution of (4) defined over  $\mathbb{R}_+$  for some absolutely continuous inputs  $u_1$  and  $u_2$ . Then any solution of (9) satisfies*

$$\lim_{t \rightarrow +\infty} \|\hat{z}_1(t) - z_1(t)\|_X = 0, \quad \lim_{t \rightarrow +\infty} \|\hat{z}_2(t) - z_2(t)\|_Y = 0$$

*and  $\|\hat{W}_{21} - W_{21}\|_{\mathcal{L}_2(X,Y)}$  and  $\|\hat{W}_{22} - W_{22}\|_{\mathcal{L}_2(X,Y)}$  remain bounded.*

*Moreover, if  $t \mapsto (S_1(z_1(t)), S_2(z_2(t)))$  is persistently exciting over  $X \times Y$ , then*

$$\lim_{t \rightarrow +\infty} \|\hat{W}_{2i}(t) - W_{2i}\|_{\mathcal{L}_2(X,Y)} = 0, \quad \forall i \in \{1, 2\}.$$

Here again, provided that condition (5) holds,  $\beta$  can always be picked large enough to fulfill (10).

### 4.3 Proof of Theorem 4.2

Consider a solution  $(x, y)$  of (1) defined on  $\mathbb{R}_+$  and the corresponding solution  $(\hat{x}, \hat{y}, \hat{B}_1, \hat{B}_2)$  of (6). The estimation error  $(\tilde{x}, \tilde{y}, \tilde{B}_1, \tilde{B}_2) := (\hat{x}, \hat{y}, \hat{B}_1, \hat{B}_2) - (x, y, B_1, B_2)$  is ruled by:

$$\begin{cases} \dot{\tilde{x}} = A_1(\hat{x}) - A_1(x) \\ \dot{\tilde{y}} = \hat{B}_1 \phi_1(\hat{x}) - B_1 \phi_1(x) + \tilde{B}_2 \phi_2(y) - \beta \tilde{y} \\ \dot{\tilde{B}}_1 = -\gamma_1 \tilde{y} \phi_1(\hat{x})^* \\ \dot{\tilde{B}}_2 = -\gamma_2 \tilde{y} \phi_2(y)^*. \end{cases} \quad (11)$$

#### 4.3.1 Proof that $(\tilde{x}, \tilde{y}) \rightarrow 0$

We endow  $H$  with the squared norm  $\|\cdot\|_H^2 = \|\cdot\|_X^2 + \|\cdot\|_Y^2 + \frac{1}{\gamma_1} \|\cdot\|_{\mathcal{L}_2(X,Y)}^2 + \frac{1}{\gamma_2} \|\cdot\|_{\mathcal{L}_2(Y,Y)}^2$ , which is equivalent to the squared norm induced by the Cartesian product  $H = X \times Y \times \mathcal{L}_2(X,Y) \times \mathcal{L}_2(Y,Y)$ . Given any initial state, denote by  $t_{\max} \in (0, +\infty]$  the maximal time of existence of  $(\tilde{x}, \tilde{y}, \tilde{B}_1, \tilde{B}_2)$ .

Using  $\hat{B}_1\phi_1(\hat{x}) - B_1\phi_1(x) = \tilde{B}_1\phi_1(\hat{x}) + B_1(\phi_1(\hat{x}) - \phi_1(x))$ , we have almost everywhere on  $[0, t_{\max})$

$$\begin{aligned} \frac{d}{dt} \|(\tilde{x}, \tilde{y}, \tilde{B}_1, \tilde{B}_2)\|_H^2 &= \langle A_1(\hat{x}) - A_1(x), \tilde{x} \rangle_X - \beta \|\tilde{y}\|_Y^2 + \langle \tilde{B}_2\phi_2(y), \tilde{y} \rangle_Y \\ &\quad + \langle \tilde{B}_1\phi_1(\hat{x}), \tilde{y} \rangle_Y + \langle B_1(\phi_1(\hat{x}) - \phi_1(x)), \tilde{y} \rangle_Y \\ &\quad - \langle \tilde{y}\phi_1(\hat{x})^*, \tilde{B}_1 \rangle_{\mathcal{L}_2(X,Y)} - \langle \tilde{y}\phi_1(y)^*, \tilde{B}_2 \rangle_{\mathcal{L}_2(X,Y)}. \end{aligned}$$

By Assumption 2.2,  $\langle A_1(\hat{x}) - A_1(x), \tilde{x} \rangle_X \leq -\alpha \|\tilde{x}\|_X^2$ . By definition of the Hilbert-Schmidt scalar product,

$$\begin{aligned} \langle \tilde{y}\phi_1(\hat{x})^*, \tilde{B}_1 \rangle_{\mathcal{L}_2(X,Y)} &= \text{Tr}(\phi_1(\hat{x})\tilde{y}^*\tilde{B}_1) = \text{Tr}(\tilde{y}^*\tilde{B}_1\phi_1(\hat{x})) \\ &= \langle \tilde{B}_1\phi_1(\hat{x}), \tilde{y} \rangle_Y \end{aligned}$$

and, similarly,

$$\langle \tilde{y}\phi_2(y)^*, \tilde{B}_2 \rangle_{\mathcal{L}_2(X,Y)} = \langle \tilde{B}_2\phi_2(y), \tilde{y} \rangle_Y.$$

Hence

$$\begin{aligned} \frac{d}{dt} \|(\tilde{x}, \tilde{y}, \tilde{B}_1, \tilde{B}_2)\|_H^2 &\leq -\alpha \|\tilde{x}\|_X^2 - \beta \|\tilde{y}\|_Y^2 + \langle B_1(\phi_1(\hat{x}) - \phi_1(x)), \tilde{y} \rangle_X. \end{aligned}$$

By Cauchy-Schwartz inequality, for all  $\varepsilon > 0$ ,

$$\begin{aligned} \langle B_1(\phi_1(\hat{x}) - \phi_1(x)), \tilde{y} \rangle_X &\leq \ell_1 \|B_1\|_{\mathcal{L}(X,Y)} \left( \frac{\varepsilon}{2} \|\tilde{x}\|_X^2 + \frac{1}{2\varepsilon} \|\tilde{y}\|_X^2 \right) \end{aligned}$$

where  $\ell_1$  is the Lipschitz constant of  $\phi_1$ . Pick  $\varepsilon = \frac{\alpha}{\ell_1 \|B_1\|_{\mathcal{L}(X,Y)}} + \frac{\ell_1 \|B_1\|_{\mathcal{L}(X,Y)}}{4\beta} > 0$ . Using condition (8), we get that  $\mu_1 := \alpha - \ell_1 \|B_1\|_{\mathcal{L}(X,Y)}\varepsilon/2 > 0$  and  $\mu_2 := \beta - \ell_1 \|B_1\|_{\mathcal{L}(X,Y)}/2\varepsilon > 0$ . Then

$$\frac{d}{dt} \|(\tilde{x}, \tilde{y}, \tilde{B}_1, \tilde{B}_2)\|_H^2 \leq -\mu_1 \|\tilde{x}\|_X^2 - \mu_2 \|\tilde{y}\|_Y^2.$$

Thus  $(\tilde{x}, \tilde{y}, \tilde{B}_1, \tilde{B}_2)$  remains bounded. Hence according to [20, Chapter IV, Theorems 4.1], we obtain  $t_{\max} = +\infty$  i.e. the state  $(\hat{x}, \hat{y}, \hat{B}_1, \hat{B}_2)$  is defined over  $\mathbb{R}_+$ . Moreover, we have  $\frac{d}{dt} \|\tilde{x}\|_X^2 \leq 0$  and

$$\begin{aligned} \frac{d}{dt} \|\tilde{y}\|_Y^2 &\leq -\beta \|\tilde{y}\|_Y^2 + \langle \tilde{B}_2\phi_2(y), \tilde{y} \rangle_Y \\ &\quad + \langle \hat{B}_1\phi_1(\hat{x}), \tilde{y} \rangle_Y - \langle B_1\phi_1(x), \tilde{y} \rangle_Y \end{aligned}$$

which is bounded since  $(\tilde{x}, \tilde{y}, \tilde{B}_1, \tilde{B}_2)$  is bounded,  $B_1$  is constant, and  $\phi_1$  and  $\phi_2$  are bounded. Hence, according to Barbalat's lemma,  $(\tilde{x}(t), \tilde{y}(t)) \rightarrow 0$  as  $t \rightarrow +\infty$ .

### 4.3.2 Proof that $(\tilde{B}_1, \tilde{B}_2) \rightarrow 0$

Now assume that  $t \mapsto (\phi_1(x(t)), \phi_2(y(t)))$  is persistently exciting over  $X \times Y$  and that  $\phi_1$  and  $\phi_2$  are differentiable with bounded derivatives. Note that  $\hat{B}_1\phi_1(\hat{x}) - B_1\phi_1(x) = \hat{B}_1(\phi_1(\hat{x}) - \phi_1(x)) + \tilde{B}_1\phi_1(x)$ . Hence the error dynamics (11) can be written as

$$\begin{cases} \dot{\tilde{y}}(t) = \tilde{B}_1(t)g_1(t) + \tilde{B}_2(t)g_2(t) + f_0(t) \\ \dot{\tilde{B}}_1(t) = f_1(t) \\ \dot{\tilde{B}}_2(t) = f_2(t), \end{cases} \quad (12)$$

where  $f_0(t) := \hat{B}_1(\phi_1(\hat{x}(t)) - \phi_1(x(t))) - \beta\tilde{y}$ ,  $f_1(t) := -\gamma_1\tilde{y}(t)\phi_1(\hat{x}(t))^*$ ,  $f_2(t) := -\gamma_2\tilde{y}(t)\phi_2(y(t))^*$ ,  $g_1(t) := \phi_1(x(t))$  and  $g_2(t) := \phi_2(y(t))$  for all  $t \geq 0$ . Since  $(\tilde{x}(t), \tilde{y}(t)) \rightarrow 0$  as  $t \rightarrow +\infty$ ,  $\hat{B}_1$  is bounded,  $\phi_1$  is globally Lipschitz and  $\phi_1$  and  $\phi_2$  are bounded, we get that  $f_i(t)$  tends toward 0 as  $t$  goes to  $+\infty$  for all  $i \in \{0, 1, 2\}$ . Set  $g := (g_1, g_2) : \mathbb{R}_+ \rightarrow X \times Y$  and  $f_{1,2}, \tilde{B} : \mathbb{R}_+ \rightarrow \mathcal{L}_2(X \times Y, Y)$ , defined by  $f_{1,2}(t)(\zeta, \xi) = f_1(t)\zeta + f_2(t)\xi$ ,  $\tilde{B}(t)(\zeta, \xi) = \tilde{B}_1(t)\zeta + \tilde{B}_2(t)\xi$  for all  $(\zeta, \xi) \in X \times Y$  and all  $t \geq 0$ , so that  $\dot{\tilde{y}}(t) = \tilde{B}(t)g(t) + f_0(t)$ . Remark that  $\|\tilde{B}(t)\|_{\mathcal{L}_2(X \times Y, Y)}^2 = \|\tilde{B}_1(t)\|_{\mathcal{L}_2(X, Y)}^2 + \|\tilde{B}_2(t)\|_{\mathcal{L}_2(Y, Y)}^2$ , so that it remains to show that  $\|\tilde{B}(t)\|_{\mathcal{L}_2(X \times Y, Y)} \rightarrow 0$  as  $t \rightarrow +\infty$  to conclude. Applying twice Duhamel's formula, we have for all  $t, \tau \geq 0$ :

$$\begin{aligned} \tilde{y}(t + \tau) &= \tilde{y}(t) + \tilde{B}(t) \int_0^\tau g(t + s) ds \\ &\quad + \int_0^\tau \int_0^s f_{1,2}(t + \sigma) g(t + s) d\sigma ds + \int_0^\tau f_0(t + s) ds. \end{aligned}$$

Define  $\mathcal{O}(t, T) := \int_0^T \|\tilde{y}(t + \tau)\|_Y^2 d\tau$  for any  $T > 0$  and  $t \geq 0$ . Since  $\tilde{y}(t) \rightarrow 0$ ,  $\mathcal{O}(t, T) \rightarrow 0$  as  $t \rightarrow +\infty$ . Moreover,

$$\begin{aligned} \mathcal{O}(t, T) &= \int_0^T \left\| \tilde{y}(t) + \int_0^\tau \int_0^s f_{1,2}(t + \sigma) g(t + s) d\sigma ds \right. \\ &\quad \left. + \int_0^\tau f_0(t + s) ds \right\|_Y^2 d\tau + \int_0^T \left\| \tilde{B}(t) \int_0^\tau g(t + s) ds \right\|_Y^2 d\tau \\ &\quad + \int_0^T \left\langle \tilde{y}(t) + \int_0^\tau \int_0^s f_{1,2}(t + \sigma) g(t + s) d\sigma ds \right. \\ &\quad \left. + \int_0^\tau f_0(t + s) ds, \tilde{B}(t) \int_0^\tau g(t + s) ds \right\rangle_Y d\tau. \end{aligned}$$

Since  $\tilde{y}(t)$  and  $f_i(t)$  tends toward 0 as  $t$  goes to  $+\infty$  for all  $i \in \{0, 1, 2\}$  and  $g$  and  $\tilde{B}$  are bounded, we get that

$$\lim_{t \rightarrow +\infty} \int_0^T \left\| \tilde{B}(t) \int_0^\tau g(t + s) ds \right\|_Y^2 d\tau = 0. \quad (13)$$

For all  $t \geq 0$ , define  $h(t, \tau) = \tilde{B}(t) \int_0^\tau g(t+s) ds$ . By (13),  $\|h(t, \cdot)\|_{L^2((0,T);Y)} \rightarrow 0$  as  $t \rightarrow +\infty$ . Note that  $\frac{\partial h}{\partial \tau}(t, \tau) = \tilde{B}(t)g(t+\tau)$  hence  $h(t, \cdot) \in W^{1,2}((0,T);Y)$  since  $g$  is bounded. Moreover,  $\dot{y}$  is bounded since  $\tilde{B}_i$  and  $g_i$  are bounded for  $i \in \{1,2\}$  and  $\dot{x} = A_1(\hat{x}) - A_1(x)$  is bounded since  $A_1$  is  $m$ -dissipative (see [16, Corollary 3.7 and Theorem 4.20]). Hence, if  $\phi_1$  and  $\phi_2$  are differentiable with bounded derivatives, then so is  $g$ . Therefore, for all  $t \geq 0$ ,  $h(t, \cdot) \in W^{2,2}((0,T);Y)$  and  $\|h(t, \cdot)\|_{W^{2,2}((0,T);Y)} \leq C_1$  for some positive constant  $C_1$  independent of  $t$ . According to the interpolation inequality (see, e.g., [21, Section II.2.1]),

$$\|h(t, \cdot)\|_{W^{1,2}((0,T);Y)}^2 \leq C_2 \|h(t, \cdot)\|_{L^2((0,T);Y)}$$

for some positive constant  $C_2$  independent of  $t$ . Thus  $\|\frac{\partial h}{\partial \tau}(t, \tau)\|_{L^2((0,T);Y)} \rightarrow 0$ , meaning that

$$\lim_{t \rightarrow +\infty} \int_0^T \left\| \tilde{B}(t)g(t+\tau) \right\|_Y^2 d\tau = 0. \quad (14)$$

Now, let  $(e_k)_{k \in \mathbb{N}}$  be a Hilbert basis of  $Y$ . Then

$$\begin{aligned} \|\tilde{B}(t)\|_{\mathcal{L}_2(X \times Y, Y)}^2 &= \|\tilde{B}^*(t)\|_{\mathcal{L}_2(Y, X \times Y)}^2 \\ &= \sum_{k \in \mathbb{N}} \|\tilde{B}(t)^* e_k\|_Y^2. \end{aligned}$$

Since  $g$  is persistently exciting, we have, for some  $T, \kappa > 0$ ,

$$\int_0^T |\langle g(t+\tau), v \rangle_{X \times Y}|^2 d\tau \geq \kappa \|v\|_{X \times Y}^2, \quad \forall t \geq 0,$$

for all  $v \in X \times Y$ . Then,

$$\begin{aligned} \kappa \|\tilde{B}(t)\|_{\mathcal{L}_2(X \times Y, Y)}^2 &\leq \sum_{k \in \mathbb{N}} \int_0^T |\langle g(t+\tau), \tilde{B}(t)^* e_k \rangle_{X \times Y}|^2 d\tau \\ &= \sum_{k \in \mathbb{N}} \int_0^T |\langle \tilde{B}(t)g(t+\tau), e_k \rangle_Y|^2 d\tau \\ &= \int_0^T \sum_{k \in \mathbb{N}} |\langle \tilde{B}(t)g(t+\tau), e_k \rangle_Y|^2 d\tau \\ &= \int_0^T \left\| \tilde{B}(t)g(t+\tau) \right\|_Y^2 d\tau. \end{aligned}$$

Thus, by (14),  $\|\tilde{B}(t)\|_{\mathcal{L}_2(X \times Y, Y)} \rightarrow 0$  as  $t \rightarrow +\infty$ , which concludes the proof.  $\blacksquare$

## 5 Numerical simulation

We provide a numerical simulation of the adaptive observer (6) in the case of a two-dimensional neural field (namely,  $n = 2$  and  $m = 1$  in Section 3.2) over the unit circle  $\Omega = \mathbb{S}^1$ . We set parameters of the system and observer as in Table 1, so that all assumptions of Corollary 4.3 are satisfied. Initial conditions are given by  $z_1(0) = z_2(0) = 1$ ,  $\hat{z}_1(0) = \hat{z}_2(0) = 0$  and  $\hat{B}_1(0) = \hat{B}_2(0) = 0$ . Kernels are given by Gaussian functions depending on the distance between  $r$  and  $r'$ , as it is frequently assumed in practice (see [7]):  $w_{ij}(r, r') = \omega_{ij} \mathbf{g}(r, r') / \|\mathbf{g}\|_{L^2(\Omega^2; \mathbb{R})}$ ,  $\mathbf{g}(r, r') = \exp(-\sigma|r - r'|^2)$  for constant parameters  $\sigma$  and  $\omega_{ij}$  given in Table 1. The inputs  $u_i$  are chosen as spatiotemporal periodic signals with irrational frequency ratio, i.e.,  $u_i(t, r) = 10^3 \sin(\lambda_i t r)$  with  $\lambda_1/\lambda_2$  irrational. This choice is made to ensure persistency of excitation of the input  $(u_1, u_2)$ , which in practice seems to be sufficient to ensure persistency of excitation of  $(S_1(z_1), S_2(z_2))$ . Note that for  $u_1 = u_2 = 0$ , the persistency of excitation assumption seems to be not guaranteed, hence the observer does not converge. However, in practice, such a persistent input is likely to occur due to exogenous signals coming from other unmodeled neuronal populations.

In numerical simulations, the system is spatially discretized over  $\Omega$  with a constant space step  $\Delta r = 1/20$ , and the resulting ordinary differential equation is solved with an explicit Runge-Kutta (4, 5) method.

$S_i(z) = \tanh(z)$	$\tau_i = 1$	$\lambda_1 = 1$	$\lambda_2 = \sqrt{2}$
$\omega_{11} = 0.1$	$\omega_{12} = 2$	$\omega_{21} = -2$	$\omega_{22} = 2$
$\beta = 1$	$\gamma_1 = 100$	$\gamma_2 = 100$	$\sigma = 60$

Table 1: System and observer parameters for the numerical simulation of Figures 1 and 2

In Figure 1, the convergence of the observer, that is proved in Corollary 4.3, is numerically verified. In Figure 2, we illustrate some iterations of the reconstructed kernel  $\hat{w}_{22}(t)$  of  $\hat{B}_2(t)$ , which converges to the kernel  $w_{22}$  of  $B_2$ .

## 6 Conclusion

In this paper, we have shown that an observer can be designed to estimate online linear operators arising in some nonlinear infinite-dimensional dynamical systems from the measurement of the state. This estimation problem is motivated by an application to kernel reconstruction for neural fields equations. The main assumption is the persistence of excitation of the system along its trajectories. Our simulations suggest that this requirement can be ensured using appropriate exogenous inputs. In future works, we wish to

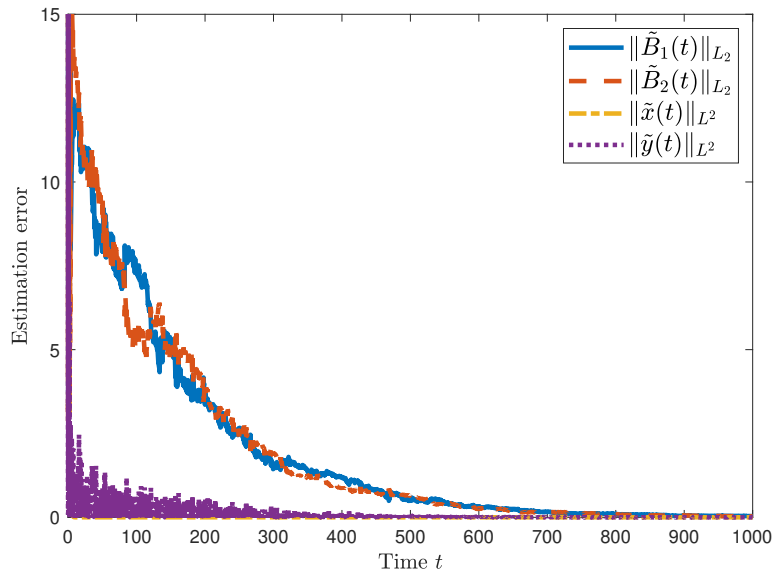


Figure 1: Evolution of the estimation errors  $\|\hat{B}_1(t) - B_1\|_{\mathcal{L}_2(X,Y)}$ ,  $\|\hat{B}_2(t) - B_2\|_{\mathcal{L}_2(X,Y)}$ ,  $\|\hat{x}(t) - x(t)\|_X$  and  $\|\hat{y}(t) - y(t)\|_Y$ , for system and observer parameters given in Table 1.

investigate this hypothesis by either designing inputs ensuring persistence, or designing observers that do not rely on this assumption (see, e.g., [22]). Moreover, the use of this estimator in closed-loop to stabilize the systems by means of dynamic output feedback will be investigated. Finally, delayed neural fields in the form of [7] could be considered, as they do not fit into the functional setting of the present paper although capturing meaningful biological processes such as non-instantaneous axonal propagation.

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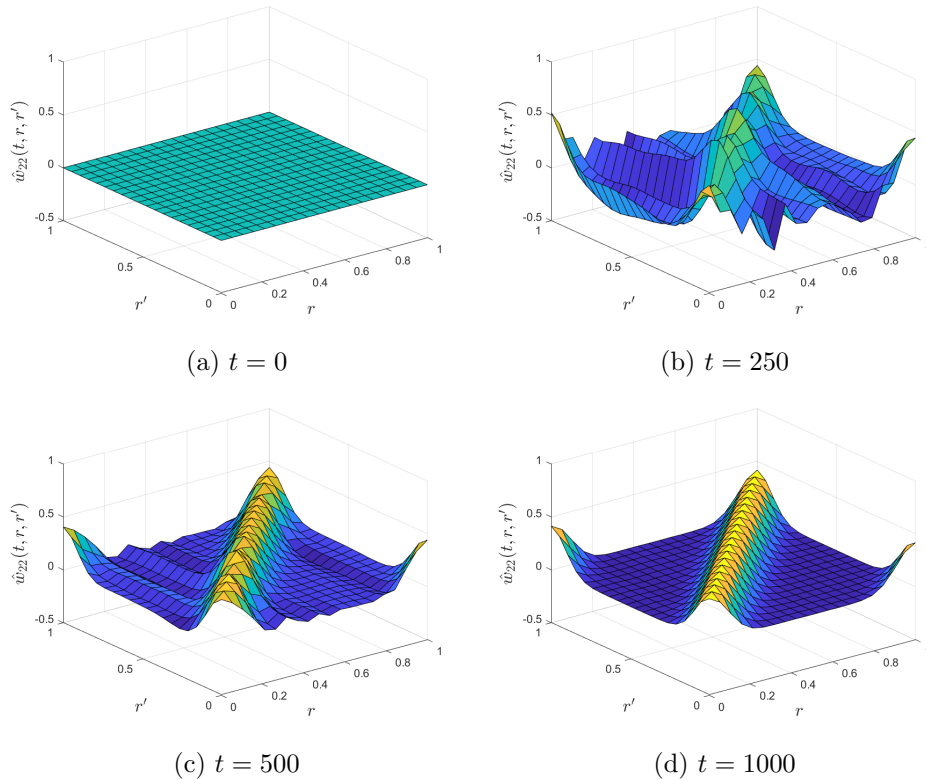


Figure 2: Evolution of the kernel  $\hat{w}_{22}(t, r, r')$  of  $\hat{B}_2(t)$  at times  $t = 0$ ,  $t = 250$ ,  $t = 500$  and  $t = 1000$ .

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