Stability properties for two coupled reaction-diffusion equations

Mathieu Bajodek, Hugo Lhachemi, Giorgio Valmorbida

Stability properties for two coupled reaction-diffusion equations

Mathieu Bajodek [*], **Hugo Lhachemi** [*], **Giorgio Valmorbida** [*]

[*] *CentraleSupelec, Bat. Bréguet, 3 Rue Joliot Curie, 91190 Gif-Sur-Yvette, France (e-mail: mathieu.bajodek@laas.fr).*

Abstract: This paper studies the stability of the interconnection of two reaction-diffusion equations. We focus on the case where the input and output operators of the interconnection are bounded. Using the spectral decomposition of both equations, we propose a sufficient condition to estimate the exponential stability decay rate of the closed-loop system. This stability test is proposed as constraints of a semidefinite programming. An extension of this condition is also outlined in the form of a Hurwitz criterion. The proposed stability analysis conditions are illustrated with two examples.

1. INTRODUCTION

Reaction-diffusion phenomena appear in many fields such as pharmaceutics or epidemiology. They belong to the class of parabolic partial differential equations (PDEs). A fundamental problem is to characterize how the trajectories behave, especially to rule on the exponential stability of the steady state.

In the literature, as a particular case of the study of linear operators, the stability analysis of linear PDEs is detailed in Pazy (2012). Modal analysis is the most used approach for studying the stability of a single reaction-diffusion equation. Spectral reduction methods have also led to the synthesis of controllers and observers for single linear parabolic PDE, see for instance Russel (1978); Lasiecka and Triggiani (2000); Morris and Levine (2010); Katz and Fridman (2022). Nevertheless, the case of two interconnected PDEs is more involved: the eigenstructures of the associated linear operator cannot be, in general, computed analytically and do not allow us to assess the stability properties of the plant. Furthermore, by summing the Lyapunov functionals of each PDEs, the application of the Lyapunov theorem leads to conservative stability conditions that are restricted to the case where both PDEs are stable (see Mironchenko and Ito (2015)).

For PDEs interconnections of type hyperbolic-hyperbolic in Di Meglio et al. (2013), hyperbolic-parabolic in Karafyllis and Krstic (2018) or parabolic-parabolic in Grüne and Meurer (2022), the stability analysis methods vary. This diversity comes from the different analysis tools and approximation scheme involved. Global exponential stability (GES) can be assessed via frequency analysis using the small-gain theorem as in Karafyllis and Krstic (2018). It can also rely on the use of Lyapunov functionals which include interconnected terms as in Lhachemi and Prieur (2022). In parallel to this last technique, an approximation is performed to obtain numerical solutions. Finite element methods as in Li et al. (1994); Burman et al. (2022), Legendre polynomials approximation as in Baudouin et al.

(2019); Bajodek et al. (2022) or modal decomposition as in Katz and Fridman (2022); Lhachemi and Prieur (2022) can be applied. Then, the application of Lyapunov's theorem leads to linear matrix inequalities (LMIs), which can be solved by linear semidefinite programming (see e.g. Valmorbida et al. (2015); Peet (2021)).

Here, the case of parabolic-parabolic interconnected PDEs is investigated. The proposed methodology consists of

- performing a modal decomposition of the Sturm-Liouville operators;
- projecting the coupling terms on this eigenbasis;
- rewriting the infinite-dimensional interconnected system as a finite-dimensional system coupled with a residual infinite-dimensional dynamics;
- applying Lyapunov's theorem on the finite-dimensional system while evaluating the impact of the residual infinite-dimensional dynamics.

The objective is to derive GES sufficient conditions and provide the exponential decay rate of the solutions.

The paper is structured as follows. Section II introduces the system description and the modal decomposition of the two Sturm-Liouville operators. An equivalent model based on this modal decomposition is proposed. Section III is concerned with the derivation of sufficient stability analysis condition. The first condition, in Theorem 8, is an LMI test and the second one, in Theorem 12, is a Hurwitz test of the approximated state matrix. Lastly, Section IV shows numerically that the coupling of two parabolic PDEs can have a either stabilizing or destabilizing effect.

Notation: Throughout this paper, notations $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^{n \times m}$ and $\mathbb{S}^n_{\succ 0(\prec 0)}$ stand for real numbers, positive numbers, matrices of size $n \times m$ and positive (negative) definite matrices of size n, respectively. Matrices I_n, $M^\top$ and $\mathrm{diag}(u_1, \ldots, u_n)$ correspond to the identity matrix of size n, transpose of matrix M in $\mathbb{R}^{n \times n}$ and diagonal matrix whose diagonal coefficients are $(u_1, \ldots, u_n)$. We also use $\mathrm{vec}(\cdot)$ as the vectorization operation and $\otimes$ as
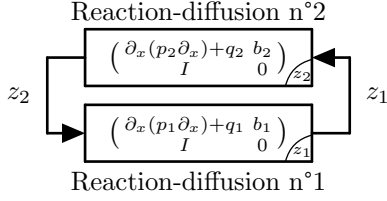
Fig. 1. Block diagram of system (1).

the Kronecker product. Moreover, $\underline{\sigma}(M)$ and $\bar{\sigma}(M)$ stand for the minimal and maximal eigenvalue of the symmetric matrix M. Matrix $M \in \mathbb{R}^{n \times n}$ is said to be μ-Hurwitz if $M + \mu I_n$ is Hurwitz. We introduce some functional spaces for real-values functions defined on $[0, 1]$. Denote by C^0 and L^2 the spaces of continuous and square-integrable functions, respectively. Denote also by C^k the set of functions with the k first derivatives in C^0 and by H^k the k-th order Sobolev space. Lastly, the Euclidian norm $|\cdot|$ and the L^2 norm $\|\cdot\|$ are defined by

$$|\cdot| := \begin{cases} \mathbb{R}^n \mapsto \mathbb{R}_+ \\ u \to \sqrt{u^\top u} \end{cases}, \quad \|\cdot\| := \begin{cases} L^2 \mapsto \mathbb{R}_+ \\ z \to \sqrt{\int_0^1 |z(x)|^2 \mathrm{d}x} \end{cases}.$$

2. REACTION-DIFFUSION SYSTEMS

In this section, we present the system under investigation and propose an equivalent model thanks to a modal based decomposition as in Lhachemi and Prieur (2022).

2.1 Interconnected plant

Consider two reaction-diffusion equations

$$\begin{cases} \partial_t z_i(t,x) = [\partial_x(p_i(x)\partial_x) + q_i(x)] z_i(t,x) + b_i(x) z_{3-i}(t,x), \\ \cos(\theta_{i0}) z_i(t,0) - \sin(\theta_{i0}) \partial_x z_i(t,0) = 0, \\ \cos(\theta_{i1}) z_i(t,1) + \sin(\theta_{i1}) \partial_x z_i(t,1) = 0, \\ z_i(0,x) = z_{i0}(x), \end{cases} \tag{1}$$

for $i \in \{1, 2\}$ and $(t,x) \in \mathbb{R}_+ \times (0,1)$. The initial conditions z_{i0} are in $L^2((0,1); \mathbb{R})$, the scalars θ_{ij} are in $[0, \pi/2]$, the diffusion functions p_i are in $C^1((0,1); \mathbb{R}_+)$ and the both reaction q_i and interconnection b_i functions are in $C^0((0,1); \mathbb{R})$. The plant (1) is represented in Fig. 1.

Remark 1. The use of functions q_1 and q_2 covers the cases where boundary conditions are subject to constant inputs, up to an appropriate change of variables.

Remark 2. The method we are going to propose applies to any couplings of the form $B_i z_{3-i}$ where $B_i \in \mathcal{L}(L^2((0,1); \mathbb{R}))$ is a bounded operator. For a fixed $\xi \in [0, 1]$, unbounded couplings as $b_i(x) z_{3-i}(t, \xi)$ on the dynamics can be addressed as well. Finally, the method can also be extended to n coupled reaction-diffusion equations.

The operator of system (1) is given by

$$\mathcal{A} = \begin{pmatrix} \partial_x(p_1 \partial_x) + q_1 & b_1 \\ b_2 & \partial_x(p_2 \partial_x) + q_2 \end{pmatrix}, \tag{2}$$

on the domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_1) \times \mathcal{D}(\mathcal{A}_2)$, where

$$\mathcal{A}_i = \partial_x(p_i \partial_x) + q_i, \tag{3}$$

$$\mathcal{D}(\mathcal{A}_i) = \{f \in H^2((0,1); \mathbb{R}), \text{ s.t. } {}^{\cos(\theta_{i0})f(0) - \sin(\theta_{i0})f'(0) = 0}_{\cos(\theta_{i1})f(1) + \sin(\theta_{i1})f'(1) = 0}\}.$$

Remark 3. In view of (2) and $-\mathcal{A}_i$ being a Sturm-Liouville operator, classical results on the bounded perturbation of generators of C_0-semigroups (see (Pazy, 2012, Chapter 3)) can be readily invoked for assessing and concluding the well-posedness of system (1), yielding solutions in $C^0(\mathbb{R}_+, L^2((0,1); \mathbb{R}) \times L^2((0,1); \mathbb{R}))$.

Definition 4. The trivial solution of system (1) is GES with exponential decay rate $\mu > 0$ if there exist $\kappa > 0$ such that the solution satisfies

$$\left\| \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \right\| \leq \kappa e^{-\mu t} \left\| \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix} \right\|, \tag{4}$$

for all t in $\mathbb{R}_+$.

In this paper, the global exponential stability (GES) of the trivial solution $(z_1, z_2) = (0, 0)$ of system (1) is studied.

2.2 Spectral decomposition

Let the two operators $\mathcal{A}_i = \partial_x(p_i \partial_x) + q_i$ be defined on the domains $\mathcal{D}(\mathcal{A}_i)$ for $i \in \{1, 2\}$ as in (3). From the properties of Sturm-Liouville operators, the eigenvalues $\{\lambda_{i,n}\}_{n \geq 1}$ of each operator $\mathcal{A}_i$ are real and ordered such that $\lambda_{i,n+1} < \lambda_{i,n}$ for any integer $n \geq 1$. For each $i \in \{1, 2\}$, $\lambda_{i,n}$ are simple, tend to $-\infty$ as n tends to infinity and satisfy (see, e.g., Orlov (2017)):

$$\lambda_{i,n} \leq -\pi^2(n-1)^2 \underline{p}_i + \bar{q}_i, \tag{5}$$

where $\underline{p}_i = \inf_{x \in [0,1]} (p_i(x))$ and $\bar{q}_i = \sup_{x \in [0,1]} (q_i(x))$.

They are associated to normalized eigenvectors $\{\varphi_{i,n}\}_{n \geq 1}$ that form a Hilbert basis of L^2.

For i in $\{1, 2\}$ and any integer $n_i \geq 1$, consider the sequence $\{\varphi_{i,1}, \ldots, \varphi_{i,n_i}\}$ and define the vectors

$$\Phi_i(x) = [\varphi_{i,1}(x) \ \cdots \ \varphi_{i,n_i}(x)]^\top \in \mathbb{R}^{n_i}. \tag{6}$$

Define also

$$\zeta_i(t) = \int_0^1 \Phi_i(x) z_i(t,x) \mathrm{d}x, \ \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \in \mathbb{R}^{n_1 + n_2}, \ \forall t \in \mathbb{R}_+, \tag{7}$$

the vectors of the n_i projections of the state z_i on the eigenfunctions sets $\{\varphi_{i,1}, \ldots, \varphi_{i,n_i}\}$. Define lastly the error state functions

$$\eta_i(t,x) = z_i(t,x) - \Phi_i^\top(x)\zeta_i(t), \ \forall(t,x) \in \mathbb{R}_+ \times (0,1). \tag{8}$$

Proposition 5. For given integers $n_i \geq 1$, the dynamics of system (1) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\zeta(t) = A\zeta(t) + \int_0^1 \begin{bmatrix} \Phi_1(x)b_1(x)\eta_2(t,x) \\ \Phi_2(x)b_2(x)\eta_1(t,x) \end{bmatrix} \mathrm{d}x, \\ \partial_t \eta_i(t,x) = [\partial_x(p_i(x)\partial_x) + q_i(x)]\eta_i(t,x) \\ \quad + b_i(x)\left(\Phi_{3-i}^\top(x)\zeta_{3-i}(t) + \eta_{3-i}(t,x)\right) \\ \quad - \Phi_i^\top(x)\displaystyle\int_0^1 \Phi_i(\xi)b_i(\xi)\left(\Phi_{3-i}^\top(\xi)\zeta_{3-i}(t) + \eta_{3-i}(t,\xi)\right)\mathrm{d}\xi, \\ \cos(\theta_{i0})\eta_i(t,0) - \sin(\theta_{i0})\partial_x \eta_i(t,0) = 0, \\ \cos(\theta_{i1})\eta_i(t,1) + \sin(\theta_{i1})\partial_x \eta_i(t,1) = 0, \end{cases} \tag{9}$$

for $i \in \{1, 2\}$ and $(t,x) \in \mathbb{R}_+ \times (0,1)$, where the matrices are given by

$$A = \begin{bmatrix} \Lambda_1 & B_1 \\ B_2 & \Lambda_2 \end{bmatrix}, \quad \begin{aligned} \Lambda_1 &= \mathrm{diag}(\lambda_{1,1}, \ldots, \lambda_{1,n_1}), \\ \Lambda_2 &= \mathrm{diag}(\lambda_{2,1}, \ldots, \lambda_{2,n_2}), \end{aligned}$$

$$B_1 = \int_0^1 (\Phi_1 b_1 \Phi_2^\top)(x)\mathrm{d}x, \quad B_2 = \int_0^1 (\Phi_2 b_2 \Phi_1^\top)(x)\mathrm{d}x. \tag{10}$$
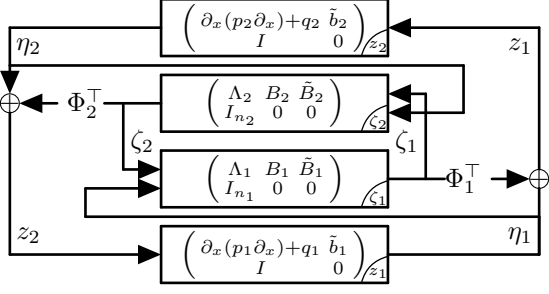
Fig. 2. Block diagram of system (9).

Proof. From one side, the dynamics of the vector ζ are obtained by replacing the state z_i by its decomposition given by (8). From the other side, the dynamics of the error η_i rewrites as follows

$$\partial_t \eta_i(t,x) = \partial_t z_i(t,x) - \Phi_i^\top(x)\frac{\mathrm{d}}{\mathrm{d}t}\zeta_i(t),$$
$$= \partial_t z_i(t,x) - \Phi_i^\top(x)\left(\Lambda_i\zeta_i(t) + \int_0^1 \Phi_i(\xi)b_i(\xi)z_{3-i}(t,\xi)\mathrm{d}\xi\right),$$
$$= [\partial_x(p_i(x)\partial_x) + q_i(x)]z_i(t,x) + b_i(x)z_{3-i}(t,x)$$
$$- \Phi_i^\top(x)\Lambda_i\zeta_i(t) - \Phi_i^\top(x)\int_0^1 \Phi_i(\xi)b_i(\xi)z_{3-i}(t,\xi)\mathrm{d}\xi,$$

which corresponds to the PDE in system (9). Finally, the boundary conditions satisfied by z_i also hold for η_i. ∎

The block diagram of this augmented system is represented in Fig 2, where $\tilde{b}_i z_{3-i} := b_i z_{3-i} - \Phi_i^\top\int_0^1(\Phi_i b_i z_{3-i})(x)\mathrm{d}x$ and $\tilde{B}_i\eta_{3-i} := \int_0^1(\Phi_i b_i \eta_{3-i})(x)\mathrm{d}x$. It can be seen as an equivalent representation of the original system in Fig.1, where the terms z_i are retrieved using (8).

3. STABILITY TEST

In this section, a GES stability condition for system (1) is derived by the use of the modal augmented model (9). In the following, the time argument will be omitted for simplicity reasons.

3.1 Lyapunov functional

For given integers $n_1, n_2 \geq 1$ and a $P \in \mathbb{S}_{\succ 0}^{n_1+n_2}$, consider the following Lyapunov functional

$$\mathcal{V}(\zeta, \eta_1, \eta_2) = \zeta^\top P\zeta + \|[\begin{smallmatrix}\eta_1\\\eta_2\end{smallmatrix}]\|^2, \tag{11}$$

where the augmented state $\zeta = \left[\begin{smallmatrix}\zeta_1\\\zeta_2\end{smallmatrix}\right] \in \mathbb{R}^{n_1+n_2}$ and error state functions $[\begin{smallmatrix}\eta_1\\\eta_2\end{smallmatrix}]$ are defined by (7) and (8), respectively. Similarly to Lhachemi and Prieur (2022), it is based on the n_i first projections of the state on the orthonormal sequences $\{\varphi_{i,n}\}_{n\geq 1}$ and to the sum of the square L^2 norms of each error functions η_i.

Remark 6. Matrix P can be decomposed as three matrices $P = \left[\begin{smallmatrix}P_1 & P_{12}\\P_{12}^\top & P_2\end{smallmatrix}\right]$, where $P_1 \succ 0$, $P_2 \succ 0$ and P_{12} is free. Note that $\zeta^\top P\zeta$ introduces interconnected terms between the two reaction-diffusion equations thanks to the matrices P_{12}. This is in contrast with Kitsos and Fridman (2022), where $P_{12} = 0$.

Along the trajectories of system (1), using the dynamics expressed by (9) and the orthonormality and completeness of the sequences $\{\varphi_{i,n}\}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2) = \zeta^\top PA\zeta + \zeta^\top P\int_0^1\left[\begin{matrix}(\Phi_1 b_1\eta_2)(x)\\(\Phi_2 b_2\eta_1)(x)\end{matrix}\right]\mathrm{d}x$$
$$+ \sum_{n=n_1+1}^\infty \lambda_{1,n}\left(\int_0^1 \varphi_{1,n}(x)\eta_1(x)\mathrm{d}x\right)^2$$
$$+ \int_0^1 \eta_1(x)b_1(x)\left(\Phi_2^\top(x)\zeta_2 + \eta_2(x)\right)\mathrm{d}x \tag{12}$$
$$+ \sum_{n=n_2+1}^\infty \lambda_{2,n}\left(\int_0^1 \varphi_{2,n}(x)\eta_2(x)\mathrm{d}x\right)^2$$
$$+ \int_0^1 \eta_2(x)b_2(x)\left(\Phi_1^\top(x)\zeta_1 + \eta_1(x)\right)\mathrm{d}x.$$

In the following, $\mathcal{V}$ in (11) and its derivatives in (12) will be used to state on the stability of system (1) in the sense of the L^2 norm as in Definition 4.

Remark 7. Note that stability tests could also be given in H^1 norm following similar developments. The Lyapunov functional to take into consideration would be

$$\mathcal{V}_{H^1}(\zeta,\eta_1,\eta_2) = \zeta^\top P\zeta + \sum_{i=1}^2\sum_{n=n_i+1}^\infty \lambda_i\left(\int_0^1 \varphi_{i,n}(x)\eta_i(x)\mathrm{d}x\right)^2. \tag{13}$$

One can refer to Lhachemi and Prieur (2022) for further details in that direction.

3.2 Main theorem

The stability of system (1) is studied exploiting (11)-(12) by formulating inequalities that guarantee the decrease of (11) along the trajectories of (1). These inequalities are cast as linear matrix inequalities (LMIs), affine on the parameter P of (11), which can suitably written as constraints of a semidefinite programming.

Theorem 8. For given orders $n_1, n_2 \geq 1$ and scalar $\mu > 0$, if there exist $P \in \mathbb{S}_{\succ 0}^{n_1+n_2}$ and scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$\left[\begin{matrix}\Gamma(P) + \left[\begin{smallmatrix}\alpha_2\|b_2\Phi_1\|^2 I_{n_1} & 0\\0 & \alpha_1\|b_1\Phi_2\|^2 I_{n_2}\end{smallmatrix}\right] & P \\ P^\top & \left[\begin{smallmatrix}-\beta_1 I_{n_1} & 0\\0 & -\beta_2 I_{n_2}\end{smallmatrix}\right]\end{matrix}\right] \prec 0, \tag{14a}$$

$$\left[\begin{matrix}\left[\begin{smallmatrix}\gamma_{1,n_1}+\beta_2\|b_2\Phi_2\|^2 & \|b_1\|+\|b_2\|\\\|b_1\|+\|b_2\| & \gamma_{2,n_2}+\beta_1\|b_1\Phi_1\|^2\end{smallmatrix}\right] & I_2 \\ I_2 & \left[\begin{smallmatrix}-\alpha_1 & 0\\0 & -\alpha_2\end{smallmatrix}\right]\end{matrix}\right] \prec 0, \tag{14b}$$

with $A \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ as in (10) and

$$\Gamma(P) = PA + A^\top P + 2\mu P,$$
$$\gamma_{1,n_1} = 2(-\pi^2 n_1^2 \underline{p}_1 + \bar{q}_1 + \mu), \quad \gamma_{2,n_2} = 2(-\pi^2 n_2^2 \underline{p}_2 + \bar{q}_2 + \mu).$$

Then, the trivial solution of system (1) is GES with the exponential decay rate μ.

Proof. Firstly, the Lyapunov functional $\mathcal{V}$ defined by (11) satisfies the following inequalities

$$\min\{\underline{\sigma}(P), 1\}\left(|\zeta|^2 + \|\eta_1\|^2 + \|\eta_2\|^2\right) \leq \mathcal{V}(\zeta,\eta_1,\eta_2),$$
$$\mathcal{V}(\zeta,\eta_1,\eta_2) \leq \max\{\bar{\sigma}(P), 1\}\left(|\zeta|^2 + \|\eta_1\|^2 + \|\eta_2\|^2\right). \tag{15}$$

Then, under the assumptions in the theorem, we show that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2) + 2\mu\mathcal{V}(\zeta,\eta_1,\eta_2) \leq 0$ along the trajectories of system (1). To do so, let us focus on each term of $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2)$ in (12). For the terms in η_i with $i \in \{1,2\}$, the ordering of the eigenvalues and the bound (5) give $\lambda_{i,n} < \lambda_{i,n+1} < -\pi^2 n_i^2 \underline{p}_i + \bar{q}_i$ for any $n \geq n_i$ so that

$$\sum_{n=n_i+1}^{\infty} \lambda_{i,n}\left(\int_0^1 \varphi_{i,n}(x)\eta_i(x)\mathrm{d}x\right)^2 \leq (-\pi^2 n_i^2 \underline{p}_i + \bar{q}_i)\|\eta_i\|^2. \tag{16}$$

For the two terms involving η_1 and η_2, applying twice Cauchy-Schwarz's inequality yields

$$\int_0^1 (b_1(x)+b_2(x))\eta_1(x)\eta_2(x)\mathrm{d}x \leq (\|b_1\|+\|b_2\|)\|\eta_1\|\|\eta_2\|, \tag{17}$$

Concerning the crossed term between η_1 and ζ_2, with any positive scalar α_1, Young's inequality leads to

$$\left|2\int_0^1 (\eta_1 b_1 \Phi_2^\top)(x)\zeta_2 \mathrm{d}x\right|$$
$$\leq \frac{1}{\alpha_1}\int_0^1 |\eta_1(x)|^2 \mathrm{d}x + \alpha_1 \int_0^1 |b_1(x)\Phi_2^\top(x)\zeta_2|^2 \mathrm{d}x,$$
$$\leq \frac{1}{\alpha_1}\|\eta_1\|^2 + \alpha_1\|b_1\Phi_2\|^2|\zeta_2|^2. \tag{18}$$

Similarly, with any positive scalar α_2, we obtain

$$2\left|\int_0^1 (\eta_2 b_2 \Phi_1^\top)(x)\zeta_1 \mathrm{d}x\right| \leq \frac{1}{\alpha_2}\|\eta_2\|^2 + \alpha_2\|b_2\Phi_1\|^2|\zeta_1|^2. \tag{19}$$

Furthermore, for $i \in \{1,2\}$ with any positive scalars β_i, the application of Cauchy–Schwarz's inequality leads to

$$\beta_i\left|\int_0^1 (\Phi_i b_i \eta_{3-i})(x)\mathrm{d}x\right|^2 \leq \beta_i\|b_i\Phi_i\|^2\|\eta_{3-i}\|^2. \tag{20}$$

From (12) and (16)-(20), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2) + 2\mu\mathcal{V}(\zeta,\eta_1,\eta_2)$$
$$\leq \xi^\top \begin{bmatrix} \Gamma(P) + \begin{bmatrix} \alpha_2\|b_2\Phi_1\|^2 I_{n_1} & 0 \\ 0 & \alpha_1\|b_1\Phi_2\|^2 I_{n_2} \end{bmatrix} & P \\ P^\top & \begin{bmatrix} -\beta_1 I_{n_1} & 0 \\ 0 & -\beta_2 I_{n_2} \end{bmatrix} \end{bmatrix} \xi$$
$$+ \eta^\top \begin{bmatrix} \gamma_{1,n_1}+\beta_2\|b_2\Phi_2\|^2+\frac{1}{\alpha_1} & \|b_1\|+\|b_2\| \\ \|b_1\|+\|b_2\| & \gamma_{2,n_2}+\beta_1\|b_1\Phi_1\|^2+\frac{1}{\alpha_2} \end{bmatrix}\eta, \tag{21}$$

where $\xi = \begin{bmatrix} \zeta \\ \int_0^1 (\Phi_1 b_1 \eta_2)(x)\mathrm{d}x \\ \int_0^1 (\Phi_2 b_2 \eta_1)(x)\mathrm{d}x \end{bmatrix}$ and $\eta = \begin{bmatrix} \|\eta_1\| \\ \|\eta_2\| \end{bmatrix}$.

If the LMIs (14) hold, then the following inequality holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\zeta,\eta_1,\eta_2) + 2\mu\mathcal{V}(\zeta,\eta_1,\eta_2) \leq 0.$$

Regarding (15) and applying the Lyapunov theorem, it implies that the following inequality holds

$$\left(|\zeta|^2 + \|[\begin{smallmatrix}\eta_1\\\eta_2\end{smallmatrix}]\|^2\right)(t) \leq \frac{\max\{\bar{\sigma}(P),1\}}{\min\{\underline{\sigma}(P),1\}}e^{-2\mu t}\left(|\zeta|^2 + \|[\begin{smallmatrix}\eta_1\\\eta_2\end{smallmatrix}]\|^2\right)(0),$$

Since the augmented state ζ and the errors (η_1,η_2) are related to the state (z_1,z_2) through relations (7) and (8), the orthonormality of the decomposition gives

$$\|z_1(t)\|^2 + \|z_2(t)\|^2 = |\zeta(t)|^2 + \|\eta_1(t)\|^2 + \|\eta_2(t)\|^2. \tag{22}$$

Therefore, the $L^2 \times L^2$ norm of the solution (z_1,z_2) vanishes with an exponential decay rate μ as in Definition 4, which concludes the proof. ∎

Remark 9. In (14) the LMIs have $N_v = \frac{(n_1+n_2)(n_1+n_2-1)}{2} + 4$ variables and $N_c = (n_1+n_2)(n_1+n_2-1)+10$ constraints. Both N_v and N_c depend quadratically on the orders n_1 and n_2. Hence, the computational complexity using current interior-point solver remains polynomial in n_1 and n_2.

Remark 10. For $(i,j) \in \{1,2\}^2$, the quantities $\|b_i\Phi_j\|^2$ are upper bounded by $\|b_i\|^2\|\Phi_j\|^2 = n_j\|b_i\|^2$, by orthonormality of the sequence Φ_j. When Φ_j are unknown, Theorem 8 can be used replacing the coefficients $\|b_i\Phi_j\|^2$ by $n_j\|b_i\|^2$.

Using the previous remark and imposing $n_1 = n_2 = n$, we simplify the stability condition from Theorem 8 as follows.

Corollary 11. For a given integer $n \geq 1$ and scalar $\mu > 0$, if there exist a $P \in \mathbb{S}_{\succ 0}^{2n}$ and scalars $\alpha, \beta > 0$ such that

$$\begin{bmatrix} \Gamma(P) + \alpha n\bar{b}^2 I_{2n} & P \\ P^\top & -\beta I_{2n} \end{bmatrix} \prec 0, \tag{23a}$$

$$\begin{bmatrix} \gamma_n + 2\bar{b} + \beta n\bar{b}^2 & 1 \\ 1 & -\alpha \end{bmatrix} \prec 0, \tag{23b}$$

with $A \in \mathbb{R}^{2n \times 2n}$ as in (10) and

$$\Gamma(P) = PA + A^\top P + 2\mu P,$$
$$\gamma_n = 2\left(-\pi^2 n^2 \min\left\{\underline{p}_1,\underline{p}_2\right\} + \max\left\{\bar{q}_1,\bar{q}_2\right\} + \mu\right),$$
$$\bar{b} = \max\left\{\|b_1\|,\|b_2\|\right\}.$$

Then, the trivial solution of system (1) is GES with the exponential decay rate μ. Furthermore, if the LMIs (23) hold for order $n+1$, then the conditions hold for order n.

In Theorem 8 and Corollary 11, in order to satisfy the LMIs conditions, the matrix A must be μ-Hurwitz (i.e. the real parts of the eigenvalues of matrix A in (10) are smaller than $-\mu$). In the next subsection, this criterion is further explored.

3.3 A simpler criterion for stability

As an extension of Corollary 11, a simple stability test based on the eigenvalues of the approximated matrix A is put forward. It relies on a particular choice of matrix P and scalars α, β with respect to the order n.

Theorem 12. For a given $\mu > 0$, assume that A in (10) is μ-Hurwitz and that the order

$$n > \left|2\bar{b}\bar{\sigma}(P_0)\right|\left|\frac{\max\{\bar{q}_1,\bar{q}_2\} + \mu + \bar{b}}{\pi^2\min\{\underline{p}_1,\underline{p}_2\}}\right|, \tag{24}$$

$$\bar{\sigma}(P_0) := \bar{\sigma}\left(-\mathrm{vec}^{-1}\left((A^\top \otimes I + I \otimes A^\top + 2\mu)^{-1}\mathrm{vec}(I)\right)\right).$$

Then, the trivial solution of system (1) is GES with the exponential decay rate μ.

Proof. Consider first the symmetric matrix P_0 solution of the Lyapunov equation

$$P_0 A + A^\top P_0 + 2\mu P_0 = -I_{2n}.$$

Assuming that A is μ-Hurwitz, the Lyapunov matrix P_0 is given by

$$P_0 = -\mathrm{vec}^{-1}\left((A^\top \otimes I_{2n} + I_{2n} \otimes A^\top + 2\mu)^{-1}\mathrm{vec}(I_{2n})\right). \tag{25}$$

For any $\rho > 0$, taking the variables

$$P = \frac{1}{\bar{\sigma}(P_0)}P_0, \quad \alpha = \frac{\sqrt{\rho}}{\bar{b}}n^{-3/2}, \quad \beta = \frac{1}{\sqrt{\rho}\bar{b}}n^{1/2}, \tag{26}$$

we note by Schur complement that

$$-\frac{1}{\bar{\sigma}(P_0)} + 2\sqrt{\rho}n^{-1/2}\bar{b} < 0, \qquad (27a)$$

$$\gamma_n + 2\bar{b} + 2\frac{1}{\sqrt{\rho}}n^{3/2}\bar{b} < 0. \qquad (27b)$$

imply that the LMIs (23a)-(23b) hold. Otherwise, inequalities (27a)-(27b) are satisfied for orders n such that

$$n > \left(2\bar{b}\bar{\sigma}(P_0)\right)^2 \rho, \qquad (28a)$$

$$n > \left(\frac{\max\{\bar{q}_1, \bar{q}_2\} + \mu + \bar{b}}{\pi^2 \min\{\underline{p}_1, \underline{p}_2\}}\right)^2 \frac{1}{\rho}. \qquad (28b)$$

Assuming now that the order satisfies (24) and selecting

$$\rho = \left|2\bar{b}\bar{\sigma}(P_0)\right|^{-1} \left|\frac{\max\{\bar{q}_1, \bar{q}_2\} + \mu + \bar{b}}{\pi^2 \min\{\underline{p}_1, \underline{p}_2\}}\right|. \qquad (29)$$

we obtain the two conditions (27a)-(27b), linked to LMIs (23a)-(23b) with the particular choice of matrix P and scalars α, β in (26). Corollary 11 yields the result. ∎

The inequality (24) gives an estimation of the minimal order n from which LMIs conditions (23) can be satisfied. Matrix A provide stability properties such as the exponential decay or growth rate as the order n goes to infinity. Similar asymptotic results have been obtained in Katz and Fridman (2022) regarding the synthesis of controllers and observers for a single reaction-diffusion equation.

4. EXAMPLES

In this section, we show that two stable reaction-diffusion systems can result in an unstable interconnection under the effect of the input and output operators. Conversely, even if it is less intuitive, two unstable reaction-diffusion systems can become stable by interconnection.

4.1 System data

Consider the following example

$$\begin{cases} \partial_t \begin{bmatrix} z_1(t,x) \\ z_2(t,x) \end{bmatrix} = \begin{bmatrix} p_1\partial_{xx}+q_1 & b_1\cos(\pi x) \\ b_2\cos(\pi x) & p_2\partial_{xx}+q_2 \end{bmatrix}\begin{bmatrix} z_1(t,x) \\ z_2(t,x) \end{bmatrix}, \\ \partial_x z_1(t,0) = 0, \quad \partial_x z_1(t,1) = 0, \\ \partial_x z_2(t,0) = 0, \quad \partial_x z_2(t,1) = 0, \end{cases} \qquad (30)$$

where p_1, p_2, q_1, q_2 are constant functions and boundary conditions are of Neumann type. In that case, we know that the Sturm-Liouville diagonal operators $-(p_i\partial_{xx} + q_i)$ are associated to the eigenvectors $\{\varphi_{i,n}\}_{n\geq 1}$ and eigenvalues $\{\lambda_{i,n}\}_{n\geq 1}$ with

$$\varphi_{i,n}(x) = \begin{cases} 1, & \text{if } n = 1, \\ \sqrt{2}\cos((n-1)\pi x), & \text{otherwise,} \end{cases} \qquad (31)$$

$$\lambda_{i,n} = -p_i((n-1)\pi)^2 + q_i.$$

We also know that $\bar{b} = \frac{1}{\sqrt{2}}\max\{b_1, b_2\}$. Furthermore, for any integers $n_1 = n_2 = n \geq 1$, the matrix A in (10) is given by

$$A = \begin{bmatrix} \Lambda_1 & b_1 B \\ b_2 B^\top & \Lambda_2 \end{bmatrix}, \begin{array}{l} \Lambda_1 = -p_1\text{diag}(1,\ldots,n^2)\pi^2 + q_1, \\ \Lambda_2 = -p_2\text{diag}(1,\ldots,n^2)\pi^2 + q_2, \end{array}$$
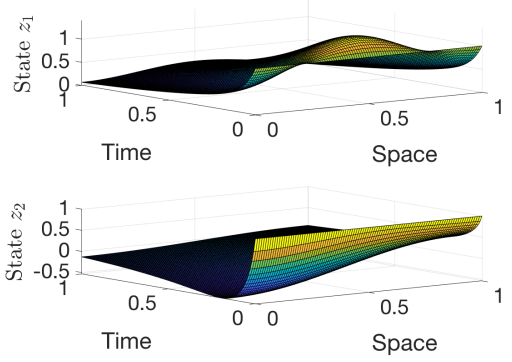

Fig. 3. Solution $(z_1, z_2)(t)$ for $(z_1, z_2)(0) = 1$.

Order n	1	2	5	10	15	20	30
Decay μ	–	0.001	1.19	1.55	1.67	1.73	1.78

Table 1. First example: Maximal exponential decay rate μ for a given n with Corollary 11.

Decay μ	$1e^{-5}$	0.1	0.5	1	1.3	1.5	1.9
Order n	3	3	4	7	11	16	471

Table 2. First example: Minimal order n to certify a given decay rate μ with Theorem 12.

$$\text{with } B = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

4.2 First example: stabilizing case

Consider the system (30) with

$$q_1 = q_2 = 1, \quad p_1 = p_2 = 2, \quad b_1 = -b_2 = 10. \qquad (32)$$

Both equation taken separately are unstable but make a stable system once interconnected (see for instance a stable solution simulated in Fig. 3).

Indeed, applying Theorem 8, the LMIs conditions (14) with $\mu = 1e^{-5}$ are satisfied for orders (n_1, n_2) such that $n_1 \geq 2$ and $n_2 \geq 2$. Applying Corollary 11, the LMIs conditions (14) are satisfied for order $n \geq 2$. The second version provides a gain in terms of computational complexity since it avoids testing pairs (n_1, n_2) by imposing $n_1 = n_2 = n$. With a bisection algorithm, it is even possible to find the maximal decay rate μ. The results are summarized in Table 1.

Moreover, Theorem 12 can be applied with $\mu = 1e^{-5}$ by increasing the order n. For $n \geq 3$, we have

$$n \geq \left|2\bar{b}\bar{\sigma}(P_0)\right| \left|\frac{\max\{q_1, q_2\} + \mu + \bar{b}}{\pi^2\max\{p_1, p_2\}}\right| \simeq 2.55. \qquad (33)$$

which allows us to conclude that the system is GES. For other values of parameter $\mu > 0$, the minimal required orders n are shown in Table 2. It gives a sketch of the order from which the LMIs conditions hold (which is not so pessimistic in that case) and allows to reduce the calculation time using a positivity test instead of LMIs. Lastly, we note that A is Hurwitz by construction from order $n = 2$ and that A is 1.91-Hurwitz when $n \to \infty$.
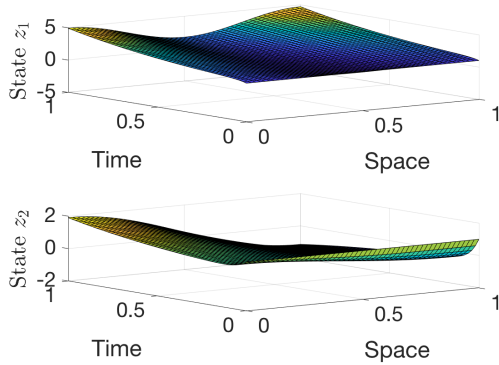
Fig. 4. Solution $(z_1, z_2)(t)$ for $(z_1, z_2)(0) = 1$.

Order n	1	2	5	10	15	20	30
Growth $\lvert\mu\rvert$	2.72	2.65	1.82	1.56	1.48	1.44	1.40

Table 3. Second example: Maximal exponential growth $-\mu$ for a given n with Corollary 11.

4.3 Second example: destabilizing case

Consider the system (30) with

$$q_1 = q_2 = -1, \quad p_1 = p_2 = 2, \quad b_1 = b_2 = 10. \qquad (34)$$

Both equation taken separately are stable but make an unstable system once interconnected. Indeed, for unitary initial conditions the state is diverging as shown in Fig. 4.

With linear matrix inequalities conditions, one cannot conclude on instability of the interconnected system. However, one can an get a bound of the exponential divergence as shown in Table 3. We note that, for any order n, A is unstable and is -1.30-Hurwitz (growth rate of 1.30), which would be certified with Corollary 11 for large values of n.

On both examples, we conjecture whether A is μ-Hurwitz asymptotically implies that the system has a decay rate (when $\mu > 0$) or growth rate (when $\mu < 0$) given by $\lvert\mu\rvert$.

5. CONCLUSIONS

This paper has discussed the stability analysis of coupled reaction-diffusion systems. With the help of the modal decomposition of each subsystem, a sufficient condition of stability has been proposed. Then, a simple test on the eigenvalues of the approximated state matrix has also been highlighted. In the numerical example section, we were able to conclude that stability properties of reaction-diffusion systems can be gained or lost through interconnection.

Future work will investigate interconnections with unbounded input and output operators. It would also be interesting to focus on nonlinear reaction-diffusion equations and determine the basin of attraction of a stable equilibrium, expressed with a finite number of state variables.

REFERENCES

Bajodek, M., Seuret, A., and Gouaisbaut, F. (2022). Stability analysis of an ordinary differential equation interconnected with the reaction–diffusion equation. *Automatica*, 145, 110515.

Baudouin, L., Seuret, A., and Gouaisbaut, F. (2019). Stability analysis of a system coupled to a heat equation. *Automatica*, 99, 195–202.

Burman, E., Durst, R., Fernández, M., and Guzmán, J. (2022). Loosely coupled, non-iterative time-splitting scheme based on Robin-Robin coupling: Unified analysis for parabolic–parabolic and parabolic–hyperbolic problems. *Journal of Numerical Mathematics*. ArXiv:2110.08181.

Di Meglio, F., Vazquez, R., and Krstic, M. (2013). Stabilization of a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input. *IEEE Transactions on Automatic Control*, 58(12), 3097–3111.

Grüne, L. and Meurer, T. (2022). Finite-dimensional output stabilization for a class of linear distributed parameter systems - A small-gain approach. *Systems & Control Letters*, 164, 105237.

Karafyllis, I. and Krstic, M. (2018). Small-gain stability analysis of certain hyperbolic–parabolic PDE loops. *Systems & Control Letters*, 118, 52–61.

Katz, R. and Fridman, E. (2022). Constructive method for finite-dimensional observer-based control of 1-D parabolic PDEs. *Automatica*, 144, 109285.

Kitsos, C. and Fridman, E. (2022). Stabilization of underactuated linear coupled reaction-diffusion PDEs via distributed or boundary actuation. ArXiv:2206.01977.

Lasiecka, I. and Triggiani, R. (2000). *Control theory for partial differential equations*, volume 1. Cambridge University Press.

Lhachemi, H. and Prieur, C. (2022). Stability analysis of reaction-diffusion PDEs coupled at the boundaries with an ODE. *Automatica*, 144, 110465.

Li, N., Steiner, J., and Tang, S. (1994). Convergence and stability analysis of an explicit finite difference method for 2-dimensional reaction-diffusion equations. *The ANZIAM Journal*, 36(2), 234–241.

Mironchenko, A. and Ito, H. (2015). Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach. *SIAM Journal on Control and Optimization*, 53(6), 3364–3382.

Morris, K. and Levine, W. (2010). Control of systems governed by partial differential equations. *The control theory handbook*.

Orlov, Y. (2017). On general properties of eigenvalues and eigenfunctions of a Sturm–Liouville operator: comments on 'ISS with respect to boundary disturbances for 1D parabolic PDEs'. *IEEE Transactions on Automatic Control*, 62, 5970–5973.

Pazy, A. (2012). *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media.

Peet, M. (2021). A Partial Integral Equation (PIE) representation of coupled linear PDEs and scalable stability analysis using LMIs. *Automatica*, 125, 109473.

Russel, D. (1978). Controllability ans stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Review*, 20(4).

Valmorbida, G., Ahmadi, M., and Papachristodoulou, A. (2015). Stability analysis for a class of partial differential equations via semidefinite programming. *IEEE Transactions on Automatic Control*, 61(6), 1649–1654.