



**HAL**  
open science

# Dynamic consensus of nonlinear oscillators under weak coupling

Anes Lazri, Elena Panteley, Antonio Loria

► **To cite this version:**

Anes Lazri, Elena Panteley, Antonio Loria. Dynamic consensus of nonlinear oscillators under weak coupling. 25th Int. Symposium on Mathematical Theory of Networks and Systems (MTNS 2022), Sep 2022, Bayreuth, Germany. hal-03754339

**HAL Id: hal-03754339**

**<https://centralesupelec.hal.science/hal-03754339v1>**

Submitted on 1 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Dynamic Consensus of Nonlinear Oscillators under Weak Coupling<sup>★</sup>

Anes Lazri<sup>\*</sup> Elena Panteley<sup>\*\*</sup> Antonio Loría<sup>\*\*</sup>

<sup>\*</sup> *Laboratoire des signaux et systèmes, Univ-Paris Saclay, CentraleSupélec, France*

<sup>\*\*</sup> *Laboratoire des signaux et systèmes, CNRS, France*

**Abstract:** Dynamic consensus is a property of networked systems that pertains to the case in which all the interconnected systems synchronise their motions and a collective behaviour arises. If the coupling strength is large such behaviour may be modelled by a single system, but if it is weak, the behaviour is best modelled by a reduced-order network. For networks of homogeneous Stuart-Landau oscillators under weak coupling, we characterise the dimension and dynamics of such reduced-order network in function of the coupling strength.

*Keywords:* Networked Control Systems, Nonlinear Systems and Control, Stability

AMS subject classification: 34D06, 93D20

## 1. CONTEXT

We analyse the dynamical behaviour of  $N$  identical Stuart-Landau oscillators interconnected via a distributed consensus-control law and with interconnection gain  $\gamma > 0$ ,

$$\dot{z}_j = f(z_j) + \mu z_j - \gamma \sum_{k=1}^N a_{kj}(z_j - z_k), \quad j \in \{1, 2, \dots, N\}, \quad (1)$$

where  $z_j, \mu \in \mathbb{C}$ , and  $f : \mathbb{C} \rightarrow \mathbb{C}$  is defined as

$$f(z) = -z|z|^2. \quad (2)$$

Equations of interconnected Stuart-Landau systems, such as (1)–(2), are often used as a universal dynamical system to model networks exhibiting oscillations, such as lasers, genetic and neuronal networks, among others (Hasty et al., 2001; Soriano et al., 2013).

More precisely, we are interested in the possible synchronised behaviour of these oscillators under the effect of the last term on the right-hand-side of (1). Two factors affect the collective behaviour of the multi-agent system. On one hand, the network's topology, which is defined by the coefficients  $a_{kj}$ , and on the other hand, the magnitude of the coupling strength. For instance, for networks of identical oscillators with an underlying undirected connected-graph topology and with  $\gamma > 0$  sufficiently large, the networked systems trajectories  $z_j(t)$  converge to the solution of an averaged dynamical system,

$$\dot{z}_m = F_m(z_m, e) \quad (3)$$

where  $z_m = \frac{1}{N} \sum_{k=1}^N z_k$  and  $e$  is a synchronisation error defined as

$$e := z - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^\top z \iff \begin{bmatrix} z_1 - z_m \\ \vdots \\ z_N - z_m \end{bmatrix} \quad (4)$$

—see (Pogromsky et al., 1999). For networks of heterogeneous oscillators, *i.e.*, with different  $\mu_j$ s in (1), in

<sup>★</sup> This work was supported by the French ANR via project HANDY, contract number ANR-18-CE40-0010 and by CEFIPRA under the grant number 6001-A.

(Maghenem et al., 2016) it is shown that for sufficiently large values of  $\gamma$  the synchronisation error defined in (4) is ultimately bounded and the solutions are frequency-synchronised. This is significant because the function  $(z_m, e) \mapsto F_m(z_m, e)$  is such that

$$F_m(z_m, 0) = -z_m |z_m|^2 + \mu_m z_m \quad (5)$$

where  $\mu_m \in \mathbb{C}$  is defined as  $\mu := \mu_R + i\mu_I$  and

$$\mu_R := \frac{1}{\sqrt{N}} \sum_1^N \mu_{Rk}, \quad \mu_I := \frac{1}{\sqrt{N}} \sum_1^N \mu_{Ik}. \quad (6)$$

In other words, for sufficiently large values of the coupling strength  $\gamma$  the response of each oscillator in (1) approaches that of a *single* emergent oscillator of the same nature,

$$\dot{z}_m = -z_m |z_m|^2 + \mu_m z_m, \quad z_m \in \mathbb{C}. \quad (7)$$

In general, for heterogeneous systems, such motion is called dynamic consensus (Panteley and Loría, 2017). For Eq. (7), the solutions are either periodic trajectories whose frequency and amplitude are defined by  $\mu$  or the unstable equilibrium point  $\{z_m = 0\}$ . Because the right-hand side of (7) corresponds to  $F_m(z_m, 0)$ , this equation describes the asymptotic collective behaviour of the networked systems, provided they enter in synchrony.

Besides the coupling gain being relatively high, the fact that the collective behaviour of the networked systems (1) may be approximated asymptotically by that of a single oscillator is also a consequence of the graph being connected and undirected. Indeed, in this case, the associated Laplacian matrix

$$L_N := [\ell_{kj}] \in \mathbb{R}^{N \times N}, \quad \ell_{kj} = \begin{cases} \sum_{l \in \mathcal{N}_k} a_{kl} & k = j \\ -a_{kj} & k \neq j, \end{cases} \quad (8)$$

has exactly one null eigenvalue and  $v_1 := \frac{1}{\sqrt{N}} \mathbf{1}_N$  is its associated left eigenvector. That is,

$$z_m := v_1^\top z \quad \text{and} \quad e = [I_N - v_1 v_1^\top] z. \quad (9)$$

It is well-documented in the literature that for relatively small values of  $\gamma$  networks of Stuart-Landau oscillators

can exhibit a rich variety of behaviours, such as chaotic motion, emergence of cluster states, and coherence resonance, among others (Golubitsky et al., 2012). Such richer behaviours cannot be captured by that of a single oscillator with dynamics as in (7). In (Tumash et al., 2019) it is shown that, in some cases, rich behaviour can be characterised by a *network* of reduced order. Specifically, the main results in (Tumash et al., 2019) only apply to networks of even dimension and a specific topology, and the reduced-order model's interconnections are nonlinear. In this extended abstract we consider networks of arbitrary dimension and with circulant-graph topology. We show that the dimension of the reduced model depends on the magnitude of the coupling strength  $\gamma$  relative to the eigenvalues of the Laplacian matrix  $L_N$ . Significantly, and in contrast to (Tumash et al., 2019), the reduced-order model has exactly the same structure as the original multiagent system, with linear interconnections.

## 2. MAIN RESULT

Consider systems modelled by (1), which we rewrite in the compact form

$$\dot{z} = F(z) + \gamma \tilde{L}_N z \quad (10)$$

where  $z = [z_1 \ z_2 \ \cdots \ z_N]^\top$ ,

$$F(z) = [f(z_1) \ f(z_2) \ \cdots \ f(z_N)]^\top$$

and

$$\tilde{L}_N := \left[ -L_N + \frac{\mu}{\gamma} I_N \right], \quad (11)$$

under the following hypothesis.

*Assumption 1.* The network is undirected and connected, has an underlying circulant-graph topology. That is, the adjacency matrix is circulant.

Assumption 1 implies that the Laplacian  $L_N$  with coefficients defined in (8) is a circulant matrix. That is, the  $(k+1)$ st row of  $L_N$  corresponds to the  $k$ th row in which the last element,  $\ell_{kN}$ , is placed first in the  $(k+1)$ st row and all other elements are shifted right. An example of a network satisfying Assumption 1 corresponds to one with an underlying ring topology, and in which the nodes may have supplementary cross-links, but with the restriction that each node has the same number of neighbours,  $m \geq 2$ . See Fig. 1 in Section 3 for an illustration.

Then, we have the following.

*Proposition 2.* (Main result). Consider a network of  $N$  Stuart-Landau oscillators with dynamics (10)–(11), such that the Laplacian  $L_N$  satisfies Assumption 1. Then, there exists  $\gamma_m > 0$  such that, for each  $\gamma > \gamma_m$ , there exists  $N_R(\gamma) < N$  and a network of reduced order  $N_R(\gamma)$ , with dynamics given by

$$\dot{z}_R = F(z_R) + \gamma \tilde{L}_R z_R, \quad (12)$$

where  $\tilde{L}_R \in \mathbb{C}^{N_R \times N_R}$ ,  $z_R \in \mathbb{C}^{N_R}$ ,  $z = [z_{R_1} \ z_{R_2} \ \cdots \ z_{R_{N_R}}]^\top$ ,

$$F(z) = [f(z_{R_1}) \ f(z_{R_2}) \ \cdots \ f(z_{R_{N_R}})]^\top$$

Moreover, the solutions of (10) satisfy

$$\lim_{t \rightarrow \infty} z(t) - \bar{z}_R(t) = 0 \quad (13)$$

for any initial conditions in  $\mathbb{C}^N$ , where  $\bar{z}_R = M z_R$  and  $M \in \mathbb{C}^{N \times N_R}$  is a matrix such that its columns generate a subspace of dimension  $N_R$ .  $\square$

In this extended abstract we do not provide a complete proof of Proposition 2, but the main rationale behind.

First, we observe that because the graph is connected (see Assumption 1),  $L_N$  has a unique zero eigenvalue and it admits the Jordan decomposition

$$L_N = U \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_2 \end{bmatrix} U^*, \quad (14)$$

where  $U^* \in \mathbb{C}^{N \times N}$  denotes the conjugate transpose of  $U$ , which is orthonormal, so  $U U^* = U^* U = I_N$ . Furthermore,  $\Lambda_2 \in \mathbb{C}^{N-1 \times N-1}$  is a diagonal matrix whose elements correspond to the nonzero eigenvalues of  $L_N$ . Then, in view of its definition—see Eq. (11),  $\tilde{L}_N$  satisfies the same decomposition as  $L_N$ . That is, denoting by  $\lambda_i(L_N)$  the eigenvalues of  $L_N$ , after (11), we have

$$\tilde{L}_N = U \begin{bmatrix} \frac{\mu}{\gamma} & 0 & \cdots & 0 \\ \gamma & -\lambda_2(L_N) + \frac{\mu}{\gamma} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -\lambda_N(L_N) + \frac{\mu}{\gamma} \end{bmatrix} U^*. \quad (15)$$

Clearly, the number of eigenvalues of the matrix above with positive real part varies from one to  $N$  depending on the value of  $\gamma$ . On the other hand, by convention, the eigenvalues  $\lambda_i$  are ordered in a way that  $0 < \Re\{\lambda_2\} \leq \Re\{\lambda_3\} \leq \cdots \leq \Re\{\lambda_N\}$ .

Furthermore, under Assumption 1, if  $N$  is odd, the non-zero eigenvalues  $\lambda_k$  come in conjugate pairs, that is  $\Re\{\lambda_k\} = \Re\{\lambda_{k+1}\}$  for all  $k \in \{2, 4, \dots, N-1\}$ . Hence, for any  $\gamma$  such that

$$\Re\{\mu/\lambda_{k+2}\} < \gamma < \Re\{\mu/\lambda_k\}, \quad k \in 2, 4, \dots, N-3$$

the matrix in (15) has necessarily an odd number  $N_R(\gamma) = k+1$  eigenvalues with positive real part, and  $N_R(\gamma) = N$  for any  $\gamma < \Re\{\mu/\lambda_{N-1}\}$ . Thus, for any  $\gamma > \gamma_m := \Re\{\mu/\lambda_N\} = \Re\{\mu/\lambda_{N-1}\}$ , the matrix in (15) has  $N_R(\gamma) < N$  eigenvalues with positive real part. Similar arguments apply to the case in which  $N$  is even, considering that  $\lambda_N \in \mathbb{R}_{>0}$ .

Now, for the sake of argument, let us disregard momentarily the nonlinear terms in (10),  $F(z)$ . For the system  $\dot{z} = \gamma \tilde{L}_N z$ , the eigenvalues with positive real part in  $\tilde{L}_N$  generate unstable modes while those with negative real part generate stable ones. That is, the solution may be written as

$$z(t) = v_1 v_1^* z(t) + v_2 v_2^* z(t) + \cdots + v_{N_R} v_{N_R}^* z(t) + e(t), \quad (16)$$

where  $v_k \in \mathbb{C}^N$ , for all  $k \in \{1, 2, \dots, N_R\}$  are eigenvectors associated with eigenvalues with positive real part and  $e(t)$  contains the contributions to the solution generated by the stable modes. As  $e(t) \rightarrow 0$  only the contributions of the unstable modes remain. This motivates the choice for the synchronisation errors,

$$e := z - U_1 U_1^* z, \quad (17)$$

where  $U_1 \in \mathbb{C}^{N \times N_R}$  is such that its columns correspond to the  $N_R$  eigenvectors associated with the  $N_R$  eigenvalues with positive real part, that is,  $v_k$  farther above. Since  $v_k$  denote the columns of  $U_1$ , note that if  $N_R = 1$  we recover the expression (4). On the other hand, on the synchronisation manifold  $\{e = 0\}$  the dynamics of the

system (10) reduces to that of reduced-order network evoked in Proposition 2. Exponential stability of  $\{e = 0\}$  can be established along the lines of (Tumash et al., 2019). Now, the statement of Proposition 2 asserts that the solution of the networked system converges asymptotically to a trajectory that we denote  $\bar{z}_R(t)$ . This trajectory, loosely speaking, may be regarded as a linear combination of the elements,  $z_{R_k}(t)$ , of the vector  $z_R(t)$ , solution of (12). Indeed, introducing the notation  $M = [v'_1 \ v'_2 \ \dots \ v'_{N_R}]$ , we see that

$$\bar{z}_R = v'_1 z_{R_1} + v'_2 z_{R_2} + \dots + v'_{N_R} z_{R_{N_R}} \quad (18)$$

and the  $N_R$  vectors  $v'_k$  are uniquely determined by the properties of the Laplacian  $L_N$  and the interconnection strength  $\gamma$ . It is important to remark that  $v'_k \neq v_k$ . As a matter of fact, it may be shown that

$$M := U_1 [U_{11}^* - U_{21}^* (U_{22}^*)^{-1} U_{12}^*],$$

where the matrices on the right-hand side above come from a suitable partition of  $U$  and its conjugate transpose,

$$U =: \begin{bmatrix} U_{11} & U_{21} \\ U_{12} & U_{22} \end{bmatrix}, \quad U^* = \begin{bmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{bmatrix}. \quad (19)$$

With these definitions, it may also be shown that  $e = z - Mz_R$  or that Eq. (16) is equivalent to  $z(t) = e(t) + Mz_R(t)$  so, in the limit, as  $e(t) \rightarrow 0$ ,  $z(t) = Mz_R(t)$ —cf. (13). In that regard, (12) is reminiscent of the zero-dynamics of the networked system (10) with respect to the converging output  $e \in \mathbb{C}^N$ . This corroborates the initial observation that, for complex networked nonlinear systems, the collective behaviour in the state of synchronisation, that is, on the manifold  $\{e = 0\}$  depends on the coupling strength  $\gamma$ . The importance of the latter observations transcend the rationale behind the proof of Proposition 2, which is omitted due to space constraints. The fact that, asymptotically,  $z(t) = Mz_R(t)$ , implies that the solutions of the (potentially) high-order network (10) may be reconstructed using the solutions of a reduced-order model, at least asymptotically, as synchronisation is reached. This is possible using a change of coordinates that leads to (12).

### 3. EXAMPLE

We have that if the coupling gain is large ( $\gamma > \gamma_M$ ) then,  $N_R = 1$ ; that is, (12) becomes, simply, Eq. (7). In this case, the oscillators synchronise globally. If  $\gamma$  is relatively small ( $\gamma_M > \gamma > \gamma_m$ ), which is the case of most interest here, there exists a network of reduced order  $N > N_R(\gamma) > 1$ .

For illustration, let us consider a network of eleven oscillators modelled as in (1), with  $\mu = 1 + 2i$  and interconnected over a graph satisfying Assumption 1, which is illustrated in Fig. 1. We consider two cases that pertain to different values of the coupling strength. First, we set  $\gamma < 1$ , which is relatively small given that  $\mu_R = 1$ . Indeed, for such  $\gamma$ ,  $\tilde{L}_N$  in (11) has three eigenvalues with positive real part. Hence, according to Proposition 2, there exists a reduced-order network of dimension  $N_R = 3$  whose behaviour approaches asymptotically that of the original one. Then, the coupling strength is increased to  $\gamma = 2.5$ , so only the first element in the diagonal of the matrix in (15) has positive real part. Consequently, it is expected that the collective behaviour of the networked systems approach asymptotically that of a single one with dynamics as in Eq. (7).

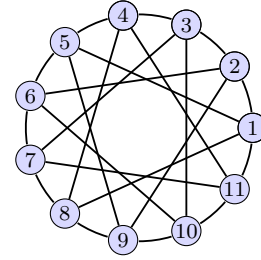


Fig. 1. Example of a ring graph in which each node has the same (even) number of undirected links

In both simulation tests the initial conditions are set to

$$\begin{cases} z_j(0) = 1 + i & \forall j \in \{1, 2, \dots, 5\}, \\ z_6(0) = 0, \\ z_j(0) = -1 - i & \forall j \in \{7, 8, \dots, 11\}. \end{cases} \quad (20)$$

The results are shown in Figs. 2–6, for a coupling strength set to  $\gamma = 0.75$ , and in Fig. 7 for  $\gamma = 2.5$ .

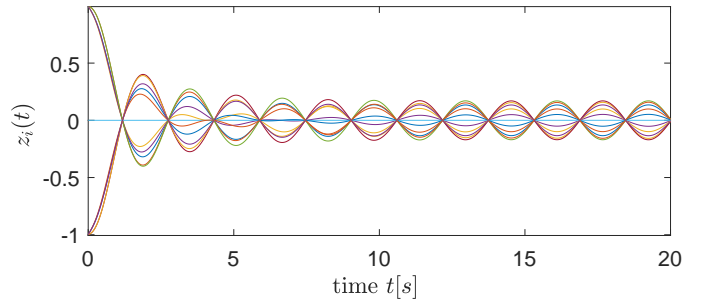


Fig. 2. Response of  $N = 11$  oscillators under relatively small coupling strength  $\gamma = 0.75$

In Fig. 2 are shown the trajectories of all oscillators. It may be appreciated that they all synchronise in frequency, but the oscillations have different amplitudes and there are two groups of in-phase oscillators. This is also appreciated from Fig. 3, which shows the evolution of the synchronisation errors as defined in (17) and from the plot on the left in Fig. 4 where the trajectories are depicted on the complex plane.

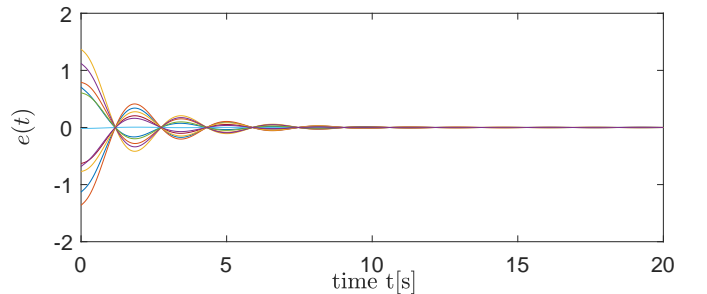


Fig. 3. Synchronisation error  $e(t)$  with  $e$  as in (17)

The behaviour of the three oscillators composing the reduced-order network (12) is depicted in Fig. 5.

Three different behaviours appear: one, generated by the mode related by the null eigenvalue of  $L_N$ , is equivalently equal to 0 since the initial condition is set at the origin, which is an equilibrium. Two other modes, related to the conjugate eigenvectors  $v_1$  and  $v_2$ , generate trajectories which are opposite to one another.

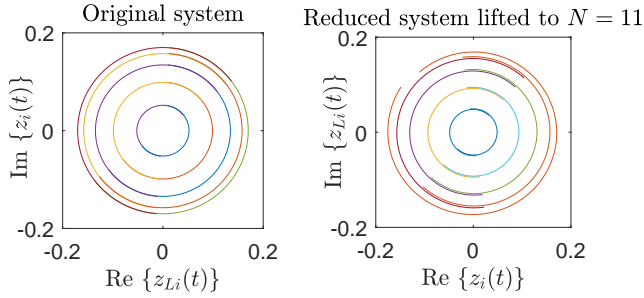


Fig. 4. Oscillators' trajectories on the complex plane for the network of 11 oscillators (left plot) and for their behaviour reconstituted from that of the reduced-order network (right plot)

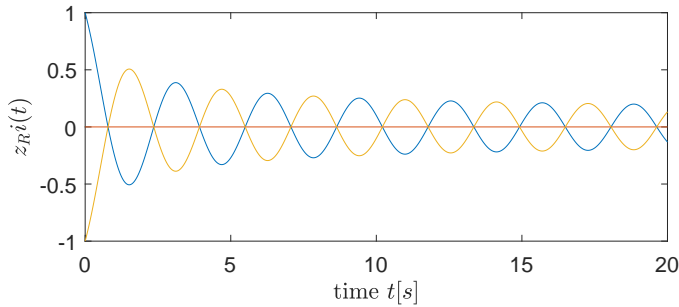


Fig. 5. Simulations results of  $N_R = 3$  reduced order system

Note that the oscillators of the original network of dimension 11 do not necessarily synchronise their behaviour with any of the three oscillators described by (12). However, the behaviour of each  $z_i(t)$  in (1) may be reconstructed, asymptotically, using the responses of the reduced-order network. The reconstituted behaviour, against time, is shown in Fig. 6 and on the complex plane in the right plot on Figure 4.

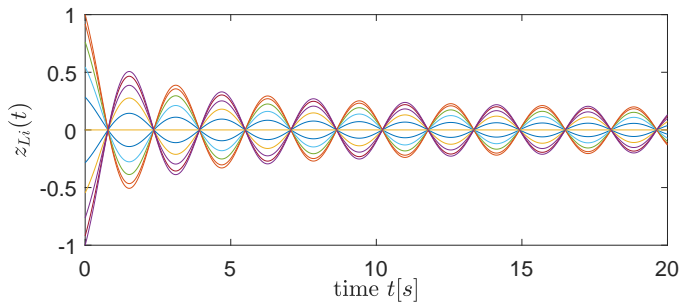


Fig. 6. Simulations results for the reduced system lifted to  $N = 11$

In Fig. 7 are shown the oscillators' responses with the initial conditions as in (20) and with high coupling strength,  $\gamma = 2.5$ . In this case, all the oscillators' trajectories converge exponentially to zero. This is explained by the fact that they approach the dynamics of the average system in (7) with zero initial condition  $z_m(0) = 0$ , which corresponds to the average of  $z_i(0)$  in (20).

Finally, to illustrate the influence of the initial conditions, in Fig. 8 we show the response of the oscillators, with coupling gain  $\gamma = 2.5$  and in which case,  $z_6(0) = 1 + i$ . Asymptotically, all the systems converge to the trajectory of one averaged oscillator, thereby achieving dynamic consensus. In addition, the asymptotic behaviour of the network may also be reconstructed from that of the averaged motion, from  $t \approx 1$ s.

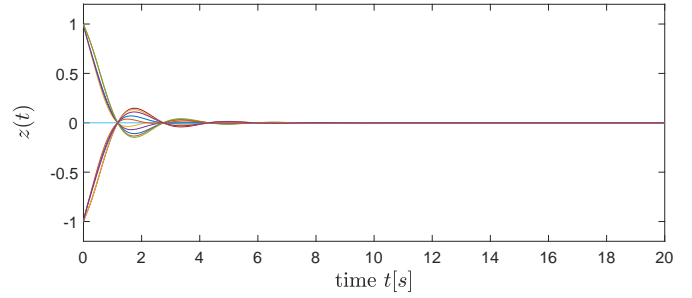


Fig. 7. Oscillators response with  $\gamma = 2.5$  with initial conditions as in (20)

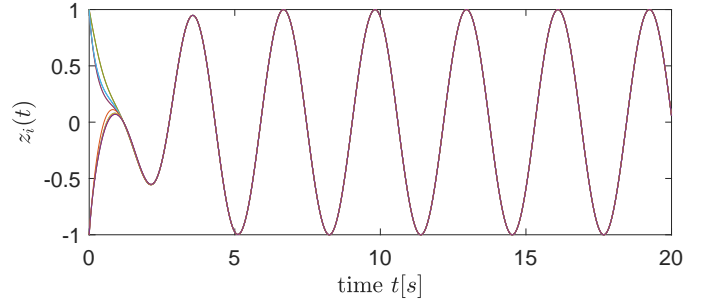


Fig. 8. Oscillators response with  $\gamma = 2.5$  with initial conditions as in (20), except for  $z_6(0) = -1$

#### 4. CLOSING REMARKS

Our results illustrate how to reconstruct the asymptotic behaviour of complex networked systems, with specific topology, via a model-reduction. Undergoing research is aimed at analysing the reduced-order model and considering other network topologies.

#### REFERENCES

- Golubitsky, M., Stewart, I., and Schaeffer, D.G. (2012). *Singularities and Groups in Bifurcation Theory: Volume II*. Springer Science & Business Media.
- Hasty, J., McMillen, D., Isaacs, F., and Collins, J.J. (2001). Computational studies of gene regulatory networks: in numero molecular biology. *Nature Reviews Genetics*, 2(4), 268–279.
- Maghenem, M., Panteley, E., and Loría, A. (2016). Singular-perturbations-based analysis of synchronization in heterogeneous networks: a case-study. In *Proc. 55th IEEE Conf. Decision and Control*, 2581–2586. Las Vegas, NV, USA.
- Panteley, E. and Loría, A. (2017). Synchronization and dynamic consensus of heterogeneous networked systems. *IEEE Trans. on Automatic Control*, 62(8), 3758–3773.
- Pogromsky, A.Y., Glad, T., and Nijmeijer, H. (1999). On diffusion driven oscillations in coupled dynamical systems. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 9(4), 629–644.
- Soriano, M.C., García-Ojalvo, J., Mirasso, C.R., and Fischer, I. (2013). Complex photonics: Dynamics and applications of delay-coupled semiconductor lasers. *Reviews of Modern Physics*, 85(1), 421–470.
- Tumash, L., Panteley, E., Zakharova, A., and Scholl, E. (2019). Synchronization patterns in Stuart-Landau networks: a reduced system approach. *The European Physical Journal B - Condensed Matter and Complex Systems*, 92(5).