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A SPECTRAL APPROACH TO ESTIMATE THE AUTOCOVARIANCE FUNCTION

CÉLINE LÉVY-LEDUC, PASCAL BONDON, AND VALDÉRIO REISEN

ABSTRACT. This paper proposes two novel alternative estimators for the autocovariance function of a short-range dependent stationary process based on a spectral approach. One is based on the classical periodogram, and the other one uses the M-periodogram. The theoretical properties of both estimators are investigated for linear processes. Monte-Carlo experiments are conducted to compare these estimators to the standard sample autocovariance function in autoregressive moving average (ARMA) models with and without additive outliers. In the non-contaminated scenario, the empirical investigation shows that the three estimators display similar performance. However, in the contaminated case, the estimator based on the M-periodogram remains unaffected in the presence of additive outliers, while the two other estimators are corrupted.

1. INTRODUCTION

The autocovariance function of a stationary process plays a central role in time series analysis. However, it is well known that the classical sample autocovariance function is very sensitive to the presence of additive outliers in the data. A small fraction of additive outliers can indeed alter the performance of the classical autocovariance estimator, see for instance Deutsch et al. (1990); Chan (1992, 1995); Maronna et al. (2006) and the references therein. Since additive outliers are common in practical situations, having an estimator of the autocovariance function which is robust to atypical data may be very helpful. Ma and Genton (2000) introduced such a robust estimator and discussed its performance on simulated and real data, and Lévy-Leduc et al. (2011) established its asymptotic properties. However, it should be pointed out that, although this estimator of the autocovariance function has interesting theoretical properties and displays reasonable performance for estimating the autocovariance of time series with additive outliers, Ma and Genton (2000) observed that their robust autocovariance estimator does not share the non-negative definiteness property with the classical sample autocovariance estimator. Due to the lack of robustness of classical sampling functions in the time domain, estimators in the frequency domain have recently emerged in the literature as alternative approaches to estimate time series models with different correlation structures see, for example, Fajardo et al. (2018); Reisen et al. (2017) and references therein.

Combining tools in the time and frequency domains, this paper proposes two alternative estimators which have the non-negative definiteness property, one of them is also not sensitive to the presence of atypical observations in the data. The theoretical roots of the estimation method discussed here are based on the diagonalization of a real symmetric circulant matrix which is discussed, for example, in (Brockwell and Davis, 1991, Section 4.5). Among many other approaches applying the circulant matrix theory in different data structures, this algebraic tool has been widely used to obtain fast simulation methods, usually called embedding

Key words and phrases. Autocovariance function, additive outliers, M-periodogram.

circulant methods, to generate stationary univariate and multivariate Gaussian time series. This method has emerged in a wide variety of domains see, for example, Percival (1992), Helgason et al. (2011), Coeurjolly and Porcu (2018), among others.

Now, let (Y_t) , $t \in \mathbb{Z}$, be a real valued stationary process with autocovariance function

$$\gamma(h) = \text{Cov}(Y_t, Y_{t+h}) = \mathbb{E}[(Y_t - m)(Y_{t+h} - m)] \quad (1)$$

where $m = \mathbb{E}(Y_t)$. In the following, we assume that γ is absolutely summable, *i.e.*

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty.$$

Then, the spectral density of (Y_t) is defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \gamma(h) e^{-ih\lambda}, \text{ for all } \lambda \in [-\pi, \pi]. \quad (2)$$

The standard estimator of $\gamma(h)$ from the N consecutive observations Y_1, \dots, Y_N is the sample autocovariance $\tilde{\gamma}_N(h)$ defined by

$$\tilde{\gamma}_N(h) = \frac{1}{N} \sum_{t=1}^{N-h} (Y_t - \bar{Y}_N)(Y_{t+h} - \bar{Y}_N), \quad h \in \{0, \dots, N-1\}, \quad (3)$$

where $\bar{Y}_N = N^{-1} \sum_{t=1}^N Y_t$. Note that $\tilde{\gamma}_N$ has the desirable property of non-negative definiteness. Following (Brockwell and Davis, 1991, Proposition 4.5.2), let

$$\begin{aligned} \mathbf{c}_j &= \sqrt{2/N} [1, \cos(\omega_j), \cos(2\omega_j), \dots, \cos((N-1)\omega_j)], \\ \mathbf{s}_j &= \sqrt{2/N} [0, \sin(\omega_j), \sin(2\omega_j), \dots, \sin((N-1)\omega_j)], \end{aligned}$$

for $j = 1, \dots, [N/2]$ where $[N/2]$ is the integer part of $N/2$ and $\omega_j = 2\pi j/N$ is the j th Fourier frequency. With $\mathbf{c}_0 = \sqrt{1/N} [1, 1, 1, \dots, 1]$, we define \mathbf{P}_N by

$$\mathbf{P}_N = \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{c}_{[N/2]} \\ \mathbf{s}_{[N/2]} \end{bmatrix}, \quad (4)$$

when N is odd, and we replace $\mathbf{c}_{N/2}$ by $2^{-1/2} \mathbf{c}_{N/2}$ in the definition of \mathbf{P}_N and drop the last row of \mathbf{P}_N when N is even. Let \mathbf{D}_N be the $N \times N$ matrix defined by

$$\mathbf{D}_N = \begin{cases} \text{diag}\{f(0), f(\omega_1), f(\omega_1), \dots, f(\omega_{[N/2]}), f(\omega_{[N/2]})\} & \text{if } N \text{ is odd,} \\ \text{diag}\{f(0), f(\omega_1), f(\omega_1), \dots, f(\omega_{(N-2)/2}), f(\omega_{(N-2)/2}), f(\omega_{N/2})\} & \text{if } N \text{ is even.} \end{cases} \quad (5)$$

The alternative to $\tilde{\gamma}_N(h)$ that we propose in this paper is defined by

$$\hat{\gamma}_N(h) = e'_{h+1} \hat{\mathbf{\Gamma}}_N e_1, \quad (6)$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)'$, the value 1 being at the i th position,

$$\hat{\mathbf{\Gamma}}_N = 2\pi \mathbf{P}'_N \hat{\mathbf{D}}_N \mathbf{P}_N, \quad (7)$$

and $\widehat{\mathbf{D}}_N$ is obtained by replacing f by an estimator \widehat{f}_N in (5). Many estimators can be proposed. Here, we consider the truncated window periodogram and the truncated window M-periodogram that will be defined below. Note that (7) implies that $\widehat{\mathbf{\Gamma}}_N$ is non-negative definite when the elements of $\widehat{\mathbf{D}}_N$ are non-negative.

The value $I_N(\omega_j)$ of the periodogram at frequency ω_j is defined by

$$I_N(\omega_j) = \frac{1}{2\pi N} \left| \sum_{t=1}^N Y_t \exp(ik\omega_j) \right|^2. \quad (8)$$

Although $I_N(\omega_j)$ is a natural estimator of $f(\omega_j)$, it is well-known that $I_N(\omega_j)$ is not a consistent estimator of $f(\omega_j)$ and that the periodogram needs to be locally averaged to be consistent, see e.g. (Priestley, 1981, Chapter 6) and (Brockwell and Davis, 1991, Chapter 10). The truncated window periodogram at frequency ω_j is given by

$$\widehat{f}_N(\omega_j) = \frac{1}{2m_N + 1} \sum_{k=-m_N}^{m_N} I_N(\omega_{j+k}), \quad (9)$$

where $m_N \rightarrow \infty$ and $m_N/N \rightarrow 0$ as $N \rightarrow \infty$. Plugging (9) into (7) gives the estimator $\widehat{\gamma}_N(h)$. Its consistency is established in Theorem 1.

In the following, another strategy for estimating the spectral density is proposed. It is based on the M-periodogram I_N^M defined in (12) which should thus lead to a robust estimator of the autocovariance function. As it is well known, M-estimation is indeed an alternative robust procedure to least-square and this estimation method has recently been the subject of intensive research for modelling linear models with absolutely summable autocovariance processes, such as ARMA models, see for example Wu (2007), Li (2010) and Li (2008) among others.

Following these ideas, Reisen et al. (2017) among others, proposed an alternative way to derive the periodogram function $I_N(\omega_j)$ by observing that the periodogram can be defined through the least-square estimates of a two-dimensional vector $\boldsymbol{\beta}' = (\beta^{(1)}, \beta^{(2)})$ in the linear regression model

$$Y_i = \beta^{(1)} \cos(i\omega_j) + \beta^{(2)} \sin(i\omega_j) + \varepsilon_i, \quad 1 \leq i \leq N, \quad (10)$$

where ε_i denotes the deviation of Y_i from $\beta^{(1)} \cos(i\omega_j) + \beta^{(2)} \sin(i\omega_j)$, $\mathbb{E}[\varepsilon_i] = 0$ and $\mathbb{E}[\varepsilon_i^2] < \infty$. More precisely, we have

$$I_N(\omega_j) = \frac{N}{8\pi} \|\widetilde{\boldsymbol{\beta}}_N(\omega_j)\|^2$$

where $\|\cdot\|$ denotes the Euclidean norm and $\widetilde{\boldsymbol{\beta}}_N(\omega_j) \in \mathbb{R}^2$ is the least-squares estimator of $\boldsymbol{\beta}$ defined by

$$\widetilde{\boldsymbol{\beta}}_N(\omega_j) = \underset{\boldsymbol{\beta} \in \mathbb{R}^2}{\operatorname{argmin}} \sum_{t=1}^N (Y_t - c'_t(\omega_j)\boldsymbol{\beta})^2, \quad (11)$$

where $c_t(\omega_j) = (\cos(t\omega_j), \sin(t\omega_j))'$. An L_p -norm periodogram was proposed by Li (2008, 2010) by replacing the L_2 -norm in (11) with an L_p -norm for $p \in [1, 2)$. Here, we replace the L_2 -norm with the Huber loss function. The so-called M-periodogram is defined by

$$I_N^M(\omega_j) = \frac{N}{8\pi} \|\widehat{\boldsymbol{\beta}}_N(\omega_j)\|^2 \quad (12)$$

where $\widehat{\boldsymbol{\beta}}_N(\omega_j) \in \mathbb{R}^2$ is given by

$$\widehat{\boldsymbol{\beta}}_N(\omega_j) = \operatorname{argmin}_{\boldsymbol{\beta} \in \mathbb{R}^2} \sum_{t=1}^N \rho(Y_t - c'_t(\omega_j)\boldsymbol{\beta}), \quad (13)$$

with $\rho(x) = (x^2 \mathbb{1}_{|x| \leq c})/2 + c(|x| - c/2) \mathbb{1}_{|x| > c}$, $c > 0$. The derivative of $\rho(x)$ is

$$\psi(x) = \max[\min(x, c), -c]. \quad (14)$$

Replacing $I_N(\omega_{j+k})$ by $I_N^M(\omega_{j+k})$ in (9), we obtained the truncated window M-periodogram $\widehat{f}_N^M(\omega_j)$ defined by

$$\widehat{f}_N^M(\omega_j) = \frac{1}{2m_N + 1} \sum_{k=-m_N}^{m_N} I_N^M(\omega_{j+k}), \quad (15)$$

where m_N tends to infinity and is such that $m_N/N \rightarrow 0$, as N tends to infinity.

Similarly as in (6), the corresponding autocovariance estimator $\widehat{\gamma}_N^M(h)$ is defined as follows:

$$\widehat{\gamma}_N^M(h) = e'_{h+1} \widehat{\boldsymbol{\Gamma}}_N^M e_1, \quad (16)$$

with

$$\widehat{\boldsymbol{\Gamma}}_N^M = 2\pi \mathbf{P}'_N \widehat{\mathbf{D}}_N^M \mathbf{P}_N, \quad (17)$$

where \mathbf{P}_N is defined in (4) and $\widehat{\mathbf{D}}_N^M$ is obtained by replacing f in (5) by $\widehat{f}_N^M(\omega_j)$ defined in (15). The asymptotic properties of $\widehat{\gamma}_N^M(h)$ defined by (16) are established in Theorem 2.

The paper is organized as follows. Theorems 1 and 2 are presented in Section 2. In Section 3, the finite sample size properties of $\widetilde{\gamma}_N(h)$, $\widehat{\gamma}_N(h)$ and $\widehat{\gamma}_N^M(h)$ are investigated in ARMA models with and without atypical observations. Concluding remarks are given in Section 4.

2. THEORETICAL RESULTS

In Theorems 1 and 2, (Y_t) is assumed to be a linear strictly stationary process defined by

$$Y_t = \sum_{j=0}^{\infty} a_j \eta_{t-j}, \quad (18)$$

where the constants a_j are real numbers and the random variables η_t are independent, identically distributed (i.i.d) and zero-mean. In Theorem 1 below, we prove that $\widehat{\gamma}_N(h)$ is a consistent estimator of $\gamma(h)$ when the sequence $(a_j \sqrt{j})$ is absolutely summable and the η_t 's have finite fourth moment, and that $\widehat{\gamma}_N(h)$ is a $\sqrt{m_N}$ -consistent estimator of $\gamma(h)$ where m_N is defined in (15) when $(a_j j^2)$ is absolutely summable. In Theorem 2, the sequence (a_j) is only assumed to be absolutely summable but the η_t 's are supposed to be Gaussian. The proofs of these theorems are given in Section 5.

Theorem 1. *Assume that (Y_t) satisfies (18) where (η_t) are i.i.d. zero-mean random variables and $\mathbb{E}(\eta_t^4) < \infty$.*

(i) Assume that $\sum_{j=0}^{\infty} |a_j| j^{1/2} < \infty$. Then, for each $h \in \{0, 1, \dots\}$, $\widehat{\gamma}_N(h)$ defined by (6) where $m_N \rightarrow \infty$ and $m_N/N \rightarrow 0$, converges in probability to $\gamma(h)$ defined by (1) as N tends to infinity.

(ii) Assume that $\sum_{j=0}^{\infty} |a_j| j^2 < \infty$. Then, for each $h \in \{0, 1, \dots\}$, $\widehat{\gamma}_N(h)$ defined by (6) where $m_N \rightarrow \infty$ and $m_N/N \rightarrow 0$, satisfies

$$\widehat{\gamma}_N(h) - \gamma(h) = O_P\left(\frac{1}{\sqrt{m_N}}\right), \text{ as } N \rightarrow \infty.$$

Assume that (Y_t) is a zero-mean Gaussian stationary process with autocovariance function γ . Since ψ defined by (14) is measurable, $(\psi(Y_t))$ is a strictly stationary process. Since ψ is bounded, $(\psi(Y_t))$ has finite moments and $\mathbb{E}(\psi(Y_t)) = 0$ for all t , since ψ is odd. Let

$$\gamma_\psi(h) = \text{Cov}(\psi(Y_t), \psi(Y_{t+h})) = \mathbb{E}(\psi(Y_t)\psi(Y_{t+h})). \quad (19)$$

Observe that

$$\gamma_\psi(h) = \mathbb{E}\left(\psi\left(\sigma \frac{Y_t}{\sigma}\right) \psi\left(\sigma \frac{Y_{t+h}}{\sigma}\right)\right) = \mathbb{E}\left(\psi_\sigma\left(\frac{Y_t}{\sigma}\right) \psi_\sigma\left(\frac{Y_{t+h}}{\sigma}\right)\right),$$

where $\sigma = \mathbb{E}(Y_t^2)^{1/2}$ and $\psi_\sigma(x) = \psi(\sigma x)$.

Furthermore, let

$$\psi_\sigma(x) = \sum_{p \geq 0} \frac{c_p}{p!} H_p(x),$$

denote the Hermite expansion of ψ_σ where H_p is the p th Hermite polynomial and

$$c_p = \mathbb{E}(\psi_\sigma(X) H_p(X)),$$

X being a standard Gaussian random variable. Since ψ_σ is odd, $c_0 = 0$ and thus

$$\psi_\sigma(x) = \sum_{p \geq 1} \frac{c_p}{p!} H_p(x). \quad (20)$$

Thus

$$\gamma_\psi(h) = \sum_{p \geq 1} \sum_{q \geq 1} \frac{c_p c_q}{p! q!} \mathbb{E}\left(H_p\left(\frac{Y_t}{\sigma}\right) H_q\left(\frac{Y_{t+h}}{\sigma}\right)\right).$$

By using the Mehler formula given in Breuer and Major (1983), we obtain that

$$\gamma_\psi(h) = \sum_{p \geq 1} \frac{c_p^2}{p!} \rho(h)^p \leq |\rho(h)| \sum_{p \geq 1} \frac{c_p^2}{p!} = |\rho(h)| \mathbb{E}(\psi_\sigma(X)^2) \leq c \left| \frac{\gamma(h)}{\gamma(0)} \right|, \quad (21)$$

where $\rho(h)$ denotes the correlation function of (Y_t) , the first inequality coming from the fact that $|\rho(h)| \leq 1$ for all h and the second from the fact that ψ is bounded by c . Hence $(\gamma_\psi(h))$ is absolutely summable and the spectral density of $(\psi(Y_t))$ can be defined by

$$f_\psi(\lambda) = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \gamma_\psi(h) e^{-ih\lambda}, \text{ for all } \lambda \in [-\pi, \pi]. \quad (22)$$

Theorem 2. Assume that (Y_t) satisfies (18) where $\sum_{j=0}^{\infty} |a_j| < \infty$ and (η_t) are i.i.d. standard Gaussian random variables. Then, for each $h \in \{0, 1, \dots\}$, $\widehat{\gamma}_N^M(h)$ defined in (16) converges in probability to $\gamma_\psi(h)/\kappa^2$ as N tends to infinity, where $\gamma_\psi(h)$ is defined in (19) and $\kappa = 2\Phi(c) - 1$, Φ being the cumulative distribution function of a standard Gaussian random variable and c is the constant appearing in (14).

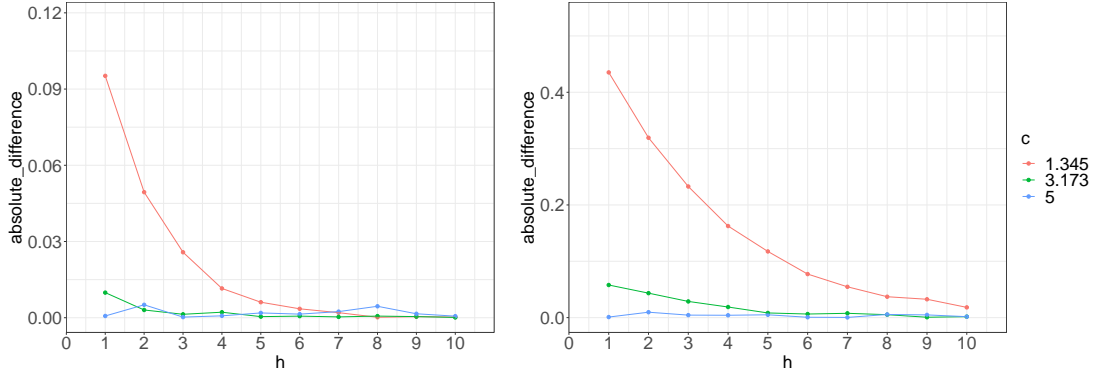


FIGURE 1. Absolute difference of $\gamma(h)$ and $\gamma_\psi(h)/\kappa^2$ for $\phi = 0.5$ (left) and $\phi = 0.7$ (right).

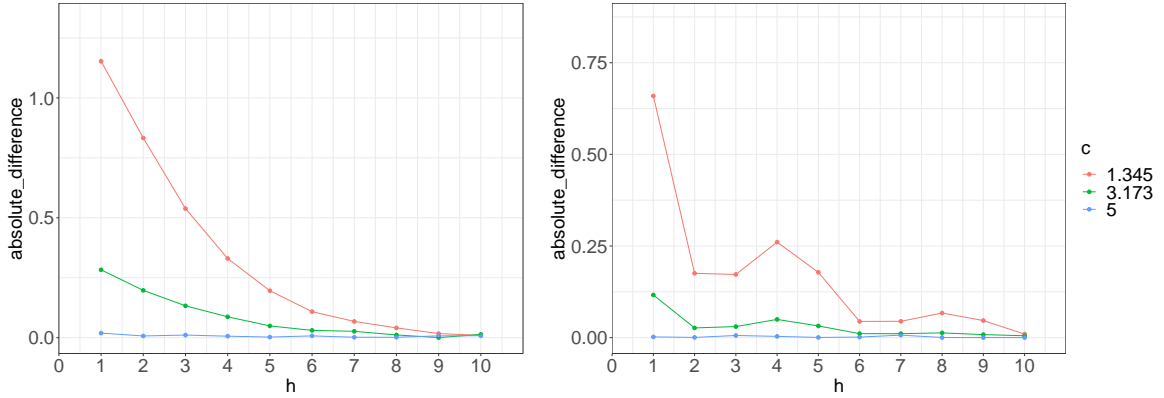


FIGURE 2. Absolute difference of $\gamma(h)$ and $\gamma_\psi(h)/\kappa^2$ in the AR(2) case for $(\phi_1, \phi_2) = (1, -0.25)$ (left) and $(\phi_1, \phi_2) = (1, -0.5)$ (right).

Remark 1. According to Theorem 2, for a Gaussian linear process, $\widehat{\gamma}_N^M(h)$ is a consistent estimator of $\gamma_\psi(h)/\kappa^2$. In Figure 1, the absolute difference between $\gamma(h)$ and $\gamma_\psi(h)/\kappa^2$ are displayed for different values of c when (Y_t) is a Gaussian AR(1) process defined by

$$Y_t = \phi Y_{t-1} + \eta_t, \quad (23)$$

where $\phi = 0.5, 0.7$ and (η_t) are i.i.d. standard Gaussian random variables. In this case, $\gamma(h) \propto \phi^{|h|}$. This figure shows that $\gamma_\psi(h)/\kappa^2$ is very close to $\gamma(h)$ independently of the values of $\gamma(h)$ when c is large enough. In Figure 2, the same quantities are displayed when (Y_t) is a Gaussian AR(2) process defined by

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \eta_t, \quad (24)$$

where (η_t) are i.i.d. standard Gaussian random variables and $(\phi_1, \phi_2) = (1, -0.25)$ and $(\phi_1, \phi_2) = (1, -0.5)$. We can also see from this figure that $\gamma_\psi(h)/\kappa^2$ is very close to $\gamma(h)$ when $c \geq 5$ independently of the values of h .

3. NUMERICAL EXPERIMENTS

In order to investigate the empirical properties of the proposed estimators, we considered finite sample sizes from AR Gaussian processes (Y_t) defined by (23) and (24) for the AR(1) and AR(2) models, respectively. The contaminated process (Z_t) is defined by

$$Z_t = Y_t + \omega\delta_t, \quad (25)$$

where ω is the magnitude of the additive outliers and (δ_t) are i.i.d random variables such that $\mathbb{P}(\delta_t = -1) = \mathbb{P}(\delta_t = 1) = p/2$, $\mathbb{P}(\delta_t = 0) = 1 - p$ with $0 < p < 1$. It follows that $\mathbb{E}(\delta_t) = 0$ and $\text{Var}(\delta_t) = p$. In (25), (Y_t) and (δ_t) are assumed to be independent.

To compare the performance of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ with and without atypical observations, $N = 1000$ observations Y_1, \dots, Y_N were generated from (23) with $\phi = 0.5, 0.7$ and from (24) with $(\phi_1, \phi_2) = (1, -0.25)$ and $(\phi_1, \phi_2) = (1, -0.5)$. The contaminated observations Z_1, \dots, Z_N were generated from (25) with $\omega = 20$, $p = 15\%$, 20% . The truncated window periodograms $\hat{f}_N(\omega_j)$ and $\hat{f}_N^M(\omega_j)$ were computed with $m_N = 25, 50, 100$ in (9) and (15). The three values $c = 1.345, 3.173, 5$ in (14) were considered. The numerical experiments were repeated 100 times and Figures 3 to 10 plot the averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ obtained from 100 replications. These figures also display the values of $\gamma(h)$ and $\gamma_\psi(h)/\kappa^2$.

In Figures 3 to 6, the estimators were calculated from the uncontaminated data Y_1, \dots, Y_N . The associated confidence intervals are displayed where the upper (resp. lower) bounds were obtained by adding (resp. subtracting) 1.96 times the standard deviation to the average. These figures show that the three estimators have similar performance and are very close to $\gamma(h)$ and $\gamma_\psi(h)/\kappa^2$ when c and m_N are well-chosen. The best results are obtained when $m_N = 25$ and $c = 3.173$, independently of the values of $\gamma(h)$.

In Figures 7 to 10, the estimators were calculated from the contaminated data Z_1, \dots, Z_N . We observe that the performance of $\tilde{\gamma}_N(h)$ and $\hat{\gamma}_N(h)$ are altered, while $\hat{\gamma}_N^M(h)$ is still reliable for well-chosen values of c and m_N . More precisely, even with 20% of atypical observations, namely 200 very large observations with respect to 800 remaining data, $\hat{\gamma}_N^M(h)$ can be considered as a competitive estimator of $\gamma(h)$, especially when $c = 3.173$ and m_N is smaller than 50. Note that we did not display the confidence intervals in these figures because the estimations obtained with $\tilde{\gamma}_N(h)$ and $\hat{\gamma}_N(h)$ are so unstable that the length of the confidence intervals is very large.

4. CONCLUSION AND FUTURE WORK

In this paper, two new estimators for the autocovariance function of short-range dependent stationary processes are proposed. Both of them are based on the diagonalization of a real symmetric circulant matrix and use spectral density estimators, namely the classical periodogram and the M-periodogram. Some consistency properties of these two estimators are established. Their performance are also investigated through Monte-Carlo simulations for different ARMA processes using contaminated with additive outliers and outlier-free time series. Our estimators display similar performance to the classical autocovariance estimator in the context of non-contaminated data. In the presence of additive outliers, the autocovariance estimator based on the M-periodogram is unaffected while the classical autocovariance estimator and the estimator based on the periodogram are altered. Therefore, the estimator based on the M-periodogram is an alternative method to estimate the autocovariance function when dealing with short-range dependent time series with and without outliers.

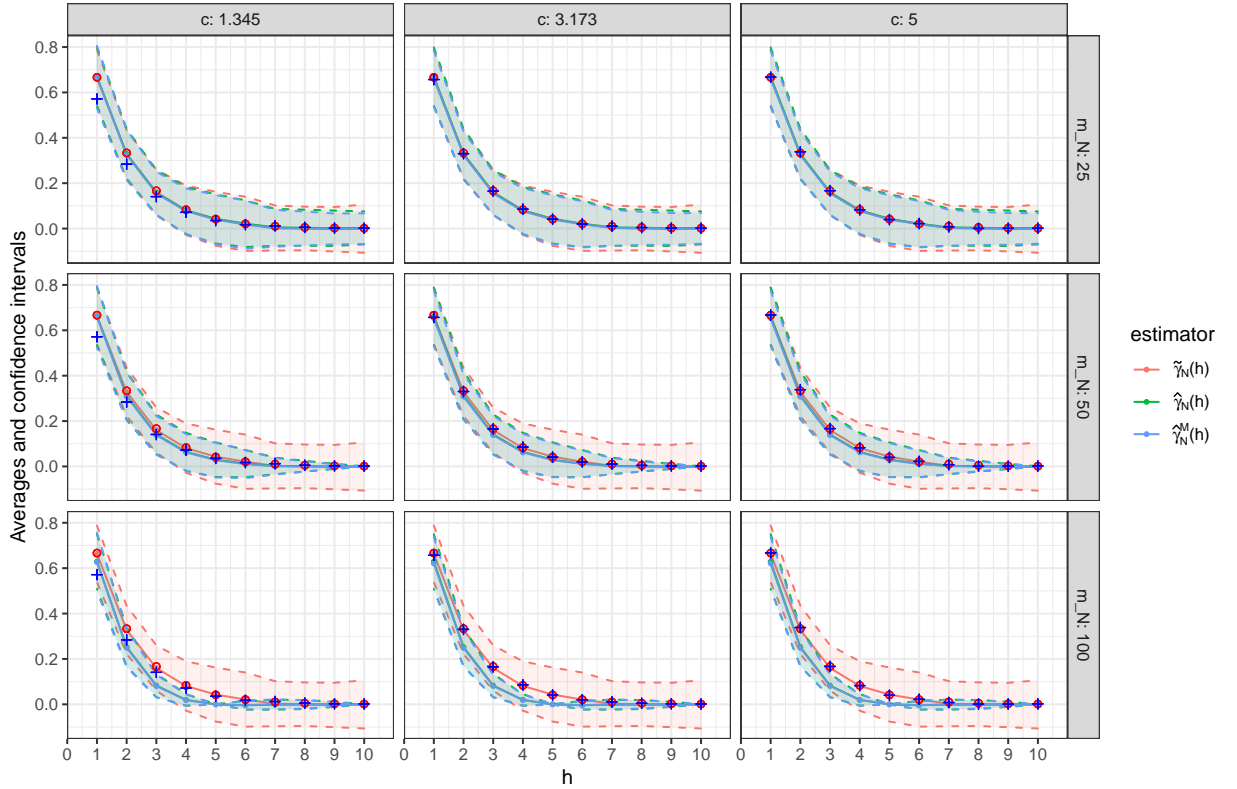


FIGURE 3. Averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ and their associated confidence intervals in the AR(1) case where $\phi = 0.5$, $\gamma(h)$ with red circles and $\gamma_\psi(h)/\kappa^2$ with blue crosses.

For future work, one idea is to use alternative spectral estimators to the ones proposed here, such as the windowed tapered or multitaper spectral estimators. As explained in Walden (2000) and Percival and Walden (2020), these estimators suffer less from leakage and would probably improve the estimation of the autocovariance function, especially when the process has a large dynamic range or when the number of samples is small. Additionally, it will be interesting to explore the impact of correlated contaminated processes on the performance of the robust autocovariance estimator defined in (16).

5. PROOFS

Lemma 1. *Assume that (Y_t) satisfies (18) where $\sum_{j=0}^{\infty} |a_j|j^{1/2} < \infty$ and (η_t) are i.i.d. centered random variables with finite second moment. Let \mathbf{P}_N and \mathbf{R}_N be defined in (4) and (38), respectively. Then, for each h in $\{0, 1, \dots\}$,*

$$e'_{h+1} \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N e_1 = o\left(\frac{1}{\sqrt{N}}\right), \text{ as } N \rightarrow \infty,$$

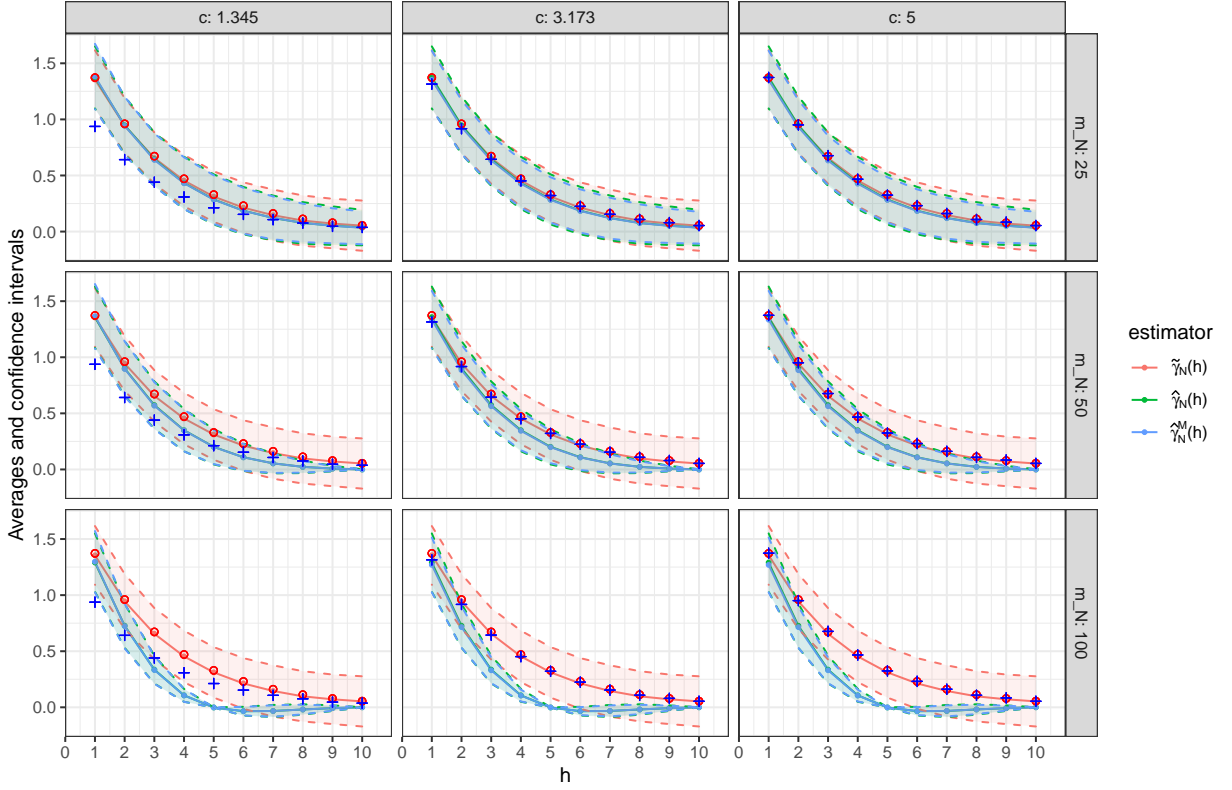


FIGURE 4. Averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ and their associated confidence intervals in the AR(1) case where $\phi = 0.7$, $\gamma(h)$ with red circles and $\gamma_\psi(h)/\kappa^2$ with blue crosses.

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)'$, the value 1 being at the i th position, if ψ appearing in (38) is equal to identity. If ψ is defined by (14) then the results holds if the η_t 's are assumed to be i.i.d. standard Gaussian random variables.

Proof of Lemma 1. By (38),

$$e'_{h+1} \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N e_1 = e'_{h+1} (\mathbf{\Gamma}_N^\psi - 2\pi \mathbf{P}'_N \mathbf{D}_N^\psi \mathbf{P}_N) e_1 = \gamma_\psi(h) - 2\pi e'_{h+1} \mathbf{P}'_N \mathbf{D}_N^\psi \mathbf{P}_N e_1.$$

Let $\mathbf{\Gamma}_N^{\psi, (s)}$ be the symmetric circulant matrix defined by

$$\mathbf{\Gamma}_N^{\psi, (s)} = \begin{bmatrix} \gamma_\psi(0) & \gamma_\psi(1) & \gamma_\psi(2) & \cdots & \gamma_\psi(2) & \gamma_\psi(1) \\ \gamma_\psi(1) & \gamma_\psi(0) & \gamma_\psi(1) & \cdots & \gamma_\psi(3) & \gamma_\psi(2) \\ \gamma_\psi(2) & \gamma_\psi(1) & \gamma_\psi(0) & \cdots & \gamma_\psi(4) & \gamma_\psi(3) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma_\psi(1) & \gamma_\psi(2) & \gamma_\psi(3) & \cdots & \gamma_\psi(1) & \gamma_\psi(0) \end{bmatrix}.$$

For each $h \in \{0, \dots, [N/2]\}$, we have

$$e'_{h+1} \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N e_1 = e'_{h+1} (\mathbf{\Gamma}_N^{\psi, (s)} - 2\pi \mathbf{P}'_N \mathbf{D}_N^\psi \mathbf{P}_N) e_1.$$

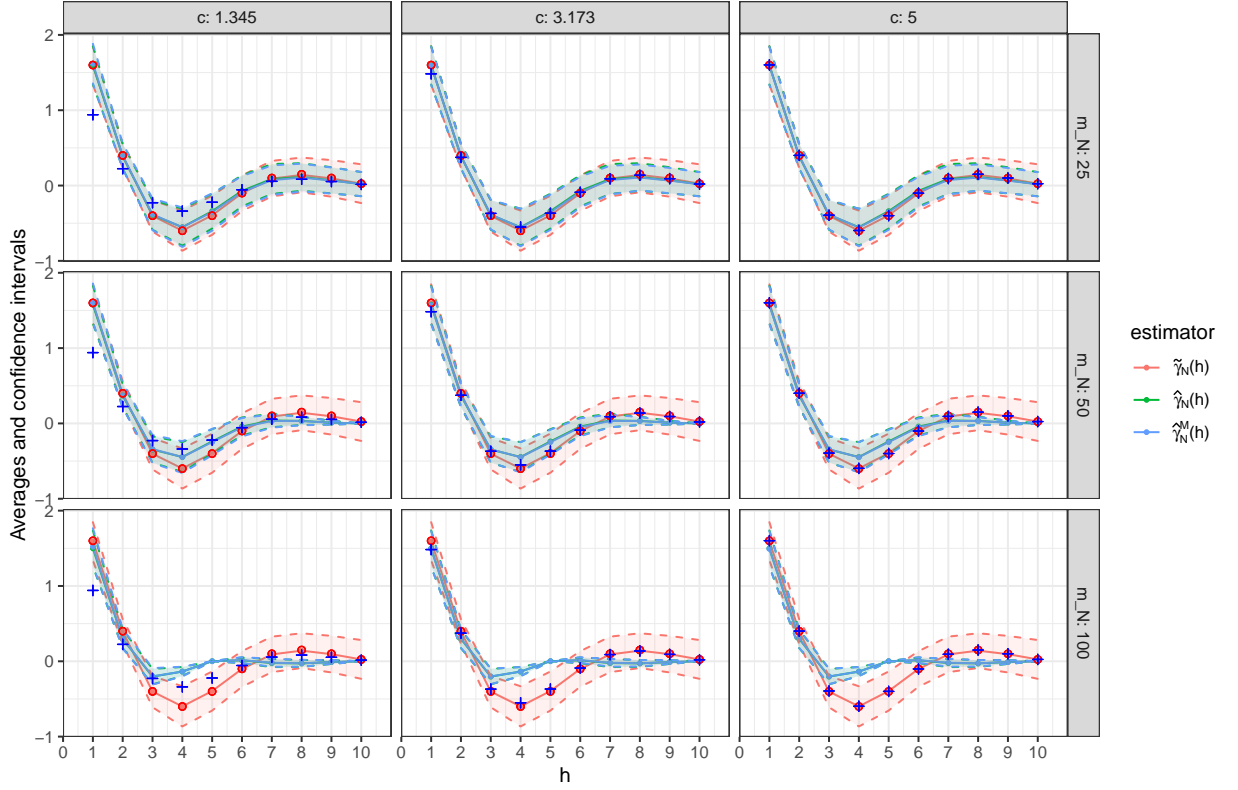


FIGURE 5. Averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ and their associated confidence intervals in the AR(2) case where $(\phi_1, \phi_2) = (1, -0.5)$, $\gamma(h)$ with red circles and $\gamma_\psi(h)/\kappa^2$ with blue crosses.

Let $\Lambda^{(s)} = \text{diag}(\lambda_0, \lambda_1, \lambda_1, \dots, \lambda_{[N/2]}, \lambda_{[N/2]})$ if N is odd, and the last row of $\Lambda^{(s)}$ is dropped if N is even, where

$$\lambda_j = \begin{cases} \sum_{|h| \leq [N/2]} \gamma_\psi(h) e^{-\frac{i2\pi jh}{N}} & \text{if } N \text{ is odd,} \\ \sum_{|h| < N/2} \gamma_\psi(h) e^{-\frac{i2\pi jh}{N}} + \gamma_\psi(N/2) e^{-i\pi j} & \text{if } N \text{ is even.} \end{cases}$$

It follows from (Brockwell and Davis, 1991, Equation (4.5.5)), that $\Gamma_N^{\psi, (s)} = \mathbf{P}'_N \Lambda^{(s)} \mathbf{P}_N$. Thus,

$$e'_{h+1} \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N e_1 = e'_{h+1} \mathbf{P}'_N (\Lambda^{(s)} - 2\pi \mathbf{D}_N^\psi) \mathbf{P}_N e_1 = \sum_{k=1}^N (\mathbf{P}_N)_{k, h+1} (\Lambda^{(s)} - 2\pi \mathbf{D}_N^\psi)_{k, k} (\mathbf{P}_N)_{k, 1}.$$

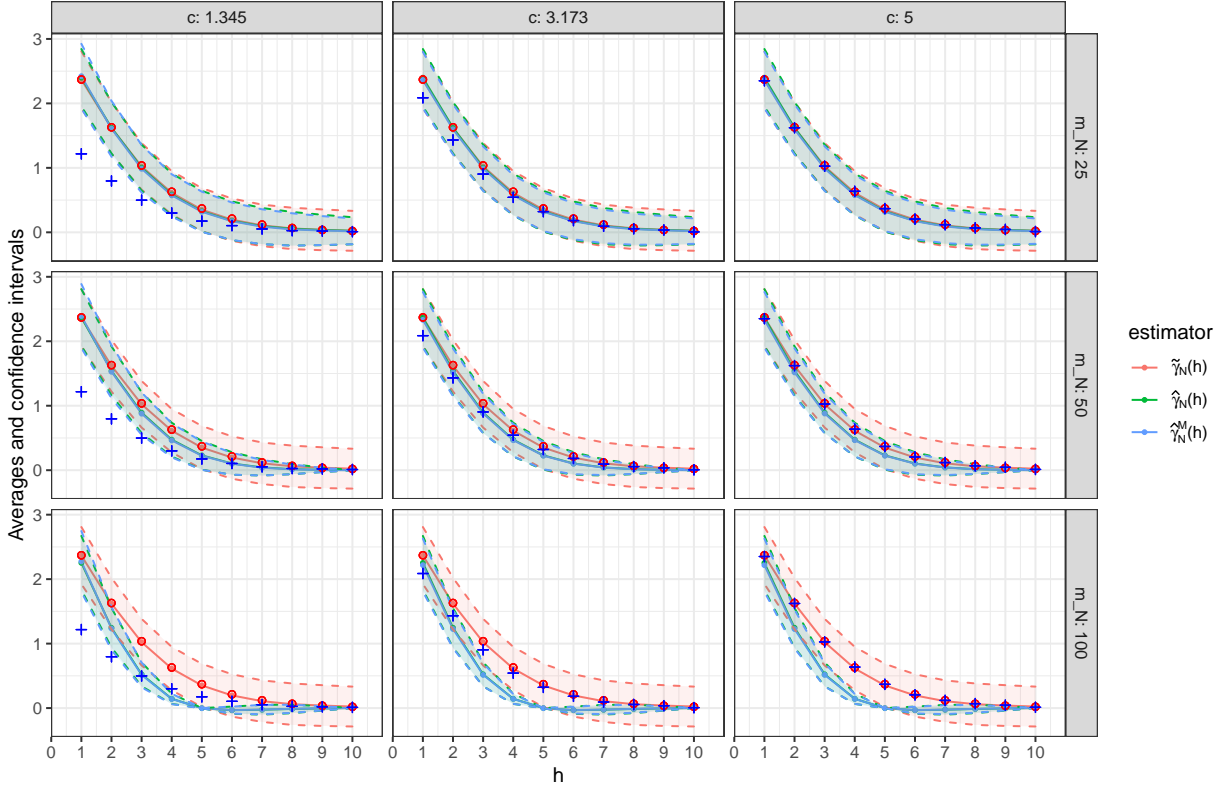


FIGURE 6. Averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ and their associated confidence intervals in the AR(2) case where $(\phi_1, \phi_2) = (1, -0.25)$, $\gamma(h)$ with red circles and $\gamma_\psi(h)/\kappa^2$ with blue crosses.

Since $|(P_N)_{i,j}| \leq \sqrt{2/N}$ for all i, j , we have

$$\begin{aligned}
 |e'_{h+1} P'_N R_N P_N e_1| &\leq \frac{2}{N} \sum_{k=1}^N \left| (\Lambda^{(s)} - 2\pi D_N^\psi)_{k,k} \right| \\
 &\leq 2 \sup_{1 \leq k \leq N} \left| (\Lambda^{(s)} - 2\pi D_N^\psi)_{k,k} \right| \\
 &\leq 2 \sum_{|h| \geq [N/2]} |\gamma_\psi(h)|,
 \end{aligned}$$

where the third inequality comes from the definition of D_N^ψ and (22). According to (21), $\sum_{h \in \mathbb{Z}} |\gamma_\psi(h)| < C \sum_{h \in \mathbb{Z}} |\gamma(h)| < \infty$, where C is a constant, thus $e'_{h+1} P'_N R_N P_N e_1 \rightarrow 0$ as $N \rightarrow \infty$. Note that if $\psi = \text{Id}$, it follows from (19) that $\gamma_\psi(h) = \gamma(h)$, thus the previous result also holds.

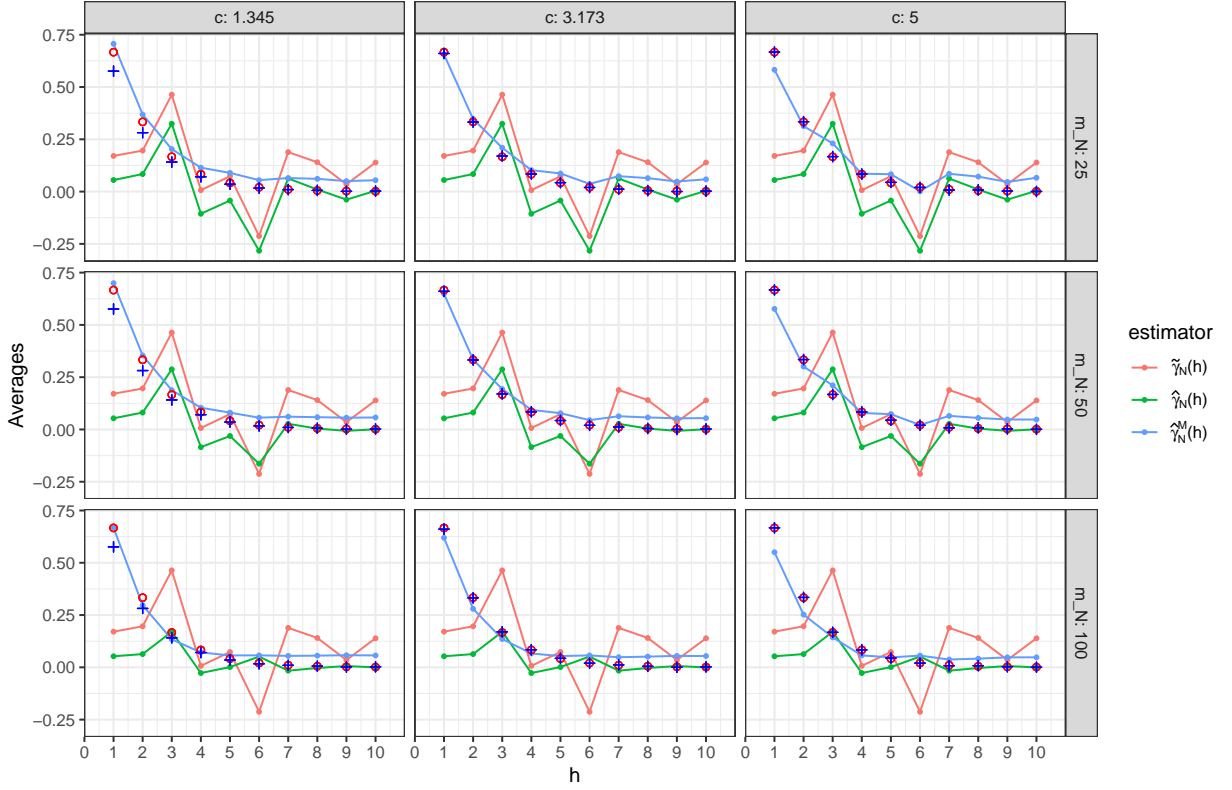


FIGURE 7. Averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ for $\phi = 0.5$, $p = 15\%$, $\omega = 20$, $\gamma(h)$ with red circles and $\gamma_\psi(h)/\kappa^2$ with blue crosses.

Moreover, since $\sum_{j \geq 0} \sqrt{j} |a_j| < \infty$ and $\gamma(h) = \sigma^2 \sum_{j \geq 0} a_j a_{j+h}$, where σ^2 is the variance of the η_k 's,

$$\begin{aligned} \sum_{|h| \geq [N/2]} |\gamma(h)| &= 2 \sum_{h \geq [N/2]} |\gamma(h)| \leq 2\sigma^2 \sum_{j \geq 0} \frac{|a_j|}{\sqrt{j + [N/2]}} \sum_{h \geq [N/2]} \sqrt{j+h} |a_{j+h}| \\ &\leq \frac{2\sigma^2}{\sqrt{[N/2]}} \left(\sum_{j \geq 0} |a_j| \right) \left(\sum_{\ell \geq [N/2]} \sqrt{\ell} |a_\ell| \right) = o\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

□

Proof of Theorem 1. Let

$$\mathbf{\Gamma}_N = (\gamma(i-j))_{1 \leq i, j \leq N}, \quad (26)$$

where γ is defined in (1). By (Brockwell and Davis, 1991, Proposition 4.5.2),

$$\mathbf{\Gamma}_N = 2\pi \mathbf{P}'_N \mathbf{D}_N \mathbf{P}_N + \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N, \quad (27)$$

where \mathbf{P}_N and \mathbf{D}_N are defined by (4) and (5), respectively, and \mathbf{R}_N converges to zero uniformly as $N \rightarrow \infty$, i.e., $\sup_{1 \leq i, j \leq N} (\mathbf{R}_N)_{(i,j)} \rightarrow 0$. For any $h \in \{0, \dots, N-1\}$, it follows from

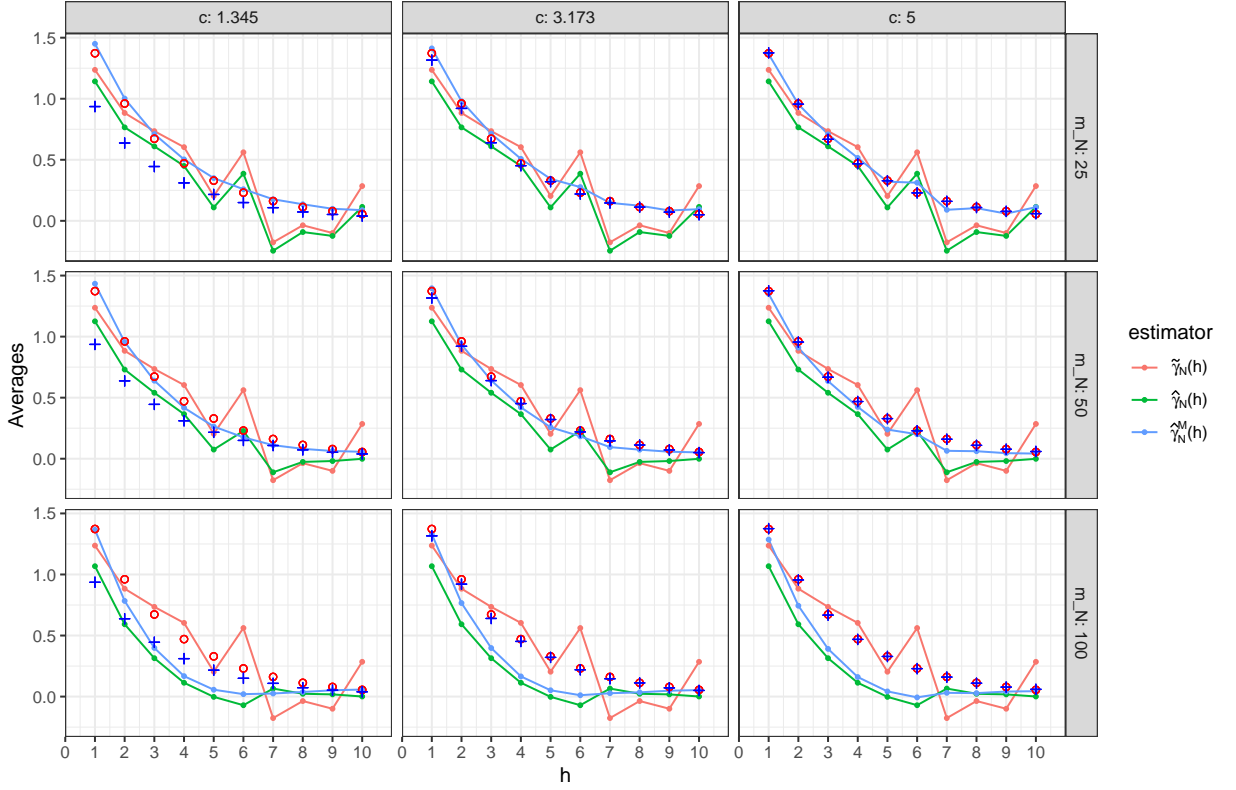


FIGURE 8. Averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ for $\phi = 0.7$, $p = 15\%$, $\omega = 20$, $\gamma(h)$ with red circles and $\gamma_\psi(h)/\kappa^2$ with blue crosses.

(6), (7), (26) and (27) that,

$$\begin{aligned} \hat{\gamma}_N(h) - \gamma(h) &= e'_{h+1} \hat{\Gamma}_N e_1 - e'_{h+1} \Gamma_N e_1 \\ &= 2\pi e'_{h+1} \mathbf{P}'_N (\hat{\mathbf{D}}_N - \mathbf{D}_N) \mathbf{P}_N e_1 - e'_{h+1} \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N e_1. \end{aligned}$$

Taking $\psi = \text{Id}$ in Lemma 1, we have $e'_{h+1} \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N e_1 = o\left(\frac{1}{\sqrt{N}}\right)$, as $N \rightarrow \infty$ and thus tends to zero. Hence, to prove (i) of Theorem 1, it is enough to prove that $e'_{h+1} \mathbf{P}'_N (\hat{\mathbf{D}}_N - \mathbf{D}_N) \mathbf{P}_N e_1 = o_P(1)$ as $N \rightarrow \infty$. Observe that

$$\begin{aligned} e'_{h+1} \mathbf{P}'_N (\hat{\mathbf{D}}_N - \mathbf{D}_N) \mathbf{P}_N e_1 &= \\ &\begin{cases} \frac{1}{N} (\hat{f}_N(0) - f(0)) + \frac{2}{N} \sum_{j=1}^{\lfloor N/2 \rfloor} (\hat{f}_N(\omega_j) - f(\omega_j)) \cos(h\omega_j) & \text{if } N \text{ is odd,} \\ \frac{1}{N} (\hat{f}_N(0) - f(0)) + \frac{2}{N} \sum_{j=1}^{N/2-1} (\hat{f}_N(\omega_j) - f(\omega_j)) \cos(h\omega_j) \\ + \frac{1}{N} (\hat{f}_N(\pi) - f(\pi)) \cos(h\pi) & \text{if } N \text{ is even.} \end{cases} \end{aligned} \quad (28)$$

By (Brockwell and Davis, 1991, Remark 1, page 353), \hat{f}_N converges in mean square to f uniformly on $[-\pi, \pi]$ as $N \rightarrow \infty$. Then $\hat{f}_N(0)$ and $\hat{f}_N(\pi)$ converge in mean square to $f(0)$ and

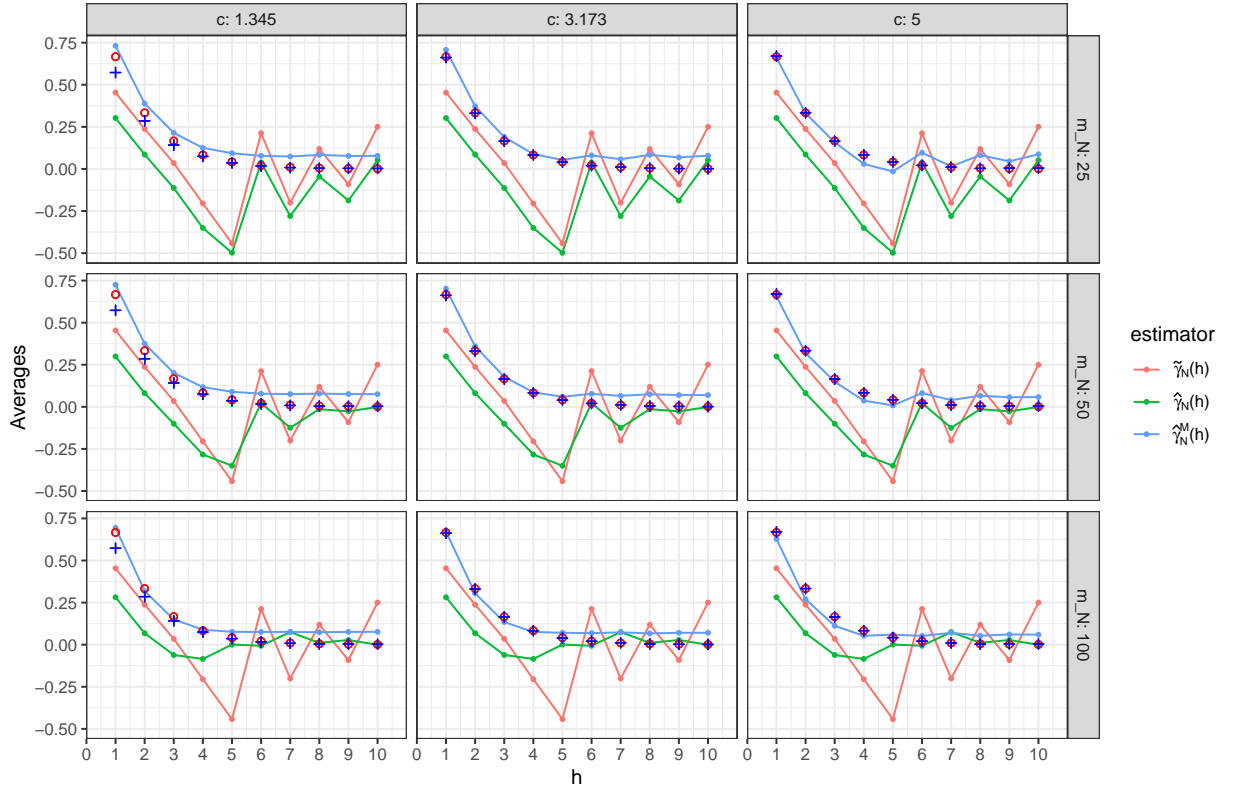


FIGURE 9. Averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ for $\phi = 0.5$, $p = 20\%$, $\omega = 20$, $\gamma(h)$ with red circles and $\gamma_\psi(h)/\kappa^2$ with blue crosses.

$f(\pi)$, respectively, as $N \rightarrow \infty$. Finally,

$$\left| \frac{2}{N} \sum_{j=1}^{\lfloor N/2 \rfloor} (\hat{f}_N(\omega_j) - f(\omega_j)) \cos(h\omega_j) \right| \leq \frac{2}{N} \sum_{j=1}^{\lfloor N/2 \rfloor} |\hat{f}_N(\omega_j) - f(\omega_j)| \leq \sup_{-\pi \leq \omega \leq \pi} |\hat{f}_N(\omega) - f(\omega)|$$

where the right-hand side converges in mean square to 0 as $N \rightarrow \infty$.

To prove (ii) of Theorem 1, it is enough to show that

$$e'_{h+1} \mathbf{P}'_N (\hat{\mathbf{D}}_N - \mathbf{D}_N) \mathbf{P}_N e_1 = O_P \left(\frac{1}{\sqrt{m_N}} \right), \text{ as } N \rightarrow \infty.$$

By (28) and the Markov inequality, it is enough to prove that

$$\frac{m_N}{N^2} \sup_{1 \leq j \leq N/2} \mathbb{E} \left[\left(\hat{f}_N(\omega_j) - f(\omega_j) \right)^2 \right] = O(1)$$

and

$$\frac{m_N}{N^2} \mathbb{E} \left[\left(\sum_{j=1}^{\lfloor N/2 \rfloor} (\hat{f}_N(\omega_j) - f(\omega_j)) \cos(h\omega_j) \right)^2 \right] = O(1).$$

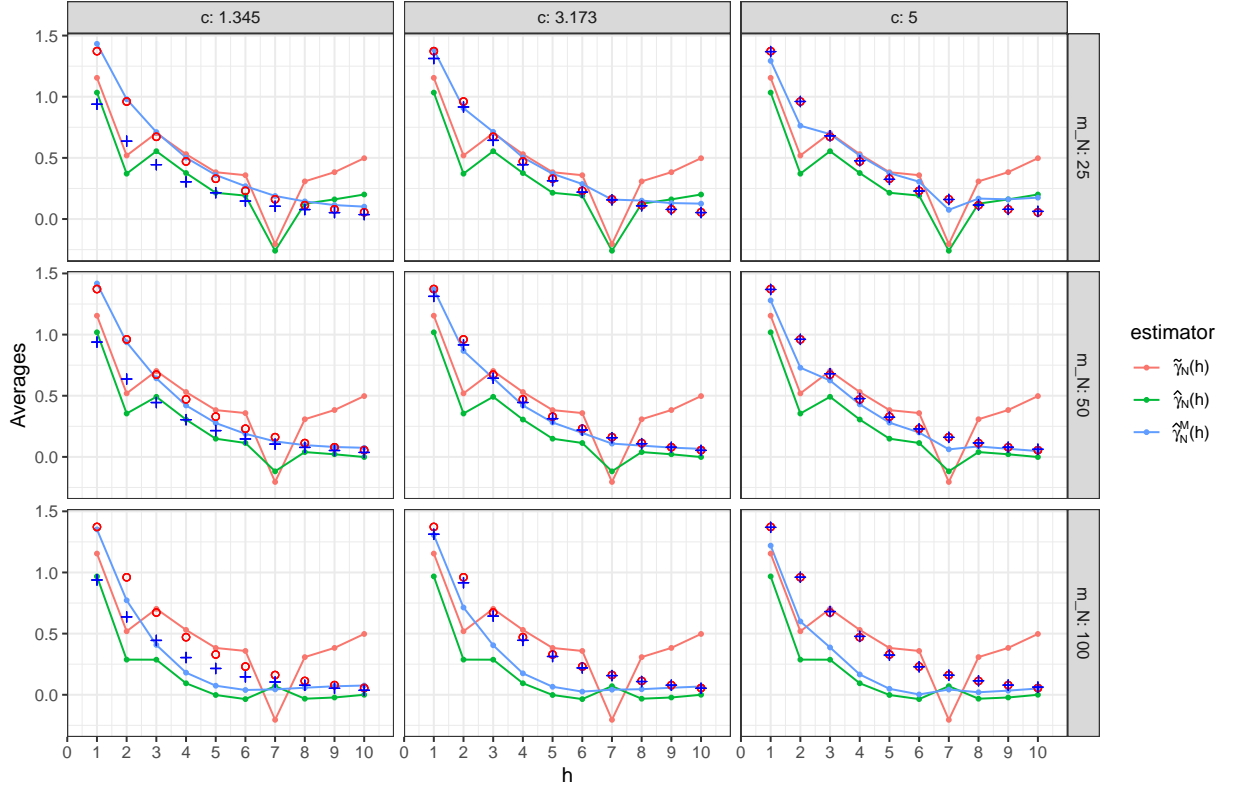


FIGURE 10. Averages of $\tilde{\gamma}_N(h)$, $\hat{\gamma}_N(h)$ and $\hat{\gamma}_N^M(h)$ for $\phi = 0.7$, $p = 20\%$, $\omega = 20$, $\gamma(h)$ with red circles and $\gamma_\psi(h)/\kappa^2$ with blue crosses.

By the Cauchy-Schwarz inequality, it is thus enough to prove that

$$m_N \sup_{1 \leq j \leq N/2} \mathbb{E} \left[\left(\hat{f}_N(\omega_j) - f(\omega_j) \right)^2 \right] = O(1). \quad (29)$$

Note that

$$\mathbb{E} \left[\left(\hat{f}_N(\omega_j) - f(\omega_j) \right)^2 \right] = \text{Var}(\hat{f}_N(\omega_j)) + \left(\mathbb{E}(\hat{f}_N(\omega_j)) - f(\omega_j) \right)^2. \quad (30)$$

Let us first prove that for each fixed j ,

$$\text{Var}(\hat{f}_N(\omega_j)) = O\left(\frac{1}{m_N}\right),$$

where O does not depend on j . By (9),

$$\begin{aligned} \text{Var}(\widehat{f}_N(\omega_j)) &= \frac{1}{(2m_N + 1)^2} \sum_{|k| \leq m_N} \sum_{|k'| \leq m_N} \text{Cov}(I_N(\omega_{j+k}), I_N(\omega_{j+k'})) \\ &= \frac{1}{(2m_N + 1)^2} \sum_{|k| \leq m_N} \text{Var}(I_N(\omega_{j+k})) \\ &\quad + \frac{1}{(2m_N + 1)^2} \sum_{|k|, |k'| \leq m_N, k \neq k'} \text{Cov}(I_N(\omega_{j+k}), I_N(\omega_{j+k'})). \end{aligned} \quad (31)$$

By (Brockwell and Davis, 1991, Theorem 10.3.2), the last term in (31) is $O(1/N)$ and $\text{Var}(I_N(\omega_{j+k})) = C f^2(\omega_{j+k}) + O(1/\sqrt{N})$. Since $f(\lambda) \leq \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} |\gamma(h)| < \infty$ for all $\lambda \in [-\pi, \pi]$, we get that

$$\text{Var}(\widehat{f}_N(\omega_j)) = O\left(\frac{1}{m_N}\right) + O\left(\frac{1}{m_N \sqrt{N}}\right) + O\left(\frac{1}{N}\right) = O\left(\frac{1}{m_N}\right), \quad (32)$$

as N tends to infinity where O does not depend on j .

Let us now focus on $\mathbb{E}(\widehat{f}_N(\omega_j)) - f(\omega_j)$. From the proof of (Brockwell and Davis, 1991, Proposition 10.3.1), we get that

$$\begin{aligned} \mathbb{E}(\widehat{f}_N(\omega_j)) - f(\omega_j) &= \frac{1}{2\pi(2m_N + 1)} \sum_{|k| \leq m_N} \left[\sum_{|\ell| < N} \left(1 - \frac{|\ell|}{N}\right) \gamma(\ell) e^{-i\ell\omega_{j+k}} - \sum_{\ell \in \mathbb{Z}} \gamma(\ell) e^{-i\ell\omega_j} \right] \\ &= \frac{1}{2\pi(2m_N + 1)} \sum_{|k| \leq m_N} \left[\sum_{|\ell| < N} \gamma(\ell) e^{-i\ell\omega_{j+k}} - \frac{1}{N} \sum_{|\ell| < N} |\ell| \gamma(\ell) e^{-i\ell\omega_{j+k}} \right. \\ &\quad \left. - \sum_{|\ell| < N} \gamma(\ell) e^{-i\ell\omega_j} - \sum_{|\ell| \geq N} \gamma(\ell) e^{-i\ell\omega_j} \right] \\ &= \frac{1}{2\pi(2m_N + 1)} \sum_{|k| \leq m_N} \left[\underbrace{\sum_{|\ell| < N} \gamma(\ell) e^{-i\ell\omega_j} (e^{-i\ell\omega_k} - 1)}_{A_{k,j}} \right. \\ &\quad \left. - \frac{1}{N} \sum_{|\ell| < N} |\ell| \gamma(\ell) e^{-i\ell\omega_{j+k}} - \sum_{|\ell| \geq N} \gamma(\ell) e^{-i\ell\omega_j} \right]. \end{aligned}$$

Observe that

$$\frac{1}{2\pi(2m_N + 1)} \left| \sum_{|k| \leq m_N} B_{k,j} \right| \leq \frac{2}{2\pi(2m_N + 1)} \sum_{|k| \leq m_N} \frac{1}{N} \sum_{\ell \in \mathbb{Z}} |\ell| |\gamma(\ell)| = O\left(\frac{1}{N}\right)$$

since $\sum_{\ell \in \mathbb{Z}} |\ell| |\gamma(\ell)| \leq \sigma^2 \sum_{\ell \in \mathbb{Z}} \sum_{j \geq 0} |a_j| |a_{j+\ell}| |\ell| \leq \sigma^2 (\sum_{j \geq 0} |a_j|) (\sum_{j \geq 0} |a_j| j) < \infty$, where O does not depend on j .

Note that

$$\begin{aligned} \left| \sum_{|k| \leq m_N} A_{k,j} \right| &\leq \sum_{|\ell| < N} |\gamma(\ell)| \left| \sum_{|k| \leq m_N} \left(e^{-i\ell\omega_k} - 1 \right) \right| = 2 \sum_{|\ell| < N} |\gamma(\ell)| \left| \sum_{k=1}^{m_N} (\cos(\ell\omega_k) - 1) \right| \\ &\leq 2 \sum_{|\ell| < N} |\gamma(\ell)| \sum_{k=1}^{m_N} |\cos(\ell\omega_k) - 1| \leq 2 \sum_{|\ell| < N} |\gamma(\ell)| \sum_{k=1}^{m_N} \frac{1}{2} \left(\frac{2\pi k\ell}{N} \right)^2 \\ &= \frac{8\pi^2}{N^2} \sum_{|\ell| < N} \ell^2 |\gamma(\ell)| \sum_{k=1}^{m_N} k^2 = O\left(\frac{m_N^3}{N^2}\right) \end{aligned}$$

since $\sum_{|\ell| \in \mathbb{Z}} \ell^2 |\gamma(\ell)| \leq \sigma^2 \sum_{\ell \in \mathbb{Z}} \sum_{j \geq 0} |a_j| |a_{j+\ell}| \ell^2 \leq \sigma^2 (\sum_{j \geq 0} |a_j|) (\sum_{j \geq 0} |a_j| j^2) < \infty$, where O does not depend on j .

Hence,

$$\left(\mathbb{E}(\widehat{f}_N(\omega_j)) - f(\omega_j) \right)^2 = \left(O\left(\frac{1}{N}\right) + O\left(\frac{m_N^2}{N^2}\right) \right)^2 = O\left(\frac{1}{N^2}\right) + O\left(\frac{m_N^2}{N^3}\right) = o\left(\frac{1}{m_N}\right), \quad (33)$$

since $m_N/N \rightarrow 0$ as N tends to infinity.

By (29), (30), (32) and (33), the result follows. \square

Lemma 2. *Under the assumptions of Theorem 2, for each $j \in \{0, 1, \dots\}$, $I_N^M(\omega_j)$ defined by (12) satisfies*

$$I_N^M(\omega_j) = \frac{1}{2\pi\kappa^2 N} \left(\sum_{i=1}^N \psi(Y_i)^2 \right) + \frac{1}{2\pi\kappa^2} \sum_{0 < |\ell| < N} \cos(\ell\omega_j) \left(\frac{1}{N} \sum_{i=1}^{N-|\ell|} \psi(Y_{i+|\ell|}) \psi(Y_i) \right) + o_P(1),$$

as $N \rightarrow \infty$, where the term $o_P(1)$ is uniform in j , $\kappa = 2\Phi(c) - 1$, Φ is the cumulative distribution function of a standard Gaussian random variable and c is the constant appearing in (14).

Proof of Lemma 2. Let j in $\{0, 1, \dots\}$ and $\widehat{\beta}_N(\omega_j) = (\widehat{\beta}_{N,1}(\omega_j), \widehat{\beta}_{N,2}(\omega_j))'$ in (13). By (Wu, 2007, Propositions 1, 4 and Example 1), the assumptions of (Wu, 2007, Theorem 1) hold. Thus, $\widehat{\beta}_{N,1}(\omega_j)$ and $\widehat{\beta}_{N,2}(\omega_j)$ satisfy

$$\begin{aligned} \widehat{\beta}_{N,1}(\omega_j) &= \frac{2}{N\kappa} \sum_{i=1}^N \psi(Y_i) \cos(i\omega_j) + o_P(1/\sqrt{N}), \\ \widehat{\beta}_{N,2}(\omega_j) &= \frac{2}{N\kappa} \sum_{i=1}^N \psi(Y_i) \sin(i\omega_j) + o_P(1/\sqrt{N}), \end{aligned} \quad (34)$$

as N tends to infinity, where $o_P(1/\sqrt{N})$ does not depend on j . Moreover,

$$\sqrt{\frac{N}{2}} \kappa \widehat{\beta}_N(\omega_j) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \Delta^{(j)}\right), \text{ as } N \rightarrow \infty.$$

where

$$\Delta^{(j)} = \sum_{k \in \mathbb{Z}} \mathbb{E}\{\psi(Y_0)\psi(Y_k)\} \Delta_k^{(j)},$$

and

$$\Delta_k^{(j)} = \lim_{N \rightarrow \infty} \frac{2}{N} \sum_{\ell=1}^{N-|k|} c_\ell(\omega_j) c_{\ell+k}(\omega_j)'$$

For each fixed $k \in \mathbb{Z}$, we have

$$\begin{aligned} \Delta_k^{(j)} &= \lim_{N \rightarrow \infty} \frac{2}{N} \sum_{\ell=1}^{N-|k|} \begin{bmatrix} \cos(\ell\omega_j) \cos((\ell+k)\omega_j) & \cos(\ell\omega_j) \sin((\ell+k)\omega_j) \\ \sin(\ell\omega_j) \cos((\ell+k)\omega_j) & \sin(\ell\omega_j) \sin((\ell+k)\omega_j) \end{bmatrix} \\ &= \lim_{N \rightarrow \infty} \frac{N-|k|}{N} \begin{bmatrix} \cos(k\omega_j) & \sin(k\omega_j) \\ -\sin(k\omega_j) & \cos(k\omega_j) \end{bmatrix} + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\ell=1}^{N-|k|} \begin{bmatrix} \cos((2\ell+k)\omega_j) & \sin((2\ell+k)\omega_j) \\ \sin((2\ell+k)\omega_j) & -\cos((2\ell+k)\omega_j) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\ell=1}^{N-|k|} \begin{bmatrix} \cos((2\ell+k)\omega_j) & \sin((2\ell+k)\omega_j) \\ \sin((2\ell+k)\omega_j) & -\cos((2\ell+k)\omega_j) \end{bmatrix}. \end{aligned}$$

Observe that

$$\begin{aligned} \frac{1}{N} \sum_{\ell=1}^{N-|k|} \cos((2\ell+k)\omega_j) &= \frac{\cos(k\omega_j)}{N} \sum_{\ell=1}^{N-|k|} \cos(2\ell\omega_j) - \frac{\sin(k\omega_j)}{N} \sum_{\ell=1}^{N-|k|} \sin(2\ell\omega_j) \\ &= \frac{\cos(k\omega_j)}{N} \cos(\omega_j(N-|k|+1)) \frac{\sin(\omega_j(N-|k|))}{\sin(\omega_j)} - \frac{\sin(k\omega_j)}{N} \sin(\omega_j(N-|k|+1)) \frac{\sin(\omega_j(N-|k|))}{\sin(\omega_j)}. \end{aligned}$$

Since $\sin(\omega_j(N-|k|))/\sin(\omega_j) \rightarrow -|k|$ as N tends to infinity, we get that

$$\frac{1}{N} \sum_{\ell=1}^{N-|k|} \cos((2\ell+k)\omega_j) \rightarrow 0, \text{ as } N \rightarrow \infty.$$

and that the same result holds for $N^{-1} \sum_{\ell=1}^{N-|k|} \sin((2\ell+k)\omega_j)$. Thus,

$$\begin{aligned} \sqrt{\frac{N}{2}} \kappa \widehat{\beta}_{N,1}(\omega_j) &\xrightarrow{d} \mathcal{N} \left(0, \sum_{k \in \mathbb{Z}} \mathbb{E}[\psi(Y_0)\psi(Y_k)] \right), \\ \sqrt{\frac{N}{2}} \kappa \widehat{\beta}_{N,2}(\omega_j) &\xrightarrow{d} \mathcal{N} \left(0, \sum_{k \in \mathbb{Z}} \mathbb{E}[\psi(Y_0)\psi(Y_k)] \right), \text{ as } N \rightarrow \infty. \end{aligned} \tag{35}$$

By (12) and (34),

$$\begin{aligned} I_N^M(\omega_j) &= \frac{N}{8\pi} \left[\left(\frac{2}{N\kappa} \sum_{i=1}^N \psi(Y_i) \cos(i\omega_j) + o_P(1/\sqrt{N}) \right)^2 + \left(\frac{2}{N\kappa} \sum_{i=1}^N \psi(Y_i) \sin(i\omega_j) + o_P(1/\sqrt{N}) \right)^2 \right] \\ &= \frac{N}{8\pi} \left[\frac{4}{N^2\kappa^2} \left(\sum_{i=1}^N \psi(Y_i) \cos(i\omega_j) \right)^2 + o_P(1/N) + \frac{4}{N^2\kappa^2} \left(\sum_{i=1}^N \psi(Y_i) \sin(i\omega_j) \right)^2 + \right. \\ &\quad \left. o_P(1/\sqrt{N}) \left(\frac{2}{N\kappa} \sum_{i=1}^N \psi(Y_i) \cos(i\omega_j) \right) + o_P(1/\sqrt{N}) \left(\frac{2}{N\kappa} \sum_{i=1}^N \psi(Y_i) \sin(i\omega_j) \right) \right], \end{aligned}$$

where, by (34) and (35),

$$\begin{aligned}\frac{2}{N\kappa} \sum_{i=1}^N \psi(Y_i) \cos(i\omega_j) &= O_P(1/\sqrt{N}), \\ \frac{2}{N\kappa} \sum_{i=1}^N \psi(Y_i) \sin(i\omega_j) &= O_P(1/\sqrt{N}), \text{ as } N \rightarrow \infty.\end{aligned}$$

Thus,

$$\begin{aligned}I_N^M(\omega_j) &= \frac{1}{2\pi\kappa^2 N} \left(\sum_{1 \leq i, i' \leq N} \psi(Y_i)\psi(Y_{i'}) [\cos(i\omega_j) \cos(i'\omega_j) + \sin(i\omega_j) \sin(i'\omega_j)] \right) + o_P(1) \\ &= \frac{1}{2\pi\kappa^2 N} \left(\sum_{i=1}^N \psi(Y_i)^2 \right) + \frac{1}{2\pi\kappa^2 N} \left(\sum_{1 \leq i \neq i' \leq N} \psi(Y_i)\psi(Y_{i'}) \cos[(i-i')\omega_j] \right) + o_P(1),\end{aligned}$$

as $N \rightarrow \infty$, which gives the result. \square

Lemma 3. For each $j \in \{0, 1, \dots, \}$,

$$\begin{aligned}\widehat{f}_N^M(\omega_j) - \frac{f_\psi(\omega_j)}{\kappa^2} &= \\ \frac{1}{2\pi\kappa^2} \sum_{0 < |\ell| < N} \left\{ \left(\frac{1}{N} \sum_{i=1}^{N-|\ell|} \psi(Y_{i+|\ell|})\psi(Y_i) \right) - \gamma_\psi(\ell) \right\} \cos(\ell\omega_j) A_N(\ell) &+ o_P(1), \text{ as } N \rightarrow \infty,\end{aligned}$$

where the term $o_P(1)$ is uniform in j and

$$A_N(\ell) = \frac{1}{2m_N + 1} \sum_{k=-m_N}^{m_N} \cos(\ell\omega_k).$$

Proof of Lemma 3. By (15) and Lemma 2,

$$\begin{aligned}\widehat{f}_N^M(\omega_j) &= \frac{1}{2m_N + 1} \sum_{k=-m_N}^{m_N} I_N^M(\omega_{j+k}) = \frac{1}{2\pi\kappa^2 N} \left(\sum_{i=1}^N \psi(Y_i)^2 \right) + \\ \frac{1}{2\pi\kappa^2} \frac{1}{2m_N + 1} \sum_{0 < |\ell| < N} \left(\frac{1}{N} \sum_{i=1}^{N-|\ell|} \psi(Y_{i+|\ell|})\psi(Y_i) \right) &\sum_{k=-m_N}^{m_N} \cos(\ell\omega_{j+k}) + o_P(1).\end{aligned}$$

Using that $\sum_{k=-m_N}^{m_N} \cos(\ell\omega_{j+k}) = \sum_{k=-m_N}^{m_N} \cos(\ell\omega_j) \cos(\ell\omega_k)$, we get that

$$\begin{aligned}\widehat{f}_N^M(\omega_j) &= \frac{1}{2\pi\kappa^2 N} \left(\sum_{i=1}^N \psi(Y_i)^2 \right) + \\ \frac{1}{2\pi\kappa^2} \sum_{0 < |\ell| < N} \left(\frac{1}{N} \sum_{i=1}^{N-|\ell|} \psi(Y_{i+|\ell|})\psi(Y_i) \right) \cos(\ell\omega_j) &\left(\frac{1}{2m_N + 1} \sum_{k=-m_N}^{m_N} \cos(\ell\omega_k) \right) + o_P(1).\end{aligned}$$

By (22),

$$f_\psi(\omega_j) = \frac{1}{2\pi} \left(\gamma_\psi(0) + \sum_{\ell \in \mathbb{Z}^*} \gamma_\psi(\ell) \cos(\ell\omega_j) \right).$$

Thus,

$$\begin{aligned} \widehat{f}_N^M(\omega_j) - \frac{f_\psi(\omega_j)}{\kappa^2} &= \frac{1}{2\pi\kappa^2} \left(\frac{1}{N} \sum_{i=1}^N \psi(Y_i)^2 - \gamma_\psi(0) \right) \\ &+ \frac{1}{2\pi\kappa^2} \sum_{0 < |\ell| < N} \left\{ \left(\frac{1}{N} \sum_{i=1}^{N-|\ell|} \psi(Y_{i+|\ell|}) \psi(Y_i) \right) - \gamma_\psi(\ell) \right\} \cos(\ell\omega_j) A_N(\ell) \\ &+ \frac{1}{2\pi\kappa^2} \sum_{0 < |\ell| < N} \gamma_\psi(\ell) \cos(\ell\omega_j) (A_N(\ell) - 1) \\ &+ \frac{1}{2\pi\kappa^2} \sum_{0 < |\ell| < N} \gamma_\psi(\ell) \cos(\ell\omega_j) - \frac{1}{2\pi\kappa^2} \sum_{\ell \in \mathbb{Z}^*} \gamma_\psi(\ell) \cos(\ell\omega_j) + o_P(1). \end{aligned} \quad (36)$$

By the ergodic theorem, the first term in the r.h.s of (36) tends to 0 in probability as N tends to infinity. By straightforward computation, we get

$$A_N(\ell) = \frac{1}{2m_N + 1} \left\{ 1 + 2 \cos\left(\frac{\pi\ell m_N}{N}\right) \frac{\sin\left(\frac{\pi\ell(m_N-1)}{N}\right)}{\sin\left(\frac{\pi\ell}{N}\right)} \right\}$$

which tends to 1 as N tends to infinity for each fixed $\ell \in \mathbb{Z}^*$. Since γ_ψ is absolutely summable the dominated convergence theorem implies that

$$\sum_{0 < |\ell| < N} \gamma_\psi(\ell) \cos(\ell\omega_j) (A_N(\ell) - 1) \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Moreover, since γ_ψ is absolutely summable, the difference between the fourth and fifth term in the r.h.s of (36) tends to 0 as $N \rightarrow \infty$, which concludes the proof. \square

Proof of Theorem 2. Let

$$\mathbf{\Gamma}_N^\psi = (\gamma_\psi(i-j))_{1 \leq i, j \leq N}, \quad (37)$$

where γ_ψ is defined by (19). By (Brockwell and Davis, 1991, Proposition 4.5.2),

$$\mathbf{\Gamma}_N^\psi = 2\pi \mathbf{P}'_N \mathbf{D}_N^\psi \mathbf{P}_N + \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N, \quad (38)$$

where \mathbf{P}_N is defined by (4), \mathbf{D}_N^ψ is defined by (5) where f is replaced by f_ψ given by (22), and $\mathbf{R}_N = \left(\mathbf{R}_N^{(i,j)} \right)_{1 \leq i, j \leq N}$ is a $N \times N$ matrix such that $\sup_{1 \leq i, j \leq N} \left| \mathbf{R}_N^{(i,j)} \right| \rightarrow 0$, as N tends to infinity. For any $h \in \{0, \dots, N-1\}$, it follows from (16), (17) (37) and (38) that,

$$\begin{aligned} \widehat{\gamma}_N^M(h) - \frac{\gamma_\psi(h)}{\kappa^2} &= e'_{h+1} \widehat{\mathbf{\Gamma}}_N^M e_1 - e'_{h+1} \frac{\mathbf{\Gamma}_N^\psi}{\kappa^2} e_1 \\ &= 2\pi e'_{h+1} \mathbf{P}'_N \left(\widehat{\mathbf{D}}_N^M - \frac{\mathbf{D}_N^\psi}{\kappa^2} \right) \mathbf{P}_N e_1 - \frac{1}{\kappa^2} e'_{h+1} \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N e_1. \end{aligned}$$

By Lemma 1, we have $e'_{h+1} \mathbf{P}'_N \mathbf{R}_N \mathbf{P}_N e_1 \rightarrow 0$, as $N \rightarrow \infty$. Thus, to prove Theorem 2, it is enough to prove that

$$e'_{h+1} \mathbf{P}'_N \left(\widehat{\mathbf{D}}_N^M - \frac{\mathbf{D}_N^\psi}{\kappa^2} \right) \mathbf{P}_N e_1 = o_P(1), \text{ as } N \rightarrow \infty.$$

We have

$$e'_{h+1} \mathbf{P}'_N \left(\widehat{\mathbf{D}}_N^M - \frac{\mathbf{D}_N^\psi}{\kappa^2} \right) \mathbf{P}_N e_1 = \begin{cases} \frac{1}{N} (\widehat{f}_N^M(0) - \frac{f_\psi(0)}{\kappa^2}) + \frac{2}{N} \sum_{j=1}^{\lfloor N/2 \rfloor} (\widehat{f}_N^M(\omega_j) - \frac{f_\psi(\omega_j)}{\kappa^2}) \cos(h\omega_j) & \text{if } N \text{ is odd,} \\ \frac{1}{N} (\widehat{f}_N^M(0) - \frac{f_\psi(0)}{\kappa^2}) + \frac{2}{N} \sum_{j=1}^{N/2-1} (\widehat{f}_N^M(\omega_j) - \frac{f_\psi(\omega_j)}{\kappa^2}) \cos(h\omega_j) \\ + \frac{1}{N} (\widehat{f}_N^M(\pi) - \frac{f_\psi(\pi)}{\kappa^2}) \cos(h\pi) & \text{if } N \text{ is even.} \end{cases}$$

By Lemma 3 and by using the same arguments as those used for proving Theorem 1, it is enough to prove that

$$\sup_{0 \leq j \leq \lfloor N/2 \rfloor} \mathbb{E} \left[\left(\sum_{0 < |\ell| < N} \left\{ \left(\frac{1}{N} \sum_{i=1}^{N-|\ell|} \psi(Y_{i+|\ell|}) \psi(Y_i) \right) - \gamma_\psi(\ell) \right\} \cos(\ell\omega_j) A_N(\ell) \right)^2 \right] \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Let

$$B_N(\ell) = \left(\frac{1}{N} \sum_{i=1}^{N-\ell} \psi(Y_{i+\ell}) \psi(Y_i) \right) - \gamma_\psi(\ell).$$

Then, it is enough to prove that

$$\sup_{0 \leq j \leq \lfloor N/2 \rfloor} \sum_{1 \leq \ell, \ell' < N} \mathbb{E} [B_N(\ell) B_N(\ell')] \cos(\ell\omega_j) \cos(\ell'\omega_j) A_N(\ell) A_N(\ell') \rightarrow 0, \text{ as } N \rightarrow \infty. \quad (39)$$

Observe that

$$\begin{aligned} \mathbb{E} [B_N(\ell) B_N(\ell')] &= \frac{1}{N^2} \sum_{1 \leq i \leq N-\ell, 1 \leq i' \leq N-\ell'} \mathbb{E} [\psi(Y_{i+\ell}) \psi(Y_i) \psi(Y_{i'+\ell'}) \psi(Y_{i'})] \\ &\quad - \left(1 - \frac{\ell}{N}\right) \gamma_\psi(\ell) \gamma_\psi(\ell') - \left(1 - \frac{\ell'}{N}\right) \gamma_\psi(\ell') \gamma_\psi(\ell) + \gamma_\psi(\ell) \gamma_\psi(\ell'). \end{aligned} \quad (40)$$

Using (20), we get that

$$\begin{aligned} &\frac{1}{N^2} \sum_{1 \leq i \leq N-\ell, 1 \leq i' \leq N-\ell'} \mathbb{E} [\psi(Y_{i+\ell}) \psi(Y_i) \psi(Y_{i'+\ell'}) \psi(Y_{i'})] \\ &= \frac{1}{N^2} \sum_{1 \leq i \leq N-\ell, 1 \leq i' \leq N-\ell'} \mathbb{E} \left[\psi_\sigma \left(\frac{Y_{i+\ell}}{\sigma} \right) \psi_\sigma \left(\frac{Y_i}{\sigma} \right) \psi_\sigma \left(\frac{Y_{i'+\ell'}}{\sigma} \right) \psi_\sigma \left(\frac{Y_{i'}}{\sigma} \right) \right] \\ &= \sum_{p_1, p_2, p_3, p_4 \geq 1} \frac{c_{p_1} c_{p_2} c_{p_3} c_{p_4}}{p_1! p_2! p_3! p_4!} \\ &\quad \left(\frac{1}{N^2} \sum_{1 \leq i \leq N-\ell, 1 \leq i' \leq N-\ell'} \mathbb{E} \left[H_{p_1} \left(\frac{Y_{i+\ell}}{\sigma} \right) H_{p_2} \left(\frac{Y_i}{\sigma} \right) H_{p_3} \left(\frac{Y_{i'+\ell'}}{\sigma} \right) H_{p_4} \left(\frac{Y_{i'}}{\sigma} \right) \right] \right). \end{aligned} \quad (41)$$

By (Rosenblatt, 1985, Equation (33), p. 69),

$$\mathbb{E} [H_{\ell_1}(X_{i_1}) \dots H_{\ell_m}(X_{i_m})] = \ell_1! \dots \ell_m! \sum_{\{\ell_1, \dots, \ell_m\}} \frac{\rho^\nu}{\nu!}, \quad (42)$$

where it is understood that $\rho^\nu = \prod_{1 \leq q < k \leq m} \rho(i_q - i_k)^{\nu_{q,k}}$, $\rho(i_q - i_k)$ denoting the correlation between X_{i_q} and X_{i_k} , $\nu! = \prod_{1 \leq q < k \leq m} \nu_{q,k}!$, and $\sum_{\{\ell_1, \dots, \ell_m\}}$ indicates that we are to sum over all symmetric matrices ν with nonnegative integer entries, $\nu_{ii} = 0$ and the row sums equal to ℓ_1, \dots, ℓ_m .

We shall prove that among all the terms in the r.h.s of (42), the leading ones in (41) correspond to the case where we have the two following pairs of equal indices in the set $\{\ell_1, \dots, \ell_4\}$: $\ell_1 = \ell_2$, $\ell_3 = \ell_4$ and $\nu_{1,2} = \ell_1$, $\nu_{3,4} = \ell_3$, the others $\nu_{i,j}$ being equal to zero. This gives:

$$\mathbb{E} [H_{\ell_1}(X_{i_1}) \dots H_{\ell_4}(X_{i_4})] = (\ell_2!)^2 (\ell_4!)^2 \frac{\rho(i_2 - i_1)^{\ell_2} \rho(i_4 - i_3)^{\ell_4}}{\ell_2! \ell_4!}.$$

The corresponding term in (41) is thus given by

$$\frac{1}{N^2} \sum_{p_2, p_4 \geq 1} \frac{c_{p_2}^2 c_{p_4}^2}{p_2! p_4!} \rho(i_2 - i_1)^{p_2} \rho(i_4 - i_3)^{p_4},$$

where $i_1 = i + \ell$, $i_2 = i$, $i_3 = i' + \ell'$, $i_4 = i'$.

Thus, in the case where $|\{p_1, p_2, p_3, p_4\}| = 2$, (41) becomes:

$$\begin{aligned} & \sum_{p_2, p_4 \geq 1} \frac{c_{p_2}^2 c_{p_4}^2}{p_2! p_4!} \left(\frac{1}{N^2} \sum_{1 \leq i \leq N - \ell, 1 \leq i' \leq N - \ell'} \rho(\ell)^{p_2} \rho(\ell')^{p_4} \right) \\ &= \left(1 - \frac{\ell}{N}\right) \left(1 - \frac{\ell'}{N}\right) \sum_{p_2, p_4 \geq 1} \frac{c_{p_2}^2 c_{p_4}^2}{p_2! p_4!} \rho(\ell)^{p_2} \rho(\ell')^{p_4} \\ &= \left(1 - \frac{\ell}{N}\right) \left(1 - \frac{\ell'}{N}\right) \gamma_\psi(\ell) \gamma_\psi(\ell'). \end{aligned}$$

In the other situations of pairs of equal indices, it is straightforward to prove that the terms can be neglected.

In the cases where $|\{p_1, p_2, p_3, p_4\}| \geq 3$ then the matrices ν that have to be considered are symmetric matrices, with nonnegative integer entries such that $\nu_{ii} = 0$ and the row sums equal to ℓ_1, \dots, ℓ_4 and such that there are at least two non null values of $\nu_{q,k}$ on a row of ν . This last point implies that in the expression of

$$\mathbb{E} \left[H_{p_1} \left(\frac{Y_{i+\ell}}{\sigma} \right) H_{p_2} \left(\frac{Y_i}{\sigma} \right) H_{p_3} \left(\frac{Y_{i'+\ell'}}{\sigma} \right) H_{p_4} \left(\frac{Y_{i'}}{\sigma} \right) \right]$$

we will have at least a product of three terms with γ which makes all of these terms negligible and gives (39). □

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