Over one Century of Spectrum Analysis in Delay Systems: An Overview and New Trends in Pole Placement Methods

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Abstract: *Pole placement* represents a classical method for controlling finite-dimensional linear time-invariant systems, largely covered in the open literature. Basically, it consists of placing the poles of the closed-loop system in some predetermined loci in the complex plane. This paper discusses some of the extensions of this method to linear systems described by delay-differential equations. Among others, the finite spectrum assignment (FSA), the continuous pole placement (CPP) and the partial pole placement (PPP) approaches are presented and illustrated through some simple low-order dynamical systems.

Keywords: Delay, Stability, Pole placement, Pole assignment.

1. INTRODUCTION

In linear time-invariant (LTI) systems, one of the simplest ideas to control the dynamical behavior of the closed-loop system is to place the poles in some desirable loci in the complex plane. Such a method is called *pole placement* (e.g., Astrom and Murray (2009) and the references therein). Roughly speaking, the pole placement's main ingredients are (i) the perfect knowledge of the state variables; (ii) some appropriate *controllability* assumptions on the system, that is, the possibility to steer a dynamical system from an arbitrary initial state to an arbitrary final state via an apropos set of admissible control laws.

If the said method is easy to understand and to apply in the control of finite-dimensional LTI systems, its extension to systems described by delay-differential equations (DDEs) seems to be more involved. More precisely, two issues need to be addressed. First, the introduction of a suitable notion of *controllability* for delay systems, and, second, the *in-depth comprehension of the location of the poles of the closed-loop system in terms of the controller's parameters*.

The focus of this overview aims at the second topic and, as indicated in the title, it covers more than one hundred years of contributions in the area of the DDEs. For a good introduction to the controllability notions in finite- and infinite-dimensional systems also including the case of dynamical systems represented by DDEs, we refer to Antoulas et al. (2001). Finally, a deeper discussion of the existing methods (*D*-decomposition, τ-decomposition) to characterize the stability regions in the parameter-space can be found in Michiels and Niculescu (2014), and the references therein.

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To the best of the authors' knowledge, the first results on the spectrum location of linear systems represented by DDEs were published one century ago. In the late 1970s, the concept of pole-placement emerged in control theory in the guise of *Finite spectrum assignment*; Olbrot (1978); Manitius and Olbrot (1979), the upshot of which was to counterbalance the effect of delay by a prediction of the state over a delay interval, thereby downsizing the closed-loop system to a finite-dimensional plant;

There is more to pole placement for delay systems than a quasipolynomial interpolation problem. As a matter of fact, in Ackermann (1972) N poles of the system are assigned to (some) desired positions in the complex plane by N feedback parameters in the same fashion as in the finite-dimensional case. Nevertheless, in order to preclude the spillover effect, it is well-known that such an interpolation is an efficient placement if, and only if, the remaining spectral values of the closed-loop system are located to the left of the rightmost of assigned poles; that is, the assignment succeeds if the latter poles are dominant. However, this feature is not ensured in general as remarked in Vyhlidal et al. (2009); see also Ram et al. (2011).

More recently, building upon the effect of multiple roots on the stability of DDEs, a novel analytical pole placement strategy called *multiplicity-induced-dominancy* (MID) was devised in Boussaada et al. (2019). In fact, quite recent works established that a real root of maximal multiplicity is necessarily the dominant root for some classes of time-delay systems; a property we call *generic multiplicity-induced-dominancy* (GMID). The property was hinted at in Pinney (1958) albeit illustrated by simple low-order cases, with no endeavour to address the general case. To the best of the authors' knowledge, very few works have tackled this issue in a systematic fashion until recently; see Boussaada et al. (2016, 2019, 2018); Ramírez et al. (2016);

Mazanti et al. (2021a,b); Benarab et al. (2020); Mazanti et al. (2020).

Henceforth, the paper is organized as follows. Section 2 presents existing pole placement paradigms. Next, each section describes an existing pole placement paradigm where the main idea as well as the advantages and the drawbacks of each are discussed. Section 3 is devoted to the Finite Spectrum Assignment. Section 4 presents the Algebraic Pole Placement. Section 5 describes the Continuous Pole Placement. Section 6 centers on the recent Partial Pole Placement. Finally, we give in Section 7 several examples which illustrate the described methods.

Notations: We denote by $\mathscr{R}[s]$ the ring of polynomials in s over $\mathscr{R}$, $\mathscr{R}(s)$ is the quotient files of $\mathscr{R}[s]$, and $\mathscr{R}[s,z]$ denotes the ring of 2-D (two-variable) polynomials in s and z over $\mathscr{R}$. For $s \in \mathbb{C}$, $\Re(s)$ and $\Im(s)$ designate the real and imaginary part of s.

2. OVERVIEW OF PIONEERING WORKS ON POLE LOCATION AND POLE PLACEMENT

To the best of the authors' knowledge, the first results devoted to the spectrum location of dynamical systems described by DDEs go back to the 1920s and are due to Pólya. In fact, in Pólya (1920), the quasipolynomial entire functions have been extensively studied and the asymptotic distribution of their zeros has been explored by some elegant geometric approaches [1] which, unfortunately, have not been sufficiently exploited and extended to higher-order equations. Rather than the geometric investigation, in Langer (1929), an analytic treatment of the location of the roots of some low-order transcendental equations is given in a more precise way. Indeed, under some conditions, the roots are located in arbitrarily small sectors, and in each of these sectors the roots are additionally confined in a finite number of strips which are asymptotically of constant width.

Later, in Pontryagin (1942), some fundamental results concerning the zeros of quasipolynomials have been obtained. In fact, necessary and sufficient conditions are given for all solutions of $P(s, \mathrm{e}^s)$ to lie in the left half-plane, where $P(s,z)$ is a bi-polynomial in s, z. These results are provided by extending the methods used to prove the Routh-Hurwitz criterion for the zeros of polynomials in order to be of negative real part.

Next, in the early 1950s, Hayes (1950) proposed an efficient way to understand the asymptotic behaviour of solutions of first-order delay-differential equations (DDEs) including a pointwise delay through the employment of the spectral method and thanks to a deep investigation of the zeros of the entire function $g : \mathbb{C} \mapsto \mathbb{C}$, $g(s) := s\,\mathrm{e}^s - a$ thereby providing a complete characterization of the spectrum distribution of such a first-order equation (see also Wright (1959) for further discussions). It should be mentioned that such remarkable properties appeared to be closely related to the well-known *Lambert−W functions* (see, for instance, Yi et al. (2010) for some applications in control theory). Later, the result is generalized to the first-order DDE of *neutral* type [2] in Wright (1961) by forging direct methods for the computation of the corresponding real spectral values and for the derivation of the least upper bound of the said spectral abscissa. More recently, several works exploited Hayes results in control problems such as in delayed feedback and in stabilization problems. Unfortunately, Hayes' approach remains complicated and natural extensions to higher-order retarded or neutral DDEs do not exist.

Afterwards, a remarkable property of the spectrum distribution of low-order quasipolynomial functions with multiple spectral values has been hinted at since the late 50's in Pinney (1958). As a matter of fact, it turns out that for the first and second-order quasipolynomials, the corresponding spectral abscissa coincides with the multiple spectral value. Regrettably, despite its pioneering character, Pinney's work has made no attempt to address the general question since the employed approach seems quite difficult to extend to higher-order equations.

A classical and a standard way to count the number of unstable roots is to apply the well-known argument principle Ahlfors (1979). The said count may also be obtained, in an easier and more elegant way, by the inspection of argument variation. Actually, the combination of the qualitative behavior of both the real and the imaginary parts of the quasipolynomial function, allows a straightforward application of the Stepan-Hassard formula Stépán (1989); Hassard (1997).

Another strategy was thoroughly explored in a more adequate algebraic framework by Brethé and Loiseau (1998) via the introduction of the ring $\mathscr{R}$, i.e., the set of all meromorphic functions in $\mathbb{C}$ generically represented as $P(s, e^{-\tau s})/Q(s)$, where Q is a polynomial in the Laplace complex variable s, P is a bivariate polynomial in s and $e^{-\tau s}$, and τ is a fixed positive real number. In fact, one pre-eminent remit regarding the algebraic design of controllers of LTI differential time-delay systems is the algorithmic investigation of $\mathscr{R}$. In addition, the limitation of this approach was observed in the early 2000s in Engelborghs et al. (2001). Namely, a delayed controller is designed to stabilize a dynamical system described by a first-order scalar differential equation, however, numerically, the closed-loop system's stability is highly sensitive to infinitesimal uncertainties.

3. FINITE SPECTRUM ASSIGNMENT (FSA)

Main idea: To the best of the authors' knowledge, the FSA approach is the oldest paradigm, it is based on a predictor able to transform an infinite dimensional system into a finite dimensional one, (see Olbrot (1978); Manitius and Olbrot (1979)). When compared to the well-known Smith-Predictor, the FSA has the advantage of arbitrarily assigning the closed-loop poles and therefore can be applied to poorly damped and unstable processes Watanabe and Ito (1981); Wang et al. (1998).

Description of the method: Consider the following linear system with an input delay

$$\dot{x}(t) = A\,x(t) + B_0\,u(t) + B_1\,u(t-\tau), \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and τ is the delay of the system $(\tau > 0)$.

In Manitius and Olbrot (1979) it is proven that the feedback of the following form: $u(t) = F\,x(t) + F\int_{-\tau}^{0} \mathrm{e}^{-(\tau+\theta)A} B_1\,u(t+\theta)\,\mathrm{d}\theta$, where F is an $m \times n$ matrix, yields a finite spectrum of the closed-loop system. The location of this spectrum can be completely controlled by the choice of F under some suitable controllability conditions. This result remains true for the more general systems governed by

$$\dot{x}(t) = A\,x(t) + \int_{-\tau}^{0} \mathrm{d}\mu(\theta)\,u(t+\theta) \tag{2}$$

where μ is an $n \times m$ matrix function of bounded variation and the corresponding feedback has the following form

[1] geometric determination of the characteristic roots' distribution

[2] see, e.g, Hale and Verduyn Lunel (1993) for the basic properties of the solutions of DDEs as well as their classification

$$u(t) = F\,x(t) + F\int_{-\tau}^{0}\int_{\sigma}^{0} \mathrm{e}^{(\sigma-\theta)A}\,\mathrm{d}\mu(\sigma)\,u(t+\theta)\,\mathrm{d}\theta \quad (3)$$

Advantages and limitations: From a practical viewpoint, the digitization of the controller generated by the FSA is subject to a discretization which unfortunately induces the loss of the control of the closed-loop spectral values. In other words, one has spectral values that exceed the range that one has assigned (see Mondié and Michiels (2003); Engelborghs et al. (2001)) yielding the Spillover phenomena (the numerical controller parameters are not exactly the same as those computed via the analytical design method). Indeed, this has been explained by the sensitivity of the design to parameter variations. Accordingly, the instability of the difference part of the control law leads to the instability of the closed-loop system's solution, see Engelborghs et al. (2001). Notice also some concern with the complexity of calculations compared to other existing methods.

4. ALGEBRAIC POLE PLACEMENT (APP)

Main idea: We consider the algebraic design paradigm developed in Brethé and Loiseau (1996); Brethé (1997); Brethé and Loiseau (1998); Loiseau (2000). The principle of the APP consists in keeping the spectral values with a real part below a chosen threshold and removing from the spectrum some undesired spectral values (typically unstable roots) via an Euclidean-like division. Its main ingredient is an appropriate division in the ring of transfer functions corresponding to pointwise or particular distributed delays, which yields fractions over $\mathbb{R}(s, e^{-\tau s})$. Furthermore, an additional set of spectral values is assigned to define the exponential decay rate of the closed-loop system's solution. Even if the origin of this algebraic approach is inspired from the FSA, its methodology differs in many ways.

Description of the method: We refer to Brethé and Loiseau (1998). Consider the single-input delay system

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - i\,\tau) + \sum_{i=0}^{k} b_i u(t - i\,\tau), \quad (4)$$

where $x \in \mathbb{R}^n$ is the state of the system and $u \in \mathbb{R}$ is the output of the system. For all $i \in \{0,...,k\}$, $A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}^{n\times 1}$ and $\tau > 0$ is the delay of the system. Consider the control law:

$$u(t) = \int_{0}^{N}\Big(f(\theta)\,u(t-\theta) + g(\theta)\,x(t-\theta)\Big)\,\mathrm{d}\theta + \sum_{i=0}^{M} p_i x(t - i\,\tau), \quad (5)$$

where $N \in \mathbb{R}_+$, $f \in L_2([0,N],\mathbb{R})$, $g \in L_2([0,N],\mathbb{R}^{1\times n})$; $M \in \mathbb{N}$ and $\forall i \in \{1,...,M\}, p_i \in \mathbb{R}$. Applying the Laplace transform with a zero initial condition to (4) and (5), one has, respectively, $s\,x = A\,x + b\,u$ and $u = F_1\,u + F_2\,x$, where $A = \sum_{i=0}^{k} A_i\,\mathrm{e}^{-\tau s i}$, $b = \sum_{i=0}^{k} b_i\,\mathrm{e}^{-\tau s i}$, and $F_1 = \int_0^N f(\theta)\,\mathrm{e}^{-\theta s}\,\mathrm{d}\theta$; $F_2 = \int_0^N g(\theta)\,\mathrm{e}^{-\theta s}\,\mathrm{d}\theta + \sum_{i=0}^{M} p_i\,\mathrm{e}^{-\tau s i}$.

In the following, we provide a definition of the finite spectrum assignability in terms of the characteristic polynomial of the closed-loop system.

Definition 1. If there exist F_1 and F_2 such that

$$\det\begin{bmatrix} s\,\mathbb{I}_n - A & -b \\ -F_2 & 1 - F_1 \end{bmatrix} = \prod_{i=1}^{n}(s - \alpha_i)$$

for any set of n complex numbers α_i such that any $\alpha_i \in \{\alpha_1,...,\alpha_n\}$ with $\Im(\alpha_i) \neq 0$ appears in conjugate pair, then the system (4) is said to be finite-spectrum-assignable.

We consider the entire function which is the finite Laplace transform of a distributed delay equation, called elementary fraction Kamen et al. (1986) and defined as

$$\theta_\sigma(s) = \frac{1 - \mathrm{e}^{(-s+\sigma)\,\tau}}{s - \sigma}, \ \sigma \in \mathbb{C}.$$

Bézout-type identities. Consider $Q \in \mathbb{R}[s, \mathrm{e}^{-s\,\tau}]^{p\times p}$, and $P \in \mathbb{R}[s, \mathrm{e}^{-s\,\tau}]^{p\times m}$, $D \in \mathbb{R}[s, \mathrm{e}^{-s\,\tau}]^{m\times m}$, $N \in \mathbb{R}[s, \mathrm{e}^{-s\,\tau}]^{p\times m}$ which satisfy $Q^{-1}\cdot P = N\cdot D^{-1}$. Matrices Q and P are required to be admissible i.e., their determinant is monic in s. We can always find matrices Q, P, D and N that satisfy such a coprimeness condition (see Morf et al. (1977)).

Lemma 2. ([Morf et al. (1977)], Theorem 5.1) Let Q and P be two 2-D left-factor-coprime matrices over $\mathbb{R}[s, \mathrm{e}^{-s\,\tau}]$. Then, there exist E a polynomial matrix in s, and two matrices X, Y over $\mathbb{R}[s, \mathrm{e}^{-s\,\tau}]$ satisfying $Q\cdot X + P\cdot Y = E$.

Theorem 3. ([Brethé and Loiseau (1998)]) If $Rank[Q|P] = p, \forall s \in \mathbb{C}$, then, $\exists\, \mathscr{X} \in \mathscr{R}[s]^{p\times p}$ and $\mathscr{Y} \in \mathscr{R}[s]^{m\times p}$ such that

$$Q\cdot\mathscr{X} + P\cdot\mathscr{Y} = \mathbb{I}_p.$$

Theorem 4. ([Brethé and Loiseau (1998)]) The system (4) is finite-spectrum-assignable if, and only if, it is spectrally controllable, i.e., $Rank\Big[s\,\mathbb{I}_n - A, b\Big] = n, \quad \forall s \in \mathbb{C}$.

Advantages and limitations: This kind of algebraic method becomes interesting when one knows beforehand the number and location of undesired spectral values. However, in general, standard complex analysis techniques, such as the argument principle, only provide the number of undesired spectral values. In addition, the standard numerical methods only produce approximations, so that, in practice, a considerable symbolic/numeric issue arises, since the method requires their exact value. Building effective algorithms to overcome the latter symbolic/numeric issue remains challenging; the reconstruction of a polynomial characterizing an exact spectral value from a polynomial characterizing its approximation represents an additional complexity for rendering the approach systematic. Besides, another challenging question related to this algebraic paradigm is the design of efficient and algorithmic calculations of the involved objects such as the ring elements derived from the corresponding Besout's identity. Solving the emphasized issues will surely break new ground for the pole placement algebraic paradigm.

5. CONTINUOUS POLE PLACEMENT (CPP)

Main idea: The CPP method introduced in Michiels et al. (2002) is the first "automated" numerical pole placement for retarded time-delay systems. The CPP paradigm consists in defining a function that represents the spectral abscissa and to exploit its dependency on the controller parameters, and the control strategy can be summarized as follows: "Shift" the unstable characteristic roots from $\mathbb{C}_+$ to $\mathbb{C}_-$ in a "quasi-continuous" way subject to the strong constraint that, during this shifting action, stable characteristic roots are not crossing the imaginary axis from $\mathbb{C}_-$ to $\mathbb{C}_+$. We refer to Michiels and Niculescu (2014) and references therein for further insights on the number of controlled characteristic roots (which is related to the available degrees of freedom induced by the controller structure) as well as the interpretation of CPP as a local strategy to solve an appropriate optimization problem where the objective function (rightmost root) is not differentiable. It is worth mentioning that

CPP, initially applied to delay systems of retarded type, was extended to neutral systems in Michiels and Vyhlidal (2005).

Description of the method: In order to describe this method, we consider the investigation of the stability of

$$\dot{x}(t) = A\,x(t) + B\,u(t-\tau), \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and τ is the positive delay. Consider the linear control law:

$$u = K^T x, \quad K \in \mathbb{R}^{n\times 1}, \tag{7}$$

where K^T is the transpose of K. This static state feedback controller reveals the link between the CPP method and the classical pole placement method.

The CPP method consists in applying slight changes to the feedback gain so as to move the unstable eigenvalues to the left half-plane. The key steps for this method may be declined as follows. First, the rightmost eigenvalues are computed; an automatic method for doing so is provided in Engelborghs and Roose (1999). Second, the sensitivity of the rightmost eigenvalues with respect to changes in the feedback gain is assessed. Next, the rightmost eigenvalues are pushed in the direction of the left half-plane by applying a slight alteration to the feedback gain, owing to the aforementioned sensitivities. Lastly, the rightmost uncontrolled eigenvalues are monitored: if necessary, the number of controlled eigenvalues shall be increased ; stop when stability is reached or when the available degrees of freedom in the controller do not allow to further reduce $\sup \Re(s)$; otherwise, resume step 2. These different steps are thoroughly detailed in Michiels et al. (2002).

Advantages and Drawbacks: Unlike FSA method, CPP approach does not render the closed-loop system finite-dimensional, but consists instead of controlling the corresponding rightmost eigenvalues. Such an idea represents a simple generalization of the pole placement for finite-dimensional systems represented by ordinary differential equations.

6. PARTIAL POLE PLACEMENT (PPP)

Main idea: The strategy of PPP consists in tuning standard controllers via the aforementioned multiplicity-induced-dominancy (MID) property. Namely, one needs to determine the conditions under which a given multiple root of a the characteristic equation is dominant.

Description of the method: The procedure of PPP is carried out in several steps. First, conditions on the system's parameters guaranteeing the existence of a multiple root are obtained. Second, an affine change of variable is required to normalize the characteristic equation. Next, a bound on the imaginary part of roots of the normalized characteristic equation in the complex right half-plane is derived. In fact, the frequency bound is the main ingredient for the proof of the dominance, for this purpose, a pseudo-code listing the instructions to be followed to target a suitable frequency bound is given in Benarab et al. (2022). The idea is to find an adequate truncation order of the exponential term appearing in the normalized quasipolynomial which depends only on the real part of its roots. By using a purely polynomial analysis, one is able to obtain a suitable bound of the imaginary part of the roots. Lastly, a certification of the dominance of the multiple root is established.

Advantages and Drawbacks: Unlike the APP method, when it comes to the PPP method, *a priori* knowledge on the number of unstable roots and/or their location is not required. On the one hand, it is reported that the PPP is easy to implement and robust to uncertain delays or the model's parameters Michiels et al. (2017) and on the other hand, it applies to retarded as well as neutral systems Boussaada et al. (2022). Furthermore, it provides a procedure to assess the critical delay, see for instance Molnar et al. (2021). The main limitation of the PPP is that the actual knowledge allows to assign the spectral abscissa only on the real axis, aside from few isolated cases, see for instance Mazanti et al. (2020)), which may not be relevant in some applications. We have yet to fathom the extent of this property, notwithstanding the fact that often small delays are required to perform the MID property which may again be a drawback in some applications.

7. ILLUSTRATIVE EXAMPLES

7.1 Example 1

In order to illustrate the FSA method, consider the system

$$\dot{x}(t) = x(t) + u(t-1). \tag{8}$$

The feedback in (3) reads in the case of system (8) as

$$u(t) = f\,x(t) + f\,z(t), \quad z(t) = \int_{-1}^{0} \mathrm{e}^{-(1+\theta)}\,u(t+\theta)\mathrm{d}\theta. \tag{9}$$

The transfer function of the new system in (9) is $\frac{\mathrm{e}^{-1}-\mathrm{e}^{-s}}{s-1}$ and the pole $s=1$ is cancelled by a zero of $\mathrm{e}^{-1}-\mathrm{e}^{-s}$. The Laplace transform of (8)-(9) yields the characteristic equation of the closed-loop system

$$\begin{pmatrix} s-1 & -\mathrm{e}^{-s} \\ -f & 1-f\,\dfrac{\mathrm{e}^{-1}-\mathrm{e}^{-s}}{s-1} \end{pmatrix} \begin{pmatrix} x(s) \\ u(s) \end{pmatrix} = 0. \tag{10}$$

Looking for the characteristic roots amounts to computing the zeros of the determinant of the system, we have $(s-1)\left(1-f\,\frac{\mathrm{e}^{-1}-\mathrm{e}^{-s}}{s-1}\right) - f\,\mathrm{e}^{-s} = 0$, which yields the zero $s = 1+\frac{f}{\mathrm{e}}$. Note that the spectrum of the closed-loop system is finite, and that the pole $s=1$ is not cancelled by a corresponding zero, but it is shifted from $s=1$ to $s=1+\frac{f}{\mathrm{e}}$ by the feedback.

7.2 Example 2

In order to illustrate the APP method, we consider the system

$$\dot{x}(t) = A_0\,x(t) + A_1\,x(t-1) + b_1\,u(t-1)$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In this example, we aim to assign the poles at -1. We refer to [Brethé and Loiseau (1998), proof of Theorem 9].

The feedback law which assigns the poles of the system at -1 is given by

$$\begin{aligned} u(t) =& 5x_1(t) + \left(1 + \frac{4\left(-1-\mathrm{e}^{-1}+\mathrm{e}^{-2}\right)}{\mathrm{e}^{-2}}\right) x_2(t-1) \\ &+ \int_0^1 \Big(\left(-\tau+4-4\,(1+\mathrm{e})\,\mathrm{e}^{\tau}\right) u(t-1-\tau) \\ &\qquad + (1-4\mathrm{e}^{\tau})\,u(t-\tau) + (-\tau+3)\,x_1(t-\tau) \Big) \mathrm{d}\tau. \end{aligned}$$

7.3 Example 3

In order to illustrate the CPP method, we consider the system

$$\dot{x}(t) = Ax(t) + bu(t-\tau), \quad u = K^T x(t), \tag{11}$$

where

$$A = \begin{pmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{pmatrix}, \quad b = \begin{pmatrix} -0.1 \\ -0.2 \\ 0.1 \end{pmatrix}, \tau = 5 \tag{12}$$

The open-loop system is unstable ($\sup \Re(s) = 0.108$) and with the feedback u in (11) where $K = (0.719\ 1.04\ 1.29)^T$. The spectral abscissa is shown in Fig. 1 as a function of the delay τ. Note that the particular control law achieves stability for $\tau = 0$, the system is unstable for the nominal delay $\tau = 5$, also, the characteristic roots must cross the imaginary axis from left to right and this occurs when $\tau_{\text{crit}} = 3.95$.

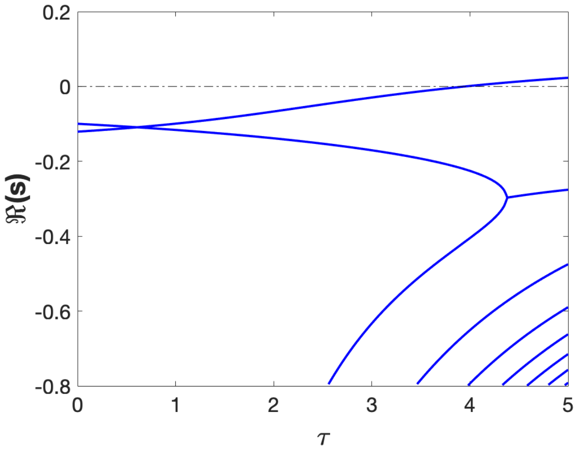

Fig. 1. Rightmost eigenvalues of the system (11)–(12) as a function of the delay τ.

7.4 Example 4

Consider the problem of stabilization of the classical harmonic oscillator, by a proportional-derivative controller

$$\ddot{x}(t) + a_0 x(t) + b_1 \dot{x}(t-\tau) + b_0 x(t-\tau) = 0,$$

for which the characteristic equation is

$$\Delta(s) = s^2 + a_0 + (b_1 s + b_0)\, e^{-\tau s}, \tag{13}$$

which admits a real spectral value at $s_\pm = \frac{1}{\tau}(-2 + \sqrt{-\tau^2 a_0 + 2})$, if, and only if, the system parameters satisfy

$$b_0 = \frac{2\left(\tau^2 a_0 + 5\tau s_\pm + 3\right) e^{\tau s_\pm}}{\tau^2}, \quad b_1 = \frac{2\left(\tau s_\pm + 1\right) e^{\tau s_\pm}}{\tau}. \tag{14}$$

Under the previous conditions (14), the spectral value $s = s_+$ is necessarily a dominant root for (13), unlike s_- which cannot be the spectral abscissa.

By substituting the gains of the controller (14) in the characteristic equation (13) with $a_0 = 1$, we obtain the particular quasipolynomial Δ_0. Now, imposing $s = 0$ to be a real root of Δ_0, one is able to obtain numerically the corresponding delay $\tau_0 \approx 1.0581$ which yields a root on the imaginary axis.

Fig. 2 distinctly illustrates the effect of the delay on the multiple root. Actually, for the delay $\tau = \tau_0 \approx 1.0581$ the multiple root is at $s = 0$, then the reduction of the value of the delay τ pushes the roots continuously from the imaginary axis to the right.

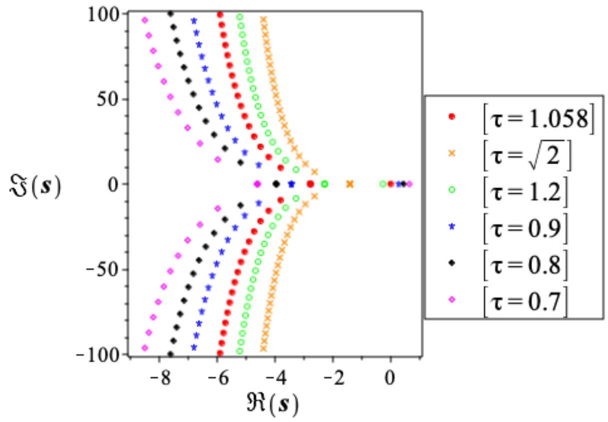

Fig. 2. Translation of the spectrum distribution of Δ_0 according to the delay change.

8. CONCLUSION

This paper discusses the existing pole placement paradigms. It gives a presentation and an illustration of finite spectrum assignment, algebraic pole placement, continuous pole placement and the partial pole placement method, via some simple dynamical systems. Nevertheless, the extension of the application of the control to infinite dimensional systems described by DDEs remains a challenging task. Indeed, the question of pole placement remains an open problem, adding to this, the problem of coprimality for quasipolynomials with multiple delays.

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