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# Rolling Balance Board Robust Stabilization: A MID-based Design

Amina Benarab<sup>\*,\*\*</sup> Csenge A. Molnar<sup>\*\*\*</sup> Islam Boussaada<sup>\*,\*\*</sup>  
Karim Trabelsi<sup>\*\*</sup> Tamas Insperger<sup>\*\*\*\*</sup> Silviu-Iulian Niculescu<sup>\*</sup>

<sup>\*</sup> *Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des signaux et systèmes (L2S), 91190 Gif-sur-Yvette, France  
(e-mail: first name.last name@centralesupelec.fr).*

<sup>\*\*</sup> *Institut Polytechnique des Sciences Avancées (IPSA), 63 boulevard de Brandebourg, 94200 Ivry-sur-Seine, France*

<sup>\*\*\*</sup> *Budapest University of Technology and Economics, Faculty of Mechanical Engineering, Department of Mechatronics, Optics and Mechanical Engineering Informatics*

<sup>\*\*\*\*</sup> *Budapest University of Technology and Economics, Faculty of Mechanical Engineering, Department of Applied Mechanics*

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## Abstract:

This paper considers the stabilization of a rolling balance board by means of the multiplicity-induced-dominancy property. A two degree-of-freedom mechanical model of a human balancing on a rolling balance board is analyzed in the sagittal plane. The human body is modeled by an inverted pendulum which connects to the balance board through the ankle joint. The system is stabilized by the ankle torque managed by the central nervous system (CNS). The action of the CNS is modeled by a delayed full state feedback: a pointwise delay stands for all latencies in the neuromechanical system (reaction time, neuromechanical lag, etc.). The aim of the paper is to achieve a good occurrence in terms of the decay rate, it shows the links with dominancy and with the exponential stability property of the solution.

*Keywords:* Human balance, exponential stability, Multiplicity-Induced-Dominancy.

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## 1. INTRODUCTION

The intrinsic brain activity is spatio-temporally structured, and there exists a few activities for which the time heterogeneity is essential. The human balancing belongs to this category. The comprehension of human balancing is a subject of growing interest since a principal contributor to falls among older adults is an age-related decline in balance. As a matter of fact, in older adults, falls are amongst the most common causes of accidental deaths, and in non-fatal cases the costs related to the treatment of fall-related sequelae are increasing as a consequence of the growing age of populations.

The passive biomechanics of human balance are unstable as they are comprised of interconnected inverted pendulums that are each unstable. The human balance system is the sensorimotor system that permits us to stand upright, walk, etc. Balance may be defined as the ability to maintain equilibrium in a gravitational field by keeping or returning the center of body mass over its base of support Horak (1987). As such, successfulness of balance may be measured by the ability to perform quiet stance, compensatory postural reactions and anticipatory postural responses. This ability is also assessed by reaction time or the controller of the central nervous system (CNS). Hence, simple balancing tasks may be investigated by a mechanical analysis. Stick balancing on fingertip Insperger

and Milton (2014), quiet standing Eurich and Milton (1996); Maurer and Peterka (2005); Kiemel et al. (2011); Nomura et al. (2013); Pasma et al. (2017), ball and beam balancing Buza et al. (2020) and standing on a balance board Chagdes et al. (2013); Cruise et al. (2017); Chumacero-Polanco and Yang (2019) have been deeply researched. Indeed, inquiries are based on the mechanical model of a single inverted pendulum, more complex tasks require multi-degree-of-freedom models.

In this paper, the mathematical model of balancing on a uniaxial rolling balance board in the sagittal plane (Fig. 1) is investigated. The balance board has a configurable geometry: the radius  $R$  of the wheels and the elevation  $h$  between the top of the wheel and the board can be adjusted as shown in Fig. 1. Preliminary computations and experiments performed by human subjects showed that the aforementioned parameters highly affect the stabilizability of the associated mathematical model. Standing on the balance board becomes harder as the wheel radius and the board elevation decrease. In case of greater radii ( $R > 100$  mm), balancing subjects use the musculature at the ankle to maintain the equilibrium, so that the human body can be considered as a single inverted pendulum. In contrast, hip strategy is dominant for smaller radii, which indicates a double inverted pendulum model for the human body. In this work, greater wheel radii are considered and therefore a two-degree-of-freedom mechanical model is analyzed involving the balance board and the human body.

Imbalance may be due to a multitude of age-associated declines in sensorimotor function, including somatosensation, vestibulo-

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lar function, vision, cognition, and strength. Stabilization of the human body and the balance board is performed by a control process governed by the CNS. Visual, vestibular and somatosensory systems obtain information about the spatial orientation of the human body. The information is delivered to the brain, CNS determines the necessary interaction to maintain the balance after processing the signals and sends an instruction to the musculature. The process described above requires definite time called reaction time. Consequently, the mathematical model of balancing tasks involves a delayed control law, here a proportional-derivative (PD) feedback with constant feedback delay Kowalczyk et al. (2012); Varszegi et al. (2016), since the visual and vestibular system perceive position and velocity, respectively. Other frequently used neuromuscular control models are for instance delayed proportional-derivative-acceleration feedback Insperger and Milton (2014); Zhang et al. (2018), intermittent control Asai et al. (2009); Cabrera and Milton (2002); Gawthrop et al. (2014); Milton et al. (2009); Nomura et al. (2013) and predictor feedback Insperger and Milton (2014); Milton et al. (2016). It was shown in Lehotzky and Tamás (2014) and Lehotzky (2016) that an integral term does not improve the stabilizability of the system in the presence of feedback delay, therefore here we concentrate only on PD feedback.

Note that even though reaction delays of different sensory systems are different, they are still at the same scale Zana and Zelei (2020). Consequently, in most of the studies related to human balancing, the delays associated with different sensory organs are assumed to be the same Asai et al. (2009); Chagdes et al. (2013); Varszegi et al. (2016). In case of quiet stance, the feedback delay is estimated to be 100-200 ms Asai et al. (2009); Goodworth and Peterka (2010). Balancing on an unstable, moving surface such as the balance board or skateboard is a more complicated task, therefore the reaction time is higher: 150-300 ms Chagdes et al. (2013); Varszegi et al. (2016); Molnar et al. (2018).

From a control theory viewpoint, a recent safe control methodology called *partial pole placement* (PPP) Boussaada et al. (2020a, 2021), based on the assignment of the closed-loop dominant solution's decay rate, shows that under appropriate conditions a multiple spectral value is the rightmost; see for instance Boussaada et al. (2018a,b); Mazanti et al. (2021, 2020); Boussaada and Niculescu (2016b,c); Boussaada et al. (2016, 2022). The PPP strategy relies on various properties of the spectrum distribution of delay systems, in particular, the *multiplicity-induced-dominancy* (MID) property which establishes the conditions under which a given multiple complex zero of a quasipolynomial is dominant. For instance, in the generic quasipolynomial case, the real root of maximal multiplicity is necessarily the dominant (GMID). However, multiple roots with intermediate admissible multiplicities may be dominant or not. Thanks to this property, an ensued control strategy is proposed in Boussaada et al. (2020b); Balogh et al. (2022); Ma et al. (2022), which consists in assigning a root with an admissible multiplicity once appropriate conditions guaranteeing its dominancy are determined. Furthermore, the MID property may be used to tune standard controllers. For instance, in Ma et al. (2022) it is applied to the systematic tuning of the stabilizing PID controller of a first order plant. Here, we aim at assigning dominant multiple real roots with admissible codimensions and the MID property is utilized for the mechanical model of human balancing on rolling balance board in the

sagittal plane. Delayed state feedback is assumed, which gives the governing equation the form of a delay differential equation (DDE). The rightmost characteristic root (called dominant root) is minimized by tuning the control gains according to the MID concept.

The main contribution of the paper is to explore the use of the MID property for the rolling balance board stabilization. It should be mentioned that the particular structure of the system's dynamics does not allow the use of any of the existing MID results straightforwardly. Thus, for instance, due to the sparsity of the open-loop transfer function, the generic MID cannot be reached and the characterization of the generic MID proposed in Mazanti et al. (2021) is not valid. Moreover, the corresponding plant is not real rooted but its roots are located on real and imaginary axis and the ideas and the approach proposed in Balogh et al. (2022) cannot apply.

The paper is organized as follows. Section 2 presents some prerequisites pertaining to quasipolynomials. In Section 3, human balancing on a rolling balance board in the sagittal plane model is described. Section 4 is devoted to the mathematical model. The main result is presented in Section 5. Finally, Section 6 is dedicated to the illustration of the obtained result.

**Notations:** In this paper, the following notations are to be used:  $\mathbb{R}$  ( $\mathbb{R}_+$ ) and  $\mathbb{C}$  denote the sets of (non-negative) real numbers, and complex numbers, respectively. We denote by  $\Re(s)$  and  $\Im(s)$ , respectively, the real and imaginary part of the complex variable  $s$ . For a (quasi)polynomial  $P(\Delta)$ ,  $\deg(P)$  denotes its degree.

## 2. PREREQUISITES AND PROBLEM FORMULATION

In the study of linear time-invariant (LTI) dynamical systems represented by DDEs, the characteristic function is expressed as a *quasipolynomial*, that is formally defined as follows.

*Definition 1.* A quasipolynomial is a particular entire function  $\Delta: \mathbb{C} \rightarrow \mathbb{C}$  which may be written as follows

$$\Delta(s) = \sum_{i=0}^{\ell} P_i(s) e^{-\tau_i s}, \quad (1)$$

where  $\ell$  is a positive integer,  $\tau_i \in \mathbb{R}_+$  ( $i = 0 \dots \ell$ ) are pairwise distinct numbers and  $P_i$  ( $i = 0 \dots \ell$ ) are polynomials of degree  $d_i \geq 0$ . Its degree  $\deg(\Delta)$  is given by

$$\deg(\Delta) = \ell + \sum_{i=0}^{\ell} \deg(P_i).$$

An important result in the literature, known as *Pólya-Szegő bound* and denoted  $PS_B$  (see; (Pólya and Szegő, 1972, Problem 206.2, page 144 and page 347), Boussaada and Niculescu (2016a)) plays an important role in characterizing the maximal allowable multiplicity of a characteristic root of a quasipolynomial. More precisely, we have the following.

*Proposition 2.* Let  $\Delta$  be a quasipolynomial of degree  $D = \deg(\Delta)$ . Then, any characteristic root  $s_0 \in \mathbb{C}$  of  $\Delta$  exhibits a multiplicity at most equal to  $D$ .

In what follows, we give a precise definition of the *dominant root*.

*Definition 3.* A characteristic root  $s_0 \in \mathbb{C}$  of  $\Delta$  is said to be a *dominant* (respectively, *strictly dominant*) *root* if for all other characteristic roots  $\hat{s} \in \mathbb{C} \setminus \{s_0\}$  of  $\Delta$ , the following inequality holds:  $\Re(\hat{s}) \leq \Re(s_0)$  (respectively,  $\Re(\hat{s}) < \Re(s_0)$ ).

It turns out that, for the characteristic equations corresponding to delay systems, the real roots of maximal multiplicity are necessarily dominant. This property is known as Generic Multiplicity-Induced-Dominancy (GMID for short) and consists in conditions under which a given real root of maximal multiplicity is necessarily dominant. However, multiple roots with intermediate admissible multiplicities may be dominant or not. As for the case of a root of strictly intermediate multiplicity, one must look for conditions on the free parameters of the system for which the former is dominant, this property is called Multiplicity-Induced-Dominancy (MID).

### 3. MECHANICAL MODEL

Human balancing on a rolling balance board in the sagittal plane is described by a two-degree-of-freedom mechanical model as shown in Fig. 1. Similar mechanical models involving a double inverted pendulum can be found in the corresponding literature for various human balancing tasks (Pinter et al., 2008; Suzuki et al., 2012; Morasso et al., 2019; Molnar et al., 2021b,a). The human body is modeled as a homogeneous rigid bar and the balance board is assumed to roll on the horizontal ground. The mass and the height of the human body are denoted by  $m_h$  and  $l$  respectively, therefore the mass moment of inertia of the human body becomes  $I_h = m_h l^2 / 12$  for the center of gravity.

The ankle height is denoted by  $f$  (see Fig. 1). The ankle joint is located on the left side of the symmetry axis of the balance board expressed by distance  $e$ . The passive damping of the muscles is neglected and the intrinsic ankle stiffness was considered as a torsional spring of constant stiffness

$$s_t = 0.8 m_h g \frac{l}{2}, \quad (2)$$

following Loram and Lakie (2002); Molnar et al. (2021b); Casadio et al. (2005).

The elements of the balance board were made from plywood with 21 mm thickness. The geometry of the balance board was designed so that two parameters, the wheel radius  $R$  and the board elevation  $h$ , can be adjusted. The difficulty of standing on the balance board can be influenced by these two parameters. The location of the center of gravity  $l_b$ , the mass  $m_b$  and the mass moment of inertia  $I_b$  of the balance board were calculated based on the actual set of the adjustable parameters  $R$  and  $h$ .

The governing equation of the motion was derived with Lagrange's equation of the second kind and were linearized about the upper unstable equilibrium. The linearized equation of motion reads

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{S}\mathbf{q}(t) = \mathbf{Q}(t), \quad (3)$$

where  $\mathbf{M}$  is the mass matrix,  $\mathbf{S}$  is the stiffness matrix, and  $\mathbf{Q}(t)$  stands for the vector of generalized forces. The vector of generalized coordinates is

$$\mathbf{q}(t) = \begin{bmatrix} \varphi(t) \\ \vartheta(t) \end{bmatrix} \quad (4)$$

where  $\varphi$  is the inclination angle of the human body and  $\vartheta$  is the inclination angle of the balance board, both measured from the equilibrium position. The vector of generalized forces reads

$$\mathbf{Q}(t) = \mathbf{H}T(t), \quad (5)$$

where

$$\mathbf{H} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (6)$$

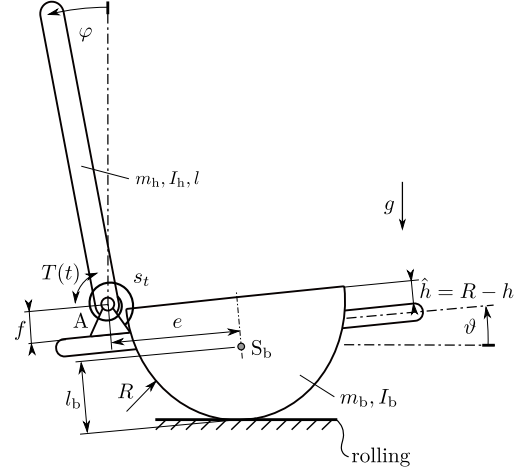


Fig. 1. Two-degree-of-freedom mechanical model of human balancing on a uniaxial rolling balance board in the sagittal plane.

The operation of the central nervous system (CNS) was modeled as full state feedback with a constant lumped delay, which involves all the latencies (reaction time, neuromuscular lag) in the feedback loop. The control torque acts at the ankle joint and becomes

$$T(t) = P_\varphi \varphi(t - \tau) + D_\varphi \dot{\varphi}(t - \tau) + \quad (7)$$

$$P_\vartheta \vartheta(t - \tau) + D_\vartheta \dot{\vartheta}(t - \tau) \quad (8)$$

where  $P_\varphi$ ,  $P_\vartheta$ ,  $D_\varphi$  and  $D_\vartheta$  are the proportional and derivative control gains for  $\varphi$  and  $\vartheta$ .

### 4. MATHEMATICAL MODEL

The system can be written in state space form by introducing  $\mathbf{x}(t) = [\mathbf{q}(t), \dot{\mathbf{q}}(t)]^T$ . Reformulation of (3) gives a compact form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad (9)$$

with

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t - \tau) \quad (10)$$

where

$$\mathbf{K} = [P_\varphi \ P_\vartheta \ D_\varphi \ D_\vartheta]. \quad (11)$$

The state matrix is

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{S} & \mathbf{0} \end{bmatrix} \quad (12)$$

where  $\mathbf{0}$  and  $\mathbf{I}$  stand for the  $2 \times 2$  zero and identity matrices and the input matrix is

$$\mathbf{B} = [0 \ 0 \ (\mathbf{M}^{-1}\mathbf{H})^T]^T \quad (13)$$

The characteristic equation of (9) can be given in the form

$$\det(\lambda \mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{K}e^{-s\tau}) = 0, \quad (14)$$

which reduces to

$$\begin{cases} \Delta(s) = P_0(s) + P_\tau(s)e^{-s\tau} & \text{with} \\ P_0(s) = s^4 + a_2s^2 + a_0 & \text{and} \\ P_\tau(s) = b_3s^3 + b_2s^2 + b_1s + b_0. \end{cases} \quad (15)$$

Note that in particular,  $P_0(s)$  is a polynomial which has either real roots or complex conjugate roots. It is important to recall that for this type of model, the approach proposed in Balogh et al. (2022) does not work since it requires a real-rooted

polynomial  $P_0$ . Since  $a_0$  is typically a negative parameter, the change of variables ( $\tilde{s} \rightarrow \sqrt[4]{-a_0} s$ ) allows to reduce the analysis to the normalized characteristic function:

$$\begin{cases} \tilde{\Delta}(s) = \tilde{P}_0(\tilde{s}) + \tilde{P}_\tau(\tilde{s})e^{-\tilde{s}\tilde{\tau}} & \text{with} \\ \tilde{P}_0(\tilde{s}) = \tilde{s}^4 + \tilde{a}_2\tilde{s}^2 - 1 & \text{and} \\ \tilde{P}_\tau(\tilde{s}) = \tilde{b}_3\tilde{s}^3 + \tilde{b}_2\tilde{s}^2 + \tilde{b}_1\tilde{s} + \tilde{b}_0, \end{cases}$$

where

$$\tilde{\tau} = \sqrt[4]{-a_0} \tau, \tilde{a}_2 = a_2/\sqrt{-a_0}, \tilde{b}_k = b_k/(-a_0)^{\frac{4-k}{4}} \text{ for } k \in \llbracket 0, 3 \rrbracket. \quad (16)$$

For the sake of simplicity, the symbol  $\tilde{\cdot}$  is omitted. In the sequel, the normalized quasipolynomial function is studied:

$$\Delta(s) = s^4 + a_2s^2 - 1 + (b_3s^3 + b_2s^2 + b_1s + b_0) e^{-s\tau} \quad (17)$$

where  $a_2$  stands for the plant parameter which contains all the stiffness and inertial terms. Coefficients  $b_k$  can be considered as control parameters since they are the linear combination of the control gains:

$$b_k = b_k(P_\varphi, P_\vartheta), \text{ for } k \in \llbracket 0, 3 \rrbracket. \quad (18)$$

Finally, the transformation (16) is used to reconstruct the appropriate stabilizing conditions for (15).

## 5. MAIN RESULTS

*Theorem 4.* Let  $\Delta$  be the quasipolynomial given in (17). If the parameters  $a_2$  and  $\tau$  are left free, then the maximal multiplicity of a given root  $s_0$  of (17) is 5.

*Proof 1.* It is recovered that the admissible multiplicity of a real spectral values of the characteristic quasipolynomial is bounded by the generic *Pólya and Szegő bound*, which is equal to the degree of the corresponding quasipolynomial. In particular, according to Definition 1, the degree of  $\Delta$  in (17) is equal to  $PS_B = 8$ . On the other hand, due to the sparsity of the open-loop polynomial ( $P_0(s) = s^4 + a_2s^2 - 1$ ), such a bound cannot be reached.

*Theorem 5.* The root  $s_0$  of (17) has multiplicity 5 if, and only if, the system parameters satisfy:

$$b_k = e^{s_0\tau} \tau^{k-4} f_k(s_0, \tau), \quad k \in \llbracket 0, 3 \rrbracket \quad (19)$$

where  $f_k$  are polynomials in  $s_0$  parametrized in  $a_2$  and  $\tau$  which will be given later, such that,  $f_0$  and  $f_1$  are of degree less than or equal to 3 while  $f_2$  and  $f_3$  are of degree 3. Moreover, if (19) is satisfied then  $s_0$  is necessarily dominant).

*Proof 2.* First, the vanishing of the quasipolynomial  $\Delta$  given in (17) yields the elimination of the exponential term as a rational function in  $s$ :

$$e^{-\tau s} = -\frac{P_0(s)}{P_\tau(s)} \quad (20)$$

where  $P_0(s) = s^4 + a_2s^2 - 1$ , and  $P_\tau(s) = b_3s^3 + b_2s^2 + b_1s + b_0$ .

Next, to investigate potential roots with algebraic multiplicity 5, one substitutes the obtained equality (20) in the ideal  $\mathcal{I}_5$  generated by the first four derivatives of  $\Delta$ , that is,  $\mathcal{I}_5 = \langle \partial_s \Delta, \partial_s^2 \Delta, \partial_s^3 \Delta, \partial_s^4 \Delta \rangle$ . This allows to investigate the variety of four algebraic equations in 7 unknowns  $a_2, s, \tau, (b_k)_{k \in \llbracket 0, 3 \rrbracket}$ :

$$\begin{aligned} P_0(s)P_\tau(s)\tau + P'_0(s)P_\tau(s) - P'_\tau(s)P_0(s) &= 0, \\ -P_0(s)P_\tau(s)\tau^2 + 2P'_\tau(s)P_0(s)\tau + P_0^{(2)}(s)P_\tau(s) & \\ -P_\tau^{(2)}(s)P_0(s) &= 0, \\ P_0(s)P_\tau(s)\tau^3 - 3P'_\tau(s)P_0(s)\tau^2 + 3P_\tau^{(2)}(s)P_0(s)\tau & \\ + 24sP_\tau(s) &= 0, \\ -P_0(s)P_\tau(s)\tau^4 + 4P'_\tau(s)P_0(s)\tau^3 - 6P_\tau^{(2)}(s)P_0(s)\tau^2 &= 0. \end{aligned}$$

The obtained system is a linear system in the unknowns  $(b_k)_{k \in \llbracket 0, 3 \rrbracket}$ . Using standard elimination techniques, one obtains a set of three solutions; the first one, asserts that  $b_k = 0$  for  $k \in \llbracket 0, 3 \rrbracket$ , the second one corresponds to  $s$  as a root of the open-loop polynomial ( $s^4 + a_2s^2 - 1 = 0$ ) with  $b_2 = -s^3b_1 - s^2b_0 - sa_2b_1 - sb_3 - a_2b_0$  and the last solution corresponds to  $s = s_0$  as a real root of an elimination polynomial of degree 4 in  $s$ , and  $b_k$  as rational functions in  $(s_0, \tau)$  given by

$$b_k = e^{s_0\tau} \tau^{k-4} f_k(s_0, \tau), \quad k \in \llbracket 0, 3 \rrbracket,$$

with

$$f_k(s_0, \tau) = \sum_{j=0}^3 \alpha_{k,j}(\tau) \tau^j s_0^j, \quad k \in \llbracket 0, 3 \rrbracket.$$

Observe that the first solution corresponds to the open-loop system while the second solution is inconsistent with respect to the transcendental-term elimination (20). So that, these two solutions are discarded. In addition, substituting conditions of the third solution in (20) yields the explicit values of the gain  $b_k, k \in \llbracket 0, 3 \rrbracket$  allowing to tune the parameters as provided in (2), which concludes the announced result.

In our approach, we derive a bound on the imaginary part of roots of the quasipolynomial in the complex right half-plane. In fact, Algorithm 1 gives an appropriate bound on the imaginary part of the characteristic roots (see Benarab et al. (2022)).

**Algorithm 1** Estimation of a frequency bound for time-delay differential equations with a single delay

- 1: **Input:**  $\tilde{\Delta}(z) = P_0(z) + P_\tau(z)e^{-z}$ ; // *Normalized quasipolynomial*
- 2: **Input:** `maxOrd`; // *Maximal order*
- 3: **Initialization:** `ord = 0`; // *ord: truncation order of the Taylor expansion  $e^{2x} = \underbrace{1}_{ord=0} + 2x + 2x^2 + \dots$*
- 4: *dominance = false*;
- 5: **While** (not *dominance*) and (`ord`  $\leq$  `maxOrd`) **do**
- 6: Set  $F(x, \omega) = |P_\tau(x + i\omega)|^2 - |P_0(x + i\omega)|^2 T_{ord}(e^{2x})$ ; //  *$T_{ord}(e^{2x})$ : Taylor expansion of  $e^{2x}$  of order = `ord`*
- 7: Set  $G(x, \Omega) = F(x, \sqrt{\Omega})$ ; //  *$G$  is a polynomial*
- 8: Set  $\Omega_k(x)$ ; // *as the  $k$ -th real root of  $G(x, \cdot)$*
- 9: **If**  $\sup_{x \geq 0} \max_k \Omega_k(x) \leq \pi^2$  **then**
- 10: *dominance = true*;
- 11: **End if**
- 12: `ord = ord + 1`;
- 13: **Output:** Frequency bound: If *dominance* is true, then  $|\omega| \leq \pi$  for every root of  $\tilde{\Delta}$  with positive real part;

## 6. ILLUSTRATION EXAMPLE

For the sake of simplicity, case  $e = 0$  is analyzed. Consequently, the upper (unstable) equilibrium becomes  $\mathbf{q}_0 = \mathbf{0}$ . The human mass and height are set to  $m_h = 70$  kg,  $l = 1.7$  m, the wheel radius is  $R = 0.25$  m, the board elevation is  $\hat{h} = 0$ .

Numerical simulations were performed in order to analyze the dynamics of the human body and the balance board for different combinations of control gains. First, the delay was fixed to 0.1 s (Pasma et al., 2017; Zelei et al., 2021) and the

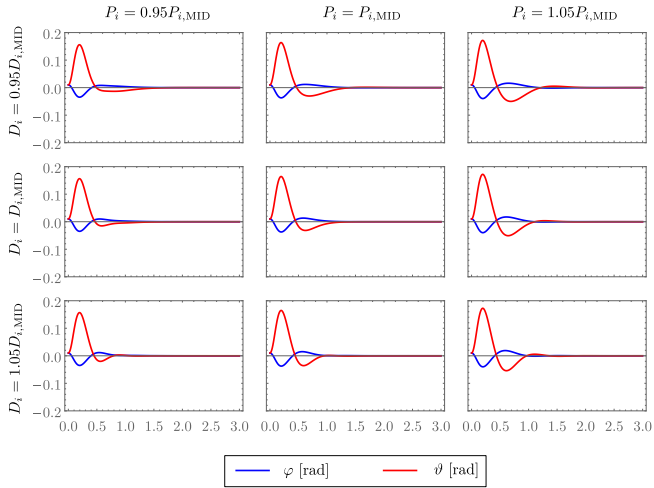


Fig. 2. Two-degree-of-freedom mechanical model of human balancing on a uniaxial rolling balance board in the sagittal plane.

Table 1. Result of MID-based control design

Parameter	Value
$P_{\varphi,MID}$	2854.4
$P_{\vartheta,MID}$	1040.34
$D_{\varphi,MID}$	1082.98
$D_{\vartheta,MID}$	335.735
$s_0$	-1.63368
$a_2$	7.421572

characteristic root  $s_0$  with multiplicity 5 was determined. Next, the corresponding control parameters  $b_0, b_1, b_2, b_3$  and also the control gains  $P_\varphi, P_\vartheta, D_\varphi, D_\vartheta$  were calculated. The results are summarized in Table 1. The control gains obtained by the MID-based control design are denoted by  $P_{i,MID}$  and  $D_{i,MID}$ , where  $i = \varphi, \vartheta$ . The corresponding time history of human body and balance board angle were determined by numerical simulation. Then, numerical simulations determined the control gains  $P_\varphi, P_\vartheta$ , and  $D_\varphi, D_\vartheta$  perturbed by  $\pm 5\%$ . The numerical results can be seen in Fig. 2.

The initial function over  $t \in (-\tau, 0)$  interval was set to the constant value of 0.01 rad both for the human body and the balance board angle. One can observe in the middle subplot of Fig. 2 that the fastest decay rate of the solution is associated with  $P_{\varphi,MID}, P_{\vartheta,MID}, D_{\varphi,MID}, D_{\vartheta,MID}$  gains obtained by the intermediate MID property.

## 7. CONCLUDING REMARKS AND FUTURE WORK

In this paper, we extend the validity of the control-oriented MID property. As a matter of fact, the result in Balogh et al. (2022) applies only for systems with real-rooted open-loop characteristic polynomial. Here, we have employed the MID property to  $P_0(s)$ , a plant not only exhibiting real roots but also complex conjugate roots. Furthermore, a biomechanical application of the MID is considered. A two degree-of-freedom mechanical model of a human balancing on a rolling balance board is analyzed in the sagittal plane. The human body is modeled by an inverted pendulum which connects to the balance board through the ankle joint. The system is stabilized by the ankle torque managed by the central nervous system (CNS). The action of the CNS is modeled by a delayed full state feedback.

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